Elements of Topology
Exam (June 2018)
Solutions

No 1. (a)

[3] (i) Def. A point $b \in X$ is adherent point for a set A if each neighborhood of b intersects A. [bookwork]

(ii) Proof.

$\Rightarrow$ Let $x \in \overline{A}$ and $\mathcal{U}$ be an open neighborhood of x. By contradiction: assume that $\mathcal{U} \cap A = \emptyset$, then $A \subset X \setminus \mathcal{U}$. Since $X \setminus \mathcal{U}$ is closed, we conclude that $\overline{A} \subset X \setminus \mathcal{U}$ and, consequently, $x \in X \setminus \mathcal{U}$, thereby contradicting the assumption that $x \in \mathcal{U}$. Thus, $\mathcal{U} \cap A \neq \emptyset$.

$\Leftarrow$ Assume that for each neighborhood $\mathcal{U}$ of x we have

[5] $\mathcal{U} \cap A \neq \emptyset$. Let (by contradiction) $x \notin \overline{A}$, i.e. $x \in X \setminus \overline{A}$. Since $X \setminus \overline{A}$ is open, $X \setminus \overline{A}$ can be treated as a neighborhood of x so that, by hypothesis, $(X \setminus \overline{A}) \cap A \neq \emptyset$, which is a contradiction with the fact that $A \subset \overline{A}$. Thus, $x \in \overline{A}$. [bookwork]

[2] (iii) The closure of $A = (0,1)$ is A itself because every ~~subsp~~ subset of a discrete topological space is closed. [standard problem]

(b) Describe the closure of:

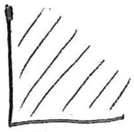
(i) $A = \{x^2 + y^2 < 1\}$ open disc

[standard question]

[2] $\bar{A} = \{x^2 + y^2 \leq 1\}$

$A \neq \bar{A} \Rightarrow A$ is not closed

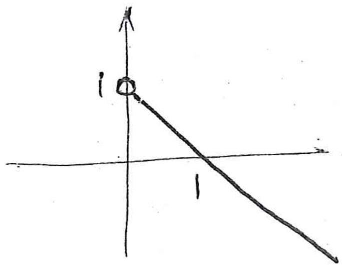
(ii) $A = \{x \leq 0,\ y \leq 0\}$



[standard question]

[2] $\bar{A} = A$ and, therefore, A is closed

(iii) $A = \{x + y = 1,\ x > 0\}$

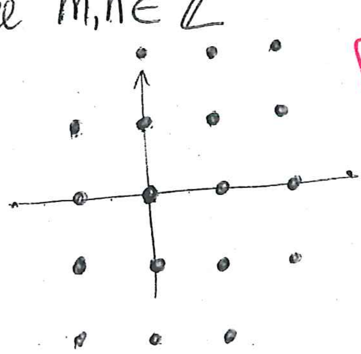


[Standard question]

[2] $\bar{A} = A \cup \{(0,1)\}$

$\bar{A} \neq A \Rightarrow A$ is not closed

(iv) $A = \{(x,y) = (m,n),$ where $m,n \in \mathbb{Z}$



[standard question]

[2] $\bar{A} = A$ and, therefore, A is closed

[2]

No 2.

(a) Def. X is a Hausdorff topological space, if for [3] any two distinct points $x, y \in X$ there exist neighborhoods $\mathcal{U}(x)$ and $V(y)$ which are disjoint, i.e., $\mathcal{U}(x) \cap V(y) = \phi$.

[bookwork]

Let $X = \mathbb{R}$ with the trivial topology $\tau = \{\phi, \mathbb{R}\}$.
[2] X is not Hausdorff, because for any two points $x, y \in \mathbb{R}$ the neighborhoods $\mathcal{U}(x)$ and $V(y)$ coincide with $\mathbb{R}$ and $\mathcal{U}(x) \cap V(y) = \mathbb{R} \neq \phi$.

(b) A compact subset of a Hausdorff topological space is closed.

[bookwork]

Proof. Let X be a Hausdorff top. space and $Y \subset X$ be a compact subset.

