Introduction to Dynamic Programming

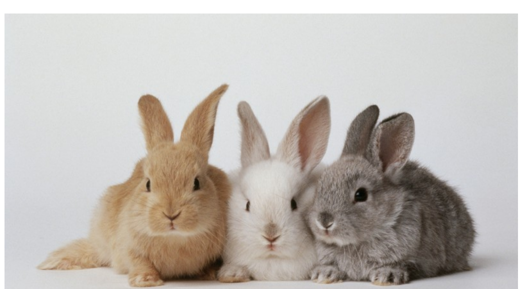
Fibonacci Numbers

Assume that there is a breed of immortal rabbits. A mature pair of these rabbits makes a new pair of baby rabbits every month (but it takes the full month to produce them). Baby rabbits mature after a single month. Assume that we go to an initially rabbit-free island and drop a pair of baby rabbits from a helicopter. Let $f(n)$ be the number of pairs of rabbits during the nth month.

$$f(0) = 0 \text{ and } f(1) = 1$$

$$f(2) = 1, f(3) = 2, f(4) = 3, f(5) = 5, f(6) = 8, \ldots$$

$$f(n) = \overset{\text{pairs from last month}}{f(n-1)} + \overset{\text{baby-producing pairs from last month}}{f(n-2)}$$

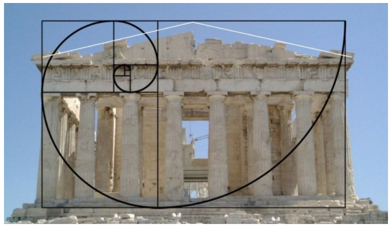


$\lim_{n\to\infty} \frac{f(n)}{f(n-1)} = \varphi \approx 1.618$

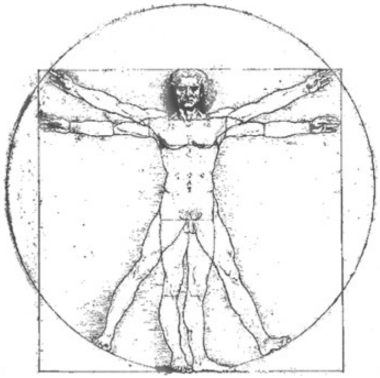
φ is the GOLDEN RATIO.

The Golden Ratio

- One Egyptian pyramid is remarkably close to a “golden pyramid”-the Great Pyramid of Giza (also known as the Pyramid of Cheops or Khufu). Its slope is extremely close to the “golden” pyramid inclination; other pyramids at Giza are also quite close. Whether the relationship to the golden ratio in these pyramids is by design or by accident remains open to speculation. Adding fuel to controversy over the architectural authorship of the Great Pyramid, Eric Temple Bell, mathematician and historian, claimed in 1950 that Egyptian mathematics would not have supported the ability to calculate the slant height of the pyramids, or the ratio to the height, except in the case of the 3:4:5 pyramid, since the 3:4:5 triangle was the only right triangle known to the Egyptians and they did not know the Pythagorean theorem nor any way to reason about irrationals.



- Golden ratio appearances in paintings: The Parthenon’s facade as well as elements of its facade and elsewhere are said by some to be circumscribed by golden rectangles. Leonardo da Vinci’s illustrations of polyhedra in De divina proportione (On the Divine Proportion) and his views that some bodily proportions exhibit the golden ratio have led some scholars to speculate that he incorporated the golden ratio in his paintings. Salvador Dalí explicitly used the golden ratio in his masterpiece, The Sacrament of the Last Supper. The dimensions of the canvas are a golden rectangle. A huge dodecahedron, in perspective so that edges appear in golden ratio to one another, is suspended above and behind Jesus and dominates the composition.

