Mab298 lecture week 12

Solutions of 2020 exam

(1) (a) $\mathbb{R}$, $\emptyset$, all subsets of the form $\mathbb{R} \setminus S$, where S is finite.

Is this a topological space?

Answer: Yes

(i) $\emptyset$, $\mathbb{R}$ belong to τ

(ii) Let $A_\alpha = \mathbb{R} \setminus S_\alpha$, $\alpha \in I$, S_α a finite set for all $\alpha \in I$.

Then $\bigcup_{\alpha \in I} A_\alpha = \mathbb{R} \setminus \underbrace{\left(\bigcap_{\alpha \in I} S_\alpha \right)}_{\text{is finite or empty}}$

$\Rightarrow \bigcup_{\alpha \in I} A_\alpha \in \tau$

(iii) Let $A_1 = \mathbb{R} \setminus S_1$ and $A_2 = \mathbb{R} \setminus S_2$ where S_1 and S_2 are finite sets.

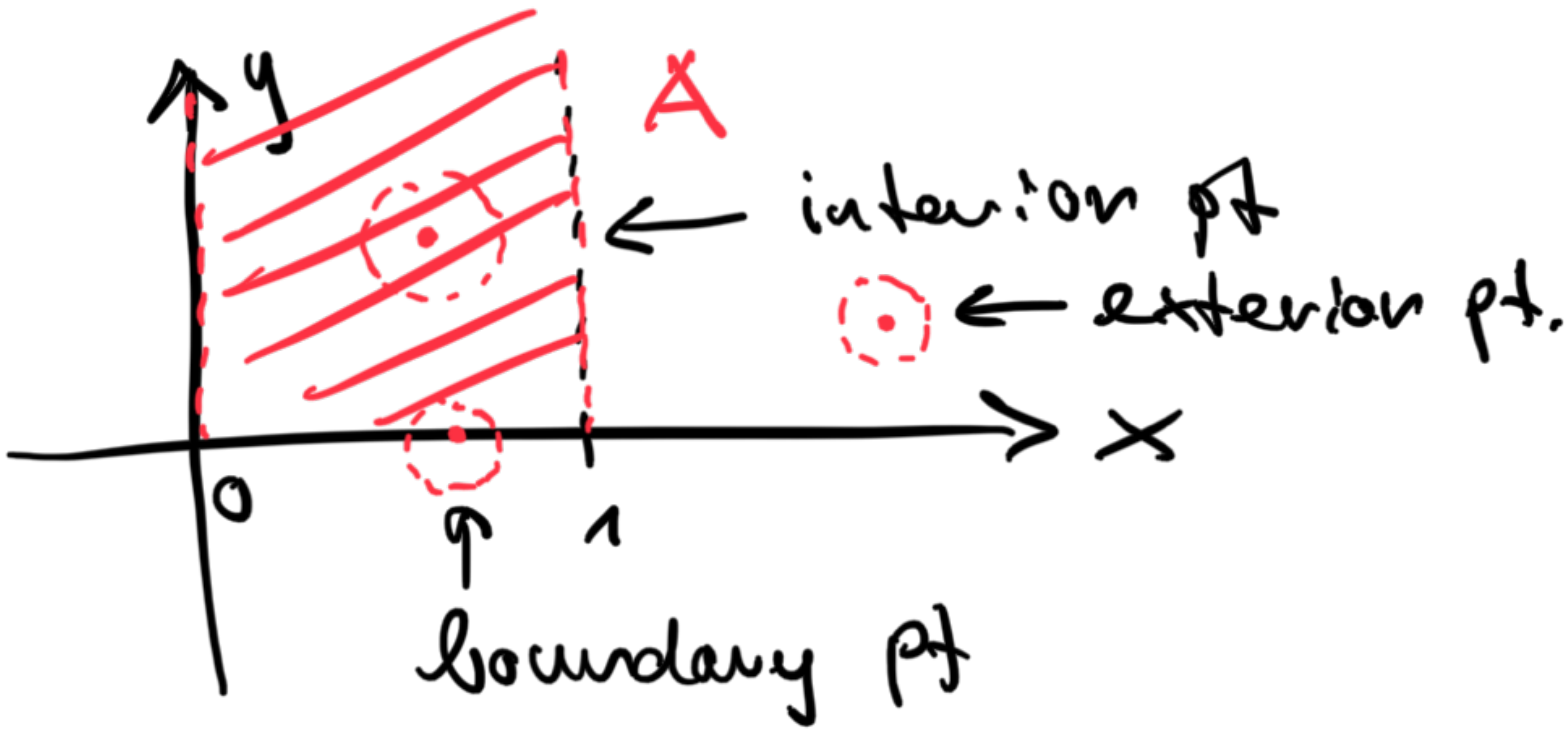
Then $A_1 \cap A_2 = \mathbb{R} \setminus \{ \underbrace{S_1 \cup S_2}_{\text{finite}} \}$.

$\Rightarrow A_1 \cap A_2 \in \tau$

Thus, the axioms (i)–(iii) are satisfied $\Rightarrow$ $(\mathbb{R}, \tau)$ is a top. space.

