Week 6 Problem Class : Problem Sheet 5

1(c) $E: y^2 = x^3 - 16x + 16$.

Calculate $T \subset E(\mathbb{Q})$ by using Nagell-Lutz.

Solution: The discriminant is

$\Delta = -4(-16)^3 - 27(16)^2 = (16)^2 \cdot 37$

$= 2^8 \cdot 37$.

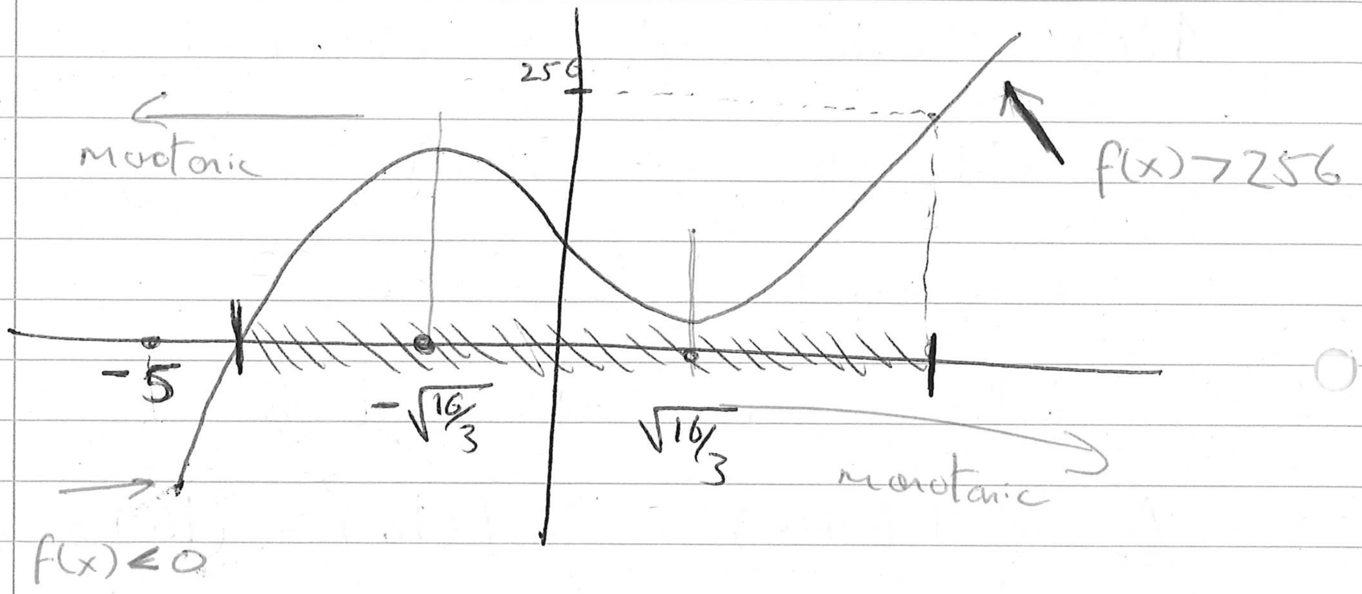
So if $y^2 \mid \Delta$ ($y \neq 0$) then $|y| = 1, 2, 4, 8,$ or 16.

$\lvert y \rvert$	0	1	2	4	8	16
y^2	0	1	4	16	64	256
x						

We need to decide whether there is a solution for $x \in \mathbb{Z}$ in each case.

Let's do it like this.

Consider the graph of $f(x) = x^3 - 16x + 16$



We want to determine the range of x-values ~~of~~ we need to consider.

$\frac{df}{dx} = 3x^2 - 16$ $\therefore$ f is monotonic when $|x| > \sqrt{16/3} > 2$

Check : $f(-5) = -29 < 0$

and $|-5| > 2$

$\therefore f(-5) < 0 \quad \forall x \leq -5$

Also $f(8) = 400$ and $|8| > 2$

$\therefore f(x) \geq 400 > 256 \quad \forall x \geq 8.$

So $f(x)$ can't equal one of our y^2 values when $x \geq 8$.

