MAB298-Elements of Topology: Solution Sheet 6

Homeomorphic or not?

1. Divide the topological spaces indicated below into classes of pairwise homeomorphic spaces:

   - $\mathbb{R}$ with discrete topology
   - $\mathbb{Z}$ with discrete topology
   - $\mathbb{R}$ with indiscrete topology
   - $\mathbb{N}$ as a subset of $\mathbb{R}$ (with the induced topology)
   - $\mathbb{R}$ (standard topology)
   - $\mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}, (a, +\infty), a \in \mathbb{R}\}$
   - $\mathbb{R}^2$
   - (a, b)
   - $[a, b)$
   - $(a, b]$
   - $[a, b]$
   - open half plane $\{(x, y) \ : \ y > 0\}$
   - closed half plane $\{(x, y) \ : \ y \geq 0\}$
   - open quadrant $\{(x, y) \ : \ x > 0, y > 0\}$
   - closed quadrant $\{(x, y) \ : \ x \geq 0, y \geq 0\}$
   - sphere $\{x^2 + y^2 + z^2 = 1\}$
   - open disc $\{x^2 + y^2 < 1\}$
   - closed disc $\{x^2 + y^2 \leq 1\}$
   - $\{(x, y) \ : \ 0 \leq y < 1\}$
   - $\{(x, y) \ : \ xy = 0\}$
   - annulus $\{1 < x^2 + y^2 < 4\}$
   - punctured plane $\mathbb{R}^2 \setminus \{(0, 0)\}$
   - punctured sphere $\{x^2 + y^2 + z^2 = 1\} \setminus \{(0, 0, 1)\}$

- $\mathbb{Q}$ as a subset in $\mathbb{R}$

It is relatively easy to check whether two given spaces from the list are homeomorphic or not. For example, $\mathbb{R}$ and $[a, b]$ are not homeomorphic because, $[a, b]$ is compact, but $\mathbb{R}$ is not. Another example: we notice that $[a, b)$ and $(a, b]$ are isomorphic because we may take the homomorphism $f : [a, b) \to (a, b]$ of the form $f(x) = a + b - x$.

However, dealing with 24 spaces, it might be more convenient to divide them into natural groups like "compact — non-compact", "connected — disconnected" and so on. Let us do it.

Compact: $\mathbb{R}$ with indiscrete topology, $[a, b]$, sphere, closed disc

Non-compact: All the others are not compact

Connected: All except for those listed below

Disconnected: $\mathbb{R}$ with discrete topology, $\mathbb{Z}$ with discrete topology, $\mathbb{Q}$, $\mathbb{N}$

Discrete: $\mathbb{R}$ with discrete topology, $\mathbb{Z}$ with discrete topology, $\mathbb{N}$ as a subset of $\mathbb{R}$

Indiscrete: $\mathbb{R}$ with indiscrete topology

Neither discrete not indiscrete: all the other spaces

Hausdorff: All except for listed below

non-Hausdorf: $\mathbb{R}$ with indiscrete topology and $\mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}, (a, +\infty)\}$

Countable: $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Q}$

Uncountable: all the others

Simply connected[1] All except for

Non-simply connected: annulus and punctured plane

[1] X is simply connected if any closed curve in X can be contracted to a single point.

Using this information we can easily divide these space into groups. Each group contains spaces with similar topological properties. It is clear that spaces from distinct groups are not homeomorphic.