(b) (i) $A = \{y \geq 0, 0 < x < 1\} \subset \mathbb{R}^2$

$\operatorname{Int} A = \{y > 0, 0 < x < 1\}$

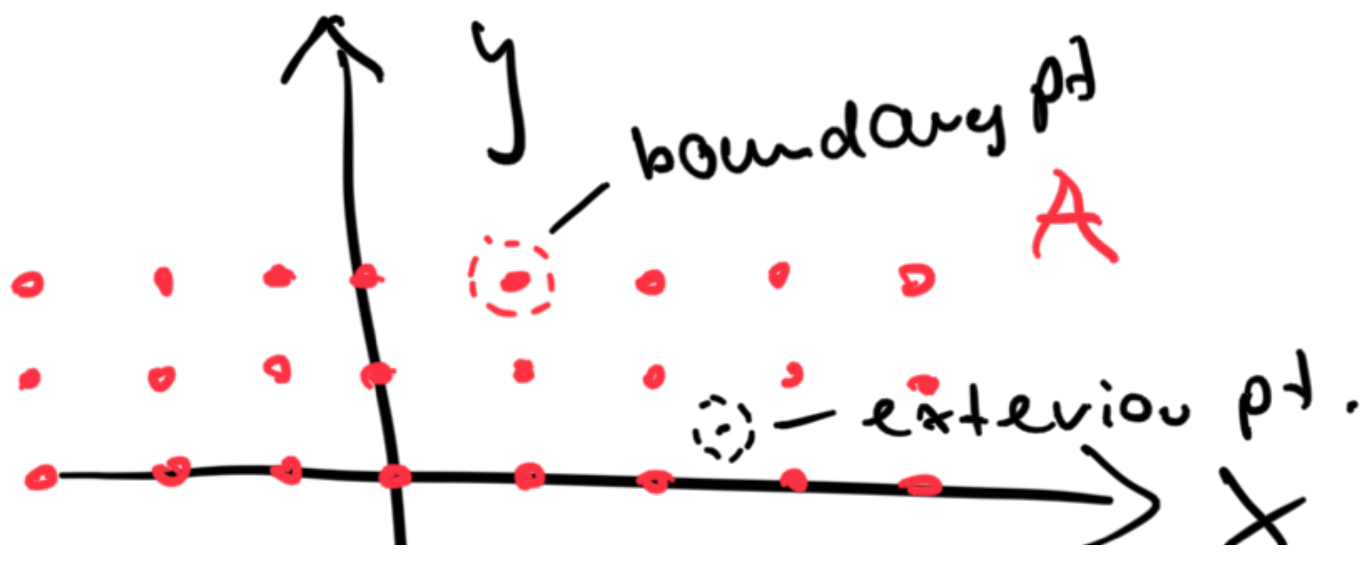


(ii) $A = \{x = 0, y \in (0, +\infty)\}$

$\text{Int}\, A = \emptyset$

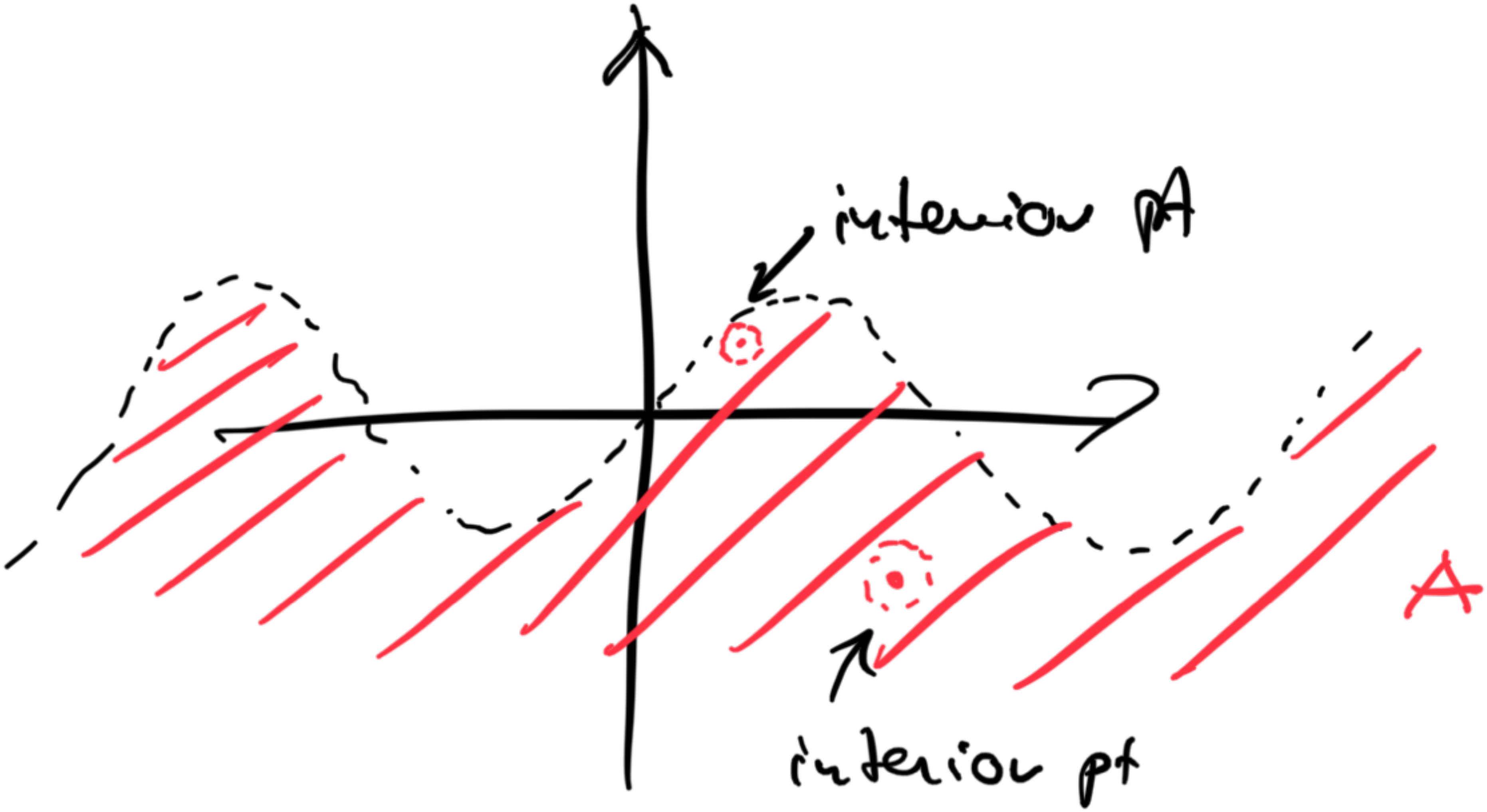
iii) $A = \{(x, y) : x \in \mathbb{Z},\ y \in \mathbb{Z}\}$

