Teaching Modeling

Discovering Bifurcation

Victor J. Donnay
Mathematics
Bryn Mawr College
101 North Merion Ave.
Bryn Mawr, PA 19010-2899
vdonnay@brynmawr.edu

Introduction

The world is facing a wide range of ecological challenges, including species extinction and the climate crisis. The mathematical concept of a *bifurcation point* underlies many of these phenomena (see Wikipedia [2022]) and hence is an important topic for our students to understand. This concept has found its way into popular culture, where it is referred to as a "tipping point."

The goal of the classroom exercise described below is to allow students to cooperate in discovering for themselves the concept of a bifurcation point. The bifurcation in question occurs in a one-parameter family of differential equations that model harvesting fish. The parameter is the number of fish harvested (or caught) each year. As the harvesting amount increases, the model undergoes a transition from a stable, self-sustaining fish population to extinction of the population. This transition occurs suddenly at one specific value of the harvesting parameter: the so-called *bifurcation value*.

For a number of different harvesting values, the students determine the equilibrium points and draw the resulting phase line diagrams. By combining multiple phase line diagrams, the students produce the bifurcation diagram of the one-parameter family of differential equations. Finally, they determine the bifurcation point: the point in the diagram at which the dynamics undergoes a sudden change in behavior.

In modeling a real-world problem, we often start with a simple model that captures some element(s) of the real-world situation. We then refine

The UMAP Journal 43 (2) (2022) 97–107.

the model by making it more complicated so it can represent more aspects of the phenomena. We illustrate this iterative process by starting with the simple exponential population model and then modifying it to produce the logistic model, which captures the notion of an ecosystem's carrying capacity. Finally, we add a harvesting term to our logistic model to produce our final model.

Prerequisites, which we review briefly, are the exponential model, the logistic equation, scaling variables, equilibrium values, and slope field and phase line diagrams. For each topic, we cite a source for further details. For bifurcation itself, further details can be found in Blanchard et al. [2011, Section 1.7].

Student worksheets, in PDF and Word, are available at the *Journal*'s page for supplementary materials:

http://www.comap.com/product/periodicals/supplements.html

Review of Prerequisites

Exponential Population Model

[Blanchard et al. 2011, Section 1.1]

A simple population model assumes that the rate of the change of the population is proportional to the population. If we denote by $Q(T)$ the population of fish as a function of time T, then that assumption translates into the differential equation

$$\frac{dQ}{dT} = rQ, \tag{1}$$

where dQ/dt is the rate of change of the fish population, r is the constant of proportionality, and we make dQ/dt proportional to the population Q.

The well-known solution to this differential equation is

$$Q(T) = Ce^{rt}.$$

Given Q_0, the initial value of Q at some initial time, and the value of Q at a second time, one can determine values for the parameters C and r.

When the rate of increase of population r is positive, the exponential model makes the unrealistic prediction that the population will grow towards infinity. To better model population growth that accounts for limited resources when a population is large, we introduce the logistic model.

Logistic Population Model

[Blanchard et al. 2011, Section 1.1]

We model a population $Q(T)$ of fish, in terms of mass of the fish, as a function of time T. The assumptions for the model are:

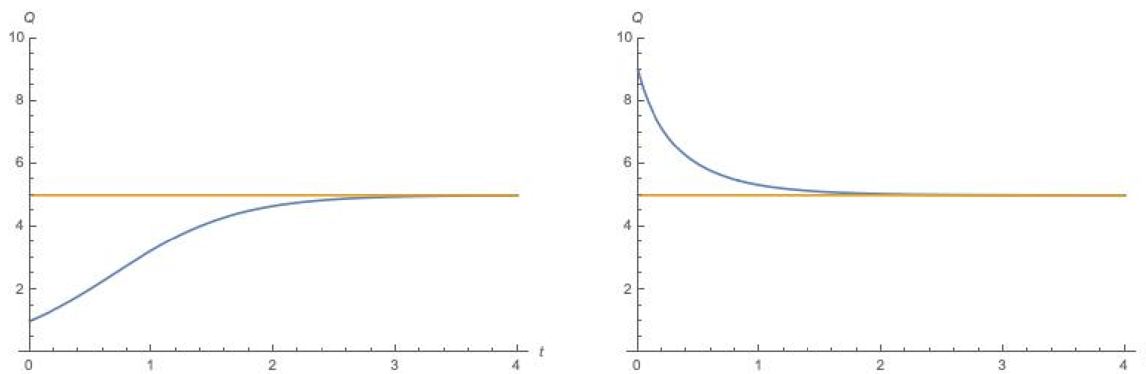
- The fish have a natural rate of increase r per unit time when there are no environmental constraints.
- Environmental constraints impose a limit N, called the *carrying capacity*, on the size of the population.
- As the population increases, the rate of increase dQ/dT decreases, reaching 0 at $Q = N$, and it is negative for $Q > N$.

