23MAC260 Elliptic Curves: Week 8

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Now we start on the third part of the module — the **analytic** theory of elliptic curves over the complex numbers $\mathbb{C}$. We will see that a complex elliptic curve can be parametrised by the points of a parallelogram in the complex plane, using a special function called the *Weierstrass $\wp$-function.*

1 Functions on the complex plane

We start by defining two special classes of complex functions. You may have seen these definitions in the Complex Analysis module, albeit in a slightly different form.

Definition 1.1 (Meromorphic, holomorphic, order)**.** *Let z_0 be a point in the complex plane.*

(i) A function f is **meromorphic at** z_0 *if it has a Laurent series expansion*

$$f(z) = a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \cdots \qquad (a_i \in \mathbb{C})$$

valid in some punctured disk $0 < |z - z_0| < \epsilon$ around the point z_0. In the above series $k \in \mathbb{Z}$ is an integer (possibly negative) and $a_k \neq 0$.

(ii) A function f is **holomorphic at** z_0 *if it is meromorphic at z_0 and $k \geq 0$ in the above series expansion.*

(iii) The integer k is called the **order** *of f at z_0, denoted $\operatorname{ord}_{z_0}(f)$. If $k < 0$ we say f has a* **pole** *of order $-k$ at z_0; if $k > 0$ we say f has a* **zero** *of order k at z_0.*

Note that if f has a pole at z_0 then f is not defined at z_0!

Examples:

1. The exponential function $\exp : \mathbb{C} \to \mathbb{C}$ is holomorphic at every point of $\mathbb{C}$. Its power series expansion at $z_0 = 0$ is:

$$\begin{aligned} \exp(z) &= 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 \cdots \\ &= \sum \frac{1}{n!}z^n \end{aligned}$$

so in this case $a_n = \frac{1}{n!}$.

2. The function $f(z) = \frac{1}{z^2}\cos(z)$ is meromorphic at $z_0 = 0$: its expansion looks like

$$\begin{aligned} f(z) &= \frac{1}{z^2}\left(1 - \frac{1}{2}z^2 + \frac{1}{4!}z^4 \cdots\right) \\ &= z^{-2} - \frac{1}{2} + \frac{1}{4!}z^2 \cdots \end{aligned}$$

So f has a pole of order 2 at 0.

1.1 Theorems about complex functions

Holomorphic functions have remarkably different properties from their real counterparts (namely, infinitely differentiable real functions). The next theorem is one of the best examples of this:

Theorem 1.2 (Liouville). *Let $f : \mathbb{C} \to \mathbb{C}$ be a function which is holomorphic at every point $z_0 \in \mathbb{C}$.*

If f is bounded, meaning there is a real number B such that

$$|f(z)| \leq B \quad \textit{for all } z \in \mathbb{C},$$

then f must be a constant function.

Integration of holomorphic and meromorphic functions also has remarkable properties. We record the following definition and result for use next time:

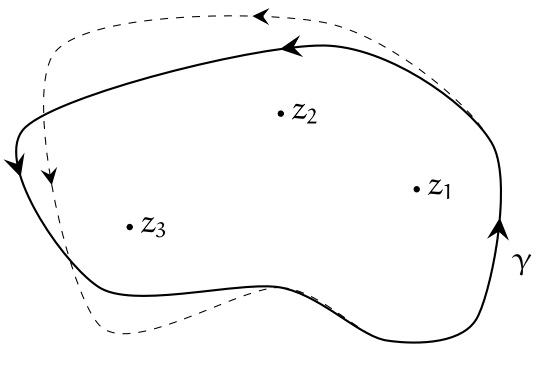
Definition 1.3. *Let f be a function which is meromorphic at $z_0 \in \mathbb{C}$. Write*

$$f(z) = a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \cdots \qquad (a_i \in \mathbb{C},\ k \in \mathbb{Z})$$

The **residue** *of f at z_0 is the coefficient a_{-1} in the above series.*

Theorem 1.4 (Residue Theorem). *Let f be a meromorphic function. Let γ be a simple closed curve in $\mathbb{C}$ that does not pass through any pole of f, and let $z_1, \ldots, z_n$ be the poles of f inside γ. Then*

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k).$$



This picture illustrates the meaning of the Residue Theorem. If f is a meromorphic function with poles at z_1, z_2, z_3, then the value of the integral $\int_\gamma f(z)dz$ does not change if we deform the simple closed curve γ.

