Elements of topology

Summer 2019

Solutions.

No 1.

(a) Def. A set $\mathcal{U}$ in a metric space (X, d) is open if and only if for any $x \in \mathcal{U}$ there exists $\delta > 0$ such that $B_\delta(x) \subset \mathcal{U}$, where [3] (bookwork)

$$B_\delta(x) = \{ y \in X : d(y, x) < \delta \}$$ (open ball of radius δ centered at x)

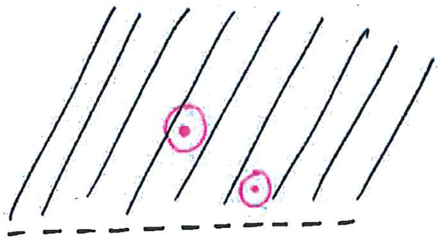
(b) Def. Let $A \subset X$. A point $a \in X$ is called interior point of A if there exists a neighborhood $\mathcal{U}$ of a such that $\mathcal{U} \subset A$. [3] (bookwork)

(c) Proof. Let $A \subset X$ and $\mathrm{Int}(A)$, the set of interior points of A.

Let $a \in \mathrm{Int}\,A$, i.e. a is an interior point. Then there is $\mathcal{U} = \mathcal{U}(a)$, a neighborhood of a, such that $\mathcal{U} \subset A$. Take any point $y \in \mathcal{U}$. Since $\mathcal{U}$ is open and contains y, then $\mathcal{U}$ can be considered as a neighborhood of y. It follows from this that y is an interior point of A and therefore $\mathcal{U} \subset \mathrm{Int}\,A$. Thus, each point $a \in \mathrm{Int}\,A$ is contained in $\mathrm{Int}\,A$ together with some neighborhood $\mathcal{U} = \mathcal{U}(a)$. This implies that $\mathrm{Int}\,A$ is open. [4] (bookwork)

(standard problem)

(d) (i) $\underbrace{\{y>0\}}_{A} \subset \underbrace{\mathbb{R}^2}_{X}$

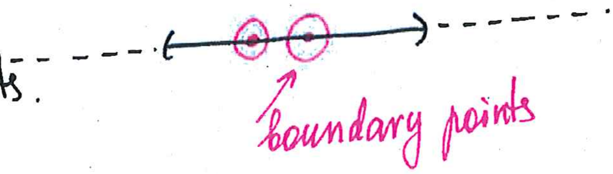


Each point $(x,y)\in A$ is interior.
Therefore, $\operatorname{Int} A = A$

[2]

(ii) $\underbrace{\{y=0,\ 0<x<1\}}_{A} \subset \underbrace{\mathbb{R}^2}_{X}$

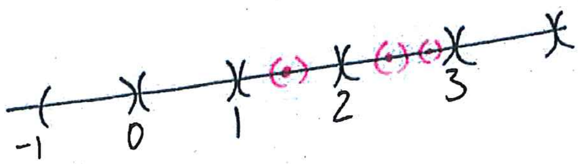
[2]



A does not have any interior points.
Therefore $\operatorname{Int} A = \emptyset$

[2]

(iii) $\underbrace{\mathbb{R}\setminus\mathbb{N}}_{A} \subset \underbrace{\mathbb{R}}_{X}$

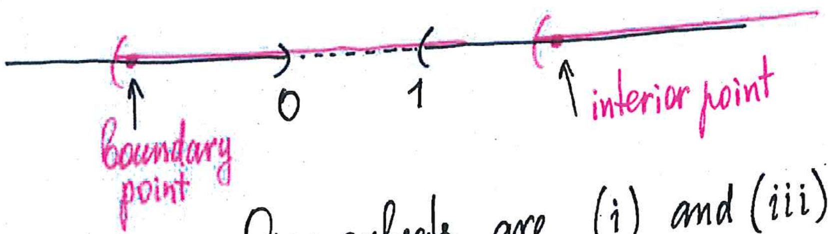


Each point $x\in A$ is interior.
Therefore, $\operatorname{Int} A = A$

[2]

(iv) $\underbrace{(-\infty,0)\cup(1,+\infty)}_{A} \subset \underbrace{\mathbb{R}_\tau}_{X}$

x is interior $\Leftrightarrow x\in(1,+\infty)$
Therefore, $\operatorname{Int} A = (1,+\infty)$.



Open subsets are (i) and (iii), as they coincides with $\operatorname{Int} A$.

[2]

No 2.

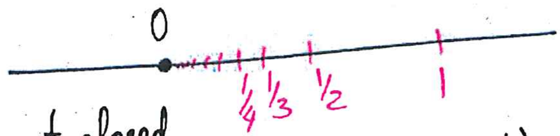
(a) Thm. A subset X of a Euclidean space $\mathbb{R}^n$ is compact if and only if it is closed and bounded. [3] (bookwork)

(b) Proof. Let $A \subset X$ be a closed subset of a compact topological space. [5] (bookwork)

Let $\mathcal{U} = \{ U_\alpha, \alpha \in I \}$ be any open cover of A,
then $\mathcal{U}' = \{ \mathcal{U}, X \setminus A \}$ is an open cover for X.
Since X is compact, we can choose a finite subcover for X.
Obviously, the same subcover (the set $X \setminus A$ should be excluded) can be considered as a finite subcover for A.
So A is compact.

