MAB298-Elements of Topology: Problem Sheet 2

Closure of a set. Interior, exterior and boundary points.

1. What is the closure of $\mathbb{Q}$ in $\mathbb{R}$?
2. Is it true that $\overline{A \cup B} = \bar{A} \cup \bar{B}$?
3. Find the closure of $\{a\}$ in $\mathbb{R}$ with the topology as in Problem Sheet 1, 3(b).
4. Find the boundary of (a, b) in $\mathbb{R}$ with the topology as in Problem Sheet 1, 3(b).
5. Prove that a set A is closed if and only if $\partial A \subset A$.
6. Find limit and isolated points of the sets $(0, 1] \cup \{2\}$ and $\{1/n \mid n \in \mathbb{N}\}$ in $\mathbb{R}$.
7. Let $X \subset \mathbb{R}$ be the subset obtained from the closed interval $[0, 1]$ by removing all the points of the form $\frac{1}{n}$, $n = 1, 2, 3, \ldots$, i.e.:

   $$X = [0, 1] \setminus \left\{ \frac{1}{n},\ n \in \mathbb{N} \right\}$$

   Is X open? closed? Describe the interior, boundary, closure, and adherent points of X?
8. Consider $A = [0, 1)$ as a subset of $\mathbb{R}$ with
   (a) the standard topology,
   (b) the discrete topology,
   (c) the indiscrete topology.

   Describe the interior, boundary, closure, adherent points of A?
9. Prove that $\partial A = \partial(X \setminus A)$.
10. Do the following equalities hold true for any sets A and B:
    (a) $\operatorname{Int}(A \cup B) = \operatorname{Int} A \cup \operatorname{Int} B$,

(b) $\text{Int}\,(A \cap B) = \text{Int}\,A \cap \text{Int}\,B$?

11. Let X be a metric space and $x \in X$. Consider the open ball of radius r
$$B(x,r) = \{y \in X \mid d(x,y) < r\}$$
and the sphere of the same radius and with the same center:
$$S_r = \{y \in X \mid d(x,y) = r\}$$
Is it true that $S_r = \partial B(x,r)$? Is it true that the closure of $B(x,r)$ equals
$$\{y \in X \mid d(x,y) \leq r\}?$$