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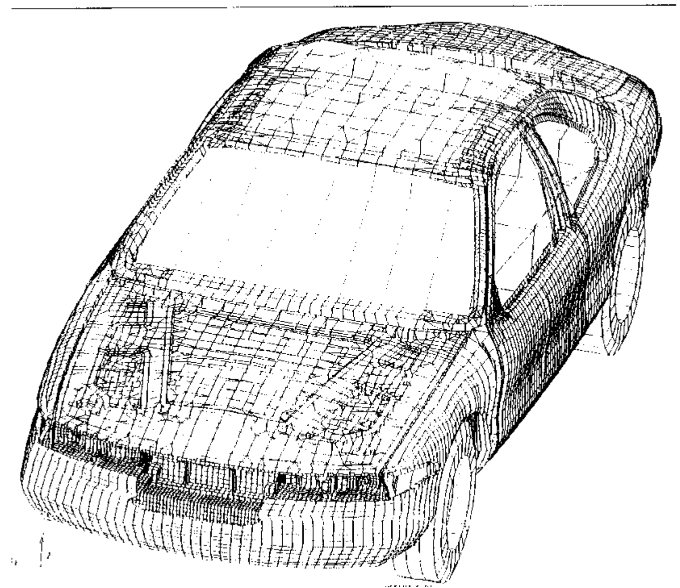
Module 718

Splines in Single and Multivariable Calculus

Yves Nievergelt



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Applications of Calculus to Computer Science, Engineering, and Typography

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INTERMODULAR DESCRIPTION SHEET: UMAP Unit 718

TITLE: Splines in Single and Multivariable Calculus

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MATHEMATICAL FIELD: Introductory single and multivariable calculus

APPLICATION FIELD: Computer science, engineering, typography

TARGET AUDIENCE: Students in a course in single or multivariable calculus.

ABSTRACT: This Module seeks to

- strengthen students' intuition in multidimensional geometry,
- provide exercises that are at a level between mechanical and theoretical, and
- show applications of calculus that are important but that rarely appear in calculus texts.

PREREQUISITES: In general, the ability to ponder a problem that does not come with a canned recipe for its solution. In particular:

- From high-school algebra: two-point and point-slope equations of straight lines, and any algorithm for the solution of small linear systems.
- From calculus: first and second derivatives for linear combinations of integer or fractional powers; some exercises also involve trigonometric functions or integrals.
- From multivariable calculus: the concepts of curves in the plane and in space, and surfaces; some examples and exercises involve Green's Theorem or Stokes's Theorem.

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Table of Contents

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS (UMAP) PROJECT

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications, to be used to supplement existing courses and from which complete courses may eventually be built.

The Project was guided by a National Advisory Board of mathematicians, scientists, and educators. UMAP was funded by a grant from the National Science Foundation and now is supported by the Consortium for Mathematics and Its Applications (COMAP), Inc., a nonprofit corporation engaged in research and development in mathematics education.

Paul J. Campbell	Editor
Solomon Garfunkel	Executive Director, COMAP

1. Introduction

In this Module, we document applications of single and multivariable calculus to the use of splines. We offer exercises that fit readily into existing calculus courses and that

- strengthen students' intuition in multidimensional geometry,
- are at a level between mechanical and theoretical, and
- show applications of calculus that are important but that rarely appear in calculus texts.

In calculus, typical curve- and surface-sketching begins with a formula and ends with computed points in the plane or in space. Yet many practical problems, especially in engineering but also in business and the sciences, require the inverse operation: start from a set of points and fit a curve or a surface through them, subject to prescribed conditions. A common method to fit curves and surfaces to data utilizes splines, which consist of pieces of polynomials or other appropriate functions, patched together into a smooth curve or surface. The resulting curve or surface has a continuous tangent, a continuous curvature, and continuous derivatives up to some specified order.

Applications of splines abound. In engineering, *natural cubic splines* pass through data and minimize the bending energy [Henrici 1982], just as do long, thin, flexible strips of wood bent along the hull of a ship, from which the name "spline" comes. Splines are used to design three-dimensional structures in automobiles [Moylan 1993] and the wings and fuselage of aircraft and spacecraft [Jones 1987], with additional dimensions representing such variables as curvature, air pressure, or stress [Barnhill et al. 1987]. In graphic arts and typography, *complete cubic splines* specify and communicate to printing devices the location, orientation, shape, size, and type of graphic logos and typographical fonts [Knuth 1979; 1980]. In medicine, splines enter into the design of algorithms for computed tomography [Natterer 1986]. In the design of computing hardware and software, other types of splines accomplish the technical tasks of approximating or interpolating special mathematical functions [Kincaid and Cheney 1991, § 6.4; Stoer & Bulirsch 1983, § 2.4; Pulskamp and Delaney 1991; Schelin 1983]. Under the name of "compound transition curves," twice-differentiable splines enabled railroad engineers to smoothe the transition from straight to circular tracks as early as the nineteenth century [Henck 1854; Searles 1882].

The general theory of splines belongs to the realm of numerical analysis, where many texts provide excellent explanations at that level, from the introductory theory of existence, uniqueness, and linear structure [Kincaid and Cheney 1991; Stoer and Bulirsch 1983] to specialized treatises on accurate and efficient automatic computations [de Boor 1978], splines in several dimensions [Chui 1988], and sophisticated theories for demanding applications [Wahba 1990].

However, some applications of splines involve only elementary concepts from the usual calculus courses, such as functions of one or two variables, first and second derivatives, extrema, and small linear systems. We do *not* aim to introduce numerical recipes and analysis of data into calculus. Rather, the examples and exercises that we propose already fit in existing calculus courses, between the usual practice exercises and theoretical proofs of additional theorems. Located between those two extremes of level of difficulty, the material of this Module aims to strengthen students' geometric intuition, challenge them with a variety of points of view on the basic concepts of calculus, and allow them to apply calculus to real problems instead of to fantasy exercises.

For other exercises on splines in calculus, see Janke [1993], which treats Bézier curves.

2. Splines in Single-Variable Calculus

Mathematical "splines" consist of functions defined piecewise by simple functions. They are used in applied mathematics and in engineering to approximate functions that are incompletely known or too complicated for practical computations.

2.1 Piecewise Affine Splines

The examples and exercises in this section merely introduce the concept of *affine spline* and require only elementary calculus and high-school algebra.

Definition 1. An *affine* function consists of the sum of a constant and a linear function. Thus, the graph of an affine function from the real numbers to the real numbers is a non-vertical straight line. (The term "affine" distinguishes such functions from the "linear" functions of linear algebra, which all must pass through the origin.)

Example 1. Income tax codes and tax tables are affine splines [Nievergelt 1989].

The simplest affine spline consists of a polynomial of degree at most one, that is, a straight-line segment, passing through two prescribed endpoints. A general affine spline consists of several consecutive straight-line segments, adjusted to form one continuous function passing through several prescribed points. Thus, the inclusion of a few exercises with affine splines in calculus or high-school algebra reinforces the ideas that a function need not consist of a single formula, but may involve several algebraic formulae and logical tests, and that such functions have practical applications.

To alleviate the need for extensive explanations of background from areas outside of calculus, we give an application to the design of algorithms used by computers to evaluate elementary mathematical functions.

The common types of digital computing hardware can perform only a few basic arithmetic operations, such as comparison, negation, addition, subtraction, multiplication, and division. How can a digital computer compute other mathematical functions, for instance, the square root? One approach replaces the square root by an approximating polynomial or rational function (quotient of two polynomials), called a *spline*, which requires only arithmetic operations.

Basic Affine Spline Problem. Build an affine function passing through two prescribed points.

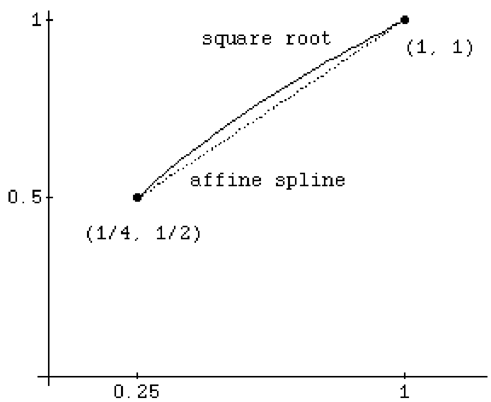

Figure 1. The affine spline (dotted straight-line segment) must pass through $(^1\!/_4, {}^1\!/_2)$ and through $(1, 1)$, as does the square root (solid curve).

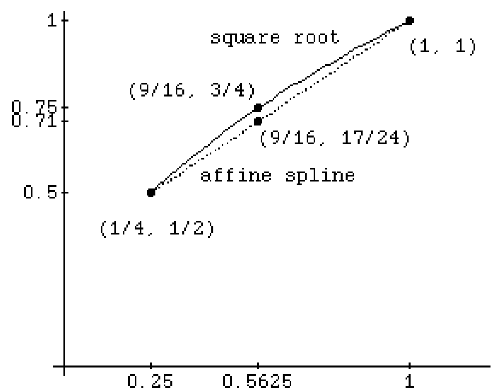

Figure 2. The maximum absolute discrepancy, $^3\!/_4 - {}^{17}\!/_{24} = {}^1\!/_{24}$, between the square root and the affine polynomial occurs at $^9\!/_{16}$.

Example 2. The square-root function passes through the point $(^1\!/_4, {}^1\!/_2)$, and through the point $(1, 1)$. To approximate the square root on the closed interval $[^1\!/_4, 1]$, consider an affine polynomial, that is, a straight line $\ell : [^1\!/_4, 1] \to \mathbb{R}$ that also passes through the points $(^1\!/_4, {}^1\!/_2)$ and $(1, 1)$, as in **Figure 1**. The two-point formula for straight lines immediately gives the formula

$$\ell(x) = 1 + \frac{^1\!/_2 - 1}{^1\!/_4 - 1}\,(x - 1) = \frac{2}{3}\,x + \frac{1}{3}.$$

To verify that ℓ meets the specifications, verify that $\ell(x) = \sqrt{x}$ at each endpoint:

$$\begin{aligned} \ell(^1\!/_4) &= (^2\!/_3)(^1\!/_4) + {}^1\!/_3 = {}^6\!/_{12} = {}^1\!/_2 \\ &= \sqrt{^1\!/_4}, \\ \ell(1) &= (^2\!/_3)(1) + {}^1\!/_3 = {}^3\!/_3 = 1 \\ &= \sqrt{1}. \end{aligned}$$

To test how closely the affine spline ℓ approximates the square root, we can select a few intermediate points and compare the values of the square root with ℓ (see **Table 1**).

Table 1.
Comparisons of the square root with the affine spline ℓ .

x	$25/49$			$9/16$			$16/25$		
$\ell(x)$	$33/49$	$=$	$0.673\ldots$	$17/24$	$=$	$0.708\ldots$	$19/25$	$=$	0.76
$\sqrt{x}$	$5/7$	$=$	$0.714\ldots$	$3/4$	$=$	0.75	$4/5$	$=$	0.8

Table 1 suggests that on the interval $[1/4, 1]$, the affine polynomial ℓ approximates the square root to about one significant digit, as shown in **Figure 2**.

For an analysis that applies to the entire interval $[1/4, 1]$, elementary calculus can determine the maximum of the discrepancy $D(x) := \sqrt{x} - \ell(x)$. To find the local extrema of D on the open interval $]1/4, 1[$ (the open square brackets avoid confusions between the open interval $]0, 1/4[$ and the point $(0, 1/4)$ in the plane), set its first derivative D' equal to zero:

$$
\begin{aligned}
D(x) &= \sqrt{x} - \ell(x) = \sqrt{x} - \left(\frac{2}{3}\,x + \frac{1}{3}\right), \\
D'(x) &= \frac{1}{2\sqrt{x}} - \frac{2}{3}, \\
0 &= \frac{1}{2\sqrt{x}} - \frac{2}{3}, \\
\frac{2}{3} &= \frac{1}{2\sqrt{x}}, \\
2\sqrt{x} &= \frac{3}{2}, \\
x &= \left(\frac{3}{4}\right)^2 = \frac{9}{16}.
\end{aligned}
$$

Thus, the discrepancy D has only one local extremum, at $x = 9/16$. However,

$$D''(x) = -\frac{1}{4}\,x^{-3/2} < 0,$$

whence $D''(9/16) < 0$ and D has a local maximum at $x = 9/16$, where

$$D(9/16) = \sqrt{9/16} - \ell(9/16) = 3/4 - 17/24 = 1/24 = 0.041\,667\ldots .$$

Because $D(1/4) = 0 = D(1)$ by design of ℓ , it follows that for every $x \in [1/4, 1]$,

$$\left|\sqrt{x} - \ell(x)\right| \leq 3/4 - 17/24 = 1/24 < 0.042. \qquad \square$$

To improve the accuracy of the approximation, use an affine spline consisting of *several* straight-line segments, as demonstrated in the following example.

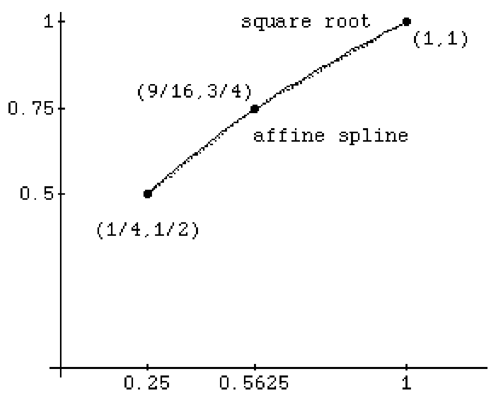

Figure 3. The affine spline (dotted straight-line segments) must pass through $({}^1\!/\!_4, {}^1\!/\!_2)$, $({}^9\!/\!_{16}, {}^3\!/\!_4)$, and $(1,1)$, as does the square root (solid curve).

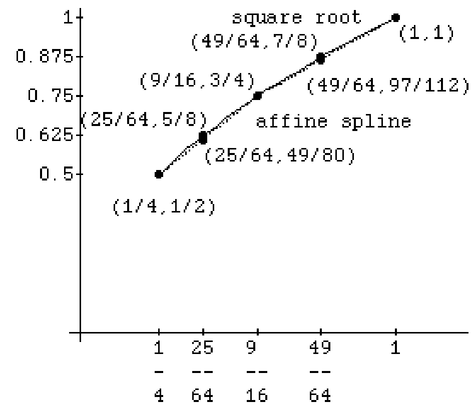

Figure 4. The maximum absolute discrepancy, ${}^5\!/\!_8 - {}^{49}\!/\!_{80} = {}^1\!/\!_{80}$, between the square root and the affine spline, occurs at ${}^{25}\!/\!_{64}$.

Example 3. The square-root function passes through the points $({}^1\!/\!_4, {}^1\!/\!_2)$, $({}^9\!/\!_{16}, {}^3\!/\!_4)$, and $(1,1)$. To approximate the square root on the closed interval $[{}^1\!/\!_4, 1]$, consider an affine spline $\ell : [{}^1\!/\!_4, 1] \to \mathbb{R}$ that joins the points $({}^1\!/\!_4, {}^1\!/\!_2)$ and $({}^9\!/\!_{16}, {}^3\!/\!_4)$ with a straight-line segment ℓ_1 and joins the points $({}^9\!/\!_{16}, {}^3\!/\!_4)$ and $(1,1)$ with another straight-line segment ℓ_2, as in **Figure 3**. The two-point formula for straight lines gives the formulae

$$\ell_1(x) = \frac{4}{5}\,x + \frac{3}{10},$$
$$\ell_2(x) = \frac{4}{7}\,x + \frac{3}{7},$$

$$\ell(x) = \begin{cases} \ell_1(x) & \text{if } {}^1\!/\!_4 \le x \le {}^9\!/\!_{16}, \\ \ell_2(x) & \text{if } {}^9\!/\!_{16} < x \le 1. \end{cases}$$

To verify that ℓ meets the specifications, verify that $\ell(x) = \sqrt{x}$ at ${}^1\!/\!_4$, ${}^9\!/\!_{16}$, and 1:

$$\begin{aligned} \ell({}^1\!/\!_4) &= \ell_1({}^1\!/\!_4) = ({}^4\!/\!_5)({}^1\!/\!_4) + {}^3\!/\!_{10} = {}^1\!/\!_2 \\ &= \sqrt{{}^1\!/\!_4}, \\ \ell({}^9\!/\!_{16}) &= \ell_1({}^9\!/\!_{16}) = ({}^4\!/\!_5)({}^9\!/\!_{16}) + {}^3\!/\!_{10} = {}^3\!/\!_4 \\ &= \sqrt{{}^9\!/\!_{16}}, \\ \ell({}^9\!/\!_{16}) &= \ell_2({}^9\!/\!_{16}) = ({}^4\!/\!_7)({}^9\!/\!_{16}) + {}^3\!/\!_7 = {}^3\!/\!_4 \\ &= \sqrt{{}^9\!/\!_{16}}, \\ \ell(1) &= \ell_2(1) = ({}^4\!/\!_7)(1) + {}^3\!/\!_7 = 1 \\ &= \sqrt{1}. \end{aligned}$$

As in **Example 2**, elementary calculus can determine the maximum of the discrepancy $D(x) = \sqrt{x} - \ell(x)$. To determine the local extrema of D on the open interval $]1/4, 1[$, set its first derivative D' equal to zero separately on each of the two open subintervals $]1/4, 9/16[$ and $]9/16, 1[$. For each subinterval, calculations similar to those in **Example 2** reveal that on the first subinterval, $\sqrt{x} - \ell_1(x)$ reaches a maximum of $1/80 = 0.0125$ at $x = 25/64$, whereas on the second subinterval, $\sqrt{x} - \ell_2(x)$ reaches a maximum of $1/112 \approx 0.008,929$ at $x = 49/64$, as shown in **Figure 4**. Consequently, the larger of the two local maxima shows that on the entire interval $[1/4, 1]$,

$$\left|\sqrt{x} - \ell(x)\right| \leq 1/80 = 0.0125. \qquad \square$$

A fixed maximum discrepancy of $1/80$ has a greater relative effect on the smallest value $\sqrt{1/4} = 1/2$ than on the largest value $\sqrt{1} = 1$. Therefore, another measure of accuracy, the *relative discrepancy*, is defined by

$$R(x) := \frac{|\sqrt{x} - \ell(x)|}{\sqrt{x}}.$$

A spline approximates the square-root function "to n significant decimal digits" if, but only if, $R \leq (1/2) \times 10^{-n}$, which means that the spline and the square-root function differ from each other by at most one half of one unit in the n^{th} digit.

Example 4. In **Example 3**, $1/2 \leq \sqrt{x}$ and $|\sqrt{x} - \ell(x)| \leq 1/80$. Consequently,

$$R(x) = \frac{|\sqrt{x} - \ell(x)|}{\sqrt{x}} \leq \frac{1/80}{1/2} = \frac{1}{40} = 0.025 < 0.05.$$

This inequality means that the relative discrepancy remains less than one half in the first decimal place, which means that the affine spline ℓ approximates the square root to one correct significant digit. $\square$

Example 5. To get an upper bound on the relative discrepancy R, **Example 4** used the maximum of the absolute discrepancy D obtained in **Example 3**. A smaller upper bound results from a direct application of calculus to R. For the functions considered here, $\sqrt{x} \geq \ell(x)$, whence $|\sqrt{x} - \ell(x)| = \sqrt{x} - \ell(x)$:

$$\begin{aligned} R(x) &= \frac{|\sqrt{x} - \ell(x)|}{\sqrt{x}} \\ &= \frac{\sqrt{x} - (ax + c)}{\sqrt{x}} \\ &= 1 - a\sqrt{x} - \frac{c}{\sqrt{x}}; \end{aligned}$$

$$\begin{aligned}
R'(x) &= \frac{-a}{2\sqrt{x}} + \frac{c}{2x\sqrt{x}};, \\
0 &= \frac{-a}{2\sqrt{x}} + \frac{c}{2x\sqrt{x}}, \\
0 &= \frac{-a}{2} + \frac{c}{2x}, \\
x &= c/a; \\
R(c/a) &= 1 - a\sqrt{c/a} - c/\sqrt{c/a} \\
&= 1 - 2\sqrt{ac}.
\end{aligned}$$

Because $R({}^1\!/_4) = 0 = R(1)$ and $R \geq 0$, the result just obtained means that the relative discrepancy R reaches its maximum value of $1 - 2\sqrt{ac}$ at $x = c/a$. For **Example 3**, this means that

$$\begin{aligned}
\frac{|\sqrt{x} - \ell_1(x)|}{\sqrt{x}} &= \frac{|\sqrt{x} - (({}^4\!/_5 x) + {}^3\!/_{10})|}{\sqrt{x}} \\
&\leq 1 - 2\sqrt{({}^4\!/_5)({}^3\!/_{10})} = 0.020\,204\,103\ldots \\
\frac{|\sqrt{x} - \ell_2(x)|}{\sqrt{x}} &= \frac{|\sqrt{x} - (({}^4\!/_7 x) + {}^3\!/_7)|}{\sqrt{x}} \\
&\leq 1 - 2\sqrt{({}^4\!/_7)({}^3\!/_7)} \\
&= 0.010\,256\,682\ldots.
\end{aligned}$$

The maximum of these two numbers shows that $R(x) < 0.021$ for every $x \in [{}^1\!/_4, 1]$. □

Another method to improve the accuracy of the approximation uses a spline consisting of several pieces of cubic polynomials, as explained in the next subsection.