2 Lattices and doubly-periodic functions

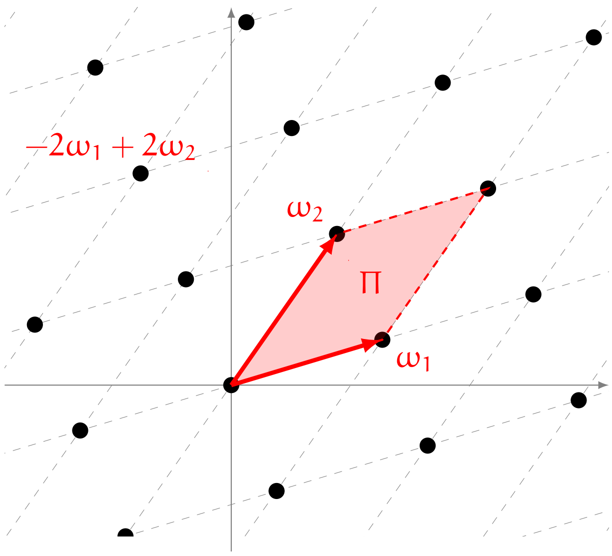
Now we move on to specific kinds of "periodic" complex functions. To say what this means precisely, we need the following definition:

Definition 2.1. *Let ω_1, ω_2 be two nonzero complex numbers such that $\omega_1/\omega_2 \notin \mathbb{R}$. The* **lattice spanned by** ω_1 **and** ω_2 *is*

$$L = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}.$$

The **fundamental parallelogram** Π *for the lattice* L *is the subset*

$$\Pi = \{x\omega_1 + y\omega_2 \mid x, y \in \mathbb{R}, 0 \leq x, y < 1\}.$$



This picture shows a lattice in the complex plane. The lattice is spanned by ω_1 and ω_2. Black dots show the points of the lattice; the shaded region (not including the dashed edges) is the fundamental parallelogram Π.

Remark: For any point $z \in \mathbb{C}$, there exist unique complex numbers $\omega \in L$ and $z_0 \in \Pi$ such that $z = z_0 + \omega$. In other words, there is a bijection between points of Π and elements of the *quotient group* $\mathbb{C}/L$.

We are familiar with the idea of periodic functions of a real variable, the most famous examples being the trigonometric functions $\sin$ and $\cos$. In this context, to say that f is periodic means that adding an integer multiple of some fixed real number p to the argument does not change the value of f:

$$f(x + np) = f(x) \quad \text{for all } n \in \mathbb{Z}.$$

We are going to consider the corresponding notion for functions of a complex variable.

Definition 2.2. *Let L be a lattice in the complex plane. A meromorphic function f is* **doubly periodic** *with respect to L if*

$$f(z + \omega) = f(z) \quad \textit{for all } z \textit{ and for all } \omega \in L$$

Again, in contrast to the case of periodic real functions, this condition is highly restrictive for holomorphic functions:

Theorem 2.3. *Let f be a function which is holomorphic at every point $z_0 \in \mathbb{C}$ and doubly-periodic with respect to a lattice L. Then f is constant.*

The proof is an easy consequence of Liouville's Theorem:

Proof. A holomorphic function is continuous, so it is bounded on the compact set

$$\overline{\Pi} = \{x\omega_1 + y\omega_2 \mid x, y \in \mathbb{R}, 0 \leq x, y \leq 1\}$$

But any $z \in \mathbb{C}$ can be written as $z = z_0 + \omega$ with $z_0 \in \Pi$ and $\omega \in L$. Hence $f(z) = f(z_0)$, and so f is bounded on the whole complex plane $\mathbb{C}$. Therefore by Liouville's Theorem, f is constant. $\square$

We'll use this next week to prove that the "most important" doubly-periodic function, called the *Weierstrass $\wp$-function*, satisfies a certain differential equation, and this will enable us to use $\wp$ to parametrise elliptic curves.

