21MAB143 Rings and Polynomials: Week 4

1 Resultant

Another computational approach to detecting common roots and multiple roots of polynomials comes from the **resultant** and **discriminant**. This week we will introduce them for polynomials of 1 variable; later in the course we will see how they work in the case of multiple variables, where their real importance becomes more clear.

In this lecture we will stick to polynomials with the coefficients in the complex numbers $\mathbf{C}$ or a subfield such as $\mathbf{R}$: when we speak of the roots of a real polynomial, we mean all its roots in the complex numbers.

Definition 1.1. *Consider two polynomials in* $\mathbf{C}[x]$*:*

$$f = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$$
$$g = b_e x^e + b_{e-1}x^{e-1} + \cdots + b_0$$

Their **resultant** $\mathrm{Res}(f,g)$ *is defined by:*

$$\mathrm{Res}(f,g) = \det \begin{pmatrix} a_d & 0 & \cdots & 0 & b_e & 0 & \cdots & 0 \\ a_{d-1} & a_d & \cdots & 0 & b_{e-1} & b_e & \cdots & 0 \\ a_{d-2} & a_{d-1} & \ddots & 0 & b_{e-2} & b_{e-1} & \ddots & 0 \\ \vdots & \vdots & \ddots & a_d & \vdots & \vdots & \ddots & b_e \\ a_0 & a_1 & \cdots & \vdots & b_0 & b_1 & \cdots & b_{e-1} \\ 0 & a_0 & \ddots & \vdots & 0 & b_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_1 & \vdots & \vdots & \ddots & b_1 \\ 0 & 0 & \cdots & a_0 & 0 & 0 & \cdots & b_0 \end{pmatrix}$$

This matrix is called the **Sylvester matrix** *of* f *and* g*. It is a square matrix of size* $d+e$*. Its first* e *columns contain the coefficients of* f *and its last* d *columns contain the coefficients of* g*. (Note that our picture of this matrix suggests that* $d = e$ *but in general this is not the case.)*

Example: Let's do a simple example to make the definition clearer. Take the polynomials

$$f = a_2x^2 + a_1x + a_0$$
$$g = b_1x + b_0$$

where a_0, a_1, a_2, b_0, b_1 are arbitrary coefficients. with both leading coeefficients a_2 and b_1 nonzero.

Here the degrees are $d = 2$ and $e = 1$. The resultant is

$$\mathrm{Res}(f, g) = \det \begin{pmatrix} a_2 & b_1 & 0 \\ a_1 & b_0 & b_1 \\ a_0 & 0 & b_0 \end{pmatrix}$$

The determinant of the Sylvester matrix is

$$\begin{aligned} & a_2 \cdot \det \begin{pmatrix} b_0 & b_1 \\ 0 & b_0 \end{pmatrix} - b_1 \cdot \det \begin{pmatrix} a_1 & b_1 \\ a_0 & b_0 \end{pmatrix} \\ &= a_2 b_0^2 - b_1 \cdot (a_1 b_0 - b_1 a_0) \\ &= a_2 b_0^2 - a_1 b_0 b_1 + a_0 b_1^2. \end{aligned}$$

Now notice that our polynomial g has a unique root at $b = -\frac{b_0}{b_1}$. We can rewrite the resultant above in terms of b as

$$\begin{aligned} & b_1^2 \left(a_2 \left(-\frac{b_0}{b_1} \right)^2 + a_1 \left(-\frac{b_0}{b_1} \right) + a_0 \right) \\ &= b_1^2 f(b). \end{aligned}$$

Since $b_1 \neq 0$, this means the resultant equals 0 if and only if $f(b) = 0$, in other words if and only if f and g have a **common root**.

What we just saw in our example was no coincidence — in fact, for any two polynomials it is true that they have a common root if and only if their resultant equals zero. To understand why, we need to look at the resultant in a different way.

Write $\mathbf{C}[x]_n$ to denote the vector space of complex polynomials of degree $< n$: you learned in the Linear Algebra module that this is a vector space of dimension n.