Exercises

1. Determine the affine polynomial that *interpolates* (passes through the same points as) the cube-root function at the two abscissae ${}^1\!/_8$ and 1. Then compute the maximum absolute discrepancy between that affine polynomial and the cube root on the interval $[{}^1\!/_8, 1]$.

2. Determine the affine spline, consisting of two straight-line segments, that interpolates the cube-root function at the three abscissae ${}^1\!/_8$, ${}^{27}\!/_{64}$, and 1. Then compute the maximum absolute discrepancy between that affine spline and the cube root on the interval $[{}^1\!/_8, 1]$.

3. Verify that if an affine function ℓ has slope a and vertical intercept c, so that $\ell(x) = ax+c$, then the discrepancy function D defined by $D(x) = \sqrt{x} - \ell(x)$ has only one local extremum, which is a local maximum and occurs at $x = 1/(4a^2)$, where $D\left(1/(4a^2)\right) = [1/(4a)] - c$.

4. Verify that if an affine function ℓ has slope a and vertical intercept c, so that $\ell(x) = ax + c$, then the discrepancy function D defined by $D(x) = \sqrt[3]{x} - \ell(x)$ has only one local extremum, which is a local maximum and occurs at $x = (3a)^{-3/2}$, where $D\left((3a)^{-3/2}\right) = [2/(3\sqrt{3a})] - c$.

5. Determine the affine spline, consisting of five straight-line segments, that interpolates the square-root function at the six abscissae

$$x_0 = \frac{1}{4}, \quad x_1 = \frac{81}{256}, \quad x_2 = \frac{25}{64}, \quad x_3 = \frac{9}{16}, \quad x_4 = \frac{49}{64}, \quad x_5 = 1.$$

Then compute the maximum relative discrepancy between that affine spline and the square root on the interval $[{}^1\!/\!_4, 1]$. Would the spline suit a two-digit calculator as an algorithm for the square root?

6. Determine the affine polynomial that interpolates the sine function at the two abscissae 0 and $\pi/8$. Then compute the maximum absolute discrepancy between that affine polynomial and the sine on the interval $[0, \pi/8]$.

7. This exercise demonstrates how to prepare *basis splines* that serve to build all splines on a given set of nodes, for example, $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$. With algebra and logic, establish formulae for the following three affine splines.

a) The affine spline $\ell^{(0)}$ must take the values $\ell^{(0)}(0) = 1$, $\ell^{(0)}(1) = 0$, and $\ell^{(0)}(2) = 0$.

b) The affine spline $\ell^{(1)}$ must take the values $\ell^{(1)}(0) = 0$, $\ell^{(1)}(1) = 1$, and $\ell^{(1)}(2) = 0$.

c) The affine spline $\ell^{(2)}$ must take the values $\ell^{(2)}(0) = 0$, $\ell^{(2)}(1) = 0$, and $\ell^{(2)}(2) = 1$.

d) Verify that for each triple of values (y_0, y_1, y_2), the affine spline $\ell := y_0\ell^{(0)} + y_1\ell^{(1)} + y_2\ell^{(2)}$ takes the prescribed values $\ell(0) = y_0$, $\ell(1) = y_1$, and $\ell(2) = y_2$.

2.2 Complete Cubic Splines

The simplest kind of cubic spline consists of one cubic polynomial, which has degree at most three, with prescribed values and prescribed slopes at two points. A general cubic spline consists of several pieces of cubic polynomials, joined to form a twice differentiable function. Thus, the inclusion of a few exercises with cubic splines in calculus reinforces the idea that one function need not consist of a single formula, but may involve several algebraic formulae and logical tests, that some problems involve not curve sketching, but building a function subject to geometric specifications, and that such functions and problems have practical applications.

Basic Spline Problem. Build a cubic polynomial passing through two prescribed points with prescribed slopes.

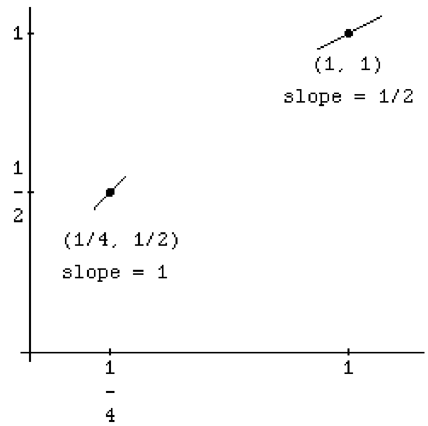

Figure 5. The cubic polynomial sought must pass through $(1/4, 1/2)$ with slope 1, and through $(1,1)$ with slope $1/2$.

Example 6. The square-root function passes through the point $(1/4, 1/2)$ with slope 1, and through the point $(1,1)$ with slope $1/2$. To approximate the square root on the closed interval $[1/4, 1]$, consider a cubic polynomial $p : [1/4, 1] \to \mathbb{R}$ that also passes through the point $(1/4, 1/2)$ with slope 1, and through the point $(1,1)$ with slope $1/2$, as in **Figure 5**. Instead of the usual expansion of p about the origin, $p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$, expand p and its derivative p' about the left-hand endpoint, in powers of $(x - 1/4)$, with yet unknown coefficients u, v, w, and y:

$$\begin{array}{rclllllll} p(x) & = & y & + & w(x - 1/4) & + & v(x - 1/4)^2 & + & u(x - 1/4)^3, \\ p'(x) & = & & & w & + & 2v(x - 1/4) & + & 3u(x - 1/4)^2. \end{array}$$

Setting $x := 1/4$ shows that p must satisfy the following two conditions (the notation $x := 1$ assigns the new value 1 to x, as opposed to $=$, which relates two previously defined quantities):

$$\begin{array}{rclrcl} p(1/4) = & 1/2 : & & y & = & 1/2, \\ p'(1/4) = & 1 : & & w & = & 1, \end{array}$$

which reveals the first two coefficients, $y = p(1/4) = 1/2$ and $w = p'(1/4) = 1$. Substituting these values and setting $x := 1$ produces two equations for the last two coefficients, v and u:

$$\begin{array}{rlllllllll} p(1) & = 1 : & 1/2 & + & 1(1 - 1/4) & + & v(1 - 1/4)^2 & + & u(1 - 1/4)^3 & = 1, \\ p'(1) & = 1/2 : & 1 & + & 2v(1 - 1/4) & + & 3u(1 - 1/4)^2 & & & = 1/2. \end{array}$$

Rearranging terms gives the linear system

$$\begin{cases} v(3/4)^2 + u(3/4)^3 = 1 - 1/2 - 3/4 = -1/4, \\ 2v(3/4) + 3u(3/4)^2 = 1/2 - 1 = -1/2. \end{cases}$$

Solving this system via Gaussian elimination or with any computing device yields $v = -2/3$ and $u = 8/27$. Consequently,

$$p(x) = \frac{1}{2} + 1(x - 1/4) - \frac{2}{3}(x - 1/4)^2 + \frac{8}{27}(x - 1/4)^3.$$

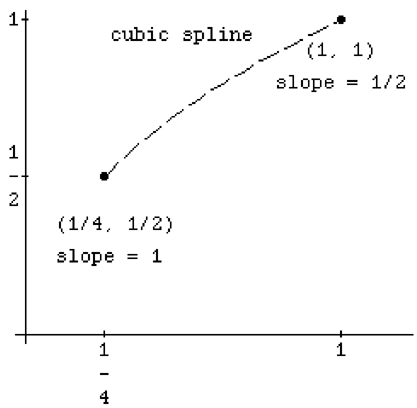

Figure 6. The cubic polynomial p passes through $(1/4, 1/2)$ with slope 1, and through $(1,1)$ with slope $1/2$.

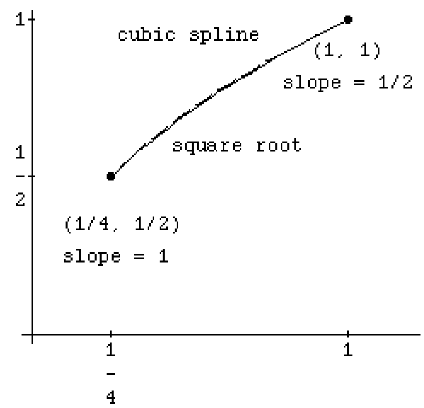

Figure 7. The cubic polynomial p (dashed curve) approximates the square root (solid curve).

Figure 6 shows that p meets the specifications, and **Figure 7** shows how p approximates the square root. To test the closeness of the fit, substituting the intermediate point $x := 9/16$ gives

$$\begin{aligned} p(9/16) &= 0.756\,438\,078\,704\ldots \\ \sqrt{9/16} &= 0.75. \end{aligned} \qquad \square$$

Remark 1. The cubic spline p of **Example 6** approximates the square root only on the interval $[1/4, 1]$, but it can also serve to approximate the square root of every positive real number, as follows. Consider first a positive real number x in the open interval $]0, 1/4[$. (The open square brackets avoid confusions between the open interval $]0, 1/4[$ and the point $(0, 1/4)$ in the plane.) Then a positive integer k exists for which $4^k x \in [1/4, 1]$. Consequently,

$$\sqrt{x} = 2^{-k} 2^k \sqrt{x} = 2^{-k}\sqrt{4^k x} \approx 2^{-k} p(4^k x).$$

Thus, multiply x by 4 as many times (k times) as necessary so that $1/4 \leq 4^k x \leq 1$, compute $p(4^k x)$, which denotes the value of p at $4^k x$, and then multiply $p(4^k x)$ by 2^{-k}.

Similarly, if $x > 1$, then divide x by a suitable power of 4, so that $4^{-k}x \in [1/4, 1]$, and then approximate $\sqrt{x}$ by

$$\sqrt{x} = 2^k \sqrt{4^{-k} x} \approx 2^k p(4^{-k} x). \qquad \square$$

Example 7. If $x := 25 > 1$, then $4^{-1}x = {}^{25}/_4$ (still greater than one), $4^{-2}x = {}^{25}/_{16}$ (still greater than one), but $4^{-3}x = {}^{25}/_{64} \in [{}^1/_4, 1]$. Consequently,

$$\sqrt{25} = 2^3\sqrt{4^{-3} \times 25} = 2^3\sqrt{{}^{25}/_{64}} \approx 2^3 p({}^{25}/_{64}) = 2^3 \times 0.628\,265\ldots \approx 5.026,$$

which approximates $\sqrt{25} = 5$ to two significant digits. Such an approximation would suffice for a one-digit calculator, which would display the same value, 5, for $\sqrt{25}$ and for the approximation $5.0\ldots$. However, p would not approximate the square root accurately enough for a two-digit calculator, which would display different values for $x = {}^9/_{16}$: $p({}^9/_{16}) \approx 0.76 \neq 0.75 = \sqrt{{}^9/_{16}}$.

□

We can achieve greater accuracy with cubic splines by subdividing the interval $[{}^1/_4, 1]$ and using two pieces of cubic polynomials, one piece on each subinterval. Though not necessarily recommended for practice in calculus courses, the following example may still serve to *illustrate* the general concept of cubic spline. Readers interested primarily in calculus may skip to the exercises at the end of the present subsection.

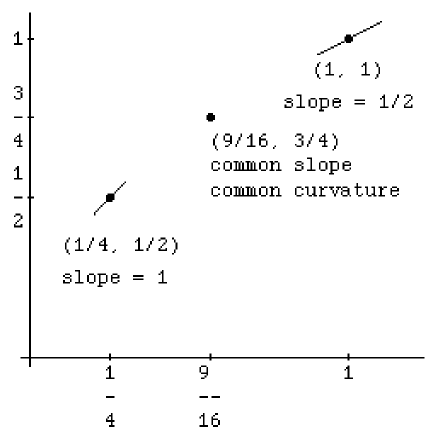

Figure 8. The cubic polynomial p_0 must pass through $({}^1/_4, {}^1/_2)$ with slope 1, while p_1 must pass through $(1, 1)$ with slope ${}^1/_2$, and both p_0 and p_1 must pass through $({}^9/_{16}, {}^3/_4)$ with common first and second derivatives.

Example 8. The square-root function passes through the three data points in **Figure 8**:

$$\begin{cases} (x_0, y_0) = ({}^1/_4, {}^1/_2) & \text{with slope } y_0' = 1, \\ (x_1, y_1) = ({}^9/_{16}, {}^3/_4), & \\ (x_2, y_2) = (1, 1) & \text{with slope } y_2' = {}^1/_2. \end{cases}$$

The reason for not using the slope at the intermediate point will become apparent shortly. To interpolate the square root, connect these data points by two pieces of cubic polynomials, $p_0 : [{}^1/_4, {}^9/_{16}] \to \mathbb{R}$ and $p_1 : [{}^9/_{16}, 1] \to \mathbb{R}$, with yet unknown coefficients: on $[{}^1\!/_4, {}^9\!/_{16}]$,

$$\begin{array}{rcl} p_0(x) &=& y_0 + w_0(x - {}^1\!/_4) + v_0(x - {}^1\!/_4)^2 + u_0(x - {}^1\!/_4)^3, \\ p_0'(x) &=& w_0 + 2v_0(x - {}^1\!/_4) + 3u_0(x - {}^1\!/_4)^2, \\ p_0''(x) &=& + 2v_0 + 6u_0(x - {}^1\!/_4)\ , \end{array}$$

whereas on $[{}^9\!/_{16}, 1]$,

$$\begin{array}{rcl} p_1(x) &=& y_1 + w_1(x - {}^9\!/_{16}) + v_1(x - {}^9\!/_{16})^2 + u_1(x - {}^9\!/_{16})^3, \\ p_1'(x) &=& w_1 + 2v_1(x - {}^9\!/_{16}) + 3u_1(x - {}^9\!/_{16})^2, \\ p_1''(x) &=& + 2v_1 + 6u_1(x - {}^9\!/_{16})\ . \end{array}$$

As in **Example 8**, the expansions about the left-hand endpoints will reduce the calculations necessary for the yet unknown coefficients to meet the following specifications. The first polynomial, p_0, must pass through $({}^1\!/_4, {}^1\!/_2)$ with slope 1, whereas the second polynomial, p_1, must pass through $(1, 1)$ with slope ${}^1\!/_2$. Moreover, both polynomials must join at the intermediate point, $({}^9\!/_{16}, {}^3\!/_4)$, as shown in **Figure 8**. Furthermore, they must meet "smoothly" there, which means that their first and second derivatives—which are also as yet unknown—must equal each other at that juncture: $p_0'({}^9\!/_{16}) = p_1'({}^9\!/_{16})$ and $p_0''({}^9\!/_{16}) = p_1''({}^9\!/_{16})$. Consequently, the cubic polynomials p_0 and p_1 must satisfy the following eight equations:

$$\begin{array}{rcl} p_0({}^1\!/_4) &=& {}^1\!/_2, \\ p_0'({}^1\!/_4) &=& 1, \\ p_0({}^9\!/_{16}) &=& {}^3\!/_4, \\ p_1({}^9\!/_{16}) &=& {}^3\!/_4, \\ p_1(1) &=& 1, \\ p_1'(1) &=& {}^1\!/_2, \\ p_0'({}^9\!/_{16}) &=& p_1'({}^9\!/_{16}), \\ p_0''({}^9\!/_{16}) &=& p_1''({}^9\!/_{16}). \end{array}$$

Written out explicitly, these eight equations become the linear system

$$\left\{\begin{array}{lllllllll} y_0 & & & & & & & = & {}^1\!/_2, \\ & & w_0 & & & & & = & 1, \\ y_0 & + & w_0({}^9\!/_{16} - {}^1\!/_4) & + & v_0({}^9\!/_{16} - {}^1\!/_4)^2 & + & u_0({}^9\!/_{16} - {}^1\!/_4)^3 & = & {}^3\!/_4, \\ y_1 & & & & & & & = & {}^3\!/_4, \\ y_1 & + & w_1(1 - {}^9\!/_{16}) & + & v_1(1 - {}^9\!/_{16})^2 & + & u_1(1 - {}^9\!/_{16})^3 & = & 1, \\ & & w_1 & + & 2v_1(1 - {}^9\!/_{16}) & + & 3u_1(1 - {}^9\!/_{16})^2 & = & {}^1\!/_2, \\ & & w_0 & + & 2v_0({}^9\!/_{16} - {}^1\!/_4) & + & 3u_0({}^9\!/_{16} - {}^1\!/_4)^2 & = & w_1, \\ & & & & 2v_0 & + & 6u_0({}^9\!/_{16} - {}^1\!/_4) & = & 2v_1. \end{array}\right.$$

Thanks to the expansions about the left-hand endpoints, the first, second, and fourth equations immediately show that $y_0 = {}^1\!/_2$, $w_0 = 1$, and $y_1 = {}^3\!/_4$.

Substituting these values into the remaining five equations, and passing w_1 and $2v_1$ to the left-hand sides of the last two equations, will produce a smaller system of five equations. Solving such a linear system by hand may consume an unacceptable amount of time, but the routines for matrix algebra on many computing devices will produce decimal approximations of the coefficients within seconds, with results in **Table 2**.

Table 2.

Coefficients of the complete cubic spline for **Example 8.**

p_0:	$y_0 = 0.5$	$w_0 = 1$	$v_0 = -0.83619047619$	$u_0 = 0.62780952381$
p_1:	$y_1 = 0.75,$	$w_1 = 0.66130952381$	$v_1 = -0.247619047618$	$u_1 = 0.0964042759945.$

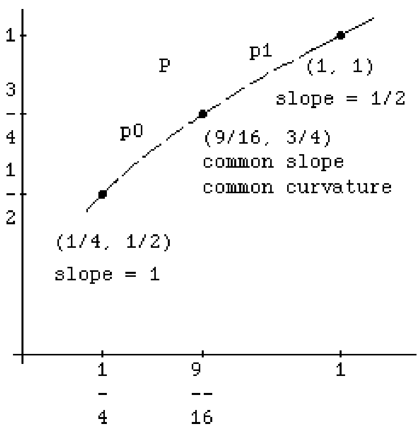

Figure 9. The cubic polynomial p_0 passes through $(1/4, 1/2)$ with slope 1, while p_1 passes through $(1, 1)$ with slope $1/2$, and both p_0 and p_1 pass through $(9/16, 3/4)$ with common first and second derivatives.

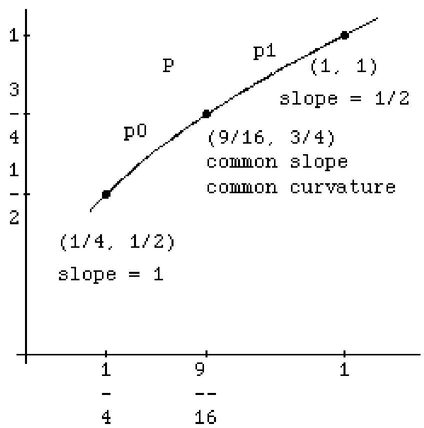

Figure 10. The cubic spline P, consisting of the two pieces of cubic polynomials p_0 and p_1, passes through $(1/4, 1/2)$ with slope 1, through $(1, 1)$ with slope $1/2$, and through $(9/16, 3/4)$ as does the square root (solid curve).

The two pieces of polynomials p_0 and p_1 combine into *a single* function $P : [1/4, 1] \to \mathbb{R}$ displayed in **Figures 9** and **10**, called a *cubic spline*, in the following manner: If $1/4 \leq x \leq 9/16$, then, to within the accuracy of the solution just obtained,

$$\begin{aligned} P(x) &= p_0(x) \\ &= 0.5 + 1(x - 1/4) - 0.83619047619(x - 1/4)^2 + 0.62780952381(x - 1/4)^3; \end{aligned}$$

whereas if $9/16 \leq x \leq 1$, then

$$\begin{aligned} P(x) &= p_1(x) \\ &= 0.75 + 0.66130952381(x - 9/16) - 0.247619047618(x - 9/16)^2 \\ &\qquad + 0.0964042759945(x - 9/16)^3. \end{aligned}$$

With such numerical results, we should conduct some verifications to test for gross errors. First, observe that the three coefficients found earlier, $y_0 = {}^1\!/_2$, $w_0 = 1$, and $y_1 = {}^3\!/_4$, all have the correct values. Second, we may verify by substitution that the other five equations also hold. (Be aware, however, that numerical rounding in a calculator or computer may occasionally start from erroneous answers and still reproduce the right-hand sides exactly. Methods to detect and correct such errors belong to the realm of numerical analysis.) Third, we may plot p_0 on the interval $[{}^1\!/_4, {}^9\!/_{16}]$ and p_1 on the interval $[{}^9\!/_{16}, 1]$ and verify visually that the polynomials pass through the given data points with the specified slopes, and with the same slope and curvature at the juncture at $x_1 = {}^9\!/_{16}$. Unfortunately, such verifications do *not* prove the answers to be right: they *might* only reveal any gross errors.