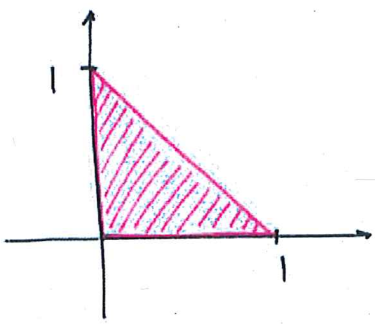
(c) Compact or not? (Standard problem)

(i) $\{ \frac{1}{n}, n \in \mathbb{N} \}$



This set is bounded, but not closed (the point 0 is adherent, but does not belong to the set) [2]
So this set is not compact.

(ii) $\{ x \geq 0, y \geq 0, x+y \leq 0 \}$

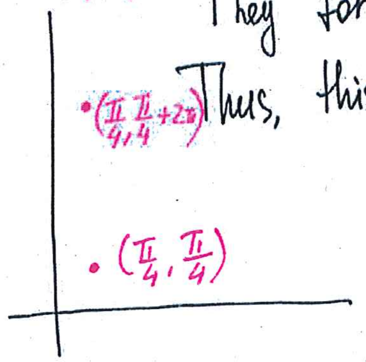


This set is compact as it is bounded and closed

[2]

(iii) $\{\sin(x-y) = \cos(x+y)\} \subset \mathbb{R}^2$

Consider the points $\left(\frac{\pi}{4}, \frac{\pi}{4} + 2\pi k\right)$.
All of them belong to the set.
They form a sequence tending to infinity (i.e. unbounded).
Thus, this set is not bounded and, therefore, is not closed.



[2]

(unseen)

(iv) $\mathbb{N} \subset X = \mathbb{R}$ indiscrete [3]

The induced topology on $\mathbb{N}$ is indiscrete.
Thus, $\mathbb{N}$ is compact (all indiscrete top. spaces are compact).

(v) $\mathbb{N} \subset X = \mathbb{R}_\tau$ $\quad \tau = \{\mathbb{R}, \emptyset, (a, +\infty), a \in \mathbb{R}\}$

[3] (unseen)

Let $\mathcal{U}$ be an open cover of $\mathbb{N}$. Consider an open subset $U_\alpha \in \mathcal{U}$ which covers the point 1.
By definition of τ, U_α must be of the form $(a, +\infty)$ where $a < 1$. It follows from this that U_α covers the whole set $\mathbb{N}$ and, therefore, can be considered as a finite subcover of $\mathbb{N}$. So $\mathbb{N}$ is compact.

No 3. (a) Def. A topological space (X, τ) is pathwise connected iff for any $x, y \in X$ there exists a continuous map $f : [0,1] \longrightarrow X$ such that $f(0) = x$, $f(1) = y$. Such a map is called a (continuous) path from x to y. [4] (bookwork)

(b) Proof. Let X be pathwise connected. [4] (bookwork)
By contradiction, assume that X is disconnected. Consider a partition of X into two disjoint non-empty open subsets: $X = A \cup B$. Take $x \in A$, $y \in B$ and a continuous path $f : [0,1] \longrightarrow X$ from x to y.
Then $[0,1] = f^{-1}(A) \cup f^{-1}(B)$ is a partition of $[0,1]$ into two disjoint non-empty open sets, which is impossible since $[0,1]$ is connected.

(c) No, X is disconnected. [4] (standard question)
As a partition of X into disjoint non-empty open subsets we can take $X = \cancel{A \cup B}$ $X^+ \cup X^-$ where

$$X^+ = \{A \in X, \det A > 0\}$$

and

$$X^- = \{A \in X, \det A < 0\}$$

$X^\pm$ are obviously disjoint and non-empty. They are both open since $\det : X \to \mathbb{R}$ is continuous function, and $X^+ = \det^{-1}(0, +\infty)$, $X^- = \det^{-1}(-\infty, 0)$, i.e. can be considered as preimages of open sets under a continuous map.

(d) $X = [0,1]$, $Y = \{x^2+y^2=1\}$

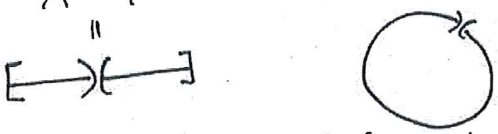
[4]
~~Standard~~
(unseen)

X and Y are not homeomorphic.

By contradiction, assume $f : X \to Y$ is a homeomorphism.

Take $x = \frac{1}{2} \in X$ and its image $f(x) \in Y$.