3 Weierstrass $\wp$-function

The most important example of a doubly-periodic meromorphic function is the following:

Definition 3.1 (Weierstrass $\wp$-function)**.** *Let* L *be a lattice in the complex plane. The* **Weierstrass $\wp$-function associated to** L *is defined as*

$$\wp_L(z) = \frac{1}{z^2} + \sum_{\omega \in L,\ \omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right). \tag{*}$$

This function is defined by an infinite series, so it is not obvious that it converges, but in fact it does for all $z \in \mathbb{C} \setminus L$. Let's sketch the proof of this fact, together with some other important properties of the Weierstrass $\wp$-function:

Theorem 3.2. *Let* L *be a lattice in the complex plane and let $\wp_L(z)$ be the Weierstrass $\wp$-function associated to* L*. Then*

1. *$\wp_L(z)$ is meromorphic, with a pole of order 2 at each point $\omega \in L$ and no other poles;*
2. *$\wp_L(z)$ is an even function of z, meaning $\wp_L(-z) = \wp_L(z)$;*
3. *$\wp_L(z)$ is doubly-periodic with respect to* L*:*

$$\wp_L(z+\omega) = \wp_L(z) \quad \textit{for all } \omega \in L.$$

Proof.

1. We sketch the proof of this part and refer to Silverman pp.165–166 for more details. Fix $z \neq 0$ and $\omega \neq 0$ such that $|z| < \frac{1}{2}|\omega|$. Then $|z-w| \geq \frac{1}{2}|\omega|$ and $|2\omega - z| \leq 3|\omega|$. So

$$\begin{aligned} \left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| &= \left| \frac{z(2\omega - z)}{\omega^2 (z-w)^2} \right| \\ &\leq \frac{3|z||\omega|}{|\omega|^2 \left(\frac{1}{2}|\omega|\right)^2} \\ &\leq \frac{12|z|}{|\omega|^3} \end{aligned}$$

   This implies that, up to a finite number of terms, the series defining $\wp_L(z)$ is bounded in absolute value by

$$12|z| \sum_{\omega \in L, \omega \neq 0} \frac{1}{|\omega|^3}$$

   This shows that the series is absolutely convergent for $z \in \mathbb{C} \setminus L$ and converges uniformly on compact subsets of $\mathbb{C} \setminus L$, hence defines a holomorphic function on $\mathbb{C} \setminus L$ by a theorem of complex analysis.

   Finally the series expansion shows that $\wp_L(z)$ has a double pole at every point $\omega \in L$.

2. To see that $\wp_L(z)$ is even, notice that for any $\omega \in L$ we also have $-\omega \in L$. Now

$$\begin{aligned}\frac{1}{(-z-\omega)^2} &= \frac{1}{(z+\omega)^2} \\ &= \frac{1}{(z-(-\omega))^2}\end{aligned}$$

and

$$\frac{1}{(-\omega)^2} = \frac{1}{\omega^2}$$

so that when we take the sum in Definition (*) over all $\omega \in L$ we obtain exactly the same terms in both the sum for $\wp_L(z)$ and $\wp_L(-z)$.

3. To prove $\wp_L(z)$ is doubly-periodic with respect to L, differentiate (*) to get:

$$\wp_L'(z) = -2\sum_{\omega \in L} \frac{1}{(z-\omega)^3} \quad \text{for all } z \in \mathbb{C} \setminus L.$$

But then for a fixed element $\omega_0 \in L$ we get

$$\wp_L'(z+\omega_0) = -2\sum_{\omega \in L} \frac{1}{(z-(\omega-\omega_0))^3} = -2\sum_{\omega \in L} \frac{1}{(z-\omega)^3} = \wp_L'(z) \qquad (**)$$

where the middle equality uses the fact that $\omega \mapsto \omega - \omega_0$ is a bijection $L \to L$.

Integrating $(**)$ we get that

$$\wp_L(z+\omega_0) = \wp_L(z) + c \quad \text{for a constant } c.$$

Now choose ω_0 so that $\frac{1}{2}\omega_0 \notin L$ and plug in $z = -\frac{1}{2}\omega_0$: we get

$$\wp_L\left(\frac{1}{2}\omega_0\right) = \wp_L\left(-\frac{1}{2}\omega_0\right) + c \qquad \text{so} \qquad c = \wp_L\left(\frac{1}{2}\omega_0\right) - \wp_L\left(-\frac{1}{2}\omega_0\right) = 0$$

since from Part (2) we know $\wp_L(z)$ is even.