The particular example under consideration here also allows for another type of verification. Because the polynomials p_0 and p_1 interpolate the square root with common slopes at the endpoints, as shown in **Figure 10**, we may expect p_0 and p_1 to remain "close" to the square root between the data points. For instance, at ${}^{25}\!/_{64} \in [{}^1\!/_4, {}^9\!/_{16}]$ and at ${}^{49}\!/_{64} \in [{}^9\!/_{16}, 1]$, the polynomials take the values in **Table 3**, displayed with the corresponding square roots for comparison:

Table 3.

Comparison of the square root with the interpolating cubic spline P.

$P({}^{25}\!/_{64}) = p_0({}^{25}\!/_{64})$	$=$	$0.625\,834\ldots$,	$P({}^{49}\!/_{64}) = p_1({}^{49}\!/_{64})$	$=$	$0.874\,919\ldots$,
$\sqrt{{}^{25}\!/_{64}}$	$=$	0.625,	$\sqrt{{}^{49}\!/_{64}}$	$=$	0.875.

Notice that p_0 and p_1 seem to remain closer to the square root than the single piece of cubic polynomial p in **Example 6**, which would give the values

$$p({}^{25}\!/_{64}) \quad = \quad 0.628\,265\ldots, \qquad\qquad p({}^{49}\!/_{64}) \quad = \quad 0.878\,997\ldots.$$

Observe also that p_0 and p_1 meet with a common slope $p_0'({}^9\!/_{16}) = p_1'({}^9\!/_{16}) = w_1 = 0.66130952381\ldots$ at the intermediate point $({}^9\!/_{16}, {}^3\!/_4)$, but that their common slope differs from the slope $\sqrt{}'$ of the square root there:

$$\sqrt{}'({}^9\!/_{16}) = \frac{1}{2\sqrt{{}^9\!/_{16}}} = 0.666\ldots \neq 0.661\ldots = w_1.$$

(The notation $\sqrt{}'$ represents the first derivative of the square root $\sqrt{}$, with $\sqrt{}'(x) = 1/(2\sqrt{x})$, just as f' represents the first derivative of a function f.)

$\square$

Remark 2. There are other methods of interpolation and approximation. For instance, one could require p_0 and p_1 to have the same slope as the square root at their juncture, but then they would not have a common second derivative there. The choice of a method of approximation depends upon the application at hand. In the preceding examples, the choice of a common second derivative came not from a goal of approximating the square root, but from our purpose, to illustrate the general concept of cubic spline. □

Definition 2. Consider a positive integer n, and consider $n+1$ distinct reals numbers $x_0, \dots, x_n$, called *nodes* and arranged in increasing order,

$$x_0 < x_1 < x_2 < \cdots < x_k < x_{k+1} < \cdots < x_{n-1} < x_n.$$

A *cubic spline* is a function

$$S : [x_0, x_n] \to \mathbb{R},$$

which on each interval $[x_k, x_{k+1}]$ (for each $k \in \{0, \dots, n-1\}$) coincides with a piece of a cubic polynomial $s_k : [x_k, x_{k+1}] \to \mathbb{R}$ of the form

$$s_k(x) = y_k + w_k(x - x_k) + v_k(x - x_k)^2 + u_k(x - x_k)^3,$$

and such that at each node x_{k+1}, the two consecutive cubic polynomials s_k and s_{k+1} that meet at that node have the same value, the same first derivative, and the same second derivative:

$$\begin{aligned} s_k(x_{k+1}) &= s_{k+1}(x_{k+1}), \\ s'_k(x_{k+1}) &= s'_{k+1}(x_{k+1}), \\ s''_k(x_{k+1}) &= s''_{k+1}(x_{k+1}). \end{aligned}$$

Remark 3. Just as the slope at a point $(x, f(x))$ on a differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ is defined as the slope of the straight line tangent to f at $(x, f(x))$, the *curvature* of a twice differentiable function is defined as the curvature (the reciprocal of the radius) of the circle that "osculates" f at $(x, f(x))$, which means that the circle touches the function there with the same first and second derivatives. As the first derivative $f'(x)$ measures the slope of f at x, the first and second derivatives together express the curvature of f at x through the formula

$$\text{curvature at } x = \frac{f''(x)}{(1 + f'(x)^2)^{3/2}},$$

a formula that we may take for granted from calculus. Consequently, if two polynomials p_0 and p_1 have the same slope at some point x, they also have the same curvature there if, but only if, they have the same second derivative: $p''_0(x) = p''_1(x)$. Thus, cubic splines have a continuous curvature, a characteristic required by some applications, for instance, the design of airfoils in the aerospace industry [Jones 1987]. □

Exercises with one piece of a cubic polynomial

8. a) Determine the coefficients y, w, v, and u of a cubic polynomial $q:\ [1/8, 1] \to \mathbb{R}$ with

$$q(x) = y + w(x - 1/8) + v(x - 1/8)^2 + u(x - 1/8)^3$$

so that q passes through $(1/8, 1/2)$ with slope $4/3$, and through $(1, 1)$ with slope $1/3$, as does the cube root.

b) Compute $q(27/64)$ and $\sqrt[3]{27/64}$.

c) Plot the polynomial q and the cube root on the interval $[1/8, 1]$.

9. a) Determine the coefficients y, w, v, and u of the piece cubic polynomial $s:\ [0, \pi/8] \to \mathbb{R}$ with

$$s(t) = y + w(t - 0) + v(t - 0)^2 + u(t - 0)^3$$

so that s has the same height and the same slope as the sine function, with argument expressed in radians, at $t_0 := 0$ and at $t_1 := \pi/8$.

b) Compute $s(\pi/12)$ and $\sin(\pi/12)$.

c) Plot the polynomial s and the sine on the interval $[0, \pi/8]$.

Exercises on the use of splines in designing algorithms

10. Assume that you have a function $C:\ [1/8, 1] \to \mathbb{R}$, that approximates the cube root on the interval $[1/8, 1]$. For instance, C may be the function q in **Exercise 8**. Explain how to use such an approximation C to approximate the cube root of any real number. Test your explanations by approximating the cube root of -27.

11. Assume that you have a function $S:\ [0, \pi/8] \to \mathbb{R}$, that approximates the sine on the interval $[0, \pi/8]$. For instance, S may be the function s in **Exercise 9**. Explain how to use such an approximation S to approximate the sine of any real number. Test your explanations by approximating the sine of $\pi/3$.

Exercises on features of splines with one piece of a polynomial

12. Let (x_0, y_0) and (x_1, y_1) represent two points in the plane, with $x_0 \neq x_1$, and let y'_0 and y'_1 denote two real numbers. Prove that exactly one cubic polynomial p exists that passes through (x_0, y_0) with slope y'_0 and through (x_1, y_1) with slope y'_1.

13. Let (x_0, y_0) and (x_1, y_1) represent two points in the plane, with $x_0 \neq x_1$, and consider the particular case where the two prescribed slopes coincide with the slope of the straight-line segment joining the two points, so that

$$y_0' = \frac{y_1 - y_0}{x_1 - x_0} = y_1'.$$

Prove that the cubic polynomial p that passes through the two points with the prescribed slopes coincides with the straight line joining both points.

Exercises with two pieces of two cubic polynomials

14. a) Determine the coefficients of two pieces, each a cubic polynomial, $q_0 : [1/8, 27/64] \to \mathbb{R}$ and $q_1 : [27/64, 1] \to \mathbb{R}$, with

$$\begin{aligned} q_0(x) &= y_0 + w_0(x - 1/8) + v_0(x - 1/8)^2 + u_0(x - 1/8)^3, \\ q_1(x) &= y_1 + w_1(x - 27/64) + v_1(x - 27/64)^2 + u_1(x - 27/64)^3, \end{aligned}$$

so that q_0 and q_1 form a function $Q : [1/8, 1] \to \mathbb{R}$ with

$$Q(x) := \begin{cases} q_0(x) & \text{if } 1/8 \leq x < 27/64, \\ q_1(x) & \text{if } 27/64 \leq x \leq 1, \end{cases}$$

which has the same height as the cube root at $x_0 := 1/8$, $x_1 := 27/64$, and $x_2 := 1$, the same slope as the cube root at the endpoints, $x_0 = 1/8$ and $x_2 = 1$, and such that $q_0'(27/64) = q_1'(27/64)$ and $q_0''(27/64) = q_1''(27/64)$.

b) Compute $Q(8/27)$, $\sqrt[3]{8/27}$, $Q(64/125)$, and $\sqrt[3]{64/125}$.

15. a) Determine the coefficients of two pieces of cubic polynomials, $s_0 : [0, \pi/16] \to \mathbb{R}$ and $s_1 : [\pi/16, \pi/8] \to \mathbb{R}$, with

$$\begin{aligned} s_0(t) &= y_0 + w_0(t - 0) + v_0(t - 0)^2 + u_0(t - 0)^3, \\ s_1(t) &= y_1 + w_1(t - \pi/16) + v_1(t - \pi/16)^2 + u_1(t - \pi/16)^3, \end{aligned}$$

so that s_0 and s_1 form a function $S : [0, \pi/8] \to \mathbb{R}$ with

$$S(t) := \begin{cases} s_0(t) & \text{if } 0 \leq t < \pi/16, \\ s_1(t) & \text{if } \pi/16 \leq t \leq \pi/8, \end{cases}$$

which has the same height as the sine at $t_0 := 0$, $t_1 := \pi/16$, and $t_2 := \pi/8$, and the same slope as the sine at the endpoints, $t_0 = 0$ and $t_2 = \pi/8$, and such that $s_0'(\pi/16) = s_1'(\pi/16)$ and $s_0''(\pi/16) = s_1''(\pi/16)$.

b) Compute $S(\pi/24)$, $\sin(\pi/24)$, $S(\pi/12)$, and $\sin(\pi/12)$.

16. This exercise demonstrates how to prepare basis splines that may then serve to build all splines on a given set of nodes, for example, $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$. With algebra and logic, establish formulae for the following three cubic splines.

a) The cubic spline $p^{(0)}$ must take the values $p^{(0)}(0) = 1$, $p^{(0)}(1) = 0$, and $p^{(0)}(2) = 0$, with second derivatives $p^{(0)\prime\prime}(0) = 0$ and $p^{(0)\prime\prime}(2) = 0$.

b) The cubic spline $p^{(1)}$ must take the values $p^{(1)}(0) = 0$, $p^{(1)}(1) = 1$, and $p^{(1)}(2) = 0$, with second derivatives $p^{(1)\prime\prime}(0) = 0$ and $p^{(1)\prime\prime}(2) = 0$.

c) The cubic spline $p^{(2)}$ must take the values $p^{(2)}(0) = 0$, $p^{(2)}(1) = 0$, and $p^{(2)}(2) = 1$ with second derivatives $p^{(2)\prime\prime}(0) = 0$ and $p^{(2)\prime\prime}(2) = 0$.

d) Verify that for each triple of values (y_0, y_1, y_2), the cubic spline $p := y_0 p^{(0)} + y_1 p^{(1)} + y_2 p^{(2)}$ takes the prescribed values $p(0) = y_0$, $p(1) = y_1$, and $p(2) = y_2$, with second derivatives $p''(0) = 0$ and $p''(2) = 0$.

2.3 Splines for Regression with Least Absolute Value

Although typical curve- and surface-sketching in calculus begins with a given formula and then produces a corresponding graph, many applications require some inverse operation: begin with geometric specifications, and then produce a formula that meets such specifications. Examples of such applications appeared in the preceding subsection, which demonstrated how to construct cubic polynomials that must pass through specified points with specified slopes. However, many applications must satisfy a yet more general kind of specification, generally called a *regression*: the construction of a curve or surface that does not necessarily pass through any data points, but which minimizes some average discrepancy between the curve or surface and the data. For instance, in calculus and in statistics, the method of *ordinary linear least squares* minimizes the average squared discrepancy between finitely many data points and a straight line. Still other methods exist, for the fitted curve need not be a straight line but may consist of polynomials or any other functions; the measure of discrepancy need not be least squares but may be the average absolute value, or the smallest maximum absolute value; also, the data may contain infinitely many points. To illustrate such concepts in the context of calculus, this subsection demonstrates how to construct a piecewise linear spline to approximate a more complicated function with the smallest maximum absolute value of the discrepancy.

Example 9. Consider the square-root function $\sqrt{\ }$ on the interval $[{}^1\!/\!_4, 1]$, and consider a straight line $\ell : [{}^1\!/\!_4, 1] \to \mathbb{R}$ defined by $\ell(x) = ax + c$, with slope $a \in \mathbb{R}$ and vertical intercept $c \in \mathbb{R}$, intended to approximate the square root on $[{}^1\!/\!_4, 1]$. Then the square root and the line differ by the discrepancy function $D : [{}^1\!/\!_4, 1] \to \mathbb{R}$ defined by $D(x) := \sqrt{x} - (ax + c)$. To *minimize the maximum of* $|D|$, adjust a and c so that D has the same absolute value at its minimum at both endpoints, and at its maximum inside $]{}^1\!/\!_4, 1[$, as justified in the next subsection.

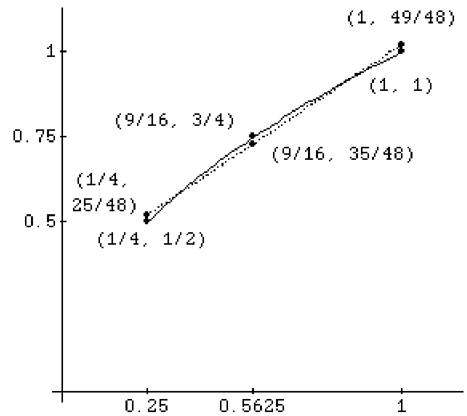

Figure 11. The dotted straight line ℓ minimizes the maximum absolute discrepancy ($3/4 - 35/48 = 1/48$) with the square root on the interval $[1/4, 1]$.

At the endpoints,

$$\begin{aligned} D(1/4) &= \sqrt{1/4} - (a(1/4) + c), \\ D(1) &= \sqrt{1} - (a + c). \end{aligned}$$

Thus, D has the same value at both endpoints if, but only if,

$$\begin{aligned} D(1/4) &= D(1), \\ \sqrt{1/4} - (a(1/4) + c) &= \sqrt{1} - (a + c), \\ 1/2 - a/4 &= 1 - a, \\ 3a/4 &= 1/2, \\ a &= 2/3. \end{aligned}$$

The common value at both endpoints becomes

$$D(1) = \sqrt{1} - (2/3 + c) = 1/3 - c.$$

At the extremum $x \in]1/4, 1[$, $D'(x) = 0$, whence

$$\begin{aligned} D'(x) &= 0, \\ \frac{1}{2\sqrt{x}} - a &= 0, \\ 2\sqrt{x} &= 1/a, \\ x &= \frac{1}{4a^2}, \end{aligned}$$

which corresponds to a maximum, because $D''(x) = -(1/4)x^{-3/2} < 0$; at that point,

$$D\left(1/(4a^2)\right) = \sqrt{\frac{1}{4a^2}} - \left(\frac{a}{4a^2} + c\right) = \frac{1}{2a} - \left(\frac{1}{4a} + c\right) = \frac{1}{4a} - c.$$

Consequently, D has the same absolute value at its maximum and at its end-points if, but only if,

$$\begin{aligned} D(1/(4a^2)) &= -D(1), \\ 1/(4a) - c &= -(1 - (a + c)), \\ 1/(4[{}^2\!/_3]) - c &= {}^2\!/_3 + c - 1, \\ {}^3\!/_8 - {}^2\!/_3 + 1 &= 2c, \\ {}^{17}\!/_{24} &= 2c, \\ {}^{17}\!/_{48} &= c. \end{aligned}$$

Therefore, the line has equation

$$\ell(x) = \frac{2}{3}\,x + \frac{17}{48} = \frac{32x + 17}{48}.$$

The maximum discrepancy between the line ℓ and the square root occurs at the endpoints ${}^1\!/_4$ and 1, and at $1/(4a^2) = 1/(4[{}^2\!/_3]^2) = {}^9\!/_{16}$, where the discrepancy D has the same absolute value,

$$|D({}^1\!/_4)| = D({}^9\!/_{16}) = |D(1)| = |1 - ({}^2\!/_3 + {}^{17}\!/_{48})| = {}^1\!/_{48}.$$

Thus, for every $x \in [{}^1\!/_4, 1]$,

$$\left|\sqrt{x} - \frac{32x + 17}{48}\right| \le \frac{1}{48} = 0.020\,833\ldots.$$

For instance, the values in **Table 4** illustrate this inequality.

Table 4.

Comparison of the square root with the line minimizing the maximum absolute discrepancy. For every $x \in [{}^1\!/_4, 1]$, $|\sqrt{x} - (({}^2\!/_3)x + {}^{17}\!/_{48})| \le {}^1\!/_{48} = 0.0208\ldots$.

x	${}^1\!/_4$	${}^{25}\!/_{64}$	${}^9\!/_{16}$	${}^{49}\!/_{64}$	1
$\sqrt{x}$	0.5	0.625	0.75	0.875	1
$({}^2\!/_3)x + {}^{17}\!/_{48}$	0.520 833 ...	0.614 583 ...	0.729 166 ...	0.864 583 ...	1.020 833 ...

Exercises

17. Determine the straight line that minimizes the maximum absolute discrepancy with the square root on the interval $[{}^1\!/_2, 1]$, and calculate the maximum discrepancy for that line.

18. Determine the straight line that minimizes the maximum absolute discrepancy with the square root on the interval $[1/4, 1/2]$, and calculate the maximum discrepancy for that line.

19. Comment on Pulskamp and Delaney's [1991] assertion that the straight line with equation $y = 0.5857864x + 0.4204951$ minimizes the discrepancy with the square root at the endpoints and at the midpoint of the interval $[1/2, 1]$.

20. Comment on Pulskamp and Delaney's [1991] assertion that the straight line with equation $y = 0.8284271x + 0.3012412$ minimizes the discrepancy with the square root at the endpoints and at the midpoint of the interval $[1/4, 1/2]$.

21. Determine the straight line that minimizes the maximum absolute discrepancy with the cube root on the interval $[1/8, 1]$, and calculate the maximum discrepancy for that straight line.

2.4 Regression with Least Absolute Value for Concave Functions

The assertion of **Example 9**, that the line obtained minimizes the absolute value of the discrepancy if the discrepancy has the same absolute value at the endpoints and at its minimum inside the interval, requires a justification, which forms the object of the following considerations. In particular, we present a theorem to justify the method. For a general theory, see [Kincaid and Cheney 1991, § 6.9]. As preparation for the theorem, the proofs of the following lemmata provide some practice with Rolle's Theorem, the Mean Value Theorem for Derivatives, and Taylor's polynomials with remainders. More importantly, the proofs of the lemmata illustrate how the *concepts*, rather than the mechanics, of calculus apply in practice. For instance, **Exercises 19** and **20** demonstrate how a theoretical understanding of **Example 9** reveals and corrects significant errors in the literature. To this end, let p, q, r, and s denote four extended real numbers with $r < p < q < s$. (The adjective "extended" means that r may be $-\infty$ and that s may be ∞; such a convention merely affords the convenience of letting $]r, s[$ stand for any of $]r, s[$, $]-\infty, s[$, $]r, \infty[$, or $]-\infty, \infty[$.)

Definition 3. A function $f :]r, s[\to \mathbb{R}$ is *concave* if, but only if, for each pair of points $p < q$ in the open interval $]r, s[\subset \mathbb{R}$, the function f remains on or above the straight line through the endpoints $(p, f(p))$ and $(q, f(q))$. In other words, f is concave if, but only if, for all $p < q$ in $]r, s[$ and for every $x \in [p, q]$,

$$f(x) \geq f(p) + \frac{f(q) - f(p)}{q - p} \cdot (x - p).$$

(The right-hand side represents the straight line through the endpoints $(p, f(p))$ and $(q, f(q))$.)

Lemma 1. *Let* $f :]r,s[\to \mathbb{R}$ *represent a function of class* C^2 *, which means that* f *, its first derivative* f' *, and its second derivative* f'' *are defined and continuous everywhere on the open interval* $]r,s[$ *. If* $f'' \le 0$ *everywhere on* $]r,s[$ *, then* f *is concave.*

Proof: For each pair of points $p < q$ in $]r,s[$, denote by L the straight line through $(p, f(p))$ and $(q, f(q))$, and consider the difference $D := f - L$.

Observe that $D(p) = 0 = D(q)$, because $D(p) = f(p) - L(p) = 0 = f(q) - L(q) = D(q)$. Consequently, Rolle's Theorem guarantees the existence of at least one point $z \in]p,q[$ where $D'(z) = 0$.