By contradiction, assume that y is a limit point of Y, but $y \notin Y$. For any point $x \in Y$ and y we can find disjoint neighborhoods $\mathcal{U}(x)$ and $V_x(y)$. Obviously, the neighborhoods $\mathcal{U}(x)$, $x \in Y$, all together form an open covering of Y. Since Y is compact, we may choose a finite subcover $\{\mathcal{U}(x_1), \dots, \mathcal{U}(x_n)\}$ so that $Y \subset \mathcal{U}(x_1) \cup \mathcal{U}(x_2) \cup \dots \cup \mathcal{U}(x_n)$. Consider the corresponding neighborhoods

$V_{x_1}(y), \dots, V_{x_n}(y)$ and take the intersection
$V(y) = V_{x_1}(y) \cap V_{x_2}(y) \cap \dots \cap V_{x_n}(y)$. Obviously, $V(y)$ is an open neighborhood which is disjoint with each of $\mathcal{U}(x_i)$ and, therefore, we have $V(y) \cap (\mathcal{U}(x_1) \cup \dots \cup \mathcal{U}(x_n)) = \emptyset$. [5]
Hence $V(y) \cap Y = \emptyset$ since Y is covered by $\mathcal{U}(x_1), \dots, \mathcal{U}(x_n)$.
Thus, we have found a neighborhood of y which contains no points of Y. This contradicts the fact that y is a limit point of Y.
Conclusion. Y contains all of its limit points and, therefore, is closed.

(c) Compact or not? [standard question partially unseen]

(i) $\mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}, (a, +\infty), a \in \mathbb{R}\}$ is not compact. [2]
Indeed the cover $\mathcal{U}_n = (-n, +\infty)$, $n \in \mathbb{N}$ does not admit any finite subcover.

[2] (ii) $\{1 - \frac{1}{n}, n \in \mathbb{N}\}$ is not compact.
This set is not closed (1 is a limit point, but $1 \notin \{1 - \frac{1}{n}, n \in \mathbb{N}\}$

[2] (iii) $\{x^2 + y^2 + z^2 \geq 1\}$ is not compact, because this set is not bounded

[2] (iv) $\{x^4 + \sin^4 y = 1\}$. This set is not bounded because it contains the points $(1, 2\pi k) \longrightarrow \infty$ as $k \to \infty$
Not compact

[2] (v) $\{(x, y) \in \mathbb{R}^2 \mid x \in [0,1], y \in [0,1]\}$ is compact, because bounded and closed

No 3

Let X and Y be topological spaces.

(a) A map $f: X \to Y$ is called continuous if [bookwork]

[3] for every open subset $B \subset Y$ its preimage $f^{-1}(B)$ is open in X.

(b) The image of a connected topological space under a continuous map is connected. [bookwork]

Proof. Let $f: X \to Y$ be continuous and X be connected.

By contrary: assume that $f(X) \subset Y$ is disconnected. Then there are open sets $A, B \subset Y$ such that

[5] $f(X) \subset A \cup B$, $A \cap B = \emptyset$, $A \cap f(X) \neq \emptyset$, $B \cap f(X) \neq \emptyset$.

Consider $f^{-1}(A)$ and $f^{-1}(B)$. They are both ~~of~~ open, as f is continuous.

Obviously, $X = f^{-1}(A) \sqcup f^{-1}(B)$ is the ~~decom~~ partition of X into two open disjoint non-empty subsets. Thus, X is disconnected.

This contradiction proves the statement.

(c) $X = \mathbb{R}$ discrete, $Y = \mathbb{R}$ standard, $f: X \to Y$, $f(x) = x$. [partially unseen]

[2] (i) X is not connected, $(-\infty, 0) \cup [0, +\infty) = \mathbb{R}$ is a partition into two open disjoint nonempty subsets.

[2] (ii) Y is connected (Theorem 6, Lecture 9)

[2] (iii) f is continuous, as X is discrete (if $B \subset Y$ is open, then $f^{-1}(B)$ is open in X as X is discrete, i.e. every map $f: X \to Y$ is continuous).

[2] (iv) no f is not a homeomorphism, as f^{-1} is not continuous.

Indeed. $\{0\}$ is ~~closed~~ open in X, but

$f(0) = (f^{-1})^{-1}(0) = \{0\}$ is not open in Y.