The groups obtained in this way are:

G1: $\mathbb{R}$ with discrete topology (discrete, uncountable, non-compact, disconnected, Hausdorff)

G2: $\mathbb{Z}$ with discrete topology (discrete, countable, non-compact, disconnected, Hausdorff)

G3: $\mathbb{R}$ with indiscrete topology (indiscrete, compact, connected, non-Hausdorff, uncountable, simply connected)

G4: $\mathbb{N}$ (Hausdorf, discrete, countable, non-compact, disconnected)

G5: $\mathbb{Q}$ (neither discrete, nor indiscrete, non-compact, disconnected, Hausdorff, countable)

G6: $\mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}, (a, +\infty)\}$ (neither discrete, nor indiscrete, connected, uncountable, non-compact, non-Hausdorff)

G7: $\mathbb{R}$, $\mathbb{R}^2$, $[a, b)$, $(a, b]$, (a, b), open half-plane $\{(x, y) : y > 0\}$, closed half-plane $\{(x, y) : y \geq 0\}$ open quadrant $\{(x, y) : x > 0, y > 0\}$, closed quadrant $\{(x, y) : x \geq 0, y \geq 0\}$, open disc $\{(x, y) : 0 \geq y < 1\}$, $\{(x, y) : 0 \leq y < 1\}$, $\{(x, y) : xy = 0\}$, punctured sphere (neither discrete, nor indiscrete, connected, uncountable, non-compact, simply connected, Hausdorff)

G8: $[a, b]$, sphere, closed disc (neither discrete, nor indiscrete, connected, uncountable, compact, simply connected, Hausdorff)

G9: annulus, puncture plane (neither discrete, nor indiscrete, connected, uncountable, non-compact, non-simply connected, Hausdorff)

Now we need to find out which spaces from the groups G7, G8 and G9 are homeomorphic.

Notice that *one-dimensional* connected spaces (like $\mathbb{R}$, $[a, b)$, $(a, b]$, (a, b) $[a, b]$, circle) and *two-dimensional* connected spaces (like $\mathbb{R}^2$, open half-plane $\{(x, y) : y > 0\}$, closed half-plane $\{(x, y) : y \geq 0\}$ open quadrant $\{(x, y) : x > 0, y > 0\}$, closed quadrant $\{(x,y) \,:\, x \geq 0, y \geq 0\}$, open disc $\{(x,y) \,:\, 0 \geq y < 1\}$, sphere, punctured sphere, closed disc, annulus, punctured plane) are not homeomorphic because of the following reason: if we remove a finite number of appropriate points from a one-dimensional space, we can get a disconnected set; whereas removing a finite number of points from a two dimensional set leaves it connected.

Another useful idea allows us to distinguish two-dimensional sets with and without boundary points. Example: the closed and open half planes are not homeomorphic to each other because of the following reason. If we remove a boundary point from the closed half plane, this half plane remains simply connected; if we remove any point from an open half plane, it becomes non-simply connected.

The idea used in Exercise 7, Problem Sheet 4 (i.e. remove an appropriate point), allows us to see that $[a, b)$ and (a, b) are not homeomorphic.

Finally, we get the following partition into classes of pairwise homeomorphic spaces (and this is the complete answer to the question):

- $\mathbb{R}$ with discrete topology
- $\mathbb{R}$ with indiscrete topology
- $\mathbb{N}$, $\mathbb{Z}$
- $\mathbb{Q}$
- $\mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}, (a, +\infty)\}$
- $\mathbb{R}$,(a, b)
- $\mathbb{R}^2$, open half-plane $\{(x,y) \,:\, y > 0\}$, open quadrant $\{(x,y) \,:\, x > 0, y > 0\}$, open disc $\{(x,y) \,:\, 0 \geq y < 1\}$, punctured sphere[2]
- $[a, b)$, $(a, b]$,

[2]These can be viewed as subsets of the complex plane $\mathbb{C}$. The Riemann mapping theorem says that any open, simply connected set is in fact holomorphically equivalent (not just homeomorphic) to the unit disk, i.e. there is a holomorphic map between the disk and any of these sets, with a holomorphic inverse.

- closed half-plane $\{(x, y) : y \geq 0\}$, closed quadrant $\{(x, y) : x \geq 0, y \geq 0\}$, $\{(x, y) : 0 \leq y < 1\}$
- $[a, b]$,
- $\{(x, y) : xy = 0\}$
- sphere,
- closed disc
- annulus, puncture plane

It is not hard to see that the spaces from the same item are indeed homeomorphic to each other.