Therefore, Taylor's polynomial about z with degree at most one and with integral remainder shows that for each $x \in]p,q[$,

$$\begin{aligned} D(x) &= D(z) + (x-z)D'(z) + \int_z^x (x-w)D''(w)\,dw \\ &= D(z) + 0 + \int_z^x (x-w)D''(w)\,dw, \end{aligned}$$

because $D'(z) = 0$. If $z \le w \le x \le q$, then $0 \le x - w \le q - w$, and multiplying throughout by $D''(w) \le 0$ reverses the inequalities to $0 \ge (x-w)D''(w) \ge (q-w)D''(w)$. Also, $(q-w)D''(w) \le 0$ for every $w \in [z,q]$. Consequently,

$$\int_z^x (x-w)D''(w)\,dw \ge \int_z^x (q-w)D''(w)\,dw \ge \int_z^q (q-w)D''(w)\,dw,$$

whence

$$\begin{aligned} D(x) &= D(z) + \int_z^x (x-w)D''(w)\,dw \\ &\ge D(z) + \int_z^q (q-w)D''(w)\,dw \\ &= D(q) \\ &= 0. \end{aligned}$$

Similarly, if $p \le x \le w \le z$, then $p - w \le x - w \le 0$, and multiplying throughout by $D''(w) \le 0$ reverses the inequalities to $(p-w)D''(w) \ge (x-w)D''(w) \ge 0$. Also, $(p-w)D''(w) \ge 0$ for every $w \in [p,z]$. Consequently,

$$\int_x^z (x-w)D''(w)\,dw \le \int_x^z (p-w)D''(w)\,dw \le \int_p^z (p-w)D''(w)\,dw;$$

hence, swapping the bounds of integration reverses the signs and the inequalities, which yields

$$\begin{aligned} D(x) &= D(z) + \int_z^x (x-w)D''(w)\,dw \\ &\ge D(z) + \int_z^p (p-w)D''(w)\,dw = D(p) = 0. \end{aligned}$$

Therefore, $f(x) - L(x) = D(x) \geq 0$ for every $x \in [p,q]$, which means that f is concave. $\square$

Example 10. The square-root function $\sqrt{\ } : \ ]0,\infty[\to \mathbb{R}$ is concave, because $\sqrt{\ }'(x) = 1/(2\sqrt{x})$ and $\sqrt{\ }''(x) = -(1/4)x^{-3/2} < 0$ for every $x \in]0,\infty[$.

Lemma 2. *Let $D : \ ]r,s[\to \mathbb{R}$ denote a concave function. Also, let m and M denote the minimum and maximum values of D on $[p,q] \subset]r,s[$. Moreover, let $L : \mathbb{R} \to \mathbb{R}$ represent the straight line through the endpoints $(p, D(p))$ and $(q, D(q))$. Then on $[p,q]$ the function $F := D + m - L$ has a maximum that does not exceed the maximum M of D, and D has a minimum that does not exceed that of F (this means that if the graph of D fits in the rectangle $[p,q] \times [m,M]$, then so does the graph of $D + m - L$):*

$$\begin{aligned} \max\{D(x) + m - L(x) : \ x \in [p,q]\} &\leq M := \max\{D(x) : \ x \in [p,q]\}, \\ m := \min\{D(x) : \ x \in [p,q]\} &\leq \min\{D(x) + m - L(x) : \ x \in [p,q]\}. \end{aligned}$$

Proof:

Step 1. By **Definition 3**, the line L remains below or on the function D; thus, $D - L \geq 0$, whence $D + m - L \geq m$, everywhere on the interval $[p,q]$.

Step 2. The two-point formula for L takes the form

$$L(x) = D(p) + \frac{D(q) - D(p)}{q - p}(x - p) = D(p)\,\frac{q - x}{q - p} + D(q)\,\frac{x - p}{q - p}.$$

At the left-hand endpoint, $L(p) = D(p) \geq m$, while at the right-hand endpoint, $L(q) = D(q) \geq m$. Because L is a straight line, it follows that $L \geq m$ everywhere on $[p,q]$:

$$L(x) = D(p)\,\frac{q - x}{q - p} + D(q)\,\frac{x - p}{q - p} \geq m\,\frac{q - x}{q - p} + m\,\frac{x - p}{q - p} = m\,\frac{q - p}{q - p} = m.$$

Step 3. The preceding step shows that $L \geq m$, whence $0 \geq m - L$ and $M \geq D \geq D + m - L$ on $[p,q]$. Thus, $M \geq D + m - L \geq m$ on $[p,q]$. $\square$

Lemma 3. *Let $H : [p,q] \to \mathbb{R}$ denote a continuous function. Also, let m and M denote the minimum and maximum values of H on $[p,q]$. Moreover, consider the function $T := H - (m + M)/2$. Then*

$$\max\{|T(x)| : \ x \in [p,q]\} = \frac{M - m}{2} \leq \max\{|H(x)| : \ x \in [p,q]\}.$$

Proof: $0 \leq (M - m)/2 \leq \max\{M, -m\} = \max\{|H(x)| : \ x \in [p,q]\}$, whence

$$-\frac{M - m}{2} = m - \frac{m + M}{2} \leq H(x) - \frac{m + M}{2} \leq M - \frac{m + M}{2} = \frac{M - m}{2}.$$

Theorem 1. *Let* $f :]r,s[\to \mathbb{R}$ *denote a concave function of class* C^2. *Also, let* $[p,q] \subset]r,s[$ *represent any closed subinterval such that* $r < p < q < s$. *Consider all affine functions* $\ell : \mathbb{R} \to \mathbb{R}$, $\ell(x) = ax + c$, *with slope* a *and vertical intercept* c. *Then exactly one such line* ℓ_f *exists, with* $\ell_f(x) = a_f x + c_f$, *such that*

$$\max\{|f(x) - (a_f x + c_f)| : x \in [p,q]\} \le \max\{|f(x) - (ax + c)| : x \in [p,q]\}$$

for every $a, c \in \mathbb{R}$. *Moreover, for the optimal line* ℓ_f *just described, the minimum discrepancy* $D = f - \ell_f$ *occurs at both endpoints* p *and* q, *and the minimum discrepancy has the same absolute value as the maximum discrepancy:*

$$\begin{aligned}
\max\{f(x) - (a_f x + c_f) : x \in [p,q]\} &= |\min\{f(x) - (a_f x + c_f) : x \in [p,q]\} \\
&= -\min\{f(x) - (a_f x + c_f) : x \in [p,q]\} \\
&= -(f(p) - (a_f p + c_f)) \\
&= -(f(q) - (a_f q + c_f)).
\end{aligned}$$

Proof: Let $\ell(x) := ax + c$, and consider the discrepancy function $D :]r,s[\to \mathbb{R}$ defined by $D(x) := f(x) - (ax + c)$. Then D is also concave of class C^2, because $D'(x) = f'(x) - a$ and $D''(x) = f''(x) \le 0$ for every $x \in]r,s[$.

Step 1. Let h denote the minimum value of D on $[p,q]$, and let L represent the straight line through $(p, D(p))$ and $(q, D(q))$. By lemma 2, the function $F := D + h - L = f - \ell + h - L$ has the same minimum as D, but F reaches its minimum at both endpoints, and the maximum of F does not exceed the maximum of D. Thus,

$$\begin{aligned}
\min\{D(x) : x \in [p,q]\} &= \min\{F(x) : x \in [p,q]\} \\
&\le \max\{F(x) : x \in [p,q]\} \le \max\{D(x) : x \in [p,q]\},
\end{aligned}$$

and, consequently, $\max\{|F(x)| : x \in [p,q]\} \le \max\{|D(x)| : x \in [p,q]\}$. This means that the line $\ell + L - h$ has an absolute discrepancy not exceeding that of ℓ.

Step 2. Let m and M denote the minimum and maximum values of F on $[p,q]$. By lemma 3, the function $H := F - (m + M)/2$ has an absolute maximum discrepancy still not exceeding that of F, with the same absolute value at the minimum, which occurs at both endpoints, as at the maximum, which occurs at some intermediate point.

Step 3. Consider two such lines ℓ_1 and ℓ_2, with the property that the discrepancy $f - \ell_j$ has the same absolute value at the minimum, which occurs at both endpoints, as at the maximum, which occurs at some intermediate point. From $f(p) - \ell_1(p) = f(q) - \ell_1(q)$ and $f(p) - \ell_2(p) = f(q) - \ell_2(q)$ it follows that $(\ell_1 - \ell_2)(p) = (\ell_1 - \ell_2)(q)$. Consequently, the straight line $\ell_1 - \ell_2$ remains horizontal, in other some words, remains some constant $K := \ell_1 - \ell_2$.

Step 4. Let $t_j \in]p,q[$ denote any point where $f(x) - \ell_j(x)$ reaches its maximum; for the optimal line ℓ_j considered here,

$$\ell_j(p) - f(p) = f(t_j) - \ell_j(t_j) = \ell_j(q) - f(q).$$

From $f(t_1) - \ell_1(t_1) \geq f(t) - \ell_1(t)$ for every $t \in [p,q]$ it follows that $f(t_1) - \ell_1(t_1) \geq f(t_2) - \ell_1(t_2)$ for $t := t_2$, whence

$$\begin{aligned} K &= \ell_1(p) - \ell_2(p) \\ &= [\ell_1(p) - f(p)] - [\ell_2(p) - f(p)] \\ &= [f(t_1) - \ell_1(t_1)] - [f(t_2) - \ell_2(t_2)] \\ &\geq [f(t_2) - \ell_1(t_2)] - [f(t_2) - \ell_2(t_2)] \\ &= \ell_2(t_2) - \ell_1(t_2) \\ &= -K. \end{aligned}$$

Thus, $K \geq -K$, which means that $K \geq 0$. Similarly, from $f(t_2) - \ell_2(t_2) \geq f(t) - \ell_2(t)$ for every $t \in [p,q]$ it follows that $f(t_2) - \ell_2(t_2) \geq f(t_1) - \ell_2(t_1)$ for $t := t_1$, whence

$$\begin{aligned} K &= \ell_1(p) - \ell_2(p) \\ &= [\ell_1(p) - f(p)] - [\ell_2(p) - f(p)] \\ &= [f(t_1) - \ell_1(t_1)] - [f(t_2) - \ell_2(t_2)] \\ &\leq [f(t_1) - \ell_1(t_1)] - [f(t_1) - \ell_2(t_1)] \\ &= \ell_2(t_1) - \ell_1(t_1) \\ &= -K. \end{aligned}$$

Thus, $K \leq -K$, which means that $K \leq 0$. Because $K \geq 0$ also, the constant $K = \ell_1 - \ell_2$ equals zero. □

Exercises

22. Consider the following definition.

> **Definition 4.** A function $g :]r,s[\to \mathbb{R}$ is *convex* if, but only if, for all $p < q$ in $]r,s[$ and for every $x \in [p,q]$,
>
> $$g(x) \leq g(p) + \frac{g(q) - g(p)}{q - p} \cdot (x - p).$$

Prove that a function $g :]r,s[\to \mathbb{R}$ is convex if, but only if, the function $-g$ is concave.

23. Prove the following theorem.

> **Theorem 2.** *Let $g :]r,s[\to \mathbb{R}$ denote a convex function. Also, let $[p,q] \subset]r,s[$ represent any closed subinterval such that $r < p < q < s$. Consider all affine functions $\ell : \mathbb{R} \to \mathbb{R}, \ell(x) = ax + c$, with slope a and vertical intercept c. Then exactly one such line ℓ_g exists, with $\ell_g(x) = a_g x + c_g$, such that*
>
> $$\max\{|g(x) - (a_g x + c_g)| : x \in [p,q]\} \leq \max\{|g(x) - (ax + c)| : x \in [p,q]\}$$

for every $a, c \in \mathbb{R}$. *Moreover, for the optimal line* ℓ_g *just described, the maximum discrepancy* $D = g - \ell_g$ *occurs at both endpoints* p *and* q, *and the minimum discrepancy has the same absolute value as the maximum discrepancy:*

$$\begin{aligned}\max\{g(x) - (a_g x + c_g) : x \in [p, q]\} &= |\min\{g(x) - (a_g x + c_g) : x \in [p, q]\}| \\ &= \max\{g(x) - (a_g x + c_g) : x \in [p, q]\} \\ &= g(p) - (a_g p + c_g) = g(q) - (a_g q + c_g).\end{aligned}$$

3. Splines in Multivariable Calculus

In several dimensions, mathematical splines consist of functions of several variables, or functions of one variable but with vectors as values, defined piecewise by simple functions, and used in applied mathematics and in engineering to approximate such functions as curves and surfaces that are incompletely known or too complicated for practical computations.

3.1 Spline Curves in the Plane and in Space

The preceding section illustrated the use of splines to approximate computationally demanding real-valued functions of one variable by simpler polynomials, especially to compute such functions to a specified accuracy. The same splines can also approximate vector-valued functions of one variable, also called *curves*, with one spline as in the preceding section for each coordinate of the curve. Such *spline curves* prove useful, for instance, in programming typefaces into printing devices [Knuth 1979; 1980], in designing car panels [Kincaid and Cheney 1991, #36, 331] [Moylan 1993], and in designing the hoses on aircraft landing gears so that the hoses do not jam as the landing gear retracts or unfolds [Grandine 1991].

To demonstrate how spline curves rely on calculus, the present section proposes the design of a planar spline to approximate the letter "S," and a spatial spline to approximate a hose clamped at two points, with prescribed ends and directions. Though spline curves of industrial grade consist of many pieces of cubic polynomials, as illustrated in the preceding section, the exercises proposed here involve only one piece of a polynomial per coordinate, which requires only elementary differential calculus, but with a point of view slightly different from that in most calculus courses.

In the following usual definitions, let r and s represent two real numbers such that $r < s$.

Definition 5. A *planar curve of class* C^k is a function

$$C : [r, s] \to \mathbb{R}^2, \quad C(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix},$$

such that the functions $X : [r,s] \to \mathbb{R}$ and $Y : [r,s] \to \mathbb{R}$ each have continuous derivatives of orders at least 0 through k. □

Definition 6. The vector *tangent* at a point $C(t)$ to a planar curve of class C^1

$$C : [r,s] \to \mathbb{R}^2, \quad C(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$$

is the vector

$$C'(t) := \begin{pmatrix} X'(t) \\ Y'(t) \end{pmatrix}.$$

Example 11. The function

$$L : [0,1] \to \mathbb{R}^2, \quad L(t) = (1-t)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} (1-t)x_0 + tx_1 \\ (1-t)y_0 + ty_1 \end{pmatrix}$$

traces the straight-line segment from the point $P_0 = (x_0, y_0)$ to the point $P_1 = (x_1, y_1)$, both included, with a constant tangent vector pointing from (x_0, y_0) to (x_1, y_1):

$$L'(t) = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix}.$$

Remark 4. The *method* employed in **Example 11** can wield great power — in the sense that it can solve difficult problems at little cost — in various contexts that require a continuous transition from one object to another. Specifically, in example 11, the problem consists of joining the point P_0 to the point P_1, and the method of solution consists of selecting two functions on $[0,1]$, $t \mapsto (1-t)$, which decrease from 1 to 0, and $t \mapsto t$, which increases from 0 to 1. Thus, at 0, $L(0) = (1-0)P_0 + 0P_1 = P_0$, whereas at 1, $L(1) = (1-1)P_0 + 1P_1 = P_1$. In general, P_0 and P_1 may represent any elements in a linear space over $\mathbb{R}$, and the same type of function L provides a continuous transition from P_0 to P_1. Examples will appear later. □

Example 12. The function

$$C : [0, 2\pi] \to \mathbb{R}^2, \quad C(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

traces the *unit circle* $\mathcal{S}^1 := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ by means of symbolically concise but computationally demanding trigonometric functions. □

Example 13. The inverse of the *stereographic projection* from the South Pole,

$$R : \,]-\infty, \infty[\to \mathbb{R}^2, \quad R(t) = \begin{pmatrix} 2t/(1+t^2) \\ (1-t^2)/(1+t^2) \end{pmatrix},$$

traces the unit circle without the South Pole $S := (0, -1)$ by means of computationally simple rational functions. □

One method to trace computationally demanding curves with simpler functions consists of approximating each coordinate of such curves with splines.

Definition 7. A *planar spline curve* is a function

$$C : [r, s] \to \mathbb{R}^2, \quad C(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix},$$

such that the functions $X : [r, s] \to \mathbb{R}$ and $Y : [r, s] \to \mathbb{R}$ are splines.

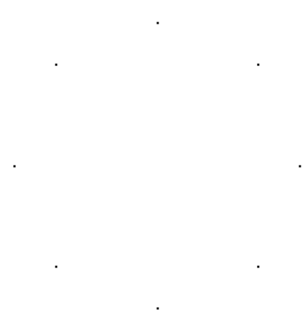

Figure 12. Points on the unit circle.

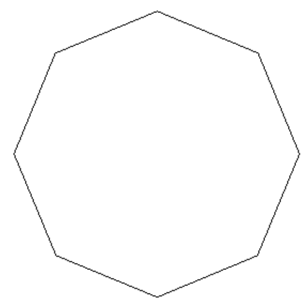

Figure 13. Planar spline approximating the unit circle.

Example 14. Approximate the unit circle by joining points on it with straight-line segments, as in **Figures 12** and **13**. For instance, consider the following nine points, where the first and last ones coincide,

(x_0, y_0)	(x_1, y_1)	(x_2, y_2)	(x_3, y_3)	(x_4, y_4)	(x_5, y_5)	(x_6, y_6)	(x_7, y_7)	(x_8, y_8)
$(1, 0)$	$\frac{(1,1)}{\sqrt{2}}$	$(0, 1)$	$\frac{(-1,1)}{\sqrt{2}}$	$(-1, 0)$	$\frac{(-1,-1)}{\sqrt{2}}$	$(0, -1)$	$\frac{(1,-1)}{\sqrt{2}}$	$(1, 0)$

and for each $i \in \{0, 8\}$, trace the line segment joining (x_i, y_i) to (x_{i+1}, y_{i+1}). As in **Example 11**, the first segment is the image of the function

$$L : [0, 1] \to \mathbb{R}^2,$$

$$L(t) = \begin{pmatrix} (1-t)x_0 + tx_1 \\ (1-t)y_0 + ty_1 \end{pmatrix} = \begin{pmatrix} (1-t)1 + t/\sqrt{2} \\ (1-t)0 + t/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 - (1 - 1/\sqrt{2})t \\ t/\sqrt{2} \end{pmatrix}.$$

To achieve greater accuracy, use more points, for instance, as in **Figures 14** and **15**. □

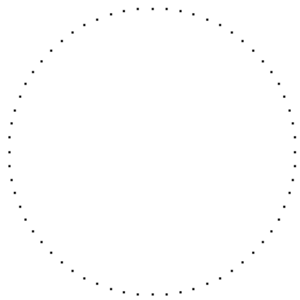

Figure 14. More points on the unit circle.

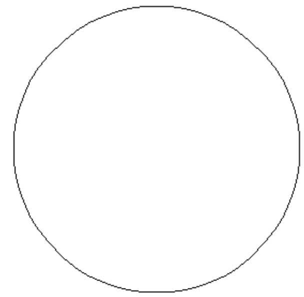

Figure 15. A better planar spline approximation to the unit circle.

Remark 5. One of the uses of planar spline curves consists of approximating the area of a planar region, which may represent a panel of an airplane or automobile, for instance, to estimate the amount of material, and hence the cost or the weight of that panel. In many such cases, engineers know or can measure only finitely many points on the perimeter of the panel. Therefore, they approximate the perimeter with a spline, for example, an affine spline, and then they calculate the area enclosed by that spline by means of the following theorem. □

Theorem 3. *Let* $P_0, P_1, \ldots, P_{n-1}, P_n = P_0$ *represent the vertices of a planar but not necessarily convex polygon* P*, numbered counterclockwise. The following formula gives the area of the polygon:*

$$\text{Area}\,(P) = \frac{1}{2}\sum_{i=1}^{n} \det(P_{i-1}, P_i),$$

where $\det(P_{i-1}, P_i)$ *stands for the determinant of the matrix with the coordinates of* P_{i-1} *in its first column and the coordinates of* P_i *in its second column, with all coordinates with respect to the canonical basis vectors* $\vec{\mathbf{e}}_1 = (1,0)$ *and* $\vec{\mathbf{e}}_2 = (0,1)$.