A model that incorporates these features is the logistic equation model,

$$\frac{dQ}{dT} = rQ\left(1 - \frac{Q}{N}\right),$$

which involves the two parameters r and N.

We depict in **Figure 1a** the growth of a population according to this model, from a very low level. **Figure 1b** shows the decline of too large a population, according to the model.



(a) Growth of a population according to a logistic model.

(b) Decline of a population according to a logistic model.

Figure 1. Growth and decline in a logistic model.

Scaling

[Fay and Joubert 2002; Ledder 2017]

It is convenient for systematic and generalizable mathematical analysis to scale the variables involved in a problem. For the logistic equation, we set $P = Q/N$, so that P measures the proportion of the carrying capacity. Since Q and N both have the dimension of mass (measured perhaps in tons), in the quotient Q/N the dimensions cancel and $P = Q/N$ is dimensionless. Such scaling is often referred to as *nondimensionalizing*.

Similarly, dQ/dT has the dimension of mass/time, while equality of dimensions on both sides of the logistic equation demands that r have the dimension of 1/time, the reciprocal of time. Thus, the quantity $t = rT$ is dimensionless.

In the logistic equation, we accordingly set $Q = NP$ and $T = t/r$, getting

$$\frac{d(NP)}{d(t/r)} = rNP\left(1 - \frac{NP}{N}\right),$$

which, taking into account that r and N can be considered constant, simplifies to

$$\frac{dP}{dt} = P(1 - P). \tag{2}$$

This form of the logistic equation generalizes and incorporates all the important aspects and behavior of models that have particular values for the parameters r and N.

Slope Field Diagram

[Blanchard et al. 2011, Section 1.3]

We can understand the behavior of solutions of a differential equation such as **(2)** by drawing a *slope field diagram*. If we have a solution curve $P(t)$ to the differential equation, then the tangent line to the solution curve at the point $\big(t, P(t)\big)$ has slope $P'(t) = dP/dt$.

We reverse this point of view: At each point of the (t, P) plane, we draw a small line segment with slope $P'(t)$. The collection of all these slope segments is called the *slope field*. A solution of the differential equation is a curve that is everywhere tangent to this slope field.

If we define $f(P) = P(1 - P)$, then at the point (t, P) the slope is given by $f(P)$. In this example, the slope does not depend on the value of t. Such a system is termed *autonomous*.

Equilibrium Values

[Blanchard et al. 2011, Section 1.6]

An *equilibrium solution* of the differential equation is one for which $P(t)$ stays constant and hence for which $P'(t) = 0$. Thus, we can find equilibrium solutions by solving

$$\frac{dP}{dt} = P(1 - P) = f(P) = 0,$$

whose solutions are $P = 0$ (no fish) and $P = 1$ (fish population at carrying capacity). An equilibrium point is either

- *attracting*, if nearby values tend toward it as time proceeds; or
- *repelling*, if nearby values trend away from it as time proceeds; or

- a *node*, if it is neither attracting or repelling.

In **Figure 2**, we graph the function $f(P) = P(1 - P)$ that gives the values of the derivative dP/dt. The equilibrium points correspond to $f(P) = dP/dt = 0$.

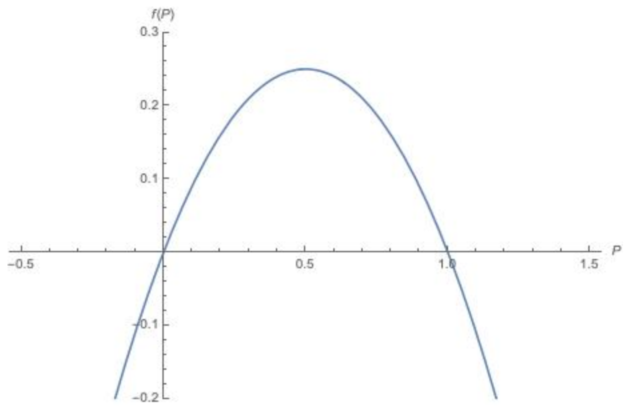


Figure 2. The graph of $f(P) = P(1 - P)$, whose zeros are equilibrium solutions and whose sign indicates whether the derivative dP/dt is positive or negative.

