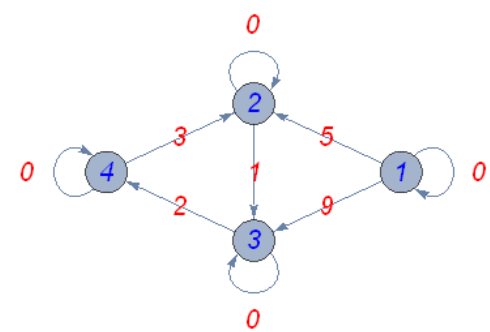
All Pairs Shortest Path

APSP

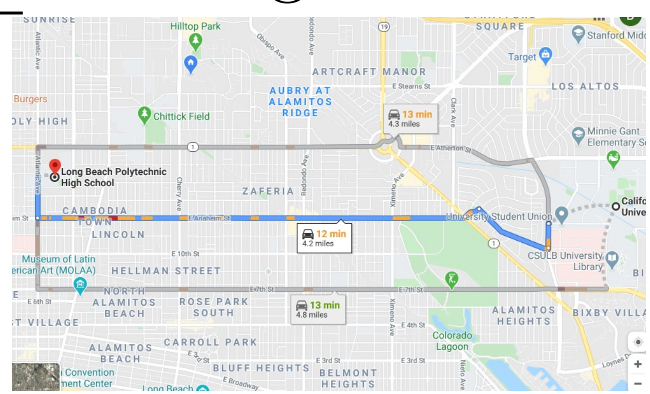
Problem: Assume that all vertices in a given directed graph are numbered from 1 to n. The goal is to build a *data structure* such that, once the data structure has been computed, given any (i, j) query, the time needed to compute the shortest path from i to j is proportional only to the length of the path.

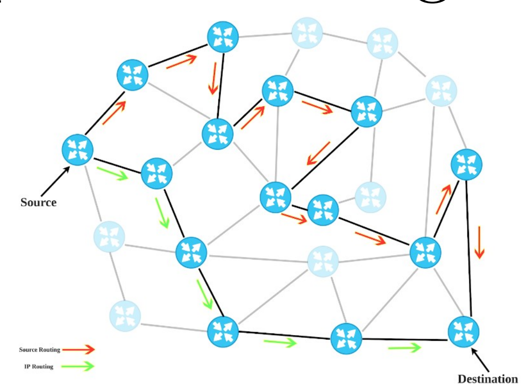


Monkey: $\nearrow$

Lower bound: (problem size)

Applications: driving directions, packet routing



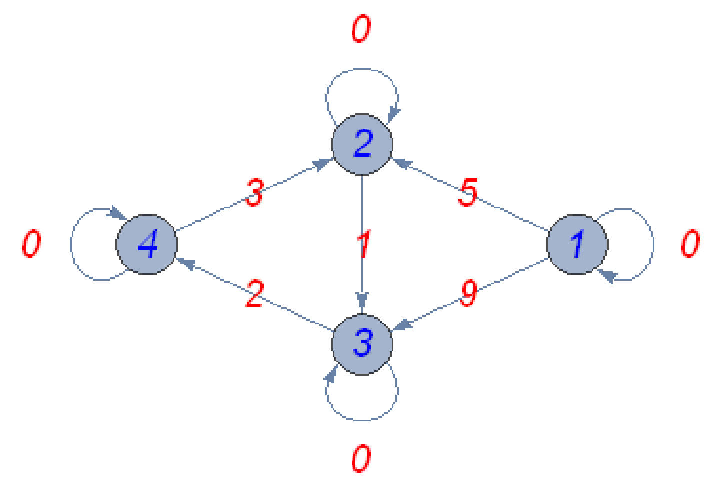


APSP

Step 1: Define $w_{ij} \equiv$ the length of the edge from i to j (possibly ∞). Define $d_{ij}^k \equiv$ the shortest distance from i to j assuming that *exactly* k edges are used in the path

Step 2: $d_{ij}^k =$

$$\begin{cases} w_{ij} & \text{if } k = 1 \\ \min_{v \in V} d_{iv}^{k-1} + w_{vj} & \text{otherwise} \end{cases}$$



