MAB298-Elements of Topology: Problem Sheet 3
Continuous maps and homeomorphisms

1. Let $f : X \to Y$ be a continuous map between two metric spaces, and (x_n) be a convergent sequence. Is it true that the sequence $y_n = f(x_n)$ converges?

2. Prove that the image of an everywhere dense set under a surjective continuous map is everywhere dense.

3. Is it true that the image of nowhere dense set under a continuous map is nowhere dense?

4. Let $X = C^1[0,1]$ be the space of continuously differentiable functions on $[0,1]$ with the metric $d(f,g) = \max_{x\in[0,1]}|f(x) - g(x)|$.

   Let $F_1 : X \to \mathbb{R}$ be defined by $F_1(f) = \int_0^1 f(x)dx$. Is F_1 continuous?

   Let $F_2 : X \to \mathbb{R}$ be defined by $F_2(f) = f'(0)$. Is F_2 continuous?

5. Prove that the following plane figures are homeomorphic to each other.

   (a) A half-plane: $\{x \geq 0\}$;
   (b) a quadrant: $\{x, y \geq 0\}$;
   (c) an angle: $\{x \geq y \geq 0\}$;
   (d) a square without two sides: $\{0 \leq x, y < 1\}$;
   (e) a disk without a boundary point: $\{x^2 + y^2 \leq 1, y \neq 1\}$;
   (f) a half-disk without the diameter: $\{x^2 + y^2 \leq 1, y > 0\}$;

6. Prove that the following plane domains are homeomorphic to each other:

   (a) punctured plane $\mathbb{R}^2 \setminus (0,0)$;
   (b) punctured open disk $B^2 \setminus (0,0) = \{0 < x^2 + y^2 < 1\}$;
   (c) annulus $\{a < x^2 + y^2 < b\}$, where $0 < a < b$;
   (d) plane without a disk: $\mathbb{R}^2 \setminus D^2$;

(e) plane without a segment: $\mathbb{R}^2 \setminus [0,1]$;

7. Let $K = \{a_1, \dots, a_n\} \subset \mathbb{R}^2$ be a finite set. The complement $\mathbb{R}^2 \setminus K$ is a plane with n punctures. Convince yourself that any two planes with n punctures are homeomorphic, i.e., the position of $a_1, \dots, a_n$ in $\mathbb{R}^2$ does not affect the topological type of $\mathbb{R}^2 \setminus \{a_1, \dots, a_n\}$.

8. Let $D_1, \dots, D_n \subset \mathbb{R}^2$ be pairwise disjoint closed disks. The complement of the union of its interiors is said to be plane with n holes. Convince yourself that any two planes with n holes are homeomorphic, i.e., the location of disks $D_1, \dots, D_n$ does not affect the topological type of $\mathbb{R}^2 \setminus \bigcup_{i=1}^n \operatorname{Int} D_i$.

9. Prove that a mug (with a handle) is homeomorphic to a doughnut.