Proof: The strategy consists of applying Green's Theorem to the perimeter ∂P of the polygon P and to the function identically equal to one:

$$\text{Area}\,(P) = \int\int_P 1\,dx\,dy = \oint_{\partial P} \frac{1}{2}\,(x\,dy - y\,dx)\,.$$

Label the coordinates of each vertex by $P_i = (x_i, y_i)$, and parametrize the edge from the vertex P_{i-1} to P_i by the affine function $C_i : [0,1] \to \mathbb{R}^2$ defined as in example 11 by $C_i(t) = (1-t)P_{i-1} + tP_i$. Substituting the parametrization just obtained into the formula for the area just derived from Green's Theorem gives

$$
\begin{aligned}
\operatorname{Area}(P) &= \int\int_P 1\,dx\,dy \\
&= \oint_{\partial P} \frac{1}{2}\,(x\,dy - y\,dx) \\
&= \sum_{i=1}^{n} \int_0^1 \frac{1}{2}\,([(1-t)x_{i-1} + tx_i](y_i - y_{i-1})dt \\
&\qquad\qquad -[(1-t)y_{i-1} + ty_i](x_i - x_{i-1})dt) \\
&= \sum_{i=1}^{n} \frac{1}{2}\left(\left[-\frac{(1-t)^2}{2}x_{i-1} + \frac{t^2}{2}x_i\right](y_i - y_{i-1})\right. \\
&\qquad\qquad \left. - \left[-\frac{(1-t)^2}{2}y_{i-1} + \frac{t^2}{2}y_i\right](x_i - x_{i-1})\right)\Bigg|_0^1 \\
&= \sum_{i=1}^{n} \frac{1}{4}\,(-x_i y_{i-1} + x_i y_i - x_{i-1}y_{i-1} + x_{i-1}y_i + y_i x_{i-1} \\
&\qquad\qquad -y_i x_i + y_{i-1}x_{i-1} - y_{i-1}x_i) \\
&= \sum_{i=1}^{n} \frac{1}{2}\,(x_{i-1}y_i - x_i y_{i-1}) \\
&= \sum_{i=1}^{n} \frac{1}{2}\,\det\begin{pmatrix} x_{i-1} & x_i \\ y_{i-1} & y_i \end{pmatrix}.
\end{aligned}
$$

Theorem 3 has the following geometric interpretation. With $\vec{\mathbf{0}}$ denoting the origin, add the areas of all the triangles with vertices $\vec{\mathbf{0}}$, P_{i-1}, and P_i that appear counterclockwise in that order, and subtract from that sum the areas of all the triangles with vertices $\vec{\mathbf{0}}$, P_{i-1}, and P_i that appear clockwise in that order. The theorem selects the suitable signs automatically by means of determinants.

The rest of the present section provides analogous concepts for spatial curves.

Definition 8. A *spatial curve of class* C^k is a function

$$
C:\ [r,s] \to \mathbb{R}^3, \quad C(t) = \begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix},
$$

such that the functions $X, Y, Z:\ [r,s] \to \mathbb{R}$ each have continuous derivatives of orders at least 0 through k. □

Definition 9. The vector *tangent* at a point $C(t)$ to a spatial curve of class C^1

$$
C:\ [r,s] \to \mathbb{R}^3, \quad C(t) = \begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix}
$$

is the vector

$$C'(t) := \begin{pmatrix} X'(t) \\ Y'(t) \\ Z'(t) \end{pmatrix}.$$

Example 15. As an illustration of **Remark 4** following **Example 11**, the function

$$L : [0,1] \to \mathbb{R}^3, \quad L(t) = (1-t)\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} (1-t)x_0 + tx_1 \\ (1-t)y_0 + ty_1 \\ (1-t)z_0 + tz_1 \end{pmatrix}$$

traces the straight-line segment from the point $P_0 = (x_0, y_0, z_0)$ to the point $P_1 = (x_1, y_1, z_1)$, both included, with tangent vector

$$L'(t) = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \\ z_1 - z_0 \end{pmatrix}.$$

Definition 10. A *spatial spline curve* is a function

$$C : [r,s] \to \mathbb{R}^3, \quad C(t) = \begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix},$$

such that the functions $X : [r,s] \to \mathbb{R}$, $Y : [r,s] \to \mathbb{R}$, and $Z : [r,s] \to \mathbb{R}$ are splines. □

Exercises

24. Establish a formula for the i^{th} segment in **Example 14**, from (x_i, y_i) to (x_{i+1}, y_{i+1}).

25. Construct a planar spline curve

$$S : [0,1] \to \mathbb{R}^2, \quad C(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$$

that approximates the shape of the letter "S," as indicated in **Figures 16** and **17**. To this end, construct cubic polynomials $X : [0,1] \to \mathbb{R}$ and $Y : [0,1] \to \mathbb{R}$ such that

$$\begin{aligned} S(0) &= (1,1), \\ S'(0) &= (10,-10), \\ S(1) &= (3,3), \\ S'(1) &= (10,-10). \end{aligned}$$

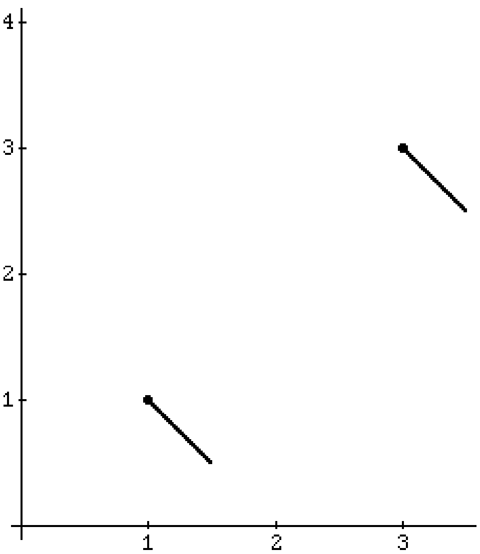

Figure 16. The cubic polynomial $S = (X, Y)$ must pass through $(1, 1)$ with tangent $(10, -10)$, and through $(3, 3)$ with tangent $(10, -10)$ (not shown to scale).

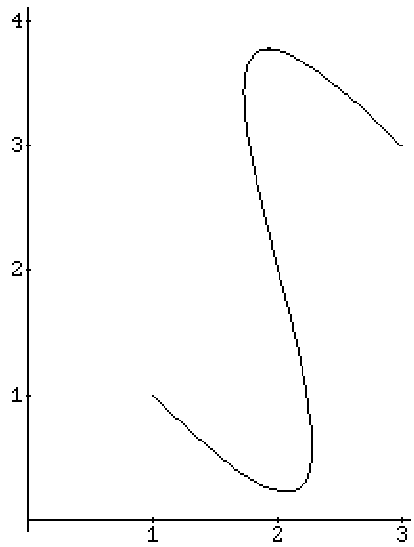

Figure 17. The cubic polynomial $S = (X, Y)$ passes through $(1, 1)$ with tangent $(10, -10)$, and through $(3, 3)$ with tangent $(10, -10)$.

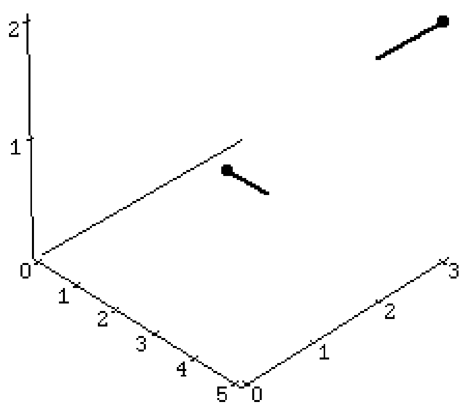

Figure 18. The cubic polynomial $H = (X, Y, Z)$ must pass through $(3, 1, 1)$ with tangent $(1, 0, 0)$, and through $(5, 3, 2)$ with tangent $(0, 1, 0)$.

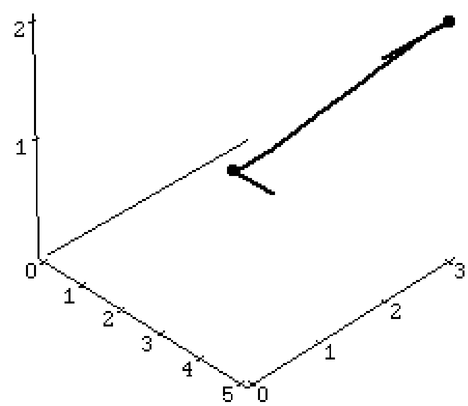

Figure 19. The cubic polynomial $H = (X, Y, Z)$ passes through $(3, 1, 1)$ with tangent $(1, 0, 0)$, and through $(5, 3, 2)$ with tangent $(0, 1, 0)$.

26. Construct a spatial spline curve, as indicated in **Figures 18** and **19**,

$$H : [0, 1] \to \mathbb{R}^3, \quad H(t) = \begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix}$$

with cubic polynomials $X : [0,1] \to \mathbb{R}$, $Y : [0,1] \to \mathbb{R}$ and $Z : [0,1] \to \mathbb{R}$ such that

$$\begin{aligned} H(0) &= (3,1,1), \\ H'(0) &= (1,0,0), \\ H(1) &= (5,3,2), \\ H'(1) &= (0,1,0). \end{aligned}$$

27. Write the formula for the area of a triangle that follows from **Theorem 3**.

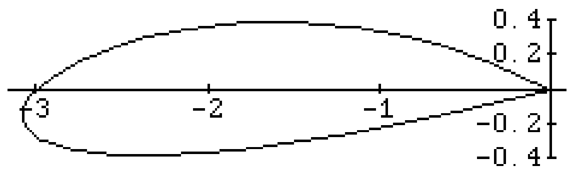

Figure 20. The image of the curve W.

28. Calculate the area enclosed by the planar curve $W : [0,2] \to \mathbb{R}^2$ shown in **Figure 20** and defined by

$$W(t) = \begin{pmatrix} t^3 - 4t \\ t^3 - 3t^2 + 2t \end{pmatrix}.$$

3.2 Spline Surfaces in Space

Definition 11. For each function

$$S : [p,q] \times [r,s] \to \mathbb{R}^3, \quad S(u,v) = \begin{pmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \end{pmatrix}$$

with differentiable coordinate functions $X, Y, Z : [p,q] \times [r,s] \to \mathbb{R}$, define the partial derivatives

$$\frac{\partial S}{\partial u} := \begin{pmatrix} \partial X/\partial u \\ \partial Y/\partial u \\ \partial Z/\partial u \end{pmatrix}, \quad \frac{\partial S}{\partial v} := \begin{pmatrix} \partial X/\partial v \\ \partial Y/\partial v \\ \partial Z/\partial v \end{pmatrix}.$$

Definition 12. A *parametric surface* of class C^k in space is a function

$$S : [p,q] \times [r,s] \to \mathbb{R}^3, \quad S(u,v) = \begin{pmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \end{pmatrix}$$

such that each function $X, Y, Z : [p, q] \times [r, s] \to \mathbb{R}$ is of class C^k, and such that the normal vector $\vec{\mathbf{n}}$ defined by the cross product

$$\vec{\mathbf{n}} := \frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v}$$

does not equal $\vec{\mathbf{0}}$ at any point $(u, v) \in [p, q] \times [r, s]$. $\square$

Example 16. Consider four points in space, $\vec{\mathbf{b}}, \vec{\mathbf{d}}, \vec{\mathbf{h}}, \vec{\mathbf{k}} \in \mathbb{R}^3$. The following parametric surface passes through all four points and contains the four edges $[\vec{\mathbf{b}}, \vec{\mathbf{d}}]$, $[\vec{\mathbf{b}}, \vec{\mathbf{h}}]$, $[\vec{\mathbf{d}}, \vec{\mathbf{k}}]$, and $[\vec{\mathbf{h}}, \vec{\mathbf{k}}]$.

$$S : [0,1] \times [0,1] \to \mathbb{R}^3,$$

$$\begin{aligned} S(u,v) &= \begin{pmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \end{pmatrix} \\ &= (1-v)\left((1-u)\vec{\mathbf{b}} + u\vec{\mathbf{d}}\right) + v\left((1-u)\vec{\mathbf{h}} + u\vec{\mathbf{k}}\right) \\ &= \begin{pmatrix} (1-v)\left((1-u)b_1 + ud_1\right) + v\left((1-u)h_1 + uk_1\right) \\ (1-v)\left((1-u)b_2 + ud_2\right) + v\left((1-u)h_2 + uk_2\right) \\ (1-v)\left((1-u)b_3 + ud_3\right) + v\left((1-u)h_3 + uk_3\right) \end{pmatrix}. \end{aligned}$$

By **Example 15**,

- for $v = 0$, the function S traces the edge P_0 from $\vec{\mathbf{b}}$ to $\vec{\mathbf{d}}$, with $P_0 : u \mapsto (1-u)\vec{\mathbf{b}} + u\vec{\mathbf{d}}$;
- for $v = 1$, the function S traces the edge P_1 from $\vec{\mathbf{h}}$ to $\vec{\mathbf{k}}$, with $P_1 : u \mapsto (1-u)\vec{\mathbf{h}} + u\vec{\mathbf{k}}$;
- for $u = 0$, the function S traces the edge Q_0 from $\vec{\mathbf{b}}$ to $\vec{\mathbf{h}}$, with $Q_0 : v \mapsto (1-v)\vec{\mathbf{b}} + v\vec{\mathbf{h}}$;
- for $u = 1$, the function S traces the edge Q_1 from $\vec{\mathbf{d}}$ to $\vec{\mathbf{k}}$, with $Q_1 : v \mapsto (1-v)\vec{\mathbf{d}} + v\vec{\mathbf{k}}$.

Again, the principle stated in **Remark 4** after **Example 11** applies: The function $v \mapsto (1-v)P_0 + vP_1$ provides a continuous transition from the edge P_0 to the edge P_1. $\square$

Example 17. The function

$$F : [-\pi, \pi] \times [-\pi/2, \pi/2] \to \mathbb{R}^3,$$

$$F(u,v) = \begin{pmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \end{pmatrix} = \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ \sin v \end{pmatrix}$$

parametrizes the unit sphere $\mathcal{S}^2$ in space $\mathbb{R}^3$,

$$\mathcal{S}^2 := \left\{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2+z^2=1\right\},$$

with the longitude $u \in [-\pi,\pi]$ and the latitude $v \in [-\pi/2,\pi/2]$. Let $N := (0,0,1) = F(0,\pi/2)$ denote the North Pole and let $S := (0,0,-1) = F(0,-\pi/2)$ denote the South Pole. Then $\vec{\mathbf{n}}(u,v) \neq \vec{\mathbf{0}}$ everywhere on the unit sphere $\mathcal{S}^2$ except at the poles N and S:

$$\begin{aligned}
\vec{\mathbf{n}}(u,v) &:= \frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}(u,v) \\
&= \begin{pmatrix} -\sin u \cos v \\ \cos u \cos v \\ 0 \end{pmatrix} \times \begin{pmatrix} cos u - \sin v \\ \sin u - \sin v \\ \cos v \end{pmatrix} \\
&= \cos v \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ \sin v \end{pmatrix}.
\end{aligned}$$

Though the image of F consists of the surface $\mathcal{S}^2$, the function F satisfies the definition of a parametric surface only on subsets of $]-\pi,\pi[\times]-\pi/2,\pi/2[$, away from the poles. □

Example 18. The *inverse stereographic projection* from the South Pole,

$$P :\ ]-\infty,\infty[\times]-\infty,\infty[\to \mathbb{R}^3,$$

$$P(u,v) = \frac{1}{u^2+v^2+1} \begin{pmatrix} 2u \\ 2v \\ u^2+v^2-1 \end{pmatrix}$$

maps the plane $\mathbb{R}^2$ onto the unit sphere without the South Pole, but with computationally simple rational functions. □

Definition 13. A *spline surface* in space is a parametric surface

$$S :\ [p,q]\times[r,s] \to \mathbb{R}^3, \quad S(u,v) = \begin{pmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \end{pmatrix}$$

such that each function $X, Y, Z :\ [p,q]\times[r,s] \to \mathbb{R}$ is a spline in each of the variables u and v.

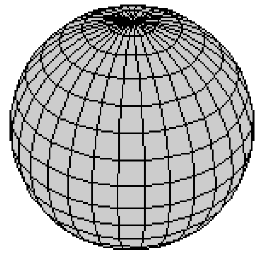

Figure 21. A continuous piecewise quadratic spline approximation of the unit sphere.

Example 19. **Figure 21** shows a spline approximation of the unit sphere. To construct such a spline, partition the intervals $[-\pi, \pi]$ and $[-\pi/2, \pi/2]$ into subintervals, $[u_i, u_{i+1}]$ in $[-\pi, \pi]$ and $[v_j, v_{j+1}]$ in $[-\pi/2, \pi/2]$. Then join the vertices $F(u_i, v_j)$, $F(u_{i+1}, v_j)$, $F(u_{i+1}, v_{j+1})$, and $F(u_i, v_{j+1})$ by a piece of surface as in **Example 16**. □

Remark 6. Spatial spline surfaces can be used to approximate the volume or the surface area of a spatial region, which may represent a part of an airplane or automobile, for instance, to estimate the amount of material, and hence the cost or the weight of that part. Usually, engineers know or measure only finitely many points on the surface of the part. They approximate the surface with, for example, an affine spline, and then calculate the volume enclosed by means of the following theorem. □

Theorem 4. *Let* $P_0, P_1, \ldots, P_{n-1}, P_n = P_0$ *represent the vertices of a spatial but not necessarily convex polyhedron* P. *The following formula gives the volume of the polyhedron* P*:*

$$\text{Volume}\,(P) = \frac{1}{6} \sum_{\text{facet }(P_h, P_k, P_\ell)}^{n} \det(P_h, P_k, P_\ell),$$

where (P_h, P_k, P_ℓ) *denotes the vertices of a triangular facet of the polyhedron, the sum extends over all facets, and* $\det(P_h, P_k, P_\ell)$ *stands for the determinant of the matrix with the coordinates of* P_h *in its first column, the coordinates of* P_k *in its second column, and the coordinates of* P_ℓ *in its third column, with all coordinates with respect to the canonical basis vectors* $\vec{\mathbf{e}}_1 = (1, 0, 0)$, $\vec{\mathbf{e}}_2 = (0, 1, 0)$, *and* $\vec{\mathbf{e}}_3 = (0, 0, 1)$.

Outline of Proof: The strategy consists of applying Stokes' Theorem to the surface ∂P of the polyhedron P and to the function identically equal to one:

$$\begin{aligned}
\text{Volume}\,(P) &= \int\!\!\int\!\!\int_P 1\, dx\, dy\, dz \\
&= \int\!\!\int_{\partial P} \frac{1}{3}\left(x\, dy\, dz - y\, dx\, dz + z\, dx\, dy\right).
\end{aligned}$$

To this end, parametrize each triangular facet, with vertices P_h, P_k, and P_ℓ, by an affine function. Substituting such a parametrization into the formula gives the stated result, through calculations similar to those for **Theorem 3**.

□

Theorem 4 has the following geometric interpretation. With $\vec{\mathbf{0}}$ denoting the origin, add the volumes of all the tetrahedra with a vertex at $\vec{\mathbf{0}}$ and with vertices P_h, P_k and P_ℓ that appear counterclockwise seen while looking toward the origin from outside the tetrahedron, and subtract from that sum the volumes of all the tetrahedra with a vertex at $\vec{\mathbf{0}}$ and with vertices P_h, P_k and P_ℓ that appear clockwise seen while looking toward the origin from inside the tetrahedron. The theorem selects the suitable signs automatically by means of determinants.

Exercises

29. Name and describe the type of surface given in **Example 16**.

30. Consider the square $[0,2] \times [0,2] = \{(x,y) : 0 \leq x, y \leq 2\}$ with the nine points, called "nodes," with integer coordinates (k,ℓ) such that $k,\ell \in \{0,1,2\}$. Establish a formula for a continuous function $f : [0,2] \times [0,2] \to \mathbb{R}$ that takes the following values. (We give no further hints, so as to illustrate how an engineer or a mathematician may have to consider and solve a problem of a new type without having an available recipe.)

$$\begin{array}{lll} f(0,2)=9, & f(1,2)=6, & f(2,2)=5, \\ f(0,1)=7, & f(1,1)=4, & f(2,1)=8, \\ f(0,0)=1, & f(1,0)=3, & f(2,0)=2. \end{array}$$

31. Consider the square $[0,2] \times [0,2] = \{(x,y) : 0 \leq x, y \leq 2\}$ with the nine points, called *nodes* with integer coordinates (k,ℓ) such that $k,\ell \in \{0,1,2\}$. Establish a formula for a function $g : [0,2] \times [0,2] \to \mathbb{R}$ of class C^2 that takes the following values.

$$\begin{array}{lll} g(0,2)=9, & g(1,2)=6, & g(2,2)=5, \\ g(0,1)=7, & g(1,1)=4, & g(2,1)=8, \\ g(0,0)=1, & g(1,0)=3, & g(2,0)=2. \end{array}$$

32. Provide specific formulae for a spline representation of a sphere as in **Figure 21** in **Example 19**.

33. Calculate the volume of the tetrahedron T with vertices at $\vec{\mathbf{0}} = (0,0,0)$, $\vec{\mathbf{e}}_1 = (1,0,0)$, $\vec{\mathbf{e}}_2 = (0,1,0)$, and $\vec{\mathbf{e}}_3 = (0,0,1)$ by means of **Theorem 4**. Then verify the result with a different method.

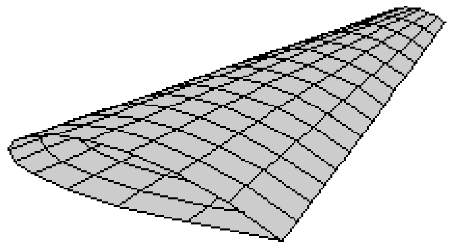

Figure 22. A wing-shaped volume.

34. Calculate the volume of a regular tetrahedron T, with four equilateral triangular faces and each of the six edges of equal length $\sqrt{2}$.