Now let $f,\ g \in \mathbf{C}[x]$ be polynomials of degrees d and e respectively. We can define a linear map

$$\begin{aligned} \varphi_{f,g} \colon \mathbf{C}[x]_e \oplus \mathbf{C}[x]_d &\to \mathbf{C}[x]_{d+e} \\ (a, b) &\mapsto af + bg \end{aligned}$$

Here the space $\mathbf{C}[x]_e \oplus \mathbf{C}[x]_d$ is called the **direct sum** of $\mathbf{C}[x]_e$ and $\mathbf{C}[x]_d$. Its elements are pairs (a, b) in which $a \in \mathbf{C}[x]_e$ and $b \in \mathbf{C}[x]_d$. It has a basis given by the set

$$\left\{ (x^{e-1}, 0),\ (x^{e-2}, 0), \ldots, (x, 0),\ (1, 0),\ (0, x^{d-1}),\ (0, x^{d-2}), \ldots, (0, x),\ (0, 1) \right\}$$

so its dimension is $d + e$.

The space $\mathbf{C}[x]_{d+e}$ also has dimension $d + e$, with a basis given by

$$\left\{ x^{d+e-1},\ x^{d+e-2}, \ldots,\ x,\ 1 \right\}$$

Having fixed these two bases we can now describe the resultant in a new way:

Proposition 1.2. *Let* $f,\ g \in \mathbf{C}[x]$ *be two polynomials. Let* $\varphi_{f,g}$ *be the linear map described above, and let* $\Phi_{f,g}$ *be the matrix representing* $\varphi_{f,g}$ *with respect to the two bases described above. Then*

$$\mathrm{Res}(f, g) = \det(\Phi_{f,g}).$$

Proof. We just need to prove that $\Phi_{f,g}$ is the Sylvester matrix of f and g, as defined in Definition 1.1. To see this, remember that the columns of $\Phi_{f,g}$ are obtained by applying the map $\varphi_{f,g}$ to the elements of our fixed basis for $\mathbf{C}[x]_e \oplus \mathbf{C}[x]_d$. So for example the first column is given by

$$\begin{aligned} \varphi_{f,g}((x^{e-1}, 0)) &= x^{e-1} \cdot f + 0 \cdot g \\ &= x^{e-1} \cdot f \end{aligned}$$

and writing this as a linear combination of elements of our basis for $\mathbf{C}[x]_{d+e}$ we get exactly the first column:

$$x^{e-1} \cdot f = a_d x^{d+e-1} + a_{d-1} x^{d+e-2} + \cdots + a_0 x^{e-1}$$

Applying $\varphi_{f,g}$ in the same way to all other elements of our basis, we get exactly the columns of the Sylvester matrix in Definition 1.1. $\square$

This new characterisation of the resultant will let us prove the key property mentioned before:

Theorem 1.3. *Two polynomials* f *and* g *have a common root if and only if* $\mathrm{Res}(f, g) = 0$.

Proof. The idea of the proof is now to focus on the linear map $\varphi_{f,g}$ instead of the Sylvester matrix $\Phi_{f,g}$. From the Linear Algebra module, you know that the determinant of a square matrix is zero if and only if the corresponding linear map is not an isomorphism. So we need to prove that f and g have a common root if and only if $\varphi_{f,g}$ is not an isomorphism.

If f and g have a common root z then every polynomial of the form $af + bg$ has a root at z. By the Factor Theorem, this means that every polynomial of the form $af + bg$ is divisible by $x - z$. In particular, this means that $\mathrm{Im}(\varphi_{f,g})$ is a proper subspace of $\mathbf{C}[x]_{d+e}$, hence $\varphi_{f,g}$ is not surjective, therefore not an isomorphism of vector spaces,

Conversely, suppose $\varphi_{f,g}$ is not an isomorphism. Then it is not injective. So there exist polynomials a, b, not both zero, with $\deg(a) < d$ and $\deg(b) < e$, such that $af + bg = 0$. Rearranging, this implies $af = -bg$. If say $a = 0$ then since $b \neq 0$ we must have $g = 0$, contradicting the fact that $\deg(g) = e$; similarly if $b = 0$. So we can assume that none of a, b f, g equals zero.

In this case, the equation $af = -bg$ implies every root of f is a root of either b or g. But $\deg(b) < \deg(f)$ so not every root of f can be a root of b. So there is at least one root of f which is a root of g, therefore f and g have a common root. $\square$

1.1 Another definition

There is a third way to describe the resultant, in terms of the roots. This form is useful to prove some properties of the resultant, though it is not practically useful for computation.

Suppose we have two polynomials f, $g \in \mathbf{C}[x]$ of degrees d and e respectively:

$$f = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0$$
$$g = b_e x^e + b_{e-1} x^{e-1} + \cdots + b_0$$

By the Fundamental Theorem of Algebra, we know that f has exactly d complex roots counted with multiplicity: call these roots $\lambda_1, \ldots, \lambda_d$. Similarly g has exactly e complex roots counted with multiplicity: call them $\mu_1, \ldots, \mu_e$.

