Loughborough University

ELEMENTS OF TOPOLOGY
(20MAB298)

Summer 2021 (1a) Exam paper

Please read this and the next page carefully.

This is a (1a) online examination, meaning you have **23 hours** in which to complete and submit this paper. How you manage your time within the window is up to you, but we expect you should only need to spend approximately **2 hours** working on it. If you have extra time or rest breaks as part of a Reasonable Adjustment, you will need to add this to the amount of time you are expected to spend on the paper.

It is your responsibility to submit your work by the deadline for this examination. You must make sure you leave yourself enough time to do so.

It is also your responsibility to check that you have submitted the correct file(s).

You may handwrite and/or word process your answers, as you see fit. Please return your work as a single PDF file. Please follow the guidance provided on how to scan and upload your work.

Answer **ALL** questions.

1. (a) How many pairwise non-homeomorphic topologies can be put on a set with two elements? Justify your answer. [2]

   (b) Let $A = [0, 1]$. Is A closed as a subset of

   (i) $X_1 = \mathbb{R}$ with the discrete topology, [2]

   (ii) $X_2 = \mathbb{R}$ with the indiscrete topology, [2]

   (iii) $X_3 = \mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}$ and $(a, +\infty),\ a \in \mathbb{R}\}$? [2]

   If A is not closed, describe the closure of A. Justify your answers. [3]

   (c) Prove that $(-\pi, \pi)$ is homeomorphic to $\mathbb{R}$. [3]

   (d) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function (here we assume that the topology is standard). Prove that the set $A = \{(x, y) \in \mathbb{R}^2 :\ -1 \leq f(x, y) \leq 1\} \subset \mathbb{R}^2$ is closed. [3]

   (e) Let $Y = \{x \in \mathbb{R} :\ \sin \frac{1}{x} = 0\}$. Is Y closed as a subset of $\mathbb{R}$ with the standard topology? Justify your answer. [3]

2. (a) Is $\mathbb{R}^3$ a Hausdorff topological space? Prove or disprove it. [3]

   (b) Let X, Y be topological spaces. Prove:

      (i) If X, Y are Hausdorff, then $X \times Y$ is Hausdorff. [2]

      (ii) If X, Y are compact, then $X \times Y$ is compact. [3]

   (c) Which of the following subsets $A \subset X$ are compact (justify your answer):

      (i) $A = \{\frac{1}{n}, n \in \mathbb{N}\} \cup [-1, 0]$ as a subset of $\mathbb{R}$ (standard topology); [2]

      (ii) $\{x^4 + y^4 - x^2 - y^2 \leq \frac{1}{2}\}$ in $\mathbb{R}^2$ (standard topology); [2]

      (iii) $\{\sin(x + y) = \cos(x - y)\}$ in $\mathbb{R}^2$ (standard topology); [2]

      (iv) $(0, 1) \subset X$, where $X = \mathbb{R}$ with indiscrete topology; [3]

      (v) $(0, 1) \subset X$, where $X = \mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}$ and $(a, +\infty), a \in \mathbb{R}\}$. [3]

3. (a) Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, $a, b \in S^1$. Find a continuous path $\gamma : [0, 1] \to S^1$ connecting a and b. [3]

   (b) Prove that pathwise connectedness is preserved under continuous maps, i.e., if X is pathwise connected and $f : X \to Y$ is continuous, then $f(X)$ is pathwise connected. [4]

   (c) Let $X = \mathbb{Z}$ with the indiscrete topology. Is there a continuous path connecting the points 0 and 1? Is X pathwise connected? [4]

   (d) Prove: If $\sim$ is an equivalence relation on a compact topological space X, then $X/\sim$ is compact. [3]

   (e) Let $X = \{x^2 + y^2 = 1\} \setminus \{N\}$ be the unit circle without the point $N = (0, 1)$ and $Y = \mathbb{R} \setminus \{0\}$ be the real line without the origin.

      (i) Is X homeomorphic to Y? [2]

      (ii) Is X homeomorphic to $\mathbb{R}$? [2]

      (iii) Is X homeomorphic to $\mathbb{R}^2$? [2]

      Justify your answers.

4. (a) Is the subset $\{xy = 0\} \subset \mathbb{R}^2$ a manifold? Justify your answer. [4]

   (b) Is the subset $\{e^{x+y} - 2x - 2y + xz + yz = 0\} \subset \mathbb{R}^3$ a manifold? Justify your answer. [4]

   (c) Is the subset $\{(x + y)(x^2 + y^2) = 0\} \subset \mathbb{R}^2$ a manifold? Justify your answer. [4]

   (d) What is the Euler characteristic of the torus? Justify your answer. [4]

(e) Consider the surface M obtained from the fundamental polygon

$$CB^{-1}AD^{-1}BDC^{-1}A^{-1}$$

by pairwise identification of its edges. Find the Euler characteristic of M and determine its topological type. [4]