Review of Asymptotics

The Force

1. constants, sin, cos, $\tan^{-1}$: 83496 and $\sin(100^{n^2})$
2. $(\log n)^{\text{constant}}$: $(\log n)^3$ and $\sqrt{\ln n}$
3. n^{constant} : n^2 and $\sqrt[3]{n}$ and $n^{2+\frac{1}{n}}$
4. constant^n : 2^n
5. $n!$ and n^n : Nobody cares.

The Force is ADDITIVE ONLY!

$n + \log n = O(n)$ but $n \log n \neq O(n)$

Growth of Functions

$$f = O(g) \Rightarrow \exists c > 0 \text{ and } N_c \text{ such that } \forall x \geq N_c,\ f(x) \leq cg(x)$$

$$5x^2 + 20 = O(x^2)$$

$$f = \Omega(g) \Rightarrow \exists c > 0 \text{ and } N_c \text{ such that } \forall x \geq N_c,\ f(x) \geq cg(x)$$

$$\frac{x^3}{3} - 9x = \Omega(x^3)$$

$$f = \Theta(g) \Rightarrow f = O(g) \text{ and } f = \Omega(g)$$

$$3\mathrm{x}^2 - 8x + 2 = \Theta(x^2)$$

$$f = o(g) \Rightarrow \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

$$\mathrm{x}^2 = o(x^3)$$

$$f = \omega(g) \Rightarrow \lim_{x \to \infty} \frac{g(x)}{f(x)} = 0$$

$$\sqrt{x} = \omega(\log x)$$

Exercises

Find a function g so that $f = \Theta(g)$.

- $f(x) = (x^3 + x^2 \log x)(\log x + 1) + (17 \log x + 19)(x^3 + 2)$
- $f(x) = \frac{x^6 - 3x + 12}{x^2 \log x + \pi x \sqrt{x}}$
- $f(x) = \frac{5x^2 \log x + x^3}{\sqrt{x}(\log x)^2 + x^3 \log x}$
- $f(x) = (2^x + x^2)(x^3 + 3^x)$
- $f(x) = x^{2^x} + x^{x^2}$
- $f(x, y) = x^2 + xy + x \log y$

- $g(x) = x^3 \log x$
- $g(x) = \frac{x^4}{\log x}$
- $g(x) = \frac{1}{\log x}$
- $g(x) = 6^x$
- $g(x) = x^{2^x}$
- $g(x, y) = x^2 + xy$

The Log Claim

Claim: If a and b are positive constants, $\log_a n = \Theta(\log_b n)$

Proof: $\log_a n = \frac{\log_b n}{\log_b a} = \left(\frac{1}{\log_b a}\right) \log_b n = \Theta(\log_b n)$

Computer scientists will often ignore the base of logarithms.

Why?

Simple Asymptotics Exercises

All functions are strictly positive and increasing to infinity. True or false:

1. $x^2 = o(x^3)$
2. $x \log x = \omega(x^2)$
3. $2^x = \omega(x^2)$
4. $x^2 = o(x^2)$

1. true
2. false
3. true
4. false

Asymptotics Exercises

All functions are strictly positive and increasing to infinity. True or false:

1. $f = o(g) \Rightarrow 2^f = o(2^g)$
2. $f = \omega(g) \Rightarrow \log(f) = \omega(\log g)$
3. $f_1 = O(g_1)$ and $f_2 = O(g_2) \Rightarrow f_1 + f_2 = O(g_1 + g_2)$
4. $f_1 = o(g_1)$ and $f_2 = o(g_2) \Rightarrow |f_1 - f_2| = o(|g_1 - g_2|)$

1. true 2. false ($f(x) = x^2, g(x) = x$) 3. true
4. false ($f_1(x) = x, g_1(x) = x^2, f_2(x) = x + 1, g_2(x) = x^2 + 1$)

A Useful Asymptotic

Claim: $\log(n!) = \Theta(n \log n)$

Proof: $\log(n!) = \log(n) + \log(n-1) + \ldots + \log(2) + \log(1) \leq$
$\log(n) + \log(n) + \ldots + \log(n) + \log(n) = n \log(n) \Rightarrow$
$\log(n!) = O(n \log n)$

$\log(n!) = \log(n) + \log(n-1) + \ldots + \log(2) + \log(1) \geq$
$\log(\frac{n}{2}) + \log(\frac{n}{2}) + \ldots + \log(\frac{n}{2}) + \log(1) + \ldots + \log(1) =$
$\frac{n}{2} \log(\frac{n}{2}) = \frac{n}{2} \log n - \frac{n}{2} \log 2 \Rightarrow$
$\log(n!) = \Omega(n \log n)$