Then the resultant of f and g can be expressed in terms of these roots:

Proposition 1.4. *For polynomials f and g as above, their resultant is*

$$\operatorname{Res}(f, g) = a_d^e b_e^d \left(\prod_{\substack{1 \le i \le d \\ 1 \le j \le e}} (\lambda_i - \mu_j) \right)$$

In other words, ignoring the leading coefficient, the resultant is the product of all possible differences between a root of f and a root of g.

We are not going to prove this formula for the resultant, but let us make a few remarks:

- If we take this formula as given, Theorem 1.3 follows immediately. A product equals zero if and only if one of its factors equals zero. We know a_0 and b_0 are nonzero, so $\operatorname{Res}(f, g)$ equals zero if and only if $\lambda_i - \mu_j = 0$ for some i and j, in other words one of the roots of f equals one of the roots of g.

- If f and g have coefficients in $\mathbf{Q}$ for example, then by Definition 1.1 it is clear that $\operatorname{Res}(f, g)$ is in $\mathbf{Q}$. But this is not at all clear from the above formula: some or all of the roots λ_i, μ_j may be irrational or even non-real, and it is not obvious that the product will turn out to be a rational number. (In fact we will be able to prove this once we learn more about symmetric functions in Week 7.)

- This formula for the resultant is not well-suited to computation. In general even if f and g have rational coefficients, the roots λ_i and μ_j can only be found approximately, and this may lead to numerical instability in the computation. Defintion 1.1 is more suitable for computation.

1.2 Example

Here is a more complicated resultant calculation. It shows one significant advantage of the resultant method over the GCD method for finding common roots: the resultant is more convenient for dealing with polynomials with **parameters**. Given two families of polynomials whose coefficients depend on some parameters, we can compute the resultant as a function of the parameters and use this to determine which members of the families have roots in common.

In our specific example, we let α and β be parameters (in other words, arbitrary constants) and consider the polynomials

$$\begin{aligned} f &= x^3 - x \\ g &= x^2 - (\alpha + \beta)x + \alpha\beta \end{aligned}$$

Now f factorises as $f = (x-1)x(x+1)$, so it has roots -1, 0, 1, while g factorises as $g = (x-\alpha)(x-\beta)$, so it has roots α and β. Therefore by Theorem 1.3 we should have that $\mathrm{Res}(f, g) = 0$ if and only if one of α or β equals one of -1, 0, 1. Let's check this is true.

The polynomial f has degree $d = 3$ and coeffcients

$$a_3 = 1,\ a_2 = 0,\ a_1 = -1,\ a_0 = 0$$

The polynomial g has degree $e = 2$ and coefficients

$$b_2 = 1,\ b_1 = -(\alpha + \beta),\ b_0 = \alpha\beta$$

So the Sylvester matrix S is

$$S = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -(\alpha+\beta) & 1 & 0 \\ -1 & 0 & \alpha\beta & -(\alpha+\beta) & 1 \\ 0 & -1 & 0 & \alpha\beta & -(\alpha+\beta) \\ 0 & 0 & 0 & 0 & \alpha\beta \end{pmatrix}$$

By definition $\mathrm{Res}(f, g) = \det(S)$, and by Theorem 1.3 this equals 0 if and only if f anf g have a common root. Let's compute $\det(S)$ and check this is true.

Expanding along the bottom row we get

$$\det(S) = \alpha\beta \det(T)$$

where T is the 4×4 matrix

$$T = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -(\alpha+\beta) & 1 \\ -1 & 0 & \alpha\beta & -(\alpha+\beta) \\ 0 & -1 & 0 & \alpha\beta \end{pmatrix}$$

Next, expanding T along the top row we get

$$\det(T) = \det(U) + \det(V)$$

where U and V are the 3×3 matrices

$$U = \begin{pmatrix} 1 & -(\alpha+\beta) & 1 \\ 0 & \alpha\beta & -(\alpha+\beta) \\ -1 & 0 & \alpha\beta \end{pmatrix} \qquad V = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -(\alpha+\beta) \\ 0 & -1 & \alpha\beta \end{pmatrix}$$

