The Master Method

Master Theorem

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined by

$$T(n) = aT(n/b) + f(n)$$

Then $T(n)$ can be bounded asymptotically as follows:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$
   and if eventually $af(n/b) \leq cf(n)$ for some constant $c < 1$,
   then $T(n) = \Theta(f(n))$.

Example

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$$T(n) = 4T(n/2) + n$$

Step 1: $a = 4, b = 2, f(n) = n$

Step 2: $f(n) = [O, \Theta, \Omega](n^{\log_b a}) \Rightarrow$

$$n = O(n^{2-\epsilon})$$

Step 3: $T(n) = \Theta(n^2)$

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$$T(n) = T(2n/3) + 1$$

Step 1: $a = 1, b = \frac{3}{2}, f(n) = 1$

Step 2: $f(n) = [O, \Theta, \Omega](n^{\log_b a}) \Rightarrow 1 = \Theta(1)$

Step 3: $T(n) = \Theta(\log n)$

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$$T(n) = 3T(n/4) + n \log_2 n$$

Step 1: $a = 3, b = 4, f(n) = n \log_2 n$

Step 2: $f(n) = [O, \Theta, \Omega](n^{\log_b a}) \Rightarrow$

$$nlog_2 n = \Omega(n^{\log_4 3 + \epsilon})$$

$$3\frac{n}{4} \log_2 \frac{n}{4} \leq cn \log_2 n \Rightarrow c = \frac{3}{4} < 1$$

Step 3: $T(n) = \Theta(n \log n)$

Example

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined by

$$T(n) = aT(n/b) + f(n)$$

Then $T(n)$ can be bounded asymptotically as follows:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
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3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$ and if eventually $af(n/b) \leq cf(n)$ for some constant $c < 1$, then $T(n) = \Theta(f(n))$.

$$T(n) = 2T(n/2) + n \log_2 n$$

Step 1: $a = 2, b = 2, f(n) = n \log_2 n$

Step 2: $f(n) = [O, \Theta, \Omega](n^{\log_b a}) \Rightarrow$

$$nlog_2 n = [O, \Theta, \Omega](n^{[1-\epsilon, 1, 1+\epsilon]})$$

You can't use the Master Theorem on this problem.

Why does it work?

$$T(n) = aT(\tfrac{n}{b}) + f(n) \Rightarrow$$
$$T(\tfrac{n}{b}) = aT(\tfrac{n}{b^2}) + f(\tfrac{n}{b}) \Rightarrow$$
$$T(\tfrac{n}{b^2}) = aT(\tfrac{n}{b^3}) + f(\tfrac{n}{b^2}) \Rightarrow \ldots$$

Start again and replace the appropriate T term every time.

$$T(n) = f(n) + aT(\tfrac{n}{b}) \Rightarrow$$
$$T(n) = f(n) + af(\tfrac{n}{b}) + a^2T(\tfrac{n}{b^2}) \Rightarrow$$
$$T(n) = f(n) + af(\tfrac{n}{b}) + a^2f(\tfrac{n}{b^2}) + a^3T(\tfrac{n}{b^3}) \Rightarrow$$
$$T(n) = f(n) + af(\tfrac{n}{b}) + \ldots + a^{k-1}f(\tfrac{n}{b^{k-1}}) + a^kT(\tfrac{n}{b^k})$$
$$T(n) = \sum_{i=0}^{k-1} a^i f(\tfrac{n}{b^i}) + a^kT(\tfrac{n}{b^k})$$

So what is k when we stop?

$$\tfrac{n}{b^k} = 1 \Rightarrow k = \log_b n$$

The second term

$T(n) = \sum_{i=0}^{k-1} a^i f(\frac{n}{b^i}) + a^k T(\frac{n}{b^k})$

If $k = \log_b n$...
$a^k T(\frac{n}{b^k}) = a^{\log_b n} T(\frac{n}{b^{\log_b n}}) = a^{\log_b n} T(1) =$
$n^{\log_b a} T(1) = \Theta(n^{\log_b a})$

The second term is $\Theta(n^{\log_b a})$.
To find $T(n)$, we now need the first term.
But it behaves differently depending on how big $f(n)$ is...

The first term (Case 1)

$$T(n) = \sum_{i=0}^{k-1} a^i f(\frac{n}{b^i}) + a^k T(\frac{n}{b^k})$$

Assume that $f(n) = O(n^{\log_b a - \epsilon})$.

Then $\exists c > 0$ s.t. eventually $f(n) \leq cn^{\log_b a - \epsilon}$.

$$\sum_{i=0}^{k-1} a^i f(n/b^i) \leq c \sum_{i=0}^{k-1} a^i (n/b^i)^{\log_b a - \epsilon} =$$

$$c \sum_{i=0}^{k-1} b^{i\epsilon} a^i n^{\log_b a - \epsilon} / a^i =$$

$$cn^{\log_b a - \epsilon} \sum_{i=0}^{k-1} b^{i\epsilon} \leq cn^{\log_b a - \epsilon} O(b^{\epsilon k}) = O(n^{\log_b a})$$

$$T(n) = O(n^{\log_b a}) + \Theta(n^{\log_b a}) = \Theta(n^{\log_b a})$$

The second case (Case 2)

$T(n) = \sum_{i=0}^{k-1} a^i f(\frac{n}{b^i}) + a^k T(\frac{n}{b^k})$

Assume that $f(n) = \Theta(n^{\log_b a})$.

Recall that $k = \log_b n$.

$\sum_{i=0}^{k-1} a^i f(n/b^i) \sim \sum_{i=0}^{k-1} a^i (n/b^i)^{\log_b a} = kn^{\log_b a} \sim n^{\log_a b} \log n$

$T(n) = \Theta(n^{\log_b a} \log n) + \Theta(n^{\log_b a}) = \Theta(n^{\log_b a} \log n)$

The third case (Case 3)

$T(n) = \sum_{i=0}^{k-1} a^i f(\frac{n}{b^i}) + a^k T(\frac{n}{b^k})$

Assume that $f(n) = \Omega(n^{\log_b a + \epsilon})$

and that eventually $\exists c < 1$ s.t. $af(\frac{n}{b}) \leq cf(n)$.

$\sum_{i=0}^{k-1} a^i f(n/b^i) \leq \sum_{i=0}^{k-1} c^i f(n) \leq \sum_{i=0}^{\infty} c^i f(n) \sim f(n)$

$T(n) = \Theta(f(n)) + \Theta(n^{\log_b a}) = \Theta(f(n))$