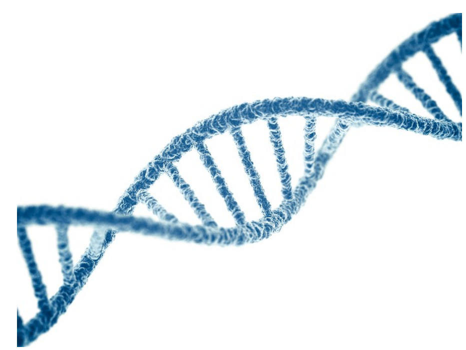
Dynamic Programming II

Longest Common Subsequence

A subsequence of a sequence X is a sequence that can be derived from X by deleting some or no elements without changing the order of the remaining elements.
Example: AURAY is a subsequence of SATURDAY.

Problem: Given two sequences X and Y, determine the longest subsequence that is common to both.
Example: CTAGGATC and TGCCATGCT
Answer: TGATC



Monkey: $\nearrow$
Lower bound: (problem size)
Applications: DNA analysis, plagiarism detection, minimum update/file difference

Longest Common Subsequence

Let $X = x_1 x_2 \ldots x_n$ and $Y = y_1 y_2 \ldots y_m$

Step 1: Define $LCS(i,j) \equiv$ the length of the longest common subsequence of $x_1 \ldots x_i$ and $y_1 \ldots y_j$

Step 2: $LCS(i,j) =$

$$\begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } x_i = y_j \\ \max\{LCS(i, j-1), LCS(i-1, j)\} & \text{otherwise} \end{cases}$$

$$\begin{matrix} x_1 \ldots \boxed{x_i} \\ y_1 \ldots \boxed{y_j} \end{matrix}$$

Are x_i and y_j the same or not?

Longest Common Subsequence

$$LCS(i,j) = \begin{cases} 0 & \text{if } i=0 \text{ or } j=0 \\ LCS(i-1,j-1)+1 & \text{if } x_i = y_j \\ \max\{LCS(i,j-1), LCS(i-1,j)\} & \text{otherwise} \end{cases}$$

LCS	Empty	"t"	"g"	"c"	"c"	"a"	"t"	"g"	"c"	"t"
Empty	0	0	0	0	0	0	0	0	0	0
"c"	0	0	0	1	1	1	1	1	1	1
"t"	0	1	1	1	1	1	2	2	2	2
"a"	0	1	1	1	1	2	2	2	2	2
"g"	0	1	2	2	2	2	2	3	3	3
"g"	0	1	2	2	2	2	2	3	3	3
"a"	0	1	2	2	2	3	3	3	3	3
"t"	0	1	2	2	2	3	4	4	4	4
"c"	0	1	2	3	3	3	4	4	5	5

<u>Answer</u>: TGATC

Matrix Chain Multiplication: Warmup

$$\begin{pmatrix} 0.362486 & 0.349886 & 0.0373537 & 0.532757 \\ 0.704704 & 0.670112 & 0.046316 & 0.989327 \end{pmatrix} \begin{pmatrix} 0.652105 & 0.905593 & 0.758684 \\ 0.257016 & 0.83652 & 0.473145 \\ 0.0355967 & 0.0583746 & 0.242247 \\ 0.611268 & 0.193707 & 0.173584 \end{pmatrix}$$

If we multiply together matrices of size $a \times b$ and $b \times c$, how large will the answer be?

<u>Answer</u>: $a \times c$

If we multiply together matrices of size $a \times b$ and $b \times c$, how many single-register multiplications does it take to get one entry in the answer?

<u>Answer</u>: b

How many total multiplications are necessary to multiply together matrices of size $a \times b$ and $b \times c$?

<u>Answer</u>: abc

Matrix Chain Multiplication

Assume that $M_1 : 10 \times 100, M_2 : 100 \times 5, M_3 : 5 \times 50$

Goal: Compute $M_1 M_2 M_3$

Which is better? $(M_1 M_2) M_3$ or $M_1 (M_2 M_3)$

Matrix multiplication associativity: As far as the answer is concerned, it doesn't matter.

$M_1 M_2 : 10 \cdot 100 \cdot 5 = 5000$ multiplications
$(M_1 M_2) M_3 : 10 \cdot 5 \cdot 50 = 2500$ multiplications
Total: 7500 multiplications

$M_2 M_3 : 100 \cdot 5 \cdot 50 = 25000$ multiplications
$M_1 (M_2 M_3) : 10 \cdot 100 \cdot 50 = 50000$ multiplications
Total: 75000 multiplications

Which is preferable? $(M_1 M_2) M_3$

Matrix Chain Multiplication

Given a sequence of matrix multiplications $M_1 M_2 \ldots M_n$, a *parenthesization* is valid iff the parenthesization could be used to program the ordering into software without running-time ambiguity.

Example: $M_1(M_2M_3)(M_4M_5)$ is invalid.

Example: $M_1((M_2M_3)(M_4M_5))$ is valid.

Problem: Given a sequence of n matrices (of valid sizes) to be multiplied together $M_1 M_2 \ldots M_n$, determine the parenthesization that uses the minimum number of single-register multiplications to get the answer.

