Loughborough University

ELEMENTS OF TOPOLOGY
(19MAC298)

Summer 2020 — Open book remote assessment (Option 1a, 23 hours)

Please read this page carefully.

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1. The expected amount of time answering the questions on this paper should be comparable to 2 hours within the given 23 hours range.

2. Answer **ALL** questions.

3. **For each question, please show all intermediate steps and comment on what you are doing to answer the question as if you were explaining your answer to your lecturer. Please write in full sentences, but be clear and concise.**
   Because of the open book nature of this assessment, it is not sufficient to merely state answers. To obtain full marks, your answer must be correct and appropriately explained, e.g. marks will be awarded for your overall working and explanation and will contribute towards 50% of the mark for each question.

4. Please return your work as a single PDF file, handwritten work is fine and accepted for submission. Please follow the guidance provided on how to scan and upload your work.

5. **Please note that your work will be checked for plagiarism and collusion.**

   Students are not permitted to contact or collaborate with other students in the process of completing this paper.

   Submissions will be subject to scrutiny in line with the University's Code of Practice on Plagiarism Detection and using the standard software applied to assessed work to detect for possible cases of cheating, plagiarism and collusion. For example by using text-matching and forensic linguistic analysis to evaluate whether the work submitted is students' own work.

   The University will also compare student performance in these assessments with prior performance and reserves the right to undertake a viva voce examination (i.e. an interview) with a student if it is felt necessary to do so to confirm the authorship of any of their submitted work.

1. (a) Consider the following collection τ of subsets of the real line $\mathbb{R}$:

   $\mathbb{R}$, $\emptyset$ and all the subsets of the form $\mathbb{R} \setminus S$, where S is a finite set.

   Is τ a topological structure? Justify your answer. [3]

   (b) Sketch the following subsets $A \subset \mathbb{R}^2$ and describe the interior of A:

   (i) $A = \{y \geq 0, \quad 0 < x < 1\}$; [2]

   (ii) $A = \{x = 0, \quad y \in (0, +\infty)\}$; [2]

   (iii) $A = \{(x, y), \quad x \in \mathbb{Z}, y \in \mathbb{Z}\}$; [2]

   (iv) $A = \{y < \sin x\}$. [2]

   Which of these subsets are open? [2]

2. (a) Prove by definition that an infinite discrete topological space is not compact. [3]

   (b) Let $A \subset \mathbb{R}$. Suppose that A is not closed. Prove that there exists a continuous function $f : A \to \mathbb{R}$ which is not bounded on A. [3]

   (c) Which of the following subsets $A \subset X$ are compact (justify your answer):

   (i) $\{\frac{1}{n^2+1} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ (standard topology); [2]

   (ii) $\{x^2 - 2y^2 = 15\}$ in $\mathbb{R}^2$ (standard topology); [2]

   (iii) $\{x^4 + y^4 + \sin x = 10\}$ in $\mathbb{R}^2$ (standard topology); [2]

   (iv) $A = [0, 1)$ as a subset of $\mathbb{R}$ with the topology $\tau = \{\mathbb{R}, \emptyset, (a, +\infty), \text{where } a \in \mathbb{R}\}$. [3]

3. (a) Prove that connectedness is preserved under continuous maps, i.e., if X is connected and $f : X \to Y$ is continuous, then $f(X)$ is connected. [4]

   (b) Consider $\mathbb{R}$ with the topology $\tau = \{\mathbb{R}, \emptyset, (a, +\infty), \text{where } a \in \mathbb{R}\}$. Is this topological space connected? Justify your answer. [4]

   (c) Let $X = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ (sphere) and $Y = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$ (circle). Are these topological spaces homeomorphic? Justify your answer. [4]

   (d) Let X be the set of all real triangular invertible 2×2 matrices viewed as a subset in the four-dimensional vector space of 2×2 matrices:

   $$X = \left\{ A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad \det A \neq 0, \ a, b, c \in \mathbb{R} \right\}.$$

   Is X connected? Justify your answer. [4]

4. (a) Is the subset $\{x^2 + y^2 - z^2 = 0\} \subset \mathbb{R}^3$ a manifold? Justify your answer. [4]

   (b) Is the subset $\{e^{x+y} - x - y + x^2 z + y^2 z = 2\} \subset \mathbb{R}^3$ a manifold? Justify your answer. [4]

   (c) What is the Euler characteristic of the sphere? Justify your answer. [3]

   (d) Consider the surface M obtained from the fundamental polygon

$$ABD^{-1}C^{-1}BACD$$

   by pairwise identification of its edges. Find the Euler characteristic of M and determine its topological type. [5]