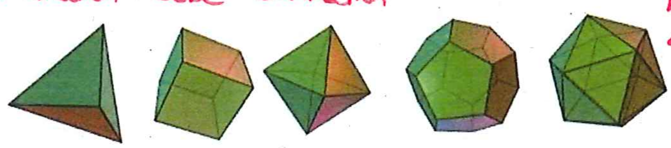
18MAA242 Lecture 18 ①

Symmetries of Platonic Solids

Theaetus (Athens, ~400BC) proved there are exactly 5 regular polyhedra ("Platonic solids"):

Tetrahedron Cube Octahedron Dodecahedron Icosahedron



T C O D I

Let $V = \#$ vertices

$E = \#$ edges

$F = \#$ faces

Then we have

	T	C	O	D	I
V	4	8	6	20	12
E	6	12	12	30	30
F	4	6	8	12	20

"Duality":

vertices $\longleftrightarrow$ faces

edges $\longleftrightarrow$ edges

$T \longleftrightarrow T$

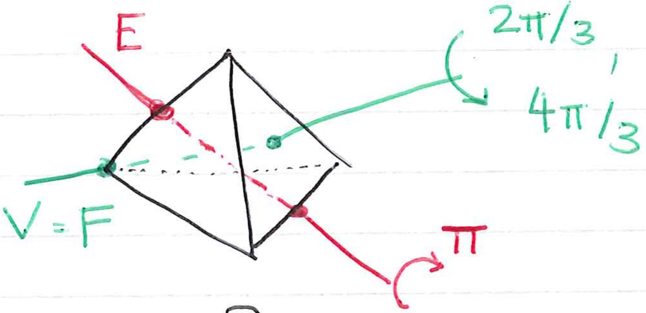
$C \longleftrightarrow O$

$D \longleftrightarrow I$

We want to study rotational symmetries of each of these shapes: rotations mapping the shape to itself.

Every rotation has an axis. We say a rotation of a solid is of type V, E, or F if the axis passes through a vertex, midpoint of an edge, or centre of a face, respectively.

Tetrahedron

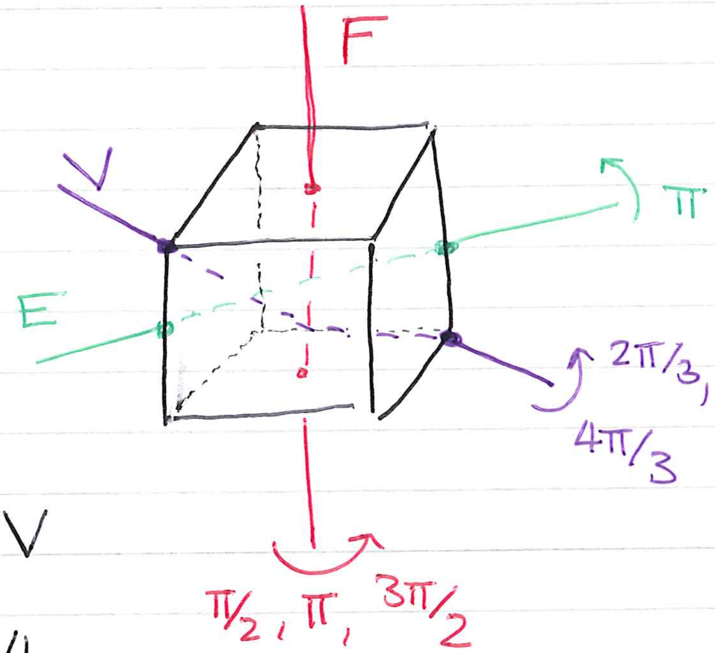


Set of symmetries T:

$\rightarrow$ identity
$\rightarrow$ $2 \times 4 = 8$ rotations of type $V \approx F$
$\rightarrow$ 3 rotations of type E

$\Rightarrow |T| = 1 + 8 + 3 = 12.$

Cube/Octahedron



Set of symmetries O:

$\rightarrow$ identity
$\rightarrow$ 3×3 rotations of type F
$\rightarrow$ 6 rotations of type E
$\rightarrow$ 2×4 rotations of type V

$\Rightarrow |O| = 1 + 9 + 6 + 8 = 24.$

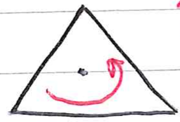
(By duality, the same is true for octahedron)

Icosahedron / Dodecahedron

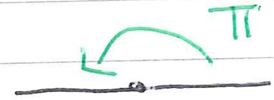
Set of symmetries I:

→ identity

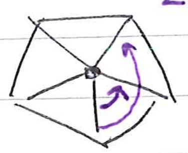
→ 2×10 rotations of type F:

10 pairs of opposite faces  $2\pi/3$, $4\pi/3$

→ 15 rotations of type E:

15 pairs of opposite edges  π

→ 4×6 rotations of type V:

6 pairs of opposite vertices  $2\pi/5$, $4\pi/5$, $6\pi/5$, $8\pi/5$

$\Rightarrow |I| = 1 + 20 + 15 + 24 = 60$

(By duality, same for dodecahedron).