[2] (v) No, because X is disconnected whereas Y is connected. ("Connectedness" is a topological property)

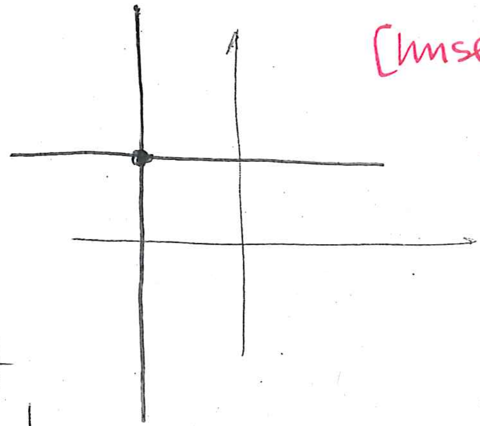
[2] (vi) The only continuous map $g: Y \to X$ is constant, e.g.,

$g(x) = 0$

(4) Manifold or not?

(i) $xy + y - x - 1 = 0$

$(y-1)(x+1) = 0$



[unseen]

No, this set is not a manifold because the point $(-1,1)$ does not have any neighborhood homeomorphic to a ball B^k. Indeed, any punctured neighborhood of this point has at least 4 connected components, whereas $B^k \setminus \{pt.\}$ is either connected (if $k>1$) or has 2 components (if $k=1$)

(ii) $\cosh x + \cosh y = 10$

[standard question]

We use the implicit function theorem

$f(x,y) = \cosh x + \cosh y$

$df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (\sinh x, \sinh y)$

if $df = 0$ then $\sinh x = \sinh y = 0$ and $x = y = 0$;

but the point $(0,0)$ does not belong to this level of f:

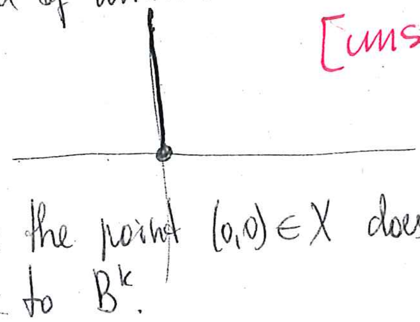
$\cosh 0 + \cosh 0 = 1+1 = 2 \neq 1'$

Thus, this set is a manifold of dimension 1.

[unseen]

(iii) $X = \{x=0, y \geq 0\}$

vertical ray



X is not a manifold, because the point $(0,0) \in X$ does not have any neighborhood homeomorphic to B^k.

(b) Classification theorem

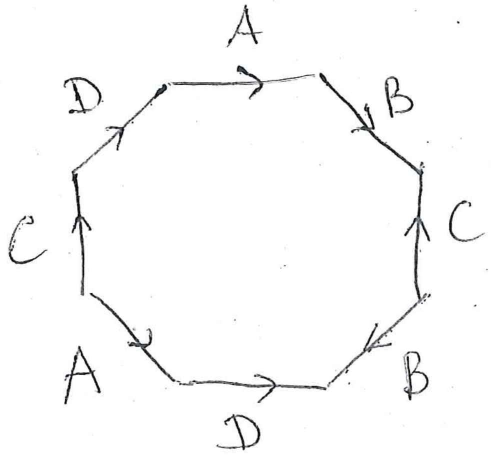
Any closed connected surface is homeomorphic either to the sphere S^2, or to the sphere with a finite number of handles added $S^2 + k \cdot h$, or

[5]

to the sphere with a finite number of Möbius strips added $S^2 + m \cdot \mu$.

[bookwork]

(c) M is obtained from the fundamental polygon



[standard question]

[5]

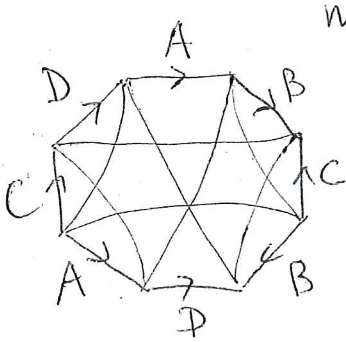
First we compute the Euler characteristic:

$$\chi = F - E + V = 1 - 4 + 1 = -2$$

number of faces $F = 1$

number of edges $E = 4$ i.e. A, B, C, D

number of vertices $V = 1$ (as all vertices of the polygon have to be identified)



M is not oriented because the word contains combination ...B...B...

Thus, M is a non-orientable surface with $\chi = -2$,

i.e. $M = S + 4 \cdot \mu$

(sphere with 4 Möbius strips)