Phase Line Diagram

[Blanchard et al. 2011, Section 1.6]

Between any two equilibrium values, the sign of the derivative is constant. We draw a vertical axis for the variable, mark the equilibrium values with dots, and designate each of the intervals between equilibrium values with

- an up-arrow $\Uparrow$ if the derivative is positive on the interval, and
- a down-arrow $\Downarrow$ if the derivative is negative on the interval.

The crucial consideration is that the arrow in an interval points toward positive infinity (unrestrained growth), or toward negative infinity (not meaningful for a biological application), or else to a particular equilibrium value that the population, as it continues to grow (or decline), will tend toward.

One can think of the phase line diagram as taking the information given in the two-dimensional (t, P) slope field diagram and projecting it onto the one-dimensional P axis.

We depict in **Figure 3** the slope field for **(2)** and the associated phase line diagram.

We include representative solutions, ones with initial conditions $0 < P_0 < 1$ and $P_0 > 1$, showing that such solutions increase (respectively, decrease) toward the attracting equilibrium solution $P(t) \equiv 1$, which corresponds to the carrying capacity. The equilibrium solutions $P(t) \equiv 0$ and $P(t) \equiv 1$

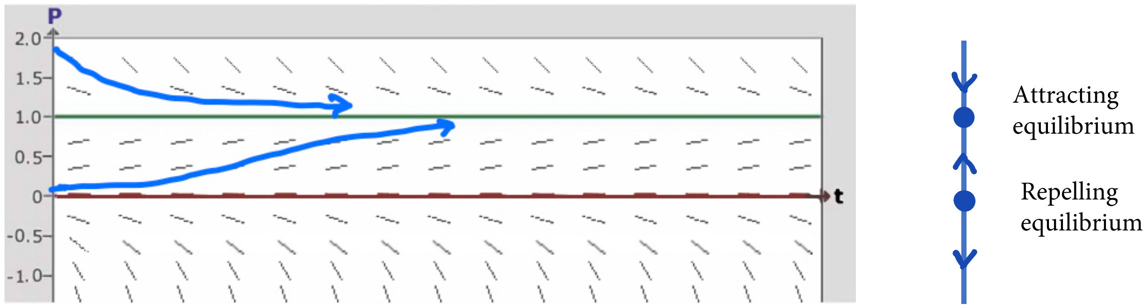


Figure 3. Slope field diagram with solutions from two initial conditions, and associated phase line diagram, for **(2)**.

are represented by points on the phase line. There are arrows in each of the two intervals $(0, 1)$, and $(1, \infty)$ that both point toward $P = 1$. The implication is that whatever the size of the population (apart from 0), it will tend toward the carrying capacity as time passes. We also include an arrow in the interval $(-\infty, 0)$ that points away from 0, showing that $P = 0$ is a repelling equilibrium. While initial conditions with $P_0 < 0$ do not make sense for the fish population, for the harvesting problem it will make sense to have initial population values less than the repelling equilibrium value.

Harvesting

[Blanchard et al. 2011, Section 1.7]

We introduce harvesting into the fish model. In particular, in a unit time interval we harvest a proportion C of the fish (notice that C is appropriately nondimensional). Our model becomes

$$\frac{dP}{dt} = P(1-P) - C = -P^2 + P - C. \tag{3}$$

When $C = 0$, no fish are harvested and the model reduces to the standard logistic model **(2)**.

Questions

The questions that we are interested in are

- What is the long-term behavior of the fish population for various harvesting levels C?
- How does this behavior change as C changes?

We have students investigate these questions in the following activity.

Directions for the Activity

Getting the Picture

We divide the class into groups of four students each. Each person in a group will examine the equation

$$\frac{dP}{dt} = P(1-P) - C = -P^2 + P - C \tag{3}$$

for two different values of C.

Here are the values of C for each of the students in a group:

Person 1:	$C = 0$,	$C = 0.25$.
Person 2:	$C = 0.05$,	$C = 0.4$.
Person 3:	$C = 0.125$,	$C = 0.3$.
Person 4:	$C = 0.2$,	$C = 0.5$.

For each C value:

- Draw the graph of the slope function $f(P) = -P^2 + P - C$ as a function of P.
- Determine the equilibrium points.
- Then draw the corresponding phase line diagram. Indicate whether an equilibrium point is attracting or repelling.

Leave your group and re-group with the people from other groups who are working with the same C values. Compare your results and agree on a solution.