35. Calculate the volume pictured in **Figure 21**, enclosed between the vertical planes with equations $y = 0$ and $y = 4$, and by the image of the parametric surface

$$W : [0,2] \times [0,4] \to \mathbb{R}^3,$$

$$W(u,v) = \begin{pmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \end{pmatrix} = \begin{pmatrix} (u^3 - 4u)(5-v)/5 \\ v \\ (u^3 - 3u^2 + 2u)(5-v)/5 \end{pmatrix}.$$

4. Splines Beyond Calculus and Linear Algebra

To demonstrate the variety of types and the breadth of uses of splines and yet remain at the level of calculus, we have restricted our treatment mainly to splines consisting of at most two pieces (e.g., two triangles) forming a continuous spline surface. Splines with a few pieces may suffice for part of an algorithm to calculate an elementary mathematical function; but for sufficiently accurate representations of automobile panels, engine parts, or aircraft wings and fuselages, a spline may have to consist of a *million* pieces, forming a twice differentiable (class C^2) surface in space. For problems on so large a scale, further mathematical theory is necessary.

On a small scale, **Exercises 7**, **16**, **30**, and **31** have already illustrated continuous but not differentiable basis splines. The problem of specifying and enforcing conditions that a surface built from a million patches should have two continuous derivatives may seem overwhelming, but basis splines provide an elementary solution. With one dimension, each basis spline $P^{(k)}$ may take the value 1 at the node x_k and 0 at almost all the other nodes x_h; such basis splines then form a basis of the linear space of splines, in the sense of linear algebra. With two dimensions, the spline $(x,y) \mapsto P^{(k)}(x)P^{(\ell)}(y)$ takes the value 1 at the node (x_k, y_ℓ) and 0 at almost all the other nodes in the rectangular grid. Furthermore, if the splines $P^{(k)}$ and $P^{(\ell)}$ are both cubic splines, then they are also both twice continuously differentiable, whence so is their product $P^{(k)}P^{(\ell)}$. Therefore, one way to design spline surfaces of class C^2 is merely to form linear combinations, called *tensor product splines*, of the form $\sum_{k,\ell} c_{k,\ell} P^{(k)} P^{(\ell)}$, with real coefficients $c_{k,\ell} \in \mathbb{R}$.

Thus, the theory of basis splines transforms a seemingly overwhelming problem into a simpler one. Further theory from numerical analysis provides efficient algorithms to solve the resulting linear systems (with a million equations) for the coefficients $c_{k,\ell}$.

Hardly anyone ever works out any concrete example of a cubic surface spline by hand. Rather, engineers and mathematicians study the initial applied problem and the theory of splines. They design, test, and prove correct algorithms to solve the problem, and finally program computers to execute the algorithms.

5. Sample Exam Problems

Each of the following problems illustrates one of the many types of problems on splines that may appear in examinations or homework at the level of calculus, amongst other problems that need not relate to splines on the same examination or homework.

5.1 Problems in One Variable

Consider the "reciprocal" function r defined for each $x \neq 0$ by $r(x) := 1/x$. The following problems focus on various methods to approximate $1/x$ without division by x, using only additions, subtractions, and multiplications, and division by selected integers.

1. Consider the function $r : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ with $r(x) = 1/x$.

- **a)** Construct a function $s : [1,2] \to \mathbb{R}$ such that $s(1) = r(1)$, $s(3/2) = r(3/2)$, and $s(2) = r(2)$, and such that s consists of a straight-line segment ℓ_1 over $[1, 3/2]$ and another straight-line segment ℓ_2 over $[3/2, 2]$. Express your result with a formula involving algebra and logic.
- **b)** Calculate the maximum absolute value of the difference between r and s over $[1,2]$.
- **c)** Sketch r and s on a common graph.

2. Consider the function $r : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ with $r(x) = 1/x$.

- **a)** Determine the slope a_r and the vertical intercept c_r of the straight-line ℓ_r that minimizes the maximum absolute value of the discrepancy $D := r - \ell_r$ over the interval $[1,2]$.
- **b)** Calculate the maximum absolute value of the discrepancy $D := r - \ell_r$ over the interval $[1,2]$.
- **c)** Sketch r and ℓ_r on a common graph.

3. Consider the function $r : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ with $r(x) = 1/x$.

- **a)** Determine the coefficients y, w, v, and u of the cubic polynomial p defined by

$$p(x) := y + w(x-1) + v(x-1)^2 + u(x-1)^3$$

  such that p has the same value and the same slope as the reciprocal function r at each of the points $x_0 := 1$ and $x_1 := 2$.
- **b)** Calculate the discrepancy $D(x) := r(x) - p(x)$ at the midpoint $x_* := 3/2$.
- **c)** Calculate the maximum absolute value of the discrepancy $D = r - p$ over the interval $[1,2]$.
- **d)** Sketch r and p on a common graph.

4. Consider the function $r : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ with $r(x) = 1/x$. Take for granted that the cubic polynomial p defined by

$$p(x) := 1 - (x-1) + \frac{3}{4}(x-1)^2 - \frac{1}{4}(x-1)^3$$

approximates the reciprocal function r so that for every $x \in [1,2]$,

$$|r(x) - p(x)| < 0.0106.$$

a) Determine a "primitive" P of p, that is, a polynomial P such that $P' = p$.

b) Approximate $\int_1^2 x^{-1}\,dx$ by computing $\int_1^2 p(x)\,dx$.

c) Find a number B such that

$$\left| \int_1^2 x^{-1}\,dx - \int_1^2 p(x)\,dx \right| < B.$$

d) Determine a polynomial L such that for every $t \in [1,2]$,

$$\left| \int_1^t x^{-1}\,dx - L(t) \right| < B.$$

5. Assume that a rational function L exists that approximates the natural logarithm $\ln$ so that $|L(t) - \ln(t)| < B$ for every $t \in [1,2]$.

a) Explain how to use the function L to approximate $\ln$ on the entire positive axis.

b) Illustrate your explanation with $\ln(17)$ and

$$L(t) := \{([4-(x-1)]\,(x-1) - 8)\,(x-1)/16 + 1\}\,(x-1).$$

5.2 Problems in Two Variables

6. In the space $\mathbb{R}^3$, consider the planar triangle S, called the *standard simplex,* with vertices at $\vec{\mathbf{e}}_1 = (1,0,0)$, $\vec{\mathbf{e}}_2 = (0,1,0)$, and $\vec{\mathbf{e}}_3 = (0,0,1)$. Parametrize the edge of the triangle S with a piecewise affine function $E : [0,3] \to \mathbb{R}^3$ as follows.

a) Parametrize the edge from $\vec{\mathbf{e}}_2$ to $\vec{\mathbf{e}}_3$ with a function $E_1 : [2,3] \to \mathbb{R}^3$ such that $E_1(2) = \vec{\mathbf{e}}_2$ and $E_1(3) = \vec{\mathbf{e}}_3$.

b) Parametrize the edge from $\vec{\mathbf{e}}_3$ to $\vec{\mathbf{e}}_1$ with a function $E_2 : [0,1] \to \mathbb{R}^3$ such that $E_2(0) = \vec{\mathbf{e}}_3$ and $E_2(1) = \vec{\mathbf{e}}_1$.

c) Parametrize the edge from $\vec{\mathbf{e}}_1$ to $\vec{\mathbf{e}}_2$ with a function $E_3 : [1,2] \to \mathbb{R}^3$ such that $E_3(1) = \vec{\mathbf{e}}_1$ and $E_3(2) = \vec{\mathbf{e}}_2$.

d) Provide a formula, involving logic and algebra, for the function E.

7. Let three noncollinear points (x_1, y_1), (x_2, y_2), and (x_3, y_3) in the plane be given $\mathbb{R}^2$, together with three numbers z_1, z_2, and z_3. Consider the function $p : \mathbb{R}^2 \to \mathbb{R}$ with $p(x, y) = ux + vy + w$ such that $p(x_i, y_i) = z_i$ for each $i \in \{1, 2, 3\}$. Thus, the graph of p consists of the plane through (x_i, y_i, z_i) for each $i \in \{1, 2, 3\}$. Write a linear system of simultaneous equations that admits (u, v, w) as its only solution. Express your system in terms of u, v, and w, and the given numbers x_i, y_i, z_i for $i \in \{1, 2, 3\}$.

8. Determine the formulae for a piecewise affine function $P : \mathbb{R}^2 \to \mathbb{R}$ that passes through the following points:

$$P(1,1) = 5, \quad P(3,1) = 7, \quad P(3,3) = 9, \quad P(1,3) = 1.$$

For instance, partition the square $[1, 3] \times [1, 3]$ into two triangles, and construct a suitable plane over each triangle. Express your formulae with algebra and with a logical test. To test your function, calculate $P(2, 2)$.

9. Construct a quadratic function L such that $L(1, 1) = 5$, $L(1, 3) = 1$, $L(3, 1) = 7$, and $L(3, 3) = 9$ as follows. Establish formulae for the lines ℓ_k defined for $k \in \{1, 2, 3, 4\}$ by the following specifications.

- **a)** $\ell_1(1) = 5$ and $\ell_1(3) = 7$.
- **b)** $\ell_2(1) = 1$ and $\ell_2(3) = 9$.
- **c)** $\ell_3(1) = 1$ and $\ell_3(3) = 0$.
- **d)** $\ell_4(1) = 0$ and $\ell_4(3) = 1$.
- **e)** Having determined the four lines ℓ_k for $k \in \{1, 2, 3, 4\}$, define the function $L : \mathbb{R}^2 \to \mathbb{R}$ by

$$L(x, y) := \ell_1(x)\ell_3(y) + \ell_2(x)\ell_4(y).$$

Verify that $L(1, 1) = 5$, $L(3, 1) = 7$, $L(3, 3) = 9$, $L(1, 3) = 1$, and calculate $L(2, 2)$.

10. Construct a surface through $\vec{\mathbf{b}} = (1, 1, 1)$, $\vec{\mathbf{d}} = (5, 3, 2)$, $\vec{\mathbf{h}} = (1, 2, 7)$, and $\vec{\mathbf{k}} = (2, 5, 9)$.

6. Solutions to the Exercises

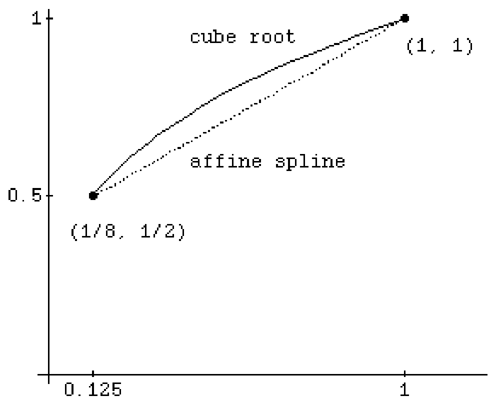

Figure 22. The affine polynomial (straight-line segment) must pass through $(1/8, 1/2)$ and through $(1, 1)$, as does the cube root.

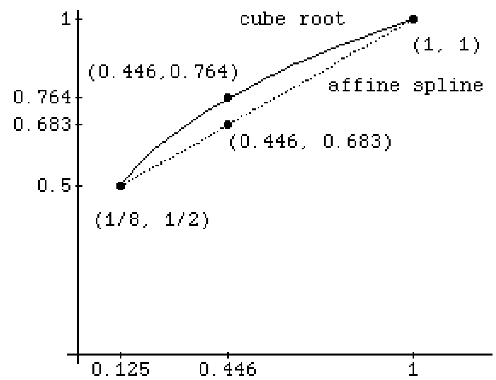

Figure 23. The maximum absolute discrepancy, $0.08\ldots$, between the cube root and the affine polynomial, occurs at $\sqrt{(7/12)^3} \approx 0.446$.

1. See **Figures 22** and **23**, with the affine spline ℓ such that

$$\ell(x) = \frac{4}{7}x + \frac{3}{7},$$

$$\left|\sqrt[3]{x} - \ell(x)\right| \leq 0.080\,604,$$

and where the maximum discrepancy of $0.080\,603\ldots$ occurs at $x = 0.445\,528\ldots$.

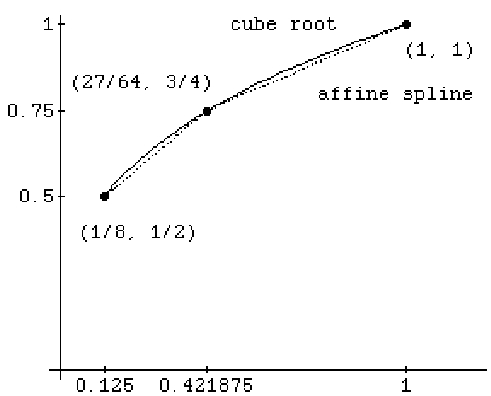

Figure 24. The affine polynomial (straight-line segment) must pass through $(1/8, 1/2)$, through $(27/64, 3/4)$, and through $(1, 1)$, as does the cube root.

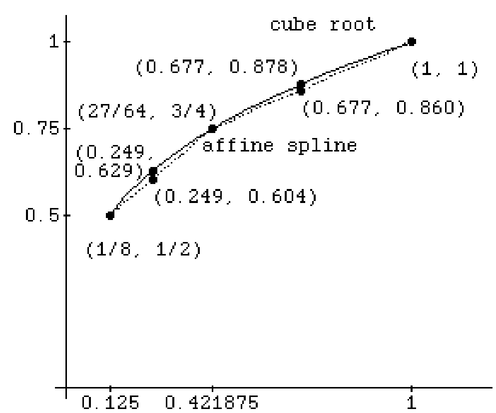

Figure 25. The maximum absolute discrepancy, $0.0247\ldots$, between the cube root and the affine polynomial, occurs at $x = 0.249\ldots$.

2. See **Figures 24** and **25**, with the affine spline ℓ such that

$$\ell(x) = \begin{cases} \frac{16}{19}x + \frac{15}{38} & \text{if } {}^1\!/_8 \leq x \leq {}^{27}\!/_{64}, \\ \\ \frac{16}{37}x + \frac{21}{37} & \text{if } {}^{27}\!/_{64} < x \leq 1, \end{cases}$$

$$\left| \sqrt[3]{x} - \ell(x) \right| \leq 0.024\,699,$$

and where the maximum discrepancy of $0.024\,698\ldots$ occurs at $x = 0.249\,039\ldots$.

3.

$$\begin{aligned} D(x) &= \sqrt{x} - (ax + c), \\ D'(x) &= \frac{1}{2\sqrt{x}} - a, \\ 0 &= \frac{1}{2\sqrt{x}} - a, \\ \sqrt{x} &= \frac{1}{2a}, \\ x &= \frac{1}{4a^2}, \\ D(1/(4a^2)) &= \sqrt{1/(4a^2)} - ([a/(4a^2)] + c) = \frac{1}{4a} - c; \\ D''(x) &= \frac{-1}{4x\sqrt{x}} < 0. \end{aligned}$$

4.

$$\begin{aligned} D(x) &= \sqrt[3]{x} - (ax + c), \\ D'(x) &= \frac{1}{3x^{2/3}} - a, \\ 0 &= \frac{1}{3x^{2/3}} - a, \\ x^{2/3} &= \frac{1}{3a}, \\ x &= \sqrt{\frac{1}{(3a)^3}} = (3a)^{-3/2}, \\ D((3a)^{-3/2}) &= \sqrt[3]{(3a)^{-3/2}} - ([a(3a)^{-3/2}] + c) = \frac{2}{3\sqrt{3a}} - c; \\ D''(x) &= \frac{-2}{9x^{5/3}}, \\ D''((3a)^{-3/2}) &= \frac{-2}{9((3a)^{-3/2})^{5/3}} \\ &= -2\sqrt{3}\,a^{5/2} < 0. \end{aligned}$$

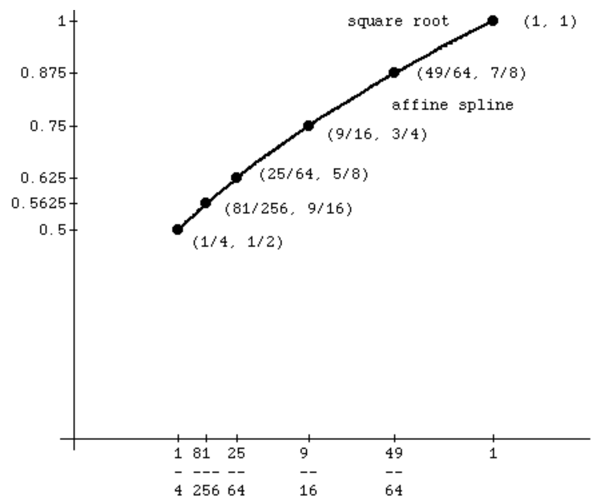

Figure 26. The affine spline (five straight-line segments) passes through the six points shown, as does the square root.

5. Figure 26 shows the affine spline ℓ, which consists of five straight-line segments:

$$\ell(x) = \begin{cases} \ell_1(x) = \frac{16}{17}x + \frac{9}{34} & \text{if } {}^1\!/_4 \le x < {}^{81}\!/_{256}, \\ & \text{with } |\sqrt{x} - \ell_1(x)| \le {}^1\!/_{1088} < 0.000\,920, \\ \ell_2(x) = \frac{16}{19}x + \frac{45}{152} & \text{if } {}^{81}\!/_{256} \le x < {}^{25}\!/_{64}, \\ & \text{with } |\sqrt{x} - \ell_2(x)| \le {}^1\!/_{1216} < 0.000\,823, \\ \ell_3(x) = \frac{8}{11}x + \frac{15}{44} & \text{if } {}^{25}\!/_{64} \le x < {}^{9}\!/_{16}, \\ & \text{with } |\sqrt{x} - \ell_3(x)| \le {}^1\!/_{352} < 0.002\,841, \\ \ell_4(x) = \frac{8}{13}x + \frac{21}{52} & \text{if } {}^{9}\!/_{16} \le x < {}^{49}\!/_{64}, \\ & \text{with } |\sqrt{x} - \ell_4(x)| \le {}^1\!/_{416} < 0.002\,404, \\ \ell_5(x) = \frac{8}{15}x + \frac{7}{15} & \text{if } {}^{49}\!/_{64} \le x \le 1, \\ & \text{with } |\sqrt{x} - \ell_5(x)| \le {}^1\!/_{480} < 0.002\,084. \end{cases}$$

Calculations as in the text confirm that $R(x) < 0.005 = (0.01)/2$, which means that the affine spline ℓ approximates the square root to two correct significant digits.

6. The affine spline ℓ must pass through the two points $(0, \sin(0))$ and $(\pi/8, \sin(\pi/8))$. Consequently, the two-point formula gives $\ell(t) = at + c$ with

$$\begin{aligned} \ell(t) &= 0 + \frac{\sin(\pi/8) - \sin(0)}{\pi/8 - 0}(t-0) \\ &= \frac{1}{\sqrt{2(\sqrt{2}+1)\sqrt{2}}}\,\frac{1}{\pi/8}\,t \\ &\approx 0.974\,495\,359t. \end{aligned}$$

With $D(t) = \sin(t) - \ell(t)$ representing the discrepancy between the sine and the affine polynomial,

$$\begin{aligned} D'(t) &= \cos(t) - a, \\ 0 &= \cos(t) - a, \\ \cos(t) &= a, \\ t &= \operatorname{Arccos}(a) = \operatorname{Arccos}\,(8\sin(\pi/8)/\pi) \approx \operatorname{Arccos}(0.974\,495\,359) \\ &= 0.226\,335\,147\ldots \\ D(t) &\approx D(0.226\,335\,147) = \sin(0.226\,335\,147) - \ell(0.226\,335\,147) \\ &\approx 0.224\,407\,656 - 0.220\,562\,550 = 0.003\,845\ldots \\ D''(t) &= -\sin(t) < 0. \end{aligned}$$

Therefore, $0 \le \sin(t) - \ell(t) < 0.004$ for every $t \in [0, \pi/8]$.

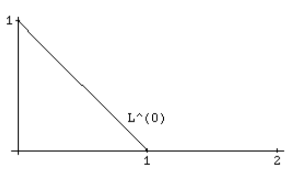

Figure 27. The basis affine spline $\ell^{(0)}$.

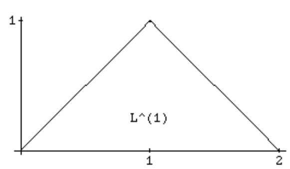

Figure 28. The basis affine spline $\ell^{(1)}$.

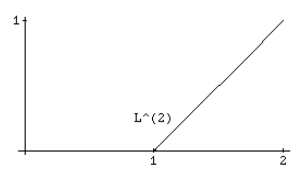

Figure 29. The basis affine spline $\ell^{(2)}$.

7. Use a straight-line segment over each subinterval, as in **Figures 27**, **28**, and **29**.

a) If $0 \le x \le 1$ then $\ell^{(0)}(x) = 1 - x$, whereas if $1 \le x \le 2$ then $\ell^{(0)}(x) = 0$.

b) If $0 \le x \le 1$ then $\ell^{(1)}(x) = x$, whereas if $1 \le x \le 2$ then $\ell^{(1)}(x) = 2 - x$.

c) If $0 \le x \le 1$ then $\ell^{(2)}(x) = 0$, whereas if $1 \le x \le 2$ then $\ell^{(2)}(x) = x - 1$.

d) $\ell(0) = y_0\ell^{(0)}(0) + y_1\ell^{(1)}(0) + y_2\ell^{(2)}(0) = y_0 \times 1 + 0 + 0 = y_0$,
$\ell(1) = y_0\ell^{(0)}(1) + y_1\ell^{(1)}(1) + y_2\ell^{(2)}(1) = 0 + y_1 \times 1 + 0 = y_1$,
$\ell(2) = y_0\ell^{(0)}(2) + y_1\ell^{(1)}(2) + y_2\ell^{(2)}(2) = 0 + 0 + y_2 \times 1 = y_2$.