If we remove x and $f(x)$ from X and Y respectively, then the restriction of f onto $X \setminus \{x\}$ will give a homeomorphism between $X \setminus \{x\}$ and $Y \setminus \{f(x)\}$



which is impossible as $X \setminus \{x\}$ is disconnected whereas $Y \setminus \{f(x)\}$ is connected.

(e) $X = \{x^2+y^2+z^2=1\}$ and $Y = \{x^2+2y^2+3z^2=1\}$ are homeomorphic.

[4]
(standard question)

Indeed, $F : X \to Y$, $F(x,y,z) = \left(x, \frac{y}{\sqrt{2}}, \frac{z}{\sqrt{3}}\right)$ is a bijective map continuous in both directions, i.e. a homeomorphism.

No 4.

(a) Thm. Let $F: \mathbb{R}^n \to \mathbb{R}$ be a smooth function, and $X = \{F(x_1, x_2, \dots, x_n) = a\}$ be one of its level sets. If $dF = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \dots, \frac{\partial F}{\partial x_n}\right) \neq 0$ at any point $P \in X$, then X is a manifold of dimension $n-1$.

[4] (bookwork)

(b) $X = \{x^3 + 3xy^2 + y^6 - z^4 = 1\} \subset \mathbb{R}^3$

[4] (standard question)

Use the IFTh.

$dF = (3x^2 + 3y^2, \; 6xy + 6y^5, \; 4z^3)$

If $dF = 0$ then $3x^2 + 3y^2 = 0$, i.e. $x = y = 0$
$4z^3 = 0$, i.e. $z = 0$

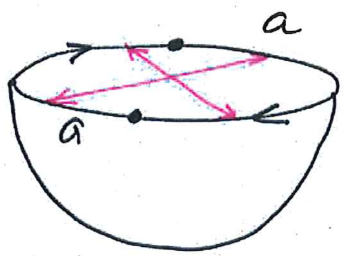
But $(0,0,0) \notin \{x^3 + 3xy^2 + y^6 - z^4 = 1\}$, i.e. dF nowhere vanishes on X, and therefore X is a manifold of dimension 2.

(c) $GL(2,\mathbb{R})$ is an open subset of $M_{2,2} = \{2\times 2 \text{ matrices}\}$ (we use the fact that det is a continuous map). This implies that for each $A \in GL(2,\mathbb{R})$ there is a δ-ball $B_\delta(A)$ s.t. $B_\delta(A) \subset GL(2,\mathbb{R})$. Thus, each point $A \in GL(2,\mathbb{R})$ possesses a neighborhood homeomorphic to a 4-dim ball. So $GL(2,\mathbb{R})$ is a manifold of dimension 4.

[4] (unseen)

(d) The Euler characteristic of $\mathbb{R}P^2$ (projective plane) is 1.

Indeed, $\mathbb{R}P^2$ can be represented as the hemisphere with opposite boundary points identified.

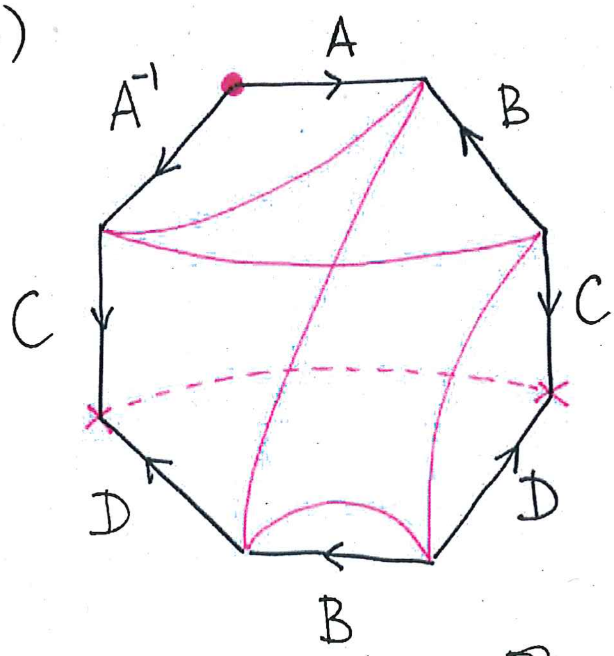


This representation can be undestood as a partition of $\mathbb{R}P^2$ into 3 cells: one face, one edge and on vertex.

Hence, $\chi(\mathbb{R}P^2) = 1 - 1 + 1 = \underline{1}$.

[3] (standard question)

(e)



number of faces $F = 1$
number of edges $E = 4$
number of vertices $V = 3$
(see Figure)

$\chi(M) = 1 - 4 + 3 = \underline{0}$

Each edge comes in combination $\ldots a \ldots a^{-1} \ldots$, therefore M is orientable.

An orientable surface with zero Euler characteristic is homeomorphic to <u>the 2-torus T^2</u>.

[5] (standard question)