Return to your original group, whose members will now combine all of your group's results into a single graph in the (C, P)-plane. Go in order of increasing C values. One at a time, each person adds their phase line picture to the overall picture. Explain what equilibrium points you calculated and describe the type (attracting or repelling). For each C, the phase line for that C value is a vertical line in the (C, P)-plane. The union of all these phase line pictures will help you understand how the fish population varies as we vary the harvesting proportion C.

Questions for Your Group

1. "Connect the dots" that represent the equilibrium values, so that you have a (rough) estimate of the equilibrium values for all C values, rather than just the few that your group calculated exactly. The connected dots form what is known as the *bifurcation diagram* for the system.
2. What happens to the fish population over the long term if the fishing level C is high?

3. What happens to the fish population over the long term if the fishing level C is low to moderate?
4. What is the critical fishing level below which the fish population will survive and above which the fish population will die out? This value is called a *tipping point* (or in mathematical terminology, a *bifurcation point*).
5. For equation **(3)**:
   (a) Determine the equilibrium points in terms of an expression involving C.
   (b) Using the formulas for the equilibrium points, give the formulas for the upper and lower curves that your group drew in the bifurcation diagram.
6. Make a recommendation: The government Department of Fisheries in partnership with business and environmental groups has set up a Fisheries Commission. The goal of the Commission is to set a quota for how many fish can be caught each year. As experts on the mathematics of fish populations, your group is being called to testify before the Fisheries Commission. Based on your mathematical analysis, what recommendation will you give to the Commission?

Alternative Approach

In the above activity, the bifurcation value, $C = 0.25$, is assigned as one of the values for the students to investigate. An alternative approach would be to assign values above and below the bifurcation value, but then challenge the students to find the value of C at which the system switches from having an attracting equilibrium solution to having the fish population go extinct.

Explanation

"Bifurcation" refers to branching into two separate paths. In the bifurcation diagram, there are no equilibrium points for $C > 0.25$; the population crashes to 0 (and, mathematically, into negative values and on toward negative infinity, with no biological significance). As we consider smaller values of C, we suddenly see an equilibrium point at $C = 0.25$ and then see two curves of equilibrium points arise as we proceed from right to left.

The upper curve consists of equilibrium points that are attracting, while the lower curve consists equilibrium points that are repelling—as we can tell by considering the directions of the arrows on either side of an equilibrium point.

One can interpret the bifurcation value both from analytic and geometric perspectives. For $C < 0.25$, when using the quadratic formula to solve the equation $f(P) = 0$ and thereby determine the equilibrium points, one finds two roots. For $C > 0.25$, the quadratic formula gives complex roots, meaning that there are no (real) equilibrium points. The bifurcation value $C = 0.25$ occurs when there is a single root, repeated with multiplicity two.

Geometrically (see **Figure 4**), when $C < 0.25$, the downward-parabola defined by $f(P)$ has a positive maximum and hence intersects the P-axis in two distinct points—the two equilibrium points. As the harvesting parameter C increases, the term $-C$ in the formula for $f(P)$ causes the graph of $f(P)$ to be shifted downward. At $C = 0.25$, the function $f(P)$ is tangent to the P-axis; the point of tangency corresponds to the single equilibrium point. For $C > 0.25$, the graph has been shifted down so far that it no longer intersects the P-axis at all; there are no equilibrium values.

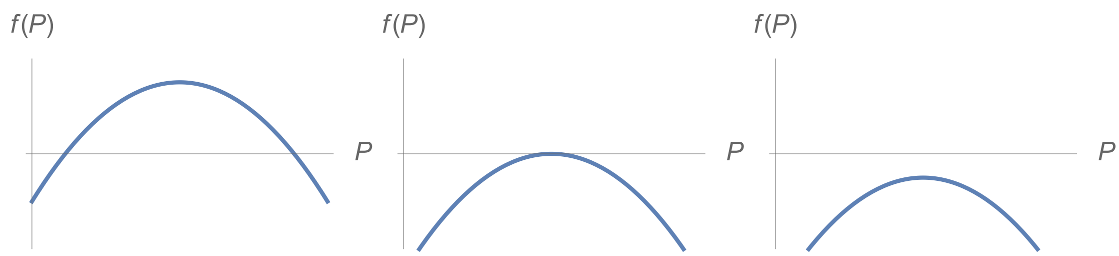


Figure 4. Graphs of $f(P)$ for $C < 0.25$, $C = 0.25$, and $C > 0.25$.

