MAC260

Week 5 Problem Class.

2 (b) E_t : $y^2 = x^3 + a(t)x + b(t)$

Assume : a, b not identically 0.

a, b no common root.

$3\deg(a) \neq 2\deg(b)$.

Fix $c \neq 0, -1728$. Want to find t s.t. $j(E_t) = c$.

Solution: $j(E_t) = 1728 \cdot \dfrac{4a(t)^3}{\Delta(t)}$

when $\Delta(t) = -4a(t)^3 - 27b(t)^2$

so we want to solve

$$1728 \frac{4a(t)^3}{\Delta(t)} = c$$

which we can rearrange as

$$4\underbrace{(c + 1728)}_{\neq 0} a(t)^3 + 27\underbrace{c}_{\neq 0} b(t)^2 = 0.$$

Note : $3\deg(a) \neq 2\deg(b) \Rightarrow$ ~~the~~ LHS is not constant polynomial $\therefore$ it has a root t_0 say.

Then we will have

$$j(E_{t_0}) = c \quad \text{unless}$$

$$\Delta(t_0) = 0$$

So we need to check that $\Delta(t_0) \neq 0$.

If t_0 is a root of both

$$4(c+1728)\,a(t)^3 + 27c\,b(t)^2 \text{ and } \Delta(t),$$

then it is a root of their gcd, which is

$a(t)^3$. But since $\Delta = -4a^3 - 27b^2$

we would get that t_0 is also a root of b

$\therefore$ common root of $a(t)$ and $b(t)$,

contradicting our assumption.

$\square$

3. Legendre form

$$E_\lambda : \quad y^2 = x(x-1)(x-\lambda) \qquad \lambda \neq 0, 1.$$

(a) Since $\lambda \neq 0, 1$ the cubic on RHS has 3 distinct roots. $\square$

(b) Weierstrass form:

First multiply out Legendre form; get

$$y^2 = x^3 + (-\lambda - 1)x^2 + \lambda x$$

To put this into Weierstrass form: put

$$X = X' - \beta/3 \qquad \text{where } \beta = (-\lambda - 1)$$

$$= X' + \left(\frac{\lambda+1}{3}\right)$$

Substitute this into the previous eq. and ~~simplify~~ expand: you get

$$y^2 = x^3 + \left(\lambda - \tfrac{1}{3}(\lambda+1)^2\right)x$$
$$+ \left(\tfrac{1}{3}\lambda(\lambda+1) - \tfrac{2}{27}(\lambda+1)^3\right).$$

(c) Show $\forall\, j \neq 0, 1728$ there are exactly 6 values of λ s.t. $j(E_\lambda) = j$.

Solution: Know

$$j(E_\lambda) = 1728 \frac{4a^3}{\Delta}$$

where

$$a = \lambda - \tfrac{1}{3}(\lambda+1)^2$$
$$= \tfrac{1}{3}\left(3\lambda - (\lambda+1)\right)^2$$

$$b = \tfrac{1}{3}\lambda(\lambda+1) - \tfrac{2}{27}(\lambda+1)^3$$
$$= \tfrac{1}{27}\left(9\lambda(\lambda+1) - 2(\lambda+1)^3\right).$$

So

$$j = \frac{-1728 \cdot 4 \cdot \frac{1}{27}\left(3\lambda-(\lambda+1)^2\right)^3}{\frac{4}{27}\left(3\lambda-(\lambda+1)^2\right)^3 + \frac{1}{27}\left(9\lambda(\lambda+1)-2(1+\lambda)^3\right)^2}$$

Cancel $\frac{1}{27}$ in numerator + denominator:

write $3\lambda - (\lambda+1)^2 = -(\lambda^2-\lambda+1)$.

Numerator becomes $\underline{-1728 \cdot 4 \cdot (\lambda^2-\lambda+1)^2}$

Denominator becomes $27\lambda^2(\lambda-1)^2$.

So we end up with

$$j = -256 \cdot \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2} \qquad (*)$$

<u>Note</u>: the equation $(*)$ with j a fixed constant becomes a polynomial of degree 6 in λ

$\therefore$ has at most 6 solutions.

When does it have exactly 6 solutions?

The idea is that $(*)$ is unchanged if we make either of the substitutions

$$\lambda \longmapsto 1/\lambda$$

or $$\lambda \longmapsto 1-\lambda$$

$$j = \frac{-256\,(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$$

E.g. if we substitute $\lambda \longmapsto 1/\lambda$ we get

$$\frac{-256\left(\left(\frac{1}{\lambda}\right)^2-\left(\frac{1}{\lambda}\right)+1\right)^3}{\left(\frac{1}{\lambda}\right)^2\left(\frac{1}{\lambda}-1\right)^2}$$

$$= \frac{-256\left(\left(\frac{1}{\lambda}\right)^2\left(1-\lambda+\lambda^2\right)\right)^3}{\left(\frac{1}{\lambda}\right)^6(\lambda^2)(1-\lambda)^2}$$

$$= \frac{-256\,(\lambda^2-\lambda+1)^3}{\lambda^2(1-\lambda)^2}.$$

~~Apply~~ Composing these substitutions we get

$$\left\{\lambda,\ 1-\lambda,\ \frac{1}{\lambda},\ \frac{1}{1-\lambda},\ \frac{\lambda}{\lambda-1},\ \frac{\lambda-1}{\lambda}\right\}$$

So: we need to understand when some of these 6 values coincide.

- Suppose $\lambda = 1-\lambda$. Then $\lambda = 1/2$

  $$\left(\begin{array}{c}\frac{1}{\lambda} = \frac{1}{1-\lambda}\\ \frac{\lambda}{\lambda-1} = \frac{\lambda-1}{\lambda}\end{array}\right)$$

  Evaluating: $j(1/2) = -1728$.

- Suppose

$$\lambda = \frac{1}{\lambda}$$

$$\left(\Rightarrow \quad 1-\lambda = \frac{\lambda - 1}{\lambda} \qquad \frac{1}{1-\lambda} = \frac{\lambda}{\lambda - 1} \right)$$

$$\Rightarrow \quad \lambda = -1$$

Evaluating again we get $j = -1728$.

Similarly if $\lambda = \frac{\lambda}{\lambda-1}$ we get $j = -1728$

$$\Rightarrow \lambda = 2.$$

- Finally $\lambda = \frac{1}{1-\lambda}$ or $\lambda = \frac{\lambda - 1}{\lambda}$

$$\Rightarrow \quad \lambda^2 - \lambda + 1 = 0 \qquad \left(\lambda = \frac{1 \pm \sqrt{3}i}{2} \right)$$

$$\Rightarrow \quad j = 0$$

Otherwise all 6 values are distinct.