Monkey: $\nearrow$

Lower bound: (problem size)

Example: $M_1 : 3 \times 5, M_2 : 5 \times 4, M_3 : 4 \times 1, M_4 : 1 \times 9$

Matrix Chain Multiplication

Step 1: Define $m(i,j) \equiv$ the minimum number of single-register multiplications needed to determine the product $M_i M_{i+1} \ldots M_j$

Step 2: $m(i,j) =$

$$\begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j} m(i,k) + m(k+1,j) + \text{cost} & \text{otherwise} \end{cases}$$

Notice that there must be a last multiplication. Assume that it occurs between M_k and M_{k+1}.

$(M_i M_{i+1} \ldots M_k)(M_{k+1} \ldots M_{j-1} M_j)$

Matrix Chain Multiplication

$M_1 : 3 \times 5, M_2 : 5 \times 4, M_3 : 4 \times 1, M_4 : 1 \times 9$

$$m(i,j) = \begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j} m(i,k) + m(k+1,j) + \text{cost} & \text{otherwise} \end{cases}$$

Step 3:

m	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	0	$k{=}1$?	$k{=}1,2$?	$k{=}1,2,3$?
$i=2$	(undefined)	0	$k{=}2$?	$k{=}2,3$?
$i=3$	(undefined)	(undefined)	0	$k{=}3$?
$i=4$	(undefined)	(undefined)	(undefined)	0

Matrix Chain Multiplication

$M_1 : 3 \times 5, M_2 : 5 \times 4, M_3 : 4 \times 1, M_4 : 1 \times 9$

$$m(i,j) = \begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j} m(i,k) + m(k+1,j) + \text{cost} & \text{otherwise} \end{cases}$$

m	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	0	k=1 60	k=1,2 ?	k=1,2,3 ?
$i=2$	(undefined)	0	k=2 20	k=2,3 ?
$i=3$	(undefined)	(undefined)	0	k=3 36
$i=4$	(undefined)	(undefined)	(undefined)	0

$m(1,2) : k = 1 \to m(1,1) + m(2,2) + 3 \cdot 5 \cdot 4 = 60$

$m(2,3) : k = 2 \to m(2,2) + m(3,3) + 5 \cdot 4 \cdot 1 = 20$

$m(3,4) : k = 3 \to m(3,3) + m(4,4) + 4 \cdot 1 \cdot 9 = 36$

Matrix Chain Multiplication

$M_1 : 3 \times 5, M_2 : 5 \times 4, M_3 : 4 \times 1, M_4 : 1 \times 9$

$$m(i, j) = \begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j} m(i, k) + m(k+1, j) + \text{cost} & \text{otherwise} \end{cases}$$

m	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$i = 1$	0	$k{=}1$ 60	$k{=}1,2$ 35	$k{=}1,2,3$?
$i = 2$	(undefined)	0	$k{=}2$ 20	$k{=}2,3$?
$i = 3$	(undefined)	(undefined)	0	$k{=}3$ 36
$i = 4$	(undefined)	(undefined)	(undefined)	0

$m(1,3) : k = 1 \rightarrow m(1,1) + m(2,3) + 3 \cdot 5 \cdot 1 = 35$

$k = 2 \rightarrow m(1,2) + m(3,3) + 3 \cdot 4 \cdot 1 = 72$

Matrix Chain Multiplication

$M_1 : 3 \times 5, M_2 : 5 \times 4, M_3 : 4 \times 1, M_4 : 1 \times 9$

$$m(i,j) = \begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j} m(i,k) + m(k+1,j) + \text{cost} & \text{otherwise} \end{cases}$$

m	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	0	k=1 60	k=1,2 35	k=1,2,3 ?
$i=2$	(undefined)	0	k=2 20	k=2,3 65
$i=3$	(undefined)	(undefined)	0	k=3 36
$i=4$	(undefined)	(undefined)	(undefined)	0

$m(2,4) : k = 2 \rightarrow m(2,2) + m(3,4) + 5 \cdot 4 \cdot 9 = 216$

$k = 3 \rightarrow m(2,3) + m(4,4) + 5 \cdot 1 \cdot 9 = 65$

Matrix Chain Multiplication

$M_1 : 3 \times 5, M_2 : 5 \times 4, M_3 : 4 \times 1, M_4 : 1 \times 9$

$$m(i, j) = \begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j} m(i, k) + m(k+1, j) + \text{cost} & \text{otherwise} \end{cases}$$

m	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$i = 1$	0	k=1 60	k=1,2 35	k=1,2,3 62
$i = 2$	(undefined)	0	k=2 20	k=2,3 65
$i = 3$	(undefined)	(undefined)	0	k=3 36
$i = 4$	(undefined)	(undefined)	(undefined)	0

$m(1, 4) : k = 1 \to m(1, 1) + m(2, 4) + 3 \cdot 5 \cdot 9 = 200$

$k = 2 \to m(1, 2) + m(3, 4) + 3 \cdot 4 \cdot 9 = 204$

$k = 3 \to m(1, 3) + m(4, 4) + 3 \cdot 1 \cdot 9 = 62$

Answer: $(M_1(M_2M_3))M_4$