The Fundumental Theorem of Calculus

Aiden Wenzel

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The Fundumental Theorem of Calculus is a theorem which links the concepts of differentiation and integration of functions. The Fundumental Theorem of Calculus is broken into two parts.

1 The First Fundamental Theorem of Calculus

1.1 Definition

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t)dt,$$

then $F'(x) = f(x)$ over $[a, b]$. **NEED CITATION FROM LIBRE TEXT**

1.2 Proof

We must prove that $F'(x) = f(x)$.

By the definition of a derivative, and the definition of $F(x)$ given in the theorem,

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \to 0} \frac{1}{h}(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt) \\ F'(x) &= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt \end{aligned} \tag{1}$$

$\frac{1}{h} \int_x^{x+h} f(t)dt$ is in the form of the average value of a function where,

$$f_{avg} = \frac{1}{b-a} \int_b^a f(x)dx$$

By the Average Value Theorem, there is some constant real number c on the interval $[x, x+h]$ such that

$$\frac{1}{h}\int_{x}^{x+h} f(t)dt = f(c)$$

Since c is between x and h, c approahes x as h approaches zero.

$$\lim_{h\to 0} f(c) = \lim_{c\to x} f(c) = f(x)$$

Finally, we can complete our proof

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{1}{h}\int_{x}^{x+h} f(t)dt \\ &= \lim_{h\to 0} f(c) \\ F'(x) &= f(x), \end{aligned} \tag{2}$$

The proof is now complete.

2 The Second Fundamental Theorem of Calculus