MAB298-Elements of Topology: Solution Sheet 1

Topological spaces, open and closed sets

1. Let X consist of four elements: $X = \{a, b, c, d\}$. Which of the following collections of its subsets generate a topology on X:

   (a) $\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}, \{a, b\}$;

   (b) $\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}$;

   (c) $\emptyset, X, \{a, c, d\}, \{b, c, d\}$?

   (a) yes, (b) yes, (c) no, because the intersection $\{a, c, d\} \cap \{b, c, d\} = \{c, d\}$ does not belong to τ.

2. Let τ be a topology in $\mathbb{R}$ such that the intervals $[a, b]$ are open for all $a < b$. Prove that this topology is discrete.

   Proof: If all the intervals of the form $[a, b]$ are open, then $[a, b] \cap [b, c] = \{b\}$ is also open. If each point b (as a one-point set) is open, then every subset $A \subset \mathbb{R}$ is open as the union of its points $(A = \cup_{b \in A}\{b\})$. Thus, the topology is discrete.

3. Consider the collection of subsets of $\mathbb{R}$ that consists of:

   (a) $\mathbb{R}$, $\emptyset$ and all infinite "closed" intervals $[a, +\infty)$, $a \in \mathbb{R}$;

   (b) $\mathbb{R}$, $\emptyset$ and all infinite "open" intervals $(a, +\infty)$, $a \in \mathbb{R}$.

   Is this topology or not?

   (a) No. Consider the union of the intervals $[\frac{1}{n}, +\infty)$. Obviously, $\cup_{n \in \mathbb{N}}[\frac{1}{n}, +\infty) = (0, +\infty)$. The set $(0, +\infty)$ does not belong to the indicated collection of sets, so the second axiom of a topology fails.

   (b) Yes. Indeed, if we take any number of intervals $(a_\alpha, +\infty)$, $\alpha \in I$ (where I is the set of indices which can be either finite or infinite), then the union

   $$\cup_{\alpha \in I}(a_\alpha, +\infty)$$

   is the interval of the form $(b, +\infty)$, where $b = \inf\{a_\alpha,\ \alpha \in I\}$, i.e is open.

If we take two intervals (a_1, ∞), (a_2, ∞) (for definiteness, let $a_1 < a_2$) then their intersection $(a_1, +\infty) \cap (a_1, +\infty)$ is $(a_2, +\infty)$ and so is open.

4. Let X be a plane. Let τ consist of $\emptyset$, X, and all open disks with center at the origin. Do X and τ define a topological space?

   Yes, all axioms of a topology are satisfied (check them!).

5. Let X be $\mathbb{R}$, and let τ consist of the empty set and all infinite subsets of $\mathbb{R}$. Do X and τ define a topological space?

   No, because the intersection of two infinite subsets can be finite. Example: $(-\infty, 0] \cap [0, +\infty) = \{0\} \notin \tau$ so that the third axiom fails.

6. List all topologies in a two-element set, say, in $\{0, 1\}$.

   There are four distinct structures:

1)	$\emptyset, \{0, 1\}$	indiscrete topology,
2)	$\emptyset, \{0, 1\}, \{0\}, \{1\}$	discrete topology,
3)	$\emptyset, \{0, 1\}, \{0\}$,	
4)	$\emptyset, \{0, 1\}, \{1\}$	

7. Let (X, τ) be a discrete topological space. Define a metric d on X such that the corresponding (metric) topology coincides with τ.

   Since each point x in a discrete space is an open set, the metric d must satisfy the following condition:

   there is a ball $B_\delta(x) = \{y \in X \mid d(x, y) < \delta\}$ of a suitable radius δ which contains the only point, its center x, i.e. $B_\delta(x) = \{x\}$.

   In other words, the distance between x and any other point is greater than $\delta > 0$, i.e., any two points are "sufficiently far" from each other. A metric with such a property can be defined, for instance, as

   $$d(x, y) = 1 \quad \text{for } x \neq y, \qquad \text{and} \qquad d(x, y) = 0 \quad \text{for } x = y.$$

   In this metric, $B_{\frac{1}{2}}(x) = \{x\}$ so that every point is open and, therefore, the topology is indeed discrete.

8. Let (X, τ) be an indiscrete topological space which contains at least two elements. Prove that there is no metric d on X such that the corresponding (metric) topology coincides with τ.

   Proof by contradiction: Let d be any metric on X and $x \neq y$ be two arbitrary distinct points in X. Let $d(x, y) = r$, then the balls $B_{r/3}(x)$ and $B_{r/3}(y)$ are non-empty open sets which do not intersect. But no such open subsets may exist in X because the only open sets are X and $\emptyset$.

9. Find examples of sets that are

   (a) both open and closed simultaneously (open-closed);

   (b) neither open, nor closed.

   (a) X and $\emptyset$ for any topological space X; any subset in a discrete space.

   (b) any non-trivial subset in an indiscrete space; $[0, 1)$ in $\mathbb{R}$ with the standard topology.

10. Give an explicit description of closed sets in

    (a) a discrete space;

    (b) an indiscrete space;

    (c) $\mathbb{R}$ with topology as in 3(a) (open sets are $\mathbb{R}$, $\emptyset$ and the infinite intervals $(a, +\infty)$).

    (a) each subset is closed,

    (b) the only closed subsets are X and $\emptyset$

    (c) $\mathbb{R}$, $\emptyset$ and infinite intervals $(-\infty, a]$, $a \in \mathbb{R}$.

11. Is a "closed" interval $[a, b]$ closed in $\mathbb{R}$ with the usual topology?

    Yes, because its complement $(-\infty, a) \cup (b, +\infty)$ is open.

    Is a "closed" interval $[a, b]$ closed in $\mathbb{R}$ with the topology defined in Ex. 3(b)?

    No, because its complement $(-\infty, a) \cup (b, +\infty)$ is not open in the sense of this topology.

12. Prove that the half-open interval $[0,1)$ is neither open nor closed in $\mathbb{R}$, but is both a union of closed sets and an intersection of open sets.

    $[0,1)$ is not open because for the point $0 \in [0,1)$ there is no neighborhood $(-\epsilon,\epsilon)$ which would be entirely contained in $[0,1)$.

    $[0,1)$ is not closed, because $1 \in \mathbb{R}$ is a limit point of $[0,1)$, but $1 \notin [0,1)$ (recall that a closed set must contain all of its limit points).

    $[0,1)$ can be presented as the intersection of open sets: $[0,1) = \bigcap_{n\in\mathbb{N}}(-\frac{1}{n},1)$.

    $[0,1)$ can be presented as the union of closed sets: $[0,1) = \bigcup_{n\in\mathbb{N}}[0,1-\frac{1}{n}]$.