$\text{Int}\, A = \emptyset$



(iv) $A = \{ y < \sin x \}$

$\operatorname{Int} A = A$



Pf: $f : \mathbb{R} \to \mathbb{R}$ $f(x) = \sin x$ is continuous

Pf: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$ is continuous

$F: \mathbb{R}^2 \to \mathbb{R}$, $F(x,y) = y - \sin x$ is continuous

$$A = \underset{\substack{\uparrow \\ \text{cont.}}}{F^{-1}}\Big(\underset{\substack{\uparrow \\ \text{open}}}{(-\infty, 0)}\Big) \quad \text{open}$$

A from (iv) is open as $\text{Int}\, A = A$, the others are not.

(2) (a) An infinite discrete top. space is not compact.

<u>Pf</u>: Let X be an infinite discrete

top. space. Consider the open cover

$$\mathcal{U} = \{ U_\alpha = \{x_\alpha\} : x_\alpha \in X \}, \text{ i.e.}$$

U_α is the one-point set which covers a single point x_α. Clearly,

$$X = \bigcup_{\alpha \in I} \{x_\alpha\},$$

i.e. $\mathcal{U}$ is an open cover of X.

($\{x_\alpha\}$ open because X is discrete).

This cover does not admit a finite subcover as any finite part

$U_{\alpha_1} \cup U_{\alpha_2} \cup \dots \cup U_{\alpha_n}$ would only cover finitely many points, whereas X is infinite. Hence, X is not compact.

$\square$

(b) $A \subset \mathbb{R}$ not closed.

<u>Claim</u>: There is a continuous function $f: A \to \mathbb{R}$ which is not bounded on A.

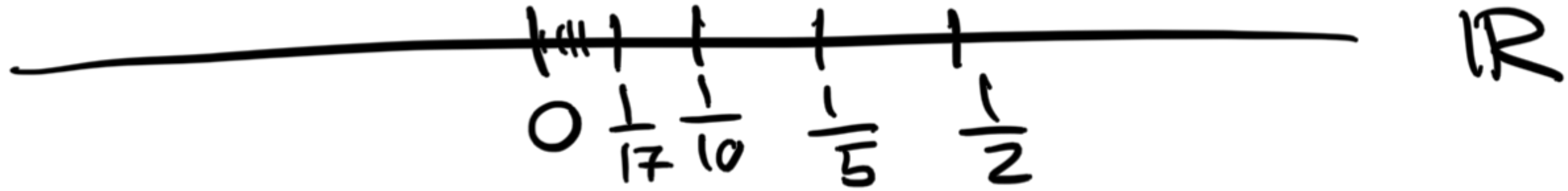
<u>Proof</u>: Let $A \subset \mathbb{R}$ be not closed. There exists an adherent point $a \in \mathbb{R}$ such that $a \notin A$. Consider the func-

tion $f(x) = \dfrac{1}{x-a}$. As $a \notin A$, f is continuous on A. On the other hand, for any $\varepsilon > 0$ there is $x \in B(a,\varepsilon) \cap A$ i.e. $|x-a| < \varepsilon$. Hence,

$$|f(x)| = \frac{1}{|x-a|} > \frac{1}{\varepsilon}.$$

Since ε can be made arbitrarily small, $\frac{1}{\varepsilon}$ can be made arbitrarily large. Therefore, f is not bounded. $\square$