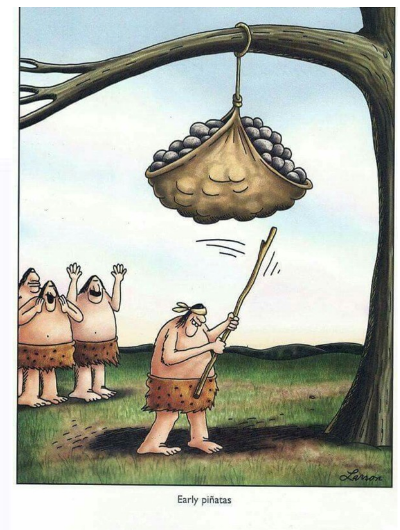


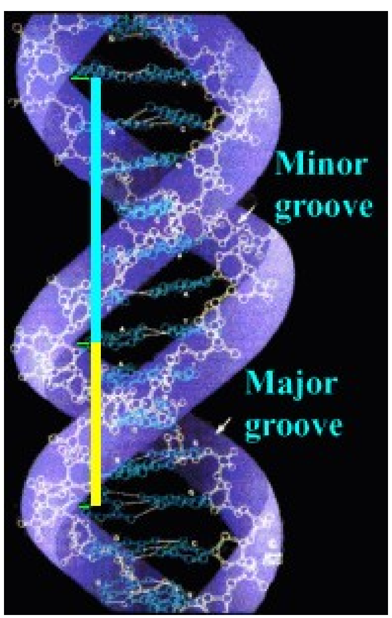
The Golden Ratio

- Some sources claim that the golden ratio is commonly used in everyday design, for example in the shapes of postcards, playing cards, posters, wide-screen televisions, photographs, light switch plates and cars.

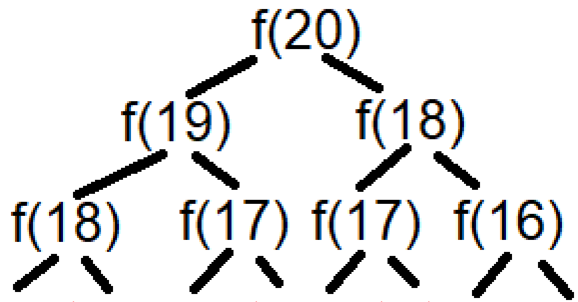
- The golden ratio expressed in the arrangement of parts such as leaves and branches along the stems of plants and of veins in leaves and the skeletons of animals and the branchings of their veins and nerves.

- Since 1991, several researchers have proposed connections between the golden ratio and human genome DNA. In 2010, the journal Science reported that the golden ratio is present at the atomic scale in the magnetic resonance of spins in cobalt niobate crystals.





Computing f(n)



$f(n) = f(n-1) + f(n-2), f(0) = 0, f(1) = 1$

Note that recursion leads to lots of repeated calculations, and the size of the computation tree will be exponential in n.

The dynamic programming approach computes from the bottom up.

$f(0) = 0, f(1) = 1, f(2) = 1, \ldots, f(20) = 6765$

This computation takes time $\sim n$.

0-1 Knapsack

Problem: A thief robs a bank and can carry a maximum of W pounds in his sack before the sack breaks. Assume that there are n items in the vault. Item i has weight w_i and can be sold for a profit of p_i. What items should we steal so as to maximize our total profit?

Example: Sack capacity: 10

Item	Weight	Profit
1	3	4
2	5	6
3	5	5
4	1	3
5	4	5

Answer: 2, 4, 5

Monkey: $\nearrow$

Lower bound: (problem size)

Applications: determining what science experiments go on the space shuttle, what subjects to study for an exam with limited study time, etc.

3 Steps to Dynamic Programming

<u>Step 1</u>: Make a clever definition.

Let $P(i,c)$ be the maximum profit to be made assuming that only items $1 \ldots i$ are available and there is only capacity c remaining in the sack.

<u>Step 2</u>: Find the recursive relation.

$$P(i,c) = \begin{cases} 0 & \text{if } i = 0 \text{ or } c = 0 \\ P(i-1,c) & \text{if } w_i > c \\ \max\{P(i-1,c), P(i-1,c-w_i)+p_i\} & \text{otherwise} \end{cases}$$

3 Steps to Dynamic Programming

Step 3: Fill out a large rectangle.

Sack capacity: 10

Item	Weight	Profit
1	3	4
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3	5	5
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$$P(i,c) = \begin{cases} 0 & \text{if } i = 0 \text{ or } c = 0 \\ P(i-1,c) & \text{if } w_i > c \\ \max\{P(i-1,c), P(i-1,c-w_i)+p_i\} & \text{otherwise} \end{cases}$$

The i variable is on the rows. The c variable is on the columns. Green indicates that we left the item. Blue indicates that we took it.

P	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	4	4	4	4	4	4	4	4
2	0	0	0	4	4	6	6	6	10	10	10
3	0	0	0	4	4	6	6	6	10	10	11
4	0	3	3	4	7	7	9	9	10	13	13
5	0	3	3	4	7	8	9	9	12	13	14

Where is the final answer?

How do we know which bricks we took?

Answer: 2, 4, 5

Good but Not Great

How long did the dynamic programming algorithm take for the 0-1 knapsack problem?

$\sim nW$

What is the input size of the parameter W?

$\log_2 W$

What does that mean about the running time of the algorithm?

$\sim nW = \sim n2^{\log_2 W}$

This is *exponential* in the size of the input!