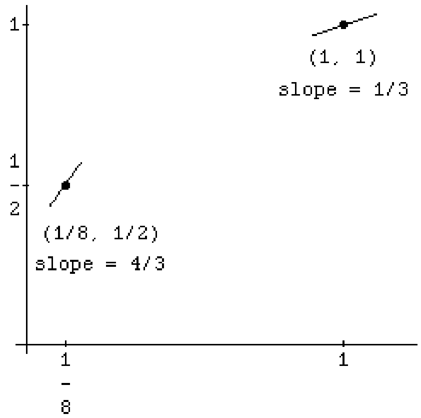

Figure 30. The cubic polynomial q must pass through $(1/8, 1/2)$ with slope $4/3$, and through $(1,1)$ with slope $1/3$.

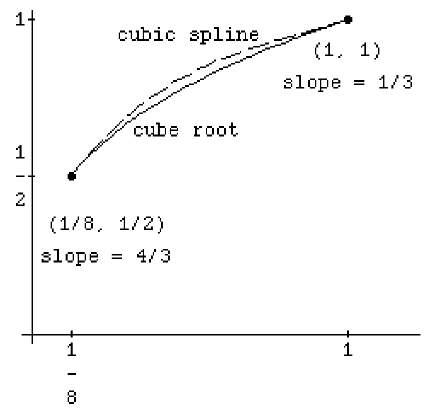

Figure 31. The cubic polynomial q (dashed curve) roughly approximates the cube root (solid curve).

8. See **Figures 30** and **31**.

$$\begin{aligned} q(x) &= \frac{1}{2} + \frac{4}{3}(x - 1/8) - \frac{216}{147}(x - 1/8)^2 + \frac{704}{1029}(x - 1/8)^3, \\ q(27/64) &= 0.784\,230\ldots = 1{,}101{,}787/1{,}404{,}928 \\ \sqrt[3]{27/64} &= 0.75 = 3/4 \end{aligned}$$

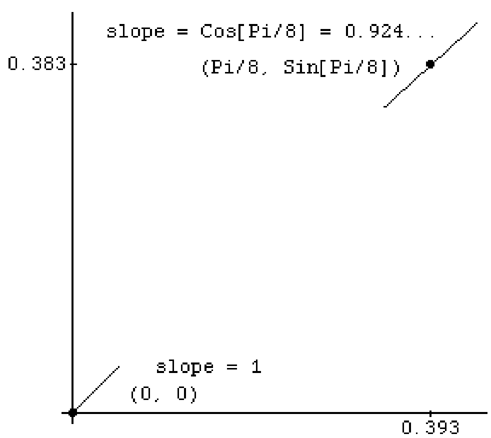

Figure 32. The cubic polynomial s must pass through $(0,0)$ with slope 1, and through $(\pi/8, \sin(\pi/8))$ with slope $\cos(\pi/8)$.

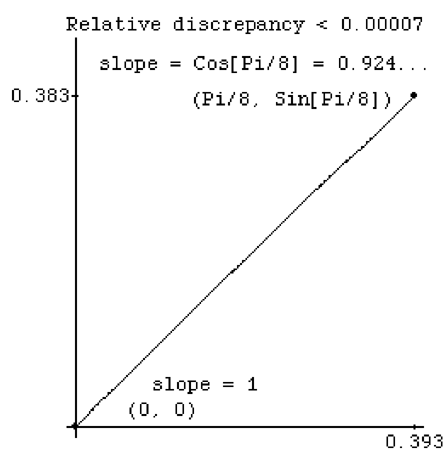

Figure 33. The cubic polynomial s (dotted curve) approximates the sine function (solid curve).

9. See **Figures 32** and **33**. With coefficients rounded to twelve significant digits,

$$s(t) = t - 0.001\,001\,930\,782t^2 - 0.162\,834\,874\,604t^3.$$

$$\begin{aligned} s(\pi/12) = s(0.261\,799\ldots) &= 0.258\,809\ldots \\ \sin(\pi/12) = \sin([\pi/6]/2) = \sqrt{\frac{1-\cos(\pi/6)}{2}} & \\ = \sqrt{\frac{1-\sqrt{3}/2}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} &= 0.258\,819\ \ldots . \end{aligned}$$

10. $\sqrt[3]{-27} = -\sqrt[3]{8^2 \times {}^{27}\!/_{64}} = -2^2\sqrt[3]{{}^{27}\!/_{64}} \approx -2^2 q({}^{27}\!/_{64}) = -3.136\ldots .$

11. Apply repeatedly the double-angle formula for the sine:

$$\begin{aligned} \sin(\pi/3) &= \sin(2[\pi/6]) = \sin(2^2[\pi/12]); \\ \sin(\pi/12) &\approx s(\pi/12) = 0.258\,809\ldots, \\ \sin(\pi/6) &= 2\sin(\pi/12)\cos(\pi/12) = 2\sin(\pi/12)\sqrt{1-[\sin(\pi/12)]^2} \\ &\approx 2 \times 0.258\,809 \times \sqrt{1-[0.258\,809]^2} \\ &\approx 0.499\,982; \\ \sin(\pi/3) &= 2\sin(\pi/6)\cos(\pi/6) = 2\sin(\pi/6)\sqrt{1-[\sin(\pi/6)]^2} \\ &\approx 2 \times 0.499\,982\sqrt{1-[0.499\,982]^2} \\ &\approx 0.866\,005; \\ \sin(\pi/3) &= 0.866\,025\ldots = \sqrt{3}/2. \end{aligned}$$

12. With $p(x) = y + w(x-x_0) + v(x-x_0)^2 + u(x-x_0)^3$, the prescribed conditions translate into the linear system

$$\begin{cases} y & & & & = y_0, \\ & w & & & = y_0', \\ y & + w(x_1-x_0) & + v(x_1-x_0)^2 & + u(x_1-x_0)^3 & = y_1, \\ & w & + 2v(x_1-x_0) & + 3u(x_1-x_0)^2 & = y_1'. \end{cases}$$

The first and second equations immediately give $y = y_0$ and $w = y_0'$. Substituting these values into the third and fourth equations give

$$\begin{cases} v(x_1-x_0)^2 + u(x_1-x_0)^3 = y_1 - y_0 - y_0'(x_1-x_0), \\ 2v(x_1-x_0) + 3u(x_1-x_0)^2 = y_1' - y_0'. \end{cases}$$

The resulting system has exactly one solution, because its determinant does not equal zero:

$$(x_1-x_0)^2 \times 3(x_1-x_0)^2 - 2(x_1-x_0)(x_1-x_0)^3 = (x_1-x_0)^4 > 0.$$

13. The preceding exercise shows that exactly one cubic polynomial passes through both points with the prescribed slopes. Yet the straight line joining both points *is* such a cubic polynomial, because it satisfies the point-slope equation

$$y = y_0 + y_0'(x-x_0) + 0(x-x_0)^2 + 0(x-x_0)^3.$$

By uniqueness, the cubic polynomial coincides with the straight line.

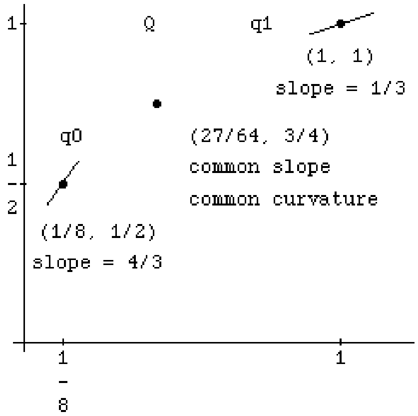

Figure 34. The cubic polynomial q_0 must pass through $(^1/_8, {^1/_2})$ with slope $^4/_3$, the cubic polynomial q_1 must pass through $(1,1)$ with slope $^1/_3$, and both q_0 and q_1 must pass through $(^{27}/_{64}, {^3/_4})$ with a common slope and a common curvature.

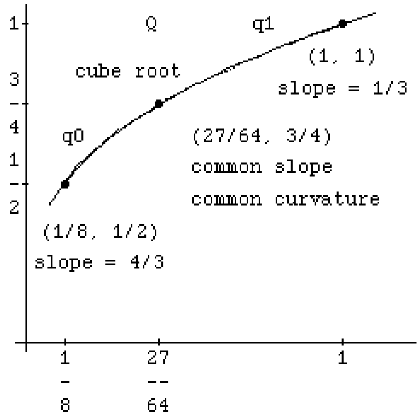

Figure 35. The cubic polynomial q_0 (dashed) passes through $(^1/_8, {^1/_2})$ with slope $^4/_3$, the cubic polynomial q_1 passes through $(1,1)$ with slope $^1/_3$, as does the cube root (solid), and both q_0 and q_1 pass through $(^{27}/_{64}, {^3/_4})$ with a common slope and a common curvature.

14. See **Figures 34** and **35**. With coefficients rounded to six significant digits,

$$\begin{aligned} q_0(x) &= {}^1\!/_2 + ({}^4\!/_3)(x - {}^1\!/_8) - 2.351127(x - {}^1\!/_8)^2 \\ &\quad + 2.345983(x - {}^1\!/_8)^3, \\ q_1(x) &= {}^3\!/_4 + 0.557640(x - {}^{27}\!/_{64}) - 0.261735(x - {}^{27}\!/_{64})^2 \\ &\quad + 0.0781153(x - {}^{27}\!/_{64})^3. \end{aligned}$$

$$\begin{aligned} Q({}^8\!/_{27}) = q_0({}^8\!/_{27}) &= 0.671\,199\ldots \\ \sqrt[3]{{}^8\!/_{27}} &= 0.666\,666\ldots \qquad \sqrt[3]{{}^{64}\!/_{125}} = 0.8. \end{aligned}$$

$$Q({}^{64}\!/_{125}) = q_1({}^{64}\!/_{125}) = 0.798\,189\ldots, \qquad \sqrt[3]{{}^{64}\!/_{125}} = 0.8$$

15. See **Figures 36** and **37**. With coefficients rounded to six significant digits,

$$\begin{aligned} s_0(t) &= 0 + t - 0.0000641766t^2 - 0.166019t^3, \\ s_1(t) &= 0.195090 + 0.980773(t - \pi/16) - 0.0978573(t - \pi/16)^2 \\ &\quad -0.159651(t - \pi/16)^3. \end{aligned}$$

$$\begin{aligned} S(\pi/24) = s_0(\pi/24) &= 0.130\,526\,225\ldots \\ \sin(\pi/24) &= 0.130\,526\,192\ldots, \\ S(\pi/12) = s_1(\pi/12) &= 0.258\,817\,824\ldots, \\ \sin(\pi/12) &= 0.258\,819\,045\ldots. \end{aligned}$$

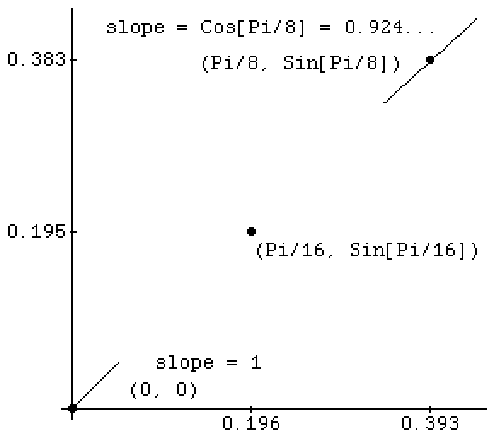

Figure 36. The cubic polynomial s_0 must pass through $(0,0)$ with slope 1, while s_1 must pass through $(\pi/8, \sin(\pi/8))$ with slope $\cos(\pi/8)$, and both s_0 and s_1 must pass through $(\pi/16, \sin(\pi/16))$ with a common slope and a common curvature.

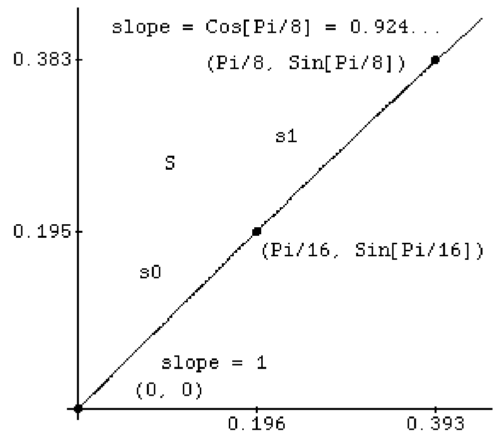

Figure 37. The cubic polynomial s_0 passes through $(0,0)$ with slope 1, while s_1 passes through $(\pi/8, \sin(\pi/8))$ with slope $\cos(\pi/8)$, as does the sine, and both s_0 and s_1 pass through $(\pi/16, \sin(\pi/16))$ with a common slope and a common curvature.

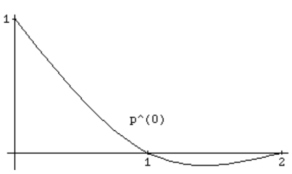

Figure 38. The basis cubic spline $p^{(0)}$.

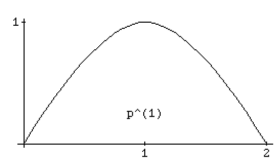

Figure 39. The basis cubic spline $p^{(1)}$.

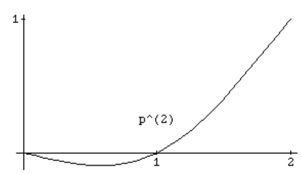

Figure 40. The basis cubic spline $p^{(2)}$.

16. See **Figures 38**, **39**, and **40**.

a) $p^{(0)}(x)$ equals $p_0^{(0)}(x) := 1 - (5/4)x + (1/4)x^3$ if $x \in [0,1]$, and $p_1^{(0)}(x) := (1/4)(x-2) - (1/4)(x-2)^3$ if $x \in [1,2]$.

b) $p^{(1)}(x)$ equals $p_0^{(1)}(x) := (3/2)x - (1/2)x^3$ if $x \in [0,1]$, and $p_1^{(1)}(x) := -(3/2)(x-2) + (1/2)(x-2)^3$ if $x \in [1,2]$.

c) $p^{(2)}(x)$ equals $p_0^{(2)}(x) := -(1/4)x + (1/4)x^3$ if $x \in [0,1]$, and $p_1^{(2)}(x) := 1 + (5/4)(x-2) - (1/4)(x-2)^3$ if $x \in [1,2]$.

17. For every $x \in [1/2, 1]$,

$$y = (2-\sqrt{2})x + (9\sqrt{2}-6)/16 \approx 0.5857864x + 0.4204951,$$
$$|\sqrt{x} - (0.5857864x + 0.4204951)| \leq 0.006281538\ldots < 0.00629.$$

18. For every $x \in [{}^1\!/_4, {}^1\!/_2]$,

$$y = 2(\sqrt{2}-1)x + 3(3-\sqrt{2})/16 \approx 0.8284271x + 0.2973350,$$
$$|\sqrt{x} - (0.8284271x + 0.2973350)| \leq 0.004441732\ldots < 0.00445.$$

19. The straight line with equation $y = 0.5857864x + 0.4204951$ minimizes the maximum absolute value of the discrepancy with the square root, which occurs at the endpoints and at an intermediate point different from the midpoint, $x = 1/(4[0.5857864]^2) = 0.728553\ldots \neq 0.75 = {}^3\!/_4$.

20. The straight line with equation $y = 0.8284271x + 0.3012412$ does not minimize the maximum absolute value of the discrepancy with the square root, because evaluating the discrepancy at the endpoints and at the midpoint gives different values, at least one of which exceeds the minimum value of 0.00445 found previously in **Exercise 18**:

$$\begin{aligned}
\left|\sqrt{{}^1\!/_4} - (0.8284271({}^1\!/_4) + 0.3012412)\right| &= 0.008347975\ldots > 0.00445,\\
\left|\sqrt{{}^3\!/_8} - (0.8284271({}^3\!/_8) + 0.3012412)\right| &= 0.0004710732\ldots,\\
\left|\sqrt{{}^1\!/_2} - (0.8284271({}^1\!/_2) + 0.3012412)\right| &= 0.008347969\ldots > 0.00445.
\end{aligned}$$

21.
$$y = \frac{4}{7}x + \frac{1}{2}\cdot\left(\frac{3}{7} + \frac{\sqrt{7}}{3\sqrt{3}}\right) \approx 0.5714286x + 0.4688733,$$

with $|\sqrt[3]{x} - (0.5714286x + 0.4688733)| \leq 0.040301824\ldots < 0.04031$ on the interval $[{}^1\!/_8, 1]$.

22. The function f is concave, if, but only if, for all $p < q$ in $]r, s[$ and for every $x \in [p, q]$, $f(x) \geq f(p) + ([f(q) - f(p)]/[q-p]) \cdot (x-p)$, which, with $f = -g$, is equivalent to $-g(x) \geq -g(p) + ([-g(q) - \{-g(p)\}]/[q-p]) \cdot (x-p)$, whence taking the additive opposite of each side reverses the inequality and yields $g(x) \leq g(p) + ([g(q) - g(p)]/[q-p]) \cdot (x-p)$, which means that g is convex.

23. If g is convex, then, by the preceding exercise, the function $f := -g$ is concave. Let $\ell_f(x) = a_f x + c_f$ correspond to the straight line ℓ_f that minimizes the maximum absolute value of the discrepancy $D = f - \ell_f$ among all straight lines, and let $\ell_g := -\ell_f$. Because $g - \ell_g = (-f) - (-\ell_f) = -(f - \ell_f) = -D$, it follows that ℓ_g minimizes the maximum absolute value of the discrepancy $g - \ell_g$ among all straight lines.

24.
$$L:\ [0,1] \to \mathbb{R}^2,$$

$$\begin{aligned} L(t) &= (1-t)\begin{pmatrix}\cos(2\pi i/8)\\ \sin(2\pi i/8)\end{pmatrix} + t\begin{pmatrix}\cos(2\pi(i+1)/8)\\ \sin(2\pi(i+1)/8)\end{pmatrix} \\ &= \begin{pmatrix}(1-t)\cos(2\pi i/8) + t\cos(2\pi(i+1)/8)\\ (1-t)\sin(2\pi i/8) + t\sin(2\pi(i+1)/8)\end{pmatrix}. \end{aligned}$$

25. See **Figures 16** and **17**, where

$$S(t) = \begin{pmatrix}1+10t-24t^2+16t^3\\ 1-10t+36t^2-24t^3\end{pmatrix}.$$

26. See **Figures 18** and **19**, where

$$H(t) = \begin{pmatrix}3+t+4t^2-3t^3\\ 1+5t^2-3t^3\\ 1+3t^2-2t^3\end{pmatrix}.$$

27. With only three vertices $P_0 = (x_0, y_0)$, $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, and $P_3 = P_0$, **Theorem 3** becomes

$$\begin{aligned} \text{Area}\,(P) &= \frac{1}{2}\sum_{i=1}^{3} \det\left(P_{i-1}, P_i\right) \\ &= \frac{1}{2}\left(\det\left(P_0, P_1\right) + \det\left(P_1, P_2\right) + \det\left(P_2, P_0\right)\right) \\ &= \frac{1}{2}\left(x_0y_1 - y_0x_1 + x_1y_2 - y_1x_2 + x_2y_0 - y_2x_0\right). \end{aligned}$$

28. Apply Green's Theorem to the boundary ∂A of the region A enclosed by the curve W, as in **Figure 20**:

$$\begin{aligned} \text{Area} &= \int\int_A dx\,dy = \frac{1}{2}\oint_{\partial A} x dy - y dx \\ &= \frac{1}{2}\int_0^2 \left(X(t)Y'(t) - Y(t)X'(t)\right)\,dt \\ &= \frac{1}{2}\int_0^2 \big((t^3-4t)(3t^2-6t+2) - (t^3-3t^2+2t)(3t^2-4)\big)\,dt \\ &= \frac{1}{2}\int_0^2 \left(3t^4 - 12t^3 + 12t^2\right)\,dt \\ &= \frac{3}{2}\,t^3\,\frac{3t^2-15t+20}{15}\Bigg|_0^2 = \frac{8}{5}. \end{aligned}$$

29. The surface is a hyperbolic paraboloid. It consists of two networks of straight line segments: segments leaning on the two edges $[\vec{\mathbf{b}}, \vec{\mathbf{d}}]$ and $[\vec{\mathbf{h}}, \vec{\mathbf{k}}]$, and segments leaning on the two edges $[\vec{\mathbf{b}}, \vec{\mathbf{h}}]$ and $[\vec{\mathbf{d}}, \vec{\mathbf{k}}]$.