Exercises: $d_{1,4}^1 = \infty, d_{1,4}^2 = 11, d_{1,4}^3 = 8$

Observation: Let the vertex v be the last vertex before j on a shortest path from i to j using exactly k edges. Then the path from i to v is a shortest path from i to v using exactly $k-1$ edges.

$$i \overset{k \text{ edges}}{\rightsquigarrow} j \Rightarrow i \overset{k-1 \text{ edges}}{\rightsquigarrow} v \rightarrow j$$

APSP

Step 3: Start with the adjacency matrix of the graph.

$$k = 1 \rightarrow \begin{pmatrix} 0 & 5 & 9 & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 2 \\ \infty & 3 & \infty & 0 \end{pmatrix}$$

$$k = 2 \rightarrow \begin{pmatrix} 0 & 5 & 9 & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 2 \\ \infty & 3 & \infty & 0 \end{pmatrix} \begin{pmatrix} 0 & 5 & 9 & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 2 \\ \infty & 3 & \infty & 0 \end{pmatrix} = \begin{pmatrix} \overset{1}{0} & \overset{2}{5} & \overset{2}{6} & \overset{3}{11} \\ \overset{4}{\infty} & \overset{2}{0} & \overset{3}{1} & \overset{3}{3} \\ \overset{4}{\infty} & \overset{4}{5} & \overset{3}{0} & \overset{4}{2} \\ \overset{4}{\infty} & \overset{4}{3} & \overset{2}{4} & \overset{4}{0} \end{pmatrix}$$

$$k = 3 \rightarrow \begin{pmatrix} 0 & 5 & 6 & 11 \\ \infty & 0 & 1 & 3 \\ \infty & 5 & 0 & 2 \\ \infty & 3 & 4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 5 & 9 & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 2 \\ \infty & 3 & \infty & 0 \end{pmatrix} = \begin{pmatrix} \overset{1}{0} & \overset{2}{5} & \overset{3}{6} & \overset{3}{8} \\ \overset{4}{\infty} & \overset{2}{0} & \overset{3}{1} & \overset{4}{3} \\ \overset{4}{\infty} & \overset{4}{5} & \overset{3}{0} & \overset{4}{2} \\ \overset{4}{\infty} & \overset{4}{3} & \overset{3}{4} & \overset{4}{0} \end{pmatrix}$$

Because the longest possible shortest path between any two vertices in this graph (with positive edges) is at most length $|V| - 1 = 3$, we can stop here.

APSP

<u>Step 3</u>: Start with the adjacency matrix of the graph.

$$k = 1 \to \begin{pmatrix} 0 & \textcolor{red}{5} & 9 & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 2 \\ \infty & 3 & \infty & 0 \end{pmatrix}$$

$$k = 2 \to \begin{pmatrix} 0 & 5 & 9 & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 2 \\ \infty & 3 & \infty & 0 \end{pmatrix} \begin{pmatrix} 0 & 5 & 9 & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 2 \\ \infty & 3 & \infty & 0 \end{pmatrix} = \begin{pmatrix} \overset{\textcolor{blue}{1}}{0} & \overset{\textcolor{blue}{2}}{5} & \overset{\textcolor{blue}{2}}{\textcolor{red}{6}} & \overset{\textcolor{blue}{3}}{11} \\ \overset{\textcolor{blue}{4}}{\infty} & \overset{\textcolor{blue}{2}}{0} & \overset{\textcolor{blue}{3}}{1} & \overset{\textcolor{blue}{3}}{3} \\ \overset{\textcolor{blue}{4}}{\infty} & \overset{\textcolor{blue}{4}}{5} & \overset{\textcolor{blue}{3}}{0} & \overset{\textcolor{blue}{4}}{2} \\ \overset{\textcolor{blue}{4}}{\infty} & \overset{\textcolor{blue}{4}}{3} & \overset{\textcolor{blue}{2}}{4} & \overset{\textcolor{blue}{4}}{0} \end{pmatrix}$$

