Loughborough University

ELEMENTS OF TOPOLOGY
(18MAC298)

Summer 2019 | 2 hours

Answer **THREE** questions.
Any calculator from the University's approved list may be used.

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1. (a) Give the definition of an open subset of a metric space. [3]

   (b) Let A be a subset of a topological space X. Give the definition of an interior point of A. [3]

   (c) Prove that the interior $\mathrm{Int}\, A$ of any subset $A \subset X$ is open. [4]

   (d) Describe the interior of the following subsets $A \subset X$:

   (i) $X = \mathbb{R}^2$ with the standard topology, $A = \{y > 0\} \subset \mathbb{R}^2$ (half plane); [2]

   (ii) $X = \mathbb{R}^2$ with the standard topology, $A = \{y = 0,\ 0 < x < 1\} \subset \mathbb{R}^2$ (interval); [2]

   (iii) $X = \mathbb{R}$ with the standard topology, $A = \mathbb{R} \setminus \mathbb{N}$, where $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ is the set of natural numbers; [2]

   (iv) $X = \mathbb{R}$ with the topology $\tau = \{\mathbb{R}, \emptyset, (a, +\infty), a \in \mathbb{R}\}$, $A = (-\infty, 0) \cup (1, +\infty)$. [2]

   Which of these subsets are open? [2]

2. (a) State the Heine-Borel theorem. [3]

   (b) Let A be a closed subset of a compact topological space X. Prove that A is compact. [5]

   (c) Which of the following subsets $A \subset X$ are compact? Justify your answer:

   (i) $\{\frac{1}{n},\ n \in N\} \subset \mathbb{R}$ (standard topology); [2]

   (ii) $\{x \geq 0, y \geq 0, x + y \leq 1\}$ in $\mathbb{R}^2$ (standard topology); [2]

   (iii) $\{\sin(x - y) = \cos(x + y)\}$ in $\mathbb{R}^2$ (standard topology); [2]

   (iv) $\mathbb{N} = \{1, 2, 3, \dots\} \subset X$, where $X = \mathbb{R}$ with indiscrete topology; [3]

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(v) $\mathbb{N} = \{1, 2, 3, \dots\} \subset X$, where $X = \mathbb{R}$ with the topology $\tau = \{\mathbb{R}, \emptyset, (a, +\infty), a \in \mathbb{R}\}$; [3]

3. (a) Give the definition of a pathwise connected topological space. [4]

(b) Prove that a pathwise connected topological space is connected. [4]

(c) Let X be the set of all real triangular invertible 2×2 matrices viewed as a subset in the four-dimensional vector space of 2×2 matrices:

$$X = \left\{ A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad \det A \neq 0, \ a, b, c \in \mathbb{R} \right\}.$$

Is X connected? Justify your answer. [4]

(d) Let $X = [0, 1]$ (closed interval) and $Y = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$ (circle).
Are these topological spaces homeomorphic? Justify your answer. [4]

(e) Let $X = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ (sphere) and
$Y = \{x^2 + 2y^2 + 3z^2 = 1\} \subset \mathbb{R}^3$ (ellipsoid).
Are these topological spaces homeomorphic? Justify your answer. [4]

4. (a) State the Implicit Function Theorem. [4]

(b) Is the subset $\{x^3 + 3xy^2 + y^6 - z^4 = 1\} \subset \mathbb{R}^3$ a manifold? Justify your answer. [4]

(c) Prove that $GL(2, \mathbb{R})$, the set of 2×2 invertible matrices, is a manifold.
What is its dimension? [4]

(d) What is the Euler characteristic of the projective plane? Justify your answer. [3]

(e) Consider the surface M obtained from the fundamental polygon

$$AB^{-1}CD^{-1}BDC^{-1}A^{-1}$$

by pairwise identification of its edges. Find the Euler characteristic of M and determine its topological type. [5]