We compute

$$\begin{aligned} \det(U) &= \alpha^2\beta^2 - \alpha^2 - \beta^2 - \alpha\beta \\ \det(V) &= \alpha\beta + 1 \end{aligned}$$

so this gives

$$\begin{aligned} \det(T) &= \det(U) + \det(V) \\ &= \alpha^2\beta^2 - \alpha^2 - \beta^2 + 1 \\ &= (\alpha^2 - 1)(\beta^2 - 1) \end{aligned}$$

and finally

$$\begin{aligned} \det(S) &= \alpha\beta \det(T) \\ &= \alpha\beta(\alpha^2 - 1)(\beta^2 - 1). \end{aligned}$$

So we see that $\det(S) = 0$ if and only if one of α or β equals one of -1, 0, -1, as claimed.

2 Discriminant of a polynomial

Last week we defined a multiple root of a polynomial f to be a common root of f and its derivative f'. Therefore we can detect the existence of such multiple roots by computing the resultant of f and f'. This leads to the following definition:

Definition 2.1. *Let $f \in \mathbf{C}[x]$ be a polynomial of degree d with leading coefficient a_d. The* **discriminant** *of f is defined as*

$$\mathrm{Disc}(f) = \left(\frac{(-1)^{d(d-1)/2}}{a_d}\right) \mathrm{Res}(f, f').$$

We will see in our analysis of examples why this apparently complicated sign is included in the definition.

Proposition 2.2. *For a polynomial $f \in \mathbf{C}[x]$, we have $\mathrm{Disc}(f) = 0$ if and only if f has a multiple root.*

Proof. From Definition 2.1 we see that $\mathrm{Disc}(f) = 0$ if and only if $\mathrm{Res}(f, f') = 0$. From Theorem 1.3 this happens if and only if f and f' have a common root, which means exactly that f has a multiple root. □

We saw in Proposition 1.4 that the resultant of two polynomials has an expression in terms of their roots. Unsurprisingly, something similar is true for the discriminant:

Proposition 2.3. *Let $f \in \mathbf{C}[x]$ be a polynomial of degree d with roots $\lambda_1, \dots, \lambda_d$ (counted with multiplicity). Let a_d denote the leading coefficient of f. Then*

$$\mathrm{Disc}(f) = a_d^{2d-2} \prod_{1 \le i < j \le d} (\lambda_i - \lambda_j)^2$$

Again we will not prove this formula. It has the same advantages and drawbacks as the root formula for the resultant (Proposition 1.4).

2.1 Real quadratics

In the next two sections we will see that for polynomials with **real** coefficients, the discriminant gives us information not just about repeated roots, but also about how many real and complex roots it has.

To start let's consider the case of a quadratic polynomial

$$f = a_2 x^2 + a_1 x + a_0.$$

Of course, from the quadratic formula, we know the condition on the coefficients that tells us if f has a repeated root, but let's check that computation of the discriminant gives the same thing. First we compute the derivative of f:

$$f' = 2a_2 x + a_1$$

and so we calculate the resultant of f and f' as:

$$
\begin{aligned}
\operatorname{Res}(f, f') &= \det \begin{pmatrix} a_2 & 2a_2 & 0 \\ a_1 & a_1 & 2a_2 \\ a_0 & 0 & a_1 \end{pmatrix} \\
&= a_2 a_1^2 - 2a_2(a_1^2 - 2a_2 a_0) \\
&= a_2 \left(-a_1^2 + 4a_2 a_0\right).
\end{aligned}
$$

Finally the discriminant of f is defined as

$$
\begin{aligned}
\operatorname{Disc}(f) &= \left(\frac{(-1)^{2(2-1)/2}}{a_2}\right) \operatorname{Res}(f, f') \\
&= \left(\frac{-1}{a_2}\right) \cdot a_2 \left(-a_1^2 + 4a_2 a_0\right) \\
&= a_1^2 - 4a_2 a_0.
\end{aligned}
$$

And indeed this is exactly the familiar "$b^2 - 4ac$" that you know from the quadratic formula, in different notation.