$$k = 3 \to \begin{pmatrix} 0 & 5 & 6 & 11 \\ \infty & 0 & 1 & 3 \\ \infty & 5 & 0 & 2 \\ \infty & 3 & 4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 5 & 9 & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 2 \\ \infty & 3 & \infty & 0 \end{pmatrix} = \begin{pmatrix} \overset{\textcolor{blue}{1}}{0} & \overset{\textcolor{blue}{2}}{5} & \overset{\textcolor{blue}{3}}{6} & \overset{\textcolor{blue}{3}}{\textcolor{red}{8}} \\ \overset{\textcolor{blue}{4}}{\infty} & \overset{\textcolor{blue}{2}}{0} & \overset{\textcolor{blue}{3}}{1} & \overset{\textcolor{blue}{4}}{3} \\ \overset{\textcolor{blue}{4}}{\infty} & \overset{\textcolor{blue}{4}}{5} & \overset{\textcolor{blue}{3}}{0} & \overset{\textcolor{blue}{4}}{2} \\ \overset{\textcolor{blue}{4}}{\infty} & \overset{\textcolor{blue}{4}}{3} & \overset{\textcolor{blue}{3}}{4} & \overset{\textcolor{blue}{4}}{0} \end{pmatrix}$$

<u>Example</u>: What is the shortest path from 1 to 4?

<u>Answer</u>: $\textcolor{red}{1 \to 2 \to 3 \to 4}$ with length $\textcolor{red}{8}$

APSP

Step 3: Start with the adjacency matrix of the graph.

$$k = 1 \to \begin{pmatrix} 0 & 5 & 9 & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 2 \\ \infty & 3 & \infty & 0 \end{pmatrix}$$

$$k = 2 \to \begin{pmatrix} 0 & 5 & 9 & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 2 \\ \infty & 3 & \infty & 0 \end{pmatrix} \begin{pmatrix} 0 & 5 & 9 & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 2 \\ \infty & 3 & \infty & 0 \end{pmatrix} = \begin{pmatrix} \overset{1}{0} & \overset{2}{5} & \overset{2}{6} & \overset{3}{11} \\ \overset{4}{\infty} & \overset{2}{0} & \overset{3}{1} & \overset{3}{3} \\ \overset{4}{\infty} & \overset{4}{5} & \overset{3}{0} & \overset{4}{2} \\ \overset{4}{\infty} & \overset{4}{3} & \overset{2}{4} & \overset{4}{0} \end{pmatrix}$$

$$k = 3 \to \begin{pmatrix} 0 & 5 & 6 & 11 \\ \infty & 0 & 1 & 3 \\ \infty & 5 & 0 & 2 \\ \infty & 3 & 4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 5 & 9 & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 2 \\ \infty & 3 & \infty & 0 \end{pmatrix} = \begin{pmatrix} \overset{1}{0} & \overset{2}{5} & \overset{3}{6} & \overset{3}{8} \\ \overset{4}{\infty} & \overset{2}{0} & \overset{3}{1} & \overset{4}{3} \\ \overset{4}{\infty} & \overset{4}{5} & \overset{3}{0} & \overset{4}{2} \\ \overset{4}{\infty} & \overset{4}{3} & \overset{3}{4} & \overset{4}{0} \end{pmatrix}$$

Example: What is the shortest path from 4 to 3?

Answer: $4 \to 2 \to 3 \to 3$ with length 4

Application: Arbitrage

Different countries use different forms of currency.

Currency	*USD*	*GBP*	*EUR*	*AUD*
USD(dollars)	1	1.56766	1.25459	0.97901
GBP(BritishPound)	0.637893	1	0.8003	0.624503
EUR(Euro)	0.797067	1.24953	1	0.780336
AUD(Australian)	1.02143	1.60127	1.28149	1