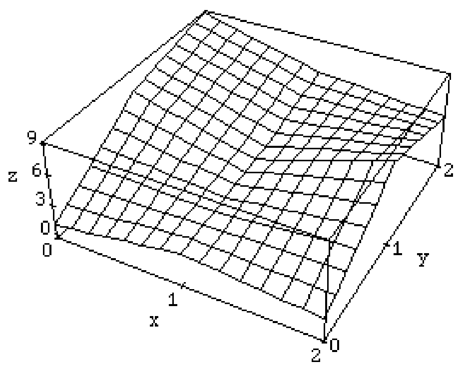

Figure 41. The quadratic spline f.

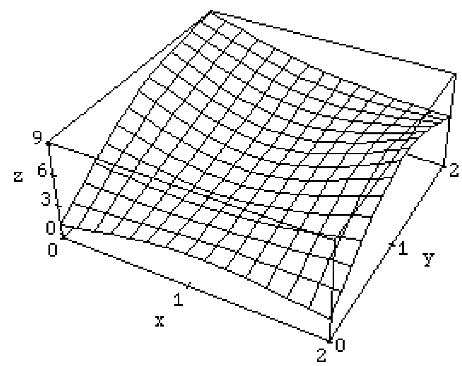

Figure 42. The cubic spline g.

30. See **Figure 41**. One solution uses products of the basis splines established in **Exercise 7**. Recall that the affine spline $\ell^{(k)}$ takes the value 1 at k and 0 at every $h \neq k$. Consequently, the product $\ell^{(p,q)}(x,y) := \ell^{(p)}(x)\ell^{(q)}(y)$ takes the value 1 at (p,q) and 0 at every other $(h,k) \neq (p,q)$. Also, the continuity of every basis spline $\ell^{(k)}$ ensures the continuity of the products $\ell^{(p,q)}(x,y) = \ell^{(p)}(x)\ell^{(q)}(y)$, because the product of two continuous functions is again continuous. Therefore, among many other solutions, the following quadratic spline $f: [0,2] \times [0,2] \to \mathbb{R}$ meets the requirements:

$$\begin{aligned} f(x,y) &= 9\ell^{(0)}(x)\ell^{(2)}(y) + 6\ell^{(1)}(x)\ell^{(2)}(y) + 5\ell^{(2)}(x)\ell^{(2)}(y) \\ &\quad +7\ell^{(0)}(x)\ell^{(1)}(y) + 4\ell^{(1)}(x)\ell^{(1)}(y) + 8\ell^{(2)}(x)\ell^{(1)}(y) \\ &\quad +1\ell^{(0)}(x)\ell^{(0)}(y) + 3\ell^{(1)}(x)\ell^{(0)}(y) + 2\ell^{(2)}(x)\ell^{(0)}(y). \end{aligned}$$

31. See **Figure 42**. One solution uses products of the basis splines established in **Exercise 16**. Recall that the cubic spline $p^{(k)}$ takes the value 1 at k and 0 at every $h \neq k$. Consequently, the product $p^{(r,s)}(x,y) := p^{(r)}(x)p^{(s)}(y)$ takes the value 1 at (r,s) and 0 at every other $(h,k) \neq (r,s)$. Also, the products $p^{(r,s)}(x,y) = p^{(r)}(x)p^{(s)}(y)$ are of class C^2, because the product of two functions of class C^2 is again of class C^2. Therefore, among many other solutions, the following cubic spline $g: [0,2] \times [0,2] \to \mathbb{R}$ meets the requirements:

$$\begin{aligned} g(x,y) &= 9p^{(0)}(x)p^{(2)}(y) + 6p^{(1)}(x)p^{(2)}(y) + 5p^{(2)}(x)p^{(2)}(y) \\ &\quad +7p^{(0)}(x)p^{(1)}(y) + 4p^{(1)}(x)p^{(1)}(y) + 8p^{(2)}(x)p^{(1)}(y) \\ &\quad +1p^{(0)}(x)p^{(0)}(y) + 3p^{(1)}(x)p^{(0)}(y) + 2p^{(2)}(x)p^{(0)}(y). \end{aligned}$$

32. Begin with the parametric surface in **Example 17** for the unit sphere,

$$F : [-\pi, \pi] \times [-\pi/2, \pi/2] \to \mathbb{R}^3,$$

$$F(u,v) = \begin{pmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \end{pmatrix} = \begin{pmatrix} \cos(u)\cos(v) \\ \sin(u)\cos(v) \\ \sin(v) \end{pmatrix}.$$

Then partition the intervals $[-\pi, \pi]$ and $[-\pi/2, \pi/2]$ into subintervals, $[u_i, u_{i+1}]$ in $[-\pi, \pi]$ for $i \in \{0, \ldots, m\}$, and $[v_j, v_{j+1}]$ in $[-\pi/2, \pi/2]$ for $j \in \{0, \ldots, n\}$. Then join the four points $\vec{\mathbf{b}} = F(u_i, v_j)$, $\vec{\mathbf{d}} = F(u_{i+1}, v_j)$, $\vec{\mathbf{h}} = F(u_i, v_{j+1})$, and $\vec{\mathbf{k}} = F(u_{i+1}, v_{j+1})$ as in **Example 16**,

$$S_{i,j} : [0,1] \times [0,1] \to \mathbb{R}^3,$$

$$\begin{aligned} S_{i,j}(u,v) &= \begin{pmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \end{pmatrix} \\ &= (1-v)[(1-u)F(u_i, v_j) + uF(u_{i+1}, v_j)] \\ &\qquad + v[(1-u)F(u_i, v_{j+1}) + uF(u_{i+1}, v_{j+1})]. \end{aligned}$$

33. Three faces of the tetrahedron contain the origin, whence the corresponding determinants vanish because they contain the column $\vec{\mathbf{0}}$. Consequently, only one nonzero determinant remains, and **Theorem 4** gives

$$\text{Volume}(T) = \frac{1}{6} \det(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3) = \frac{1}{6}.$$

As a verification, a triple integral gives

$$\begin{aligned} \text{Volume}(T) &= \int\int\int_T dx\,dy\,dz \\ &= \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-(x+y)} dz\,dy\,dx \\ &= \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} (1-[x+y])\; dy\,dx \\ &= \int_{x=0}^{x=1} \frac{1}{2}(1-x)^2\,dx = -\frac{1}{6}(1-x)^3 \Big|_0^1 = \frac{1}{6}. \end{aligned}$$

34. Place one vertex at the origin, $\vec{\mathbf{0}} = (0,0,0)$, and the other three vertices at selected vertices of the unit cube: $\vec{\mathbf{v}}_1 := (1,0,1)$, $\vec{\mathbf{v}}_2 := (1,1,0)$, and $\vec{\mathbf{v}}_3 := (0,1,1)$. Because every edge of the resulting tetrahedron consists of a diagonal of a face of the unit cube, all edges have the specified length. Also, because one of the vertices lies at the origin, three determinants contains the column $\vec{\mathbf{0}}$ and thus vanish. Therefore, theorem 4 yields

$$\text{Volume}\,(T) = \frac{1}{6}\,\det\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 1 & 0 & 1\end{pmatrix} = \frac{1}{6}\times 2 = \frac{1}{3}.$$

35. Apply Stokes' Theorem to the volume V enclosed by the surface ∂V, and use the result of **Exercise 28** to calculate the area of the vertical cross-sections:

$$\begin{aligned}
\text{Volume} &= \int\!\!\int\!\!\int_V dx\,dy\,dz\\
&= \frac{1}{2}\int\!\!\int_{\partial V} x\,dy\,dz - z\,dy\,dx\\
&= \frac{1}{2}\int\left(\int_{\partial V} x\,dz - z\,dx\right)\,dy\\
&= \int_0^4 \frac{8}{5}\left(\frac{5-v}{5}\right)^2\,dv = \frac{8}{5}\times\frac{124}{75} = \frac{992}{375} = 2.645\,333\ldots,
\end{aligned}$$

because multiplying by $([5-v]/5)^2$ the calculations in **Exercise 28**, but here with x and z each multiplied by $([5-v]/5)$ instead of only x and y, gives $\int_{\partial V} x\,dz - z\,dx = \cdots = ([5-v]/5)^2 \times 8/5$.

7. Solutions to Sample Exam Problems

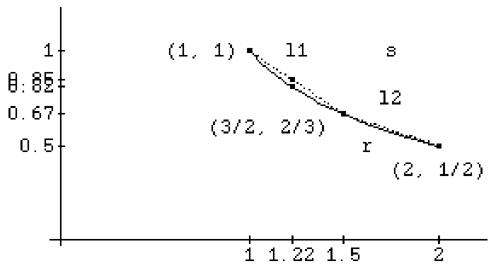

Figure 43. The affine spline s (straight-line segments ℓ_1 and ℓ_2) pass through $(1,1)$, $(3/2, 2/3)$, and $(2, 1/2)$, as does the reciprocal function r: $|r(x) - s(x)| < 0.034$ for every $x \in [1,2]$.

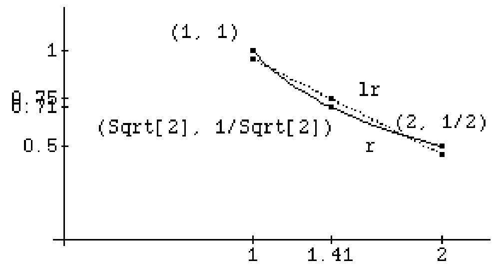

Figure 44. The straight line ℓ_r minimizes the maximum absolute value of the discrepancy, $|r - \ell|$, among all straight lines ℓ over $[1,2]$: $|r(x) - \ell_r(x)| < 0.043$ for every $x \in [1,2]$.

1. a)

$$s(x) = \begin{cases} \ell_1(x) := (5-2x)/3 & \text{if } 1 \le x \le 3/2, \\ \ell_2(x) := (7-2x)/6 & \text{if } 3/2 \le x \le 2. \end{cases}$$

b) $D(x) = r(x) - s(x)$ reaches its maximum absolute value at $x = \sqrt{3/2}$, where $D(\sqrt{3/2}) = -0.033\,673\ldots$.

c) See **Figure 43**.

2. a) $a_r = -1/2$ and $c_r = (3+2\sqrt{2})/4$, whence $\ell_r(x) = -(1/2)x + (3+2\sqrt{2})/4$.

b) The discrepancy $D := r - \ell_r$ reaches its maximum absolute value at $x = \sqrt{2}$, where $D(1) = -D(\sqrt{2}) = D(2) = 0.042\,893\ldots$.

c) See **Figure 44**.

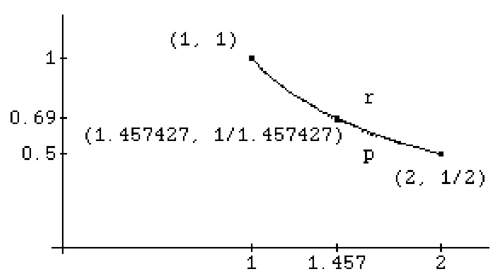

Figure 45. The cubic polynomial p passes through $(1,1)$ and $(2, 1/2)$ with the same slope as does the reciprocal function r: $|r(x) - p(x)| < 0.011$ for every $x \in [1,2]$.

3. a) $y = 1$, $w = -1$, $v = 3/4$, and $u = -1/4$, whence

$$p(x) := 1 - (x-1) + (3/4)(x-1)^2 - (1/4)(x-1)^3.$$

b) $D(3/2) := r(3/2) - p(3/2) = 0.0104\ldots$.

c) Examine the endpoints, where D vanishes, and solve $D'(x) = 0$ numerically. The quantity $D = r - p$ reaches its maximum absolute value over the interval $[1, 2]$ at $x = 1.457\,427\,107\,756\ldots$, where
$D(1.457\,427\,107\,756\ldots) = 0.010\,566\ldots$.

d) See **Figure 45**.

4. a) $P(x) = (x-1) - (1/2)(x-1)^2 + (1/4)(x-1)^3 - (1/16)(x-1)^4$.

b) $\int_1^2 p(x)\,dx = P(2) - P(1) = 11/16 - 0 = 11/16 = 0.6875$.

c) $\left|\int_1^2 x^{-1}\,dx - \int_1^2 p(x)\,dx\right| \le \int_1^2 \left|x^{-1} - p(x)\right|\,dx < \int_1^2 0.0106\,dx = 0.0106$.

d) $L = P$.

5. a) For each $x > 0$, multiply or divide x by a suitable power of 2 so that $2^{-k}x \in [1, 2]$. Then $\ln(x) = \ln(2^k 2^{-k} x) = k\ln(2) + \ln(2^{-k}x)$. Because $2^{-k}x \in [1, 2]$ use the approximation $\ln(2^{-k}x) \approx L(2^{-k}x)$; to estimate $\ln(2)$, borrow the approximation $\ln(2) \approx 0.6875$ from **Problem 4**. Thus, $\ln(x) = \ln(2^k 2^{-k}x) = k\ln(2) + \ln(2^{-k}x) \approx 0.6875k + L(2^{-k}x)$.

b) $17 = 2^4 \times 17/2^4$, whence $\ln(17) = 4\ln(2) + \ln(17/16) \approx 0.6875 \times 4 + L(17/16) = 2.810\ldots$. Compare with $\ln(17) = 2.833\ldots$.

6. a) $E_1(t) = (1-[t-2])\vec{\mathbf{e}}_2 + (t-2)\vec{\mathbf{e}}_3 = (3-t)(0,1,0) + (t-2)(0,0,1) = (0, 3-t, t-2)$.

b) $E_2(t) = (1-t)\vec{\mathbf{e}}_3 + t\vec{\mathbf{e}}_1 = (1-t)(0,0,1) + t(1,0,0) = (t, 0, 1-t)$.

c) $E_3(t) = (1-[t-1])\vec{\mathbf{e}}_1 + (t-1)\vec{\mathbf{e}}_2 = (2-t)(1,0,0) + (t-1)(0,1,0) = (2-t, t-1, 0)$.

d)

$$E(t) = \begin{cases} E_2(t) = (t, 0, 1-t) & \text{if } 0 \le t \le 1, \\ E_3(t) = (2-t, t-1, 0) & \text{if } 1 \le t \le 2, \\ E_1(t) = (0, 3-t, t-2) & \text{if } 2 \le t \le 3. \end{cases}$$

7.

$$\begin{cases} p(x_1, y_1) = z_1 & : & x_1 u & + & y_1 v & + & w & = & z_1, \\ p(x_2, y_2) = z_2 & : & x_2 u & + & y_2 v & + & w & = & z_2, \\ p(x_3, y_3) = z_3 & : & x_3 u & + & y_3 v & + & w & = & z_3. \end{cases}$$

8. There are two ways to partition the square $S := [1,3] \times [1,3]$ into two triangles, one way for each diagonal. The two partitions give two functions, P and Q, which need not coincide with each other.

The first way partitions S through the diagonal from $(1,1)$ to $(3,3)$ and gives the two triangles

T_1 with vertices at $(1,1)$, $(3,1)$, and $(3,3)$, and

T_2 with vertices at $(1,1)$, $(3,3)$, and $(1,3)$.

On T_1, the function P reduces to an affine function p_1 with $p_1(x,y) = u_1 x + v_1 y + w_1$ with

$$\begin{cases} 1u_1 & + & 1v_1 & + & w_1 & = & 5, \\ 3u_1 & + & 1v_1 & + & w_1 & = & 7, \\ 3u_1 & + & 3v_1 & + & w_1 & = & 9. \end{cases}$$

Solving the system yields $u_1 = 1$, $v_1 = 1$, and $w_1 = 3$, so that $p_1(x,y) = x + y + 3$.

On T_2, the function P reduces to an affine function p_2 with $p_2(x,y) = u_2 x + v_2 y + w_2$ with

$$\begin{cases} 1u_2 & + & 1v_2 & + & w_2 & = & 5, \\ 1u_2 & + & 3v_2 & + & w_2 & = & 1, \\ 3u_2 & + & 3v_2 & + & w_2 & = & 9. \end{cases}$$

Solving the system yields $u_2 = 4$, $v_2 = -2$, and $w_2 = 3$, so that $p_1(x,y) = 4x - 2y + 3$. At $(2,2)$, $P(2,2) = p_1(2,2) = 7 = p_2(2,2)$.

The second way partitions S through the diagonal from $(1,3)$ to $(3,1)$ and gives the two triangles

T_3 with vertices at $(1,1)$, $(3,1)$, and $(1,3)$, and

T_4 with vertices at $(1,3)$, $(3,1)$, and $(3,3)$.

On T_3, the function Q reduces to an affine function q_3 with $q_3(x,y) = u_3 x + v_3 y + w_3$ with

$$\begin{cases} 1u_3 & + & 1v_3 & + & w_3 & = & 5, \\ 3u_3 & + & 1v_3 & + & w_3 & = & 7, \\ 1u_3 & + & 3v_3 & + & w_3 & = & 1. \end{cases}$$

Solving the system yields $u_3 = 1$, $v_3 = -2$, and $w_3 = 6$, so that $q_3(x,y) = x - 2y + 6$.

On T_4, the function Q reduces to an affine function q_4 with $q_4(x,y) = u_4 x + v_4 y + w_4$ with

$$\begin{cases} 3u_4 & + & 3v_4 & + & w_4 & = & 9, \\ 3u_4 & + & 1v_4 & + & w_4 & = & 7, \\ 1u_4 & + & 3v_4 & + & w_4 & = & 1. \end{cases}$$

Solving the system yields $u_4 = 4$, $v_4 = 1$, and $w_4 = -6$, so that $q_4(x,y) = 4x + y - 6$.

At $(2,2)$, $Q(2,2) = q_3(2,2) = 4 = q_4(2,2)$. Notice that $P(2,2) = 7 \neq 4 = Q(2,2)$, even though P and Q interpolate the same data.

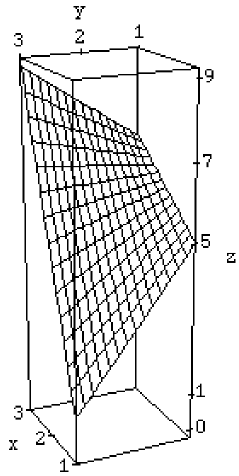

Figure 46. The quadratic function L shown here has values $L(1,1) = 5$, $L(1,3) = 1$, $L(3,1) = 7$, and $L(3,3) = 9$.

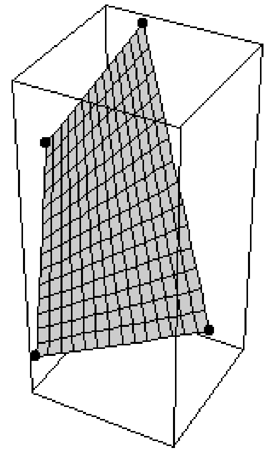

Figure 47. The quadratic spline surface patch exhibited here passes through $\vec{\mathbf{b}} = (1,1,1)$, $\vec{\mathbf{d}} = (5,3,2)$, $\vec{\mathbf{h}} = (1,2,7)$, and $\vec{\mathbf{k}} = (2,5,9)$.

9. **Figure 46** shows the function L, obtained as in **Remark 4** and **Exercises 7** and **30**.

a) $\ell_1(t) = t + 4$.

b) $\ell_2(t) = 4t - 3$.

c) $\ell_3(t) = (3 - t)/2$.

d) $\ell_4(t) = (t - 1)/2$.

e)
$$\begin{aligned} L(x,y) &= \ell_1(x)\ell_3(y) + \ell_2(x)\ell_4(y) \\ &= (1/2)\,\{(x+4)(3-y) + (4x-3)(y-1)\} \\ &= (1/2)\,\{3xy - x - 7y + 15\}. \end{aligned}$$

$L(1,1) = 5, \quad L(1,3) = 1, \quad L(3,1) = 7, \quad L(3,3) = 9. \;\; L(2,2) = 11/2$.

10. The following function $S : \mathbb{R}^2 \to \mathbb{R}^3$, shown in **Figure 47**, passes through the four points:

$$\begin{aligned}
S(u,v) &= (1-v)\left((1-u)\vec{\mathbf{b}} + u\vec{\mathbf{d}}\right) + v\left((1-u)\vec{\mathbf{h}} + u\vec{\mathbf{k}}\right) \\
&= \begin{pmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \end{pmatrix} \\
&= \begin{pmatrix} (1-v)\left((1-u)b_1 + ud_1\right) + v\left((1-u)h_1 + uk_1\right) \\ (1-v)\left((1-u)b_2 + ud_2\right) + v\left((1-u)h_2 + uk_2\right) \\ (1-v)\left((1-u)b_3 + ud_3\right) + v\left((1-u)h_3 + uk_3\right) \end{pmatrix} \\
&= \begin{pmatrix} (1-v)\left(1(1-u) + 5u\right) + v\left(1(1-u) + 2u\right) \\ (1-v)\left(1(1-u) + 3u\right) + v\left(2(1-u) + 5u\right) \\ (1-v)\left(1(1-u) + 2u\right) + v\left(7(1-u) + 9u\right) \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} -3uv & + & 4u & & & + & 1 \\ uv & + & 2u & + & v & + & 1 \\ uv & + & u & + & 6v & + & 1 \end{pmatrix}.$$

$S(0,0) = (1,1,1)$, $S(1,0) = (5,3,2)$, $S(0,1) = (1,2,7)$, $S(1,1) = (2,5,9)$.

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