At this point we have seen that $f = a_2x^2 + a_1x + a_0$ has a repeated root if any only if $a_1^2 - 4a_2a_0 = 0$. This is true for all quadratic polynomials in $\mathbf{C}[x]$. But in the special case of **real** quadratics, the discriminant also gives us more refined information about the nature of the roots:

Proposition 2.4. *Let* $f \in R[x]$ *be a real quadratic polynomial:*

$$f = a_2x^2 + a_1x + a_0$$

and let $\operatorname{Disc}(f) = a_1^2 - 4a_2a_0$. *Then*

- $\operatorname{Disc}(f) = 0$ *if and only if* f *has a double real root.*
- $\operatorname{Disc}(f) > 0$ *if and only if* f *has 2 distinct real roots.*
- $\operatorname{Disc}(f) < 0$ *if and only if* f *has 2 distinct non-real roots (which must be complex conjugate).*

Again, one possible proof comes from the quadratic formula. However we will give a proof that generalises to higher-degree polynomials, where formulae for the roots are more complicated or non-existent. For this we use the form of the discriminant in Proposition 2.3.

Proof. Suppose the roots of f are λ_1 and λ_2. According to Proposition 2.3 we have

$$\operatorname{Disc}(f) = a_2^2(\lambda_1 - \lambda_2)^2$$

Notice that the leading factor a_2^2 does not affect the sign of f.

There are 3 cases to consider:

Case 1. $\lambda_1 = \lambda_2$. By the formula above this is equivalent to $\mathrm{Disc}(f) = 0$.

Case 2. $\lambda_1 \neq \lambda_2$ and both λ_1 and λ_2 are real. In this case, the formula shows that $\mathrm{Disc}(f)$ is the square of a real number, hence $\mathrm{Disc}(f) > 0$.

Case 3. $\lambda_1 \neq \lambda_2$ and one root is not real. In this case we know that both are non-real, and they must be complex conjugates: $\lambda_2 = \overline{\lambda_1}$. Then $\lambda_1 - \lambda_2$ is purely imaginary: that is,

$$\begin{aligned}\lambda_1 - \lambda_2 &= \lambda_1 - \overline{\lambda_1} \\ &= 2ir\end{aligned}$$

where r is the imaginary part of λ_1. So in this case $\mathrm{Disc}(f) = a_2^2(2ir)^2 = -4a_2^2r^2 < 0$.

□

2.2 Real cubics

Now let's consider the case of a real cubic polynomial:

$$f = a_3x^3 + a_2x^2 + a_1x + a_0 \quad (a_i \in \mathbf{R})$$

Similar to the case of quadratics, we can characterise the roots of f in terms of the sign of the discriminant:

Proposition 2.5. *Let $f \in \mathbf{R}[x]$ be a real cubic as above. Then*

- $\mathrm{Disc}(f) = 0$ *if and only if* f *has a repeated root.*
- $\mathrm{Disc}(f) > 0$ *if and only if* f *has 3 distinct real roots.*
- $\mathrm{Disc}(f) < 0$ *if and only if* f *has 1 real root and 2 non-real complex conjugate roots.*

Proof. Suppose the roots of f are $\lambda_1, \lambda_2, \lambda_3$. Then Proposition 2.3 tells us that

$$\mathrm{Disc}(f) = a_3^4(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2$$

Again the leading coefficient a_3^4 does not change the sign of $\mathrm{Disc}(f)$.

Case 1. $\lambda_i = \lambda_j$ for some $i \neq j$. Again it is clear that $\mathrm{Disc}(f) = 0$ in this case.

Case 2. The λ_i are all distinct, and all real. Then $\lambda_i - \lambda_j$ is nonzero and real for all i and j, so $\mathrm{Disc}(f)$ is the product of squares of nonzero real numbers, hence $\mathrm{Disc}(f) > 0$.

Case 3. The λ_i are all distinct, and one is non-real. Again, non-real roots of real polynomials occur in conjugate pairs, so this means two of the λ_i are non-real and complex conjugate, and the other is real. We may assume that λ_1 is real, λ_2 and λ_3 are not, and $\lambda_3 = \overline{\lambda_2}$. Then

$$\begin{aligned}
\mathrm{Disc}(f) &= a_3^4(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2 \\
&= a_3^4(\lambda_1-\lambda_2)^2(\lambda_1-\overline{\lambda_2})^2(\lambda_2-\overline{\lambda_2})^2 \\
&= a_3^4\left((\lambda_1-\lambda_2)\overline{(\lambda_1-\lambda_2)}\right)^2(2ir)^2 \\
&= a_3^4\,|\lambda_1-\lambda_2|^4\,(2ir)^2 \\
&= -4a_3^4\,|\lambda_1-\lambda_2|^4\,r^2 < 0
\end{aligned}$$

where again r is the imaginary part of λ_2.

