Loughborough University

ELEMENTS OF TOPOLOGY
(21MAB298)

Summer 2022 In-Person Exam paper

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

Help during the exam

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

You may **not** use a calculator for this exam.

Answer 3 questions.

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1. (a) Give the definition of an open set in a metric space. [4]

   (b) Sketch the following subsets of $\mathbb{R}^2$. Which of them are open?

      (i) $A_1 = \{x \in (-1, 1),\ y \in (-1, 0) \cup (1, 2)\}$, [2]

      (ii) $A_2 = \{y = 3,\ \ x \in (-\infty, +\infty)\}$, [2]

      (iii) $A_3 = \{0 < x^2 + y^2 < 1\}$. [2]

   (c) Give one example of a Hausdorff topological space and one example of a space that is not Hausdorff. [4]

   (d) Let (X, d) be a metric space.

      (i) Prove that X is a Hausdorff topological space. [3]

      (ii) Let $x \in X$ be a point. Prove that $\{x\}$ is closed. [3]

2. (a) Give the definition of compactness. [3]

   (b) Prove that a closed subset of a compact topological space is compact. [5]

   (c) Which of the following sets are compact (justify your answer):

      (i) $[0, 1]$ as a subset of $\mathbb{R}$ (standard topology); [2]

      (ii) $(0, 1)$ with the indiscrete topology; [2]

      (iii) $[0, +\infty) \subset \mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}$ and $(a, +\infty), a \in \mathbb{R}\}$; [2]

      (iv) $\left\{\frac{1}{4^n},\ n \in \mathbb{N}\right\}$ as a subset of $\mathbb{R}$ (standard topology); [2]

      (v) $\{x^2 - y^2 = 1\}$ as a subset of $\mathbb{R}^2$ (standard topology); [2]

      (vi) $\{1 < x^4 + y^4 \leq 2\}$ as a subset of $\mathbb{R}^2$ (standard topology). [2]

3. (a) Give the definition of a pathwise connected topological space. [4]

   (b) Prove that a pathwise connected topological space is connected. [4]

   (c) Give the definition of a homeomorphism. [3]

   (d) Which of the following maps are homeomorphisms? Justify your answers.

      (i) $f : \mathbb{R} \to (0,1],\ \ f(x) = e^{-x^2}$; [2]

      (ii) $f : [0, 2\pi) \to S^1,\ \ f(x) = e^{ix}$; [2]

      (iii) $f : (0, +\infty) \to (0, +\infty),\ f(x) = x^2$. [2]

   (e) Let X be the real line $\mathbb{R}$ endowed with the discrete topology and Y be the real line with the standard topology. Consider the map

$$f : X \to Y, \quad f(x) = x.$$

   Is f continuous? Is f invertible? Is $f^{-1} : Y \to X$ continuous? Justify your answers. [3]

4. (a) State the implicit function theorem. [4]

   (b) Is the subset $\left\{\frac{1}{4}x^4 + \frac{1}{2}xy^2 + xy = 100\right\} \subset \mathbb{R}^2$ a manifold? Justify your answer. [4]

   (c) Is the subset $\{x > 0,\ 0 < y < 1\} \subset \mathbb{R}^2$ a manifold? Justify your answer. [4]

   (d) What is the Euler characteristic of the Klein bottle? Justify your answer. [4]

   (e) Consider the surface M obtained from the fundamental polygon

$$AB^{-1}D^{-1}CB^{-1}CA^{-1}D$$

   by pairwise identification of its edges. Find the Euler characteristic of M and determine its topological type. [4]