So we can restrict our attention to

$-4 \leq x \leq 7 \quad , x \in \mathbb{Z}.$

For these x we compute the values of

$f(x)$:

$\underset{x=-4}{\underline{16}}, 37, 40, 31, \underset{x=0}{\underline{16}}, \underset{x=1}{\underline{1}}, -8, -5, \underset{x=4}{\underline{16}}, 61, 136, 247$

Which ones match y^2 values in our list?

So we get candidate torsion points

$\pm P_1 = (0, \pm 4)$

$\pm P_2 = (-4, \pm 4)$

$\pm P_3 = (4, \pm 4)$

$\pm P_4 = (1, \pm 1)$

together with $\mathcal{O}$.

Now we need to decide which (if any) are actual torsion points.

To do this: compute multiples.

Start with $P_1 = (0,4)$

we find

$$x(2P_1) = \left(\frac{3x^2-16}{2y}\right)^2\Bigg|_{P_1} - 2x(P_1)$$

$$= \left(\frac{-16}{8}\right)^2 - 2(0)$$

$$= 4.$$

and $y(2P_1) = 4$

So $2P_1 = (4,4) = P_3$.

Also $3P_1 = 2P_1 + P_1$

$$= (-4,-4) = -P_2$$

$4P_1 = 2(2P_1)$

$$= (8,-20).$$

Since $(-20)^2$ does not divide Δ,

Nagell-Lutz $\Rightarrow$ $4P_1$ does not have finite order $\Rightarrow$ P_1 doesn't either.

So no multiple of P_1 can have finite order

$\therefore$ $\pm P_1$, $\pm P_2$, $\pm P_3$ don't have finite order.

It remains to consider $\pm P_4 = (1, \pm 1)$.

Again, use addition formula:

$$x(2P_4) = \left(\frac{3x^2-16}{2y}\right)^2\Bigg|_{P_4} - 2x(P_4)$$

$$= \left(\frac{3-16}{2}\right)^2 - 2$$

$$= \frac{13^2}{4} - 2 = \frac{161}{4} \notin \mathbb{Z}.$$

$\therefore$ By Integrality Theorem, $2P_4$ does not have finite order $\therefore$ neither does $\pm P_4$.

So finally $T = \{O\}$.

Comments: in 1(a) and 1(b)

there is an easier way to decide if $\exists$

a solution $x \in \mathbb{Z}$ for given y^2.

E.g. 1(b) $y^2 = x^3 + 4x = x(x^2+4)$.

Nagell-Lutz says $y = 0$ or $|y| = 1, 2, 4, 8, 16$.

$\lvert y\rvert$	0	1	2	4	8	16
$x(x^2+4) = y^2$	0	1	4	16	64	256
x	0	—	—	2	—	—

So we need to decide if ~~each~~ a given y^2 is of the form $x(x^2+4)$. To do this:

- note $x(x^2+4) \geq 0 \iff x \geq 0$
- note $x(x^2+4) \geq x^3$ for $x \geq 0$

so we only need to look as far as $\sqrt[3]{y^2}$.

When $y^2 = 256$ then x is at most $\sqrt[3]{256} \approx 6.34$ $\therefore$ $x \leq 6$ since $x \in \mathbb{Z}$

When $y = 0$ get $x = 0$
$|y| = 4$ get $x = 2$
} only solutions.

So here candidate torsion points

are $\{\mathcal{O}, (0,0), (2,\pm 4)\}$.

Finally: $(0,0)$ has order 2 $\therefore \in T$

What about $(2,\pm 4)$?

Let $P = (2,4)$. Compute multiples of P:

find $x(2P) = 0$ $\therefore y(2P) = 0$

so $2P = (0,0)$ which has order 2

$\therefore$ $4P = \mathcal{O}$ $\therefore$ $P \in T$ also.

So $T = \{\mathcal{O}, (0,0), (2,\pm 4)\}$.

This is a group with 4 elements:

could be $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

Not every element of T has order 1 or 2

$\therefore$ $T \cong \mathbb{Z}_4$.

$\square$.