□

The key point here is that computing the roots of a given cubic can be complicate, but we can calculate $\mathrm{Disc}(f)$ in terms of the coefficients of f without finding the roots. So this proposition tells us something about the nature of the roots without having to find them explicitly.

2.3 Example

Unfortunately, in the case of cubics the general formula for $\mathrm{Disc}(f)$ in terms of the coefficients a_i is quite complicated. To see how Proposition 2.5 works in practice, let's focus on a simpler special case.

Let $f \in R[x]$ be a real cubic in the form

$$f = x^3 + a_1x + a_0$$

Let's compute its discriminant using Defintion 2.1 and see how the nature of the roots depends on a_1 and a_0.

Note for cubics in this form we have $a_3 = 1$ and $a_2 = 0$. Also, here the degree is $d = 3$ so the sign in the definition of discriminant is

$$\begin{aligned}
(-1)^{d(d-1)/2} &= (-1)^{3\cdot 2/2} \\
&= -1.
\end{aligned}$$

Since $a_3 = 1$, Definition 2.1 then gives us

$$\mathrm{Disc}(f) = -\mathrm{Res}(f, f').$$

Let's compute this using the Sylvester matrix: we have

$$\begin{aligned}
f &= x^3 + a_1x + a_0 \\
f' &= 3x^2 + a_1
\end{aligned}$$

hence

$$\operatorname{Res}(f, f') = \det \begin{pmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ a_1 & 0 & a_1 & 0 & 3 \\ a_0 & a_1 & 0 & a_1 & 0 \\ 0 & a_0 & 0 & 0 & a_1 \end{pmatrix}$$

Computing this determinant we find

$$\operatorname{Res}(f, f') = 4a_1^3 + 27a_0^2$$

and hence

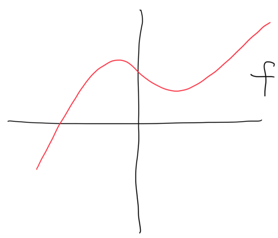
$$\begin{aligned} \operatorname{Disc}(f) &= -\operatorname{Res}(f, f') \\ &= -4a_1^3 - 27a_0^2. \end{aligned}$$

Let's look at some specific examples to see how this results agrees with Proposition 2.5.

(a) Suppose we have a cubic of the form

$$f = x^3 + a \quad (a > 0)$$

The graph of f looks like this:



Using our formula above we find

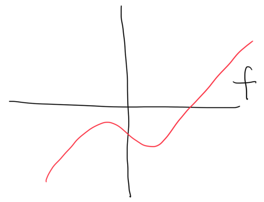
$$\operatorname{Disc}(f) = -27a^2 < 0$$

so f has 1 real root, as the picture suggests.

(b) Similarly for a cubic

$$f = x^3 + a \quad (a < 0)$$

we have the following graph



Again we find

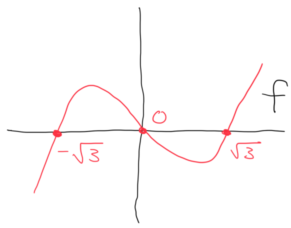
$$\mathrm{Disc}(f) = -27a^2 < 0$$

so again f has 1 real root.

(c) For the cubic

$$f = x^3 - 3x$$

we have $a_1 = -3$, $a_0 = 0$. The graph looks like



This time we find

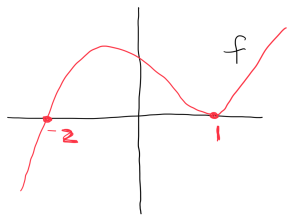
$$\begin{aligned}\mathrm{Disc}(f) &= -4a_1^3 - 27a_0^2 < 0 \\ &= -4(-3)^3 > 0\end{aligned}$$

confirming that f has 3 distinct real roots.

(d) Finally, for the cubic

$$f = x^3 - 3x + 2$$

we have $a_1 = -3$, $a_0 = 2$. The graph looks like



The discriminant is

$$\begin{aligned}
\mathrm{Disc}(f) &= -4a_1^3 - 27a_0^2 < 0 \\
&= -4(-3)^3 + 27(2)^2 = 0
\end{aligned}$$

which means that f has a double root. We verify this by computing:

$$f'(x) = 3x^2 - 3$$

so we have

$$\begin{aligned}
f(1) &= 1^3 - 3 \cdot 1 + 2 = 0 \\
f'(1) &= 3 \cdot 1^2 - 3 = 0
\end{aligned}$$

So $x = 1$ is a double root.