Week 9 - Problem Sheet 8

1. $$f(z) = \frac{3\cos^2(z)}{\sin^2(2z)}$$

(a) $\mathrm{ord}(f, 0)$:

Use series expansion for cos and sin around 0 :

$$\cos(z) = 1 - \frac{1}{2}z^2 + \cdots$$

$$\sin(z) = z - \frac{1}{6}z^3 + \cdots$$

$$\Rightarrow \quad 3\cos^2(z) = 3 - 3z^2 + \cdots$$

$$\sin^2(2z) = 4z^2 - \frac{16}{3}z^4 + \cdots$$

So $$f(z) = \frac{3 - 3z^2 + \cdots}{4z^2 - \frac{16}{3}z^4 + \cdots}$$

$$= \frac{1}{z^2}\left(\frac{3 - 3z^2 + \cdots}{4 - \frac{16}{3}z^2 + \cdots}\right)$$

Write this as $\frac{1}{z^2}h(z)$

Then $h(0) = 3/4$ so expansion of $h(z)$ around $z=0$ looks like

$$h(z) = 3/4 + (\text{higher order terms})$$

So $f(z) = 3/4\, z^{-2} + $ (higher order terms)

$\therefore$ ord$(f, 0) = -2$.

(b) $\text{Res}(f, 0)$

By def $\text{Res}(f, 0)$ means:

coefficient of z^{-1} in the expansion

of f around 0.

Remark: remember $f(z) =$

$$\frac{1}{z^2}\left(\frac{3 - 3z^2 + \cdots}{4 - \frac{16}{3}z^2 + \cdots}\right)$$

Everything appearing only has even powers

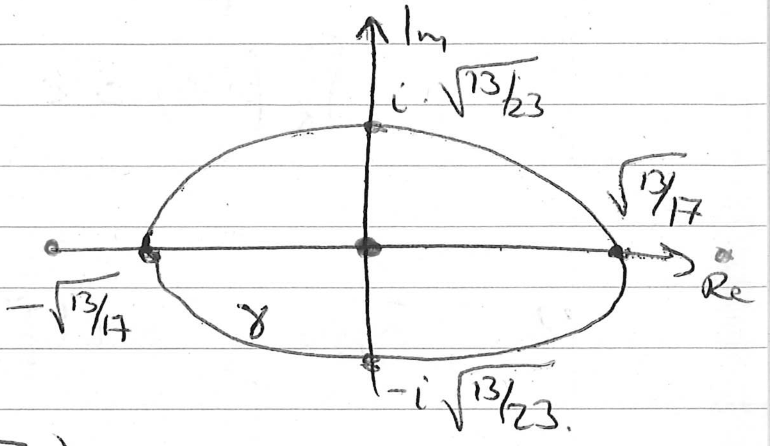
of z. $\therefore$ coefficient of z^{-1} in the

series expansion equals 0.

$\therefore$ $\text{Res}(f, 0) = 0$.

(c) Calculate $\int_\gamma f(z)\,dz$

γ is defined by $17\,\mathrm{Re}(z)^2 + 23\,\mathrm{Im}(z)^2 = 13$



Residue Thm:

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{i=1}^{n} \mathrm{Res}(f, z_i)$$

where $z_1, \dots, z_n$ are the poles of f inside γ.

What are the poles of f?

$$f = \frac{3\cos^2(z)}{\sin^2(2z)}$$

Only (possible) poles of f are where

$\sin^2(2z) = 0$.

$\iff \sin(2z) = 0$

$\iff z = \frac{k}{2}\pi \quad k \in \mathbb{Z}$

If $k \neq 0$ then $|\frac{k}{2}\pi| > 1 \therefore$ z is not inside γ

So the only pole of f inside γ

is at $z=0$, and we saw

$\text{Res}(f,0)=0$

$\therefore \int_\gamma f(z)\,dz = 0$.

$\square$.

2. Suppose $\text{ord}(f, z_0) = k$

So:

$$f(z) = a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + \dots$$

with $a_k \neq 0$.

So $f'(z) = k a_k (z-z_0)^{k-1} + (k+1)(a_{k+1})(z-z_0)^k + \dots$

So

$$g = \frac{f'(z)}{f(z)} = \frac{k a_k (z-z_0)^{k-1} + (k+1)(a_{k+1})(z-z_0)^k + \dots}{a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + \dots}$$

Divide top & bottom by $a_k(z-z_0)^k$ to get:

$$g(z) = \frac{k(z-z_0)^{-1} + \dots}{1 + \frac{a_{k+1}}{a_k}(z-z_0) + \dots}$$

The function $\dfrac{1}{1+\frac{a_{k+1}}{a_k}(z-z_0)+\cdots} = h(z)$ is holomorphic at z_0 with value 1 there.

So if we expand it around z_0 we get

$$1 + b_k(z-z_0) + \cdots\cdots$$

So we get $g(z) = \left(k(z-z_0)^{-1} + \cdots\right)h(z)$

$$= \left(k(z-z_0)^{-1} + \cdots\right)\left(1 + b_k(z-z_0) + \cdots\right)$$

$$= k(z-z_0)^{-1} + \cdots\cdots$$

$\therefore$ $\mathrm{Res}(g, z_0) = k = \mathrm{ord}(f, z_0)$ $\square$.

3. Solution.

ω_1, ω_2

$m, n, r, s \in \mathbb{Z}$

$\tau_1 = m\omega_1 + n\omega_2$

$\tau_2 = r\omega_1 + s\omega_2$

$$L = \{ a\omega_1 + b\omega_2 \mid a, b \in \mathbb{Z} \}$$

$$L' = \{ a\tau_1 + b\tau_2 \mid a, b \in \mathbb{Z} \}$$

- Suppose $L = L'$.

Want to show $\det \begin{pmatrix} m & n \\ r & s \end{pmatrix} = \pm 1$.

Since $L = L'$ and $\omega_1, \omega_2 \in L$

we have $\omega_1, \omega_2 \in L'$ too.

$\therefore \exists\ \mu, \nu, \rho, \sigma$ s.t.

$$\omega_1 = \mu\tau_1 + \nu\tau_2$$

$$\omega_2 = \rho\tau_1 + \sigma\tau_2$$

Substituting the given expressions for the τ_i in terms of the ω_i we get

$$\omega_1 = \mu(\overbrace{m\omega_1 + n\omega_2}^{\tau_1}) + \nu(\overbrace{r\omega_1 + s\omega_2}^{\tau_2})$$

$$\omega_2 = \rho(m\omega_1 + n\omega_2) + \sigma(r\omega_1 + s\omega_2)$$

Simplify: get

$$\omega_1 = (\mu m + \nu r)\omega_1 + (\mu n + \nu s)\omega_2$$

$$\omega_2 = (\rho m + \sigma r)\omega_1 + (\rho n + \sigma s)\omega_2$$

Write as a matrix eq:

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix} \cdot M \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

If we write $\omega_1 = x_1 + iy_1$,
$\omega_2 = x_2 + iy_2$

compare real + img. parts we get

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix} \cdot M \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$$

Since ω_1 & ω_2 are not real multiples of each other we have $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ is invertible.

$$\therefore \begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix} \cdot M = \mathrm{Id}_2$$

$$\therefore \underset{\mathbb{Z}}{\underset{\cap}{\det(M)}} \cdot \underset{\mathbb{Z}}{\underset{\cap}{\det\begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix}}} = 1$$

$$\therefore \det(M) = \pm 1.$$

Converse: see typed solutions.