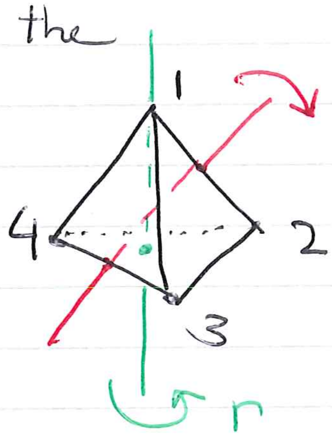
In each case the set of symmetries is a group, with operation of composition.

Theorem: The groups of rotational symmetries of platonic solids are:

$$T \cong A_4, \quad O \cong S_4, \quad I \cong A_5$$

T: tetrahedron; O: cube + octahedron; I: dodecahedron + icosahedron

<u>Proof</u>: (T) Label the vertices of the tetrahedron by 1, 2, 3, 4 and consider the map



$$\varphi : T \longrightarrow S_4$$
$$\text{symmetry} \longmapsto \text{induced permutation of vertices}$$

Rotations of type V=F give 3-cycles:

$r \mapsto (2\,4\,3)$
$r^2 \mapsto (2\,3\,4)$

Rotations of type E give $(12)(34)$, $(13)(24)$, $(14)(23)$.

So all elements in the image of φ are <u>even</u>.

So $\varphi : T \longrightarrow A_4$.

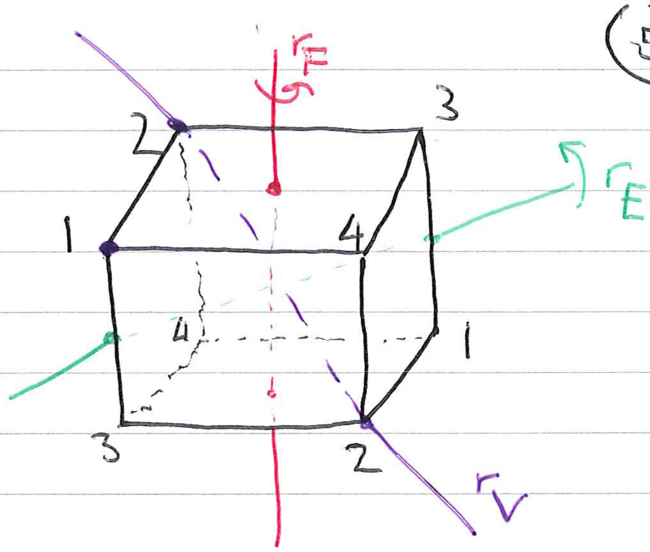
Now φ is injective since no symmetry fixes all vertices. So in fact φ is a bijection, because $|T| = 12 = \frac{4!}{2} = |A_4|$.

Since the operation on both T and A_4 is composition, property (*) is satisfied, so in fact φ is an isomorphism.

(O) Want to prove

$O \cong S_4$.

Label the diagonals of the cube by 1, 2, 3, 4.



Again we have a map

$$\varphi : O \longrightarrow S_4$$

defined by sending a symmetry to the induced permutation of the diagonals.

E.g. $\quad r_E \longmapsto (1\,3) \qquad$ (2-cycles)

$\qquad r_V \longmapsto (1\,3\,4) \qquad$ (3-cycles)

$\qquad r_F \longmapsto (1\,4\,3\,2) \qquad$ (4-cycles)

Again φ is injective. Now

$|O| = 24 = 4! = S_4$ so

$\varphi : O \longrightarrow S_4$ is bijective,

hence an isomorphism.

(I) 5 cubes hidden in a dodecahedron:

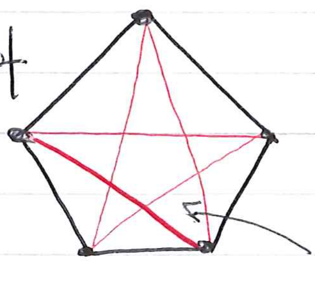
label them 1, 2, 3, 4, 5.

Again, define $\varphi : I \to S_5$ by taking a symmetry to the induced permutation of the cubes. Can check that corresponding permutations are even:

{3-cycles, 5-cycles, product of disjoint transpositions.

So we have $\varphi : I \to A_5$, injective, and again since $|I| = 60 = \frac{5!}{2} = |A_5|$ we get that φ is a bijection, hence an isomorphism. $\square$.

Face of D



edge of cube: 5 possibilities.