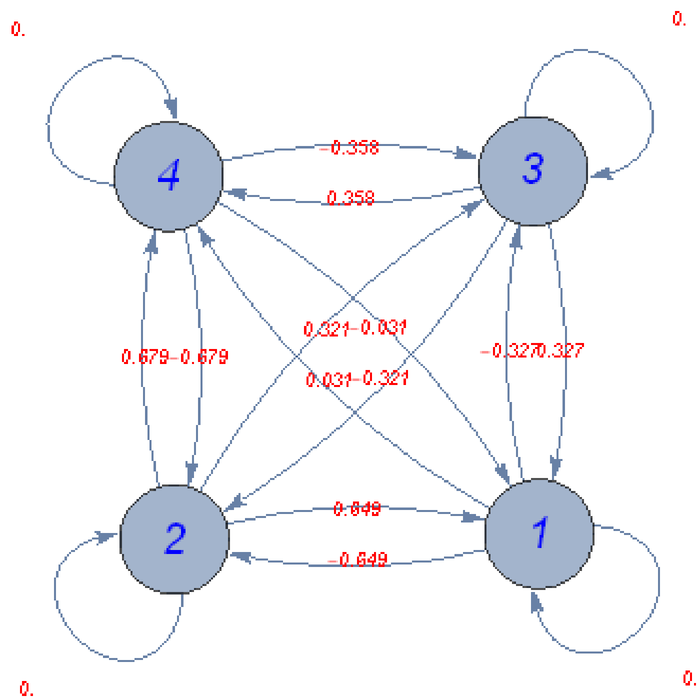
This should be read in the following way: 1 row unit converts to so many column units. For example, 1 USD buys 1.56766 GBP's.

Arbitrage

$Currency$	USD	GBP	EUR	AUD
$USD(dollars)$	1	1.56766	1.25459	0.97901
$GBP(BritishPound)$	0.637893	1	0.8003	0.624503
$EUR(Euro)$	0.797067	1.24953	1	0.780336
$AUD(Australian)$	1.02143	1.60127	1.28149	1

Make a complete graph out of these currencies. The weight of each edge $c_1 \to c_2$ should be $-\log_2$(amount of c_2 that 1 unit of c_1 will buy).

Key: $USD \to 1, GBP \to 2, EUR \to 3, AUD \to 4$



$$\begin{pmatrix} 0 & -0.648613 & -0.327216 & 0.0306045 \\ 0.648614 & 0 & 0.321387 & 0.67922 \\ 0.327227 & -0.321386 & 0 & 0.357833 \\ -0.0305903 & -0.679217 & -0.357822 & 0 \end{pmatrix}$$

Make sure that this graph makes sense: Why should the self-loops be 0? Why should a path from any node back to itself be very close to 0?

Arbitrage

Run the APSP min-plus matrix multiplication algorithm...

$$k = 1 \rightarrow \begin{pmatrix} 0 & -0.648613 & -0.327216 & 0.0306045 \\ 0.648614 & 0 & 0.321387 & 0.67922 \\ 0.327227 & -0.321386 & 0 & 0.357833 \\ -0.0305903 & -0.679217 & -0.357822 & 0 \end{pmatrix}$$

$$k = 2 \rightarrow \begin{pmatrix} \overset{1}{0} & \overset{2}{-0.648613} & \overset{2}{-0.327226} & \overset{4}{0.0306045} \\ \overset{2}{0.648614} & \overset{2}{0} & \overset{3}{0.321387} & \overset{1}{0.679218} \\ \overset{3}{0.327227} & \overset{1}{-0.321386} & \overset{3}{0} & \overset{1}{0.357832} \\ \overset{2}{-0.0306029} & \overset{4}{-0.679217} & \overset{2}{-0.357829} & \overset{4}{0} \end{pmatrix}$$

$$k = 3 \rightarrow \begin{pmatrix} \overset{1}{0} & \overset{2}{-0.648613} & \overset{3}{-0.327226} & \overset{4}{0.0306045} \\ \overset{2}{0.648614} & \overset{2}{0} & \overset{3}{0.321387} & \overset{1}{0.679218} \\ \overset{3}{0.327227} & \overset{1}{-0.321386} & \overset{3}{0} & \overset{1}{0.357832} \\ \overset{2}{-0.0306029} & \overset{4}{-0.679217} & \overset{2}{-0.357829} & \overset{4}{0} \end{pmatrix}$$

...and then it never changes no matter how many steps we take. Why won't the shortest path computations yield any further changes? What is the significance of the fact that the diagonal is 0?