Sample Answers to the Questions

1. See **Figure 5**.

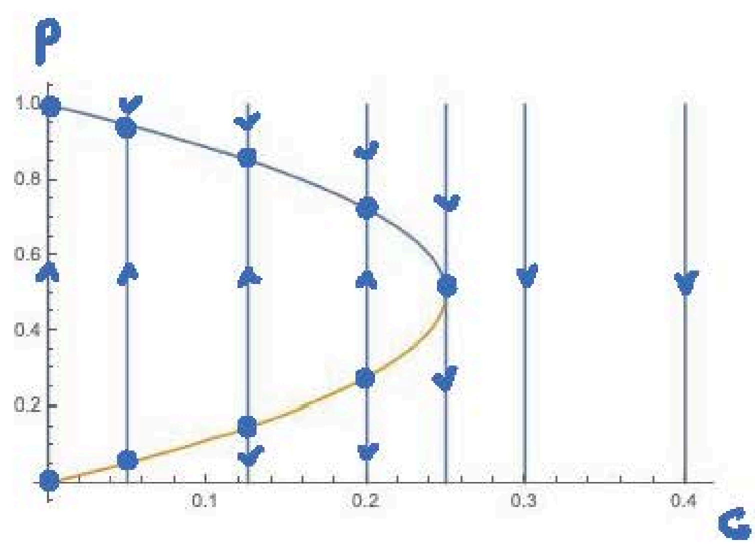


Figure 5. Bifurcation diagram for **(3)**.

2. The population dies off.

3. The population tends toward the attracting upper equilibrium point in the phase line diagram.

4. $C = 0.25$.

5. (a) The equilibrium points are $P = \frac{-1\pm\sqrt{1-4C}}{-2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - C}$.

   (b) The upper curve is $P^{+}(C) = \frac{1}{2} + \sqrt{\frac{1}{4} - C}$.
   The lower curve is $P^{-}(C) = \frac{1}{2} - \sqrt{\frac{1}{4} - C}$.

6. Answers will vary.

Further Reading

Examples of tipping points in the climate system can be found in Lenton et al. [2019], with a more general discussion available at Wikipedia [2022]

Is our climate headed for a mathematical tipping point? You may want to watch an animated TED-Ed video that uses analogies with chaotic billiard systems to suggest how tipping points can impact climate change [Donnay 2014], and/or a description of climate tipping points that could change the Earth forever [Grist 2021].

Further examples of how mathematics connects to issues of sustainability can be found at the website for Mathematics Awareness Month 2013 [Joint Policy Board for Mathematics 2013].

References

Blanchard, Paul, Robert L. Devaney, and Glen R. Hall. 2011. *Differential Equations.* 4th ed. Boston, MA: Cengage Learning.

Donnay, Victor J. 2014. Is our climate headed for a mathematical tipping point? https://www.youtube.com/watch?v=EoYSToa2Yfw.

Fay, Temple H., and Stephan V. Joubert. 2002. Dimensional analysis: An elegant technique for facilitating the teaching of mathematical modelling. *International Journal of Mathematical Education in Science and Technology* 33 (2): 280–293. http://dx.doi.org/10.1080/002073902753586382.

Grist. 2021. The climate tipping points that could change the Earth forever. https://www.youtube.com/watch?v=j3q7j8igyN4.

Joint Policy Board for Mathematics. 2013. Mathematics Awareness Month—April 2013. Mathematics of Sustainability. https://ww2.amstat.org/mam/2013/.

Ledder, Glenn. 2017. Scaling for dynamical systems in biology. *Bulletin of Mathematical Biology* 79: 2747–2772. https://www.math.unl.edu/~gledder1/Papers/Ledder_BMB.pdf.

Lenton, Timothy M., et al. 2019. Climate tipping points—too risky to bet against. *Nature* 575 (28 November 2019): 592–595. https://www.nature.com/articles/d41586-019-03595-0.

Wikipedia. 2022. Tipping points in the climate system. https://en.wikipedia.org/wiki/Tipping_points_in_the_climate_system.

Acknowledgments

The author would like to thank journal editors Kayla Blyman and Paul Campbell for helping him develop this article and an anonymous reviewer for useful feedback.

About the Author

Victor Donnay is Professor of Mathematics at Bryn Mawr College, where he holds the William Kennan Jr. Chair. He enjoys thinking of creative ways to bring the beauty and fun of mathematics to a general audience. As Director of the Philadelphia Regional Institute for STEM Educators (PRISE), he works to support STEM teachers at all stages of their careers. His mathematical research focuses on developing new examples of dynamical systems that exhibit chaotic motion. His favorite hobby is golf, which provides plenty of opportunity to investigate the phenomena of sensitive dependence on initial conditions.