Arbitrage

Occasionally, the banks can make a small mistake...

$Currency$	USD	GBP	EUR	AUD
$USD(dollars)$	1	1.56766	1.25459	0.979015
$GBP(BritishPound)$	0.637893	1	0.8003	0.624503
$EUR(Euro)$	0.797067	1.24953	1	0.780336
$AUD(Australian)$	1.02143	1.60127	1.28149	1

...which leads to the adjacency matrix...

$$\begin{pmatrix} 0 & -0.648613 & -0.327216 & 0.0305971 \\ 0.648614 & 0 & 0.321387 & 0.67922 \\ 0.327227 & -0.321386 & 0 & 0.357833 \\ -0.0305903 & -0.679217 & -0.357822 & 0 \end{pmatrix}$$

Arbitrage

Run the APSP min-plus matrix multiplication algorithm...

$$k = 1 \rightarrow \begin{pmatrix} 0 & -0.648613 & -0.327216 & 0.0305971 \\ 0.648614 & 0 & 0.321387 & 0.67922 \\ 0.327227 & -0.321386 & 0 & 0.357833 \\ -0.0305903 & -0.679217 & -0.357822 & 0 \end{pmatrix}$$

$$k = 2 \rightarrow \begin{pmatrix} \overset{1}{0} & \overset{4}{-0.648619} & \overset{2}{-0.327226} & \overset{4}{0.0305971} \\ \overset{2}{0.648614} & \overset{2}{0} & \overset{3}{0.321387} & \overset{1}{0.679211} \\ \overset{3}{0.327227} & \overset{1}{-0.321386} & \overset{3}{0} & \overset{1}{0.357824} \\ \overset{2}{-0.0306029} & \overset{4}{-0.679217} & \overset{2}{-0.357829} & \overset{4}{0} \end{pmatrix}$$

$$k = 3 \rightarrow \begin{pmatrix} \overset{2}{-5.81158 \times 10^{-6}} & \overset{4}{-0.648619} & \overset{2}{-0.327232} & \overset{4}{0.0305971} \\ \overset{2}{0.648614} & \overset{4}{-5.81158 \times 10^{-6}} & \overset{3}{0.321387} & \overset{4}{0.679211} \\ \overset{3}{0.327227} & \overset{4}{-0.321392} & \overset{3}{0} & \overset{4}{0.357824} \\ \overset{2}{-0.0306029} & \overset{4}{-0.679217} & \overset{3}{-0.357829} & \overset{1}{-5.81158 \times 10^{-6}} \end{pmatrix}$$

What do you notice about the main diagonal?

Arbitrage

Currency	*USD*	*GBP*	*EUR*	*AUD*
USD(dollars)	1	1.56766	1.25459	0.979015
GBP(BritishPound)	0.637893	1	0.8003	0.624503
EUR(Euro)	0.797067	1.24953	1	0.780336
AUD(Australian)	1.02143	1.60127	1.28149	1

$$k = 3 \rightarrow \begin{pmatrix} \overset{2}{-5.81158 \times 10^{-6}} & \overset{4}{-0.648619} & \overset{2}{-0.327232} & \overset{4}{0.0305971} \\ \overset{2}{0.648614} & \overset{4}{-5.81158 \times 10^{-6}} & \overset{3}{0.321387} & \overset{4}{0.679211} \\ \overset{3}{0.327227} & \overset{4}{-0.321392} & \overset{3}{0} & \overset{4}{0.357824} \\ \overset{2}{-0.0306029} & \overset{4}{-0.679217} & \overset{3}{-0.357829} & \overset{1}{-5.81158 \times 10^{-6}} \end{pmatrix}$$

Example: What is the shortest path from 1(USD) to 1(USD)?

Answer: $1 \rightarrow 4 \rightarrow 2 \rightarrow 1$ (which corresponds to USD to AUS to GBP to USD)

Execute the trades:

1 USD $\rightarrow$ 0.979015 AUD $\rightarrow \approx$ 1.56767 GBP $\approx$ 1.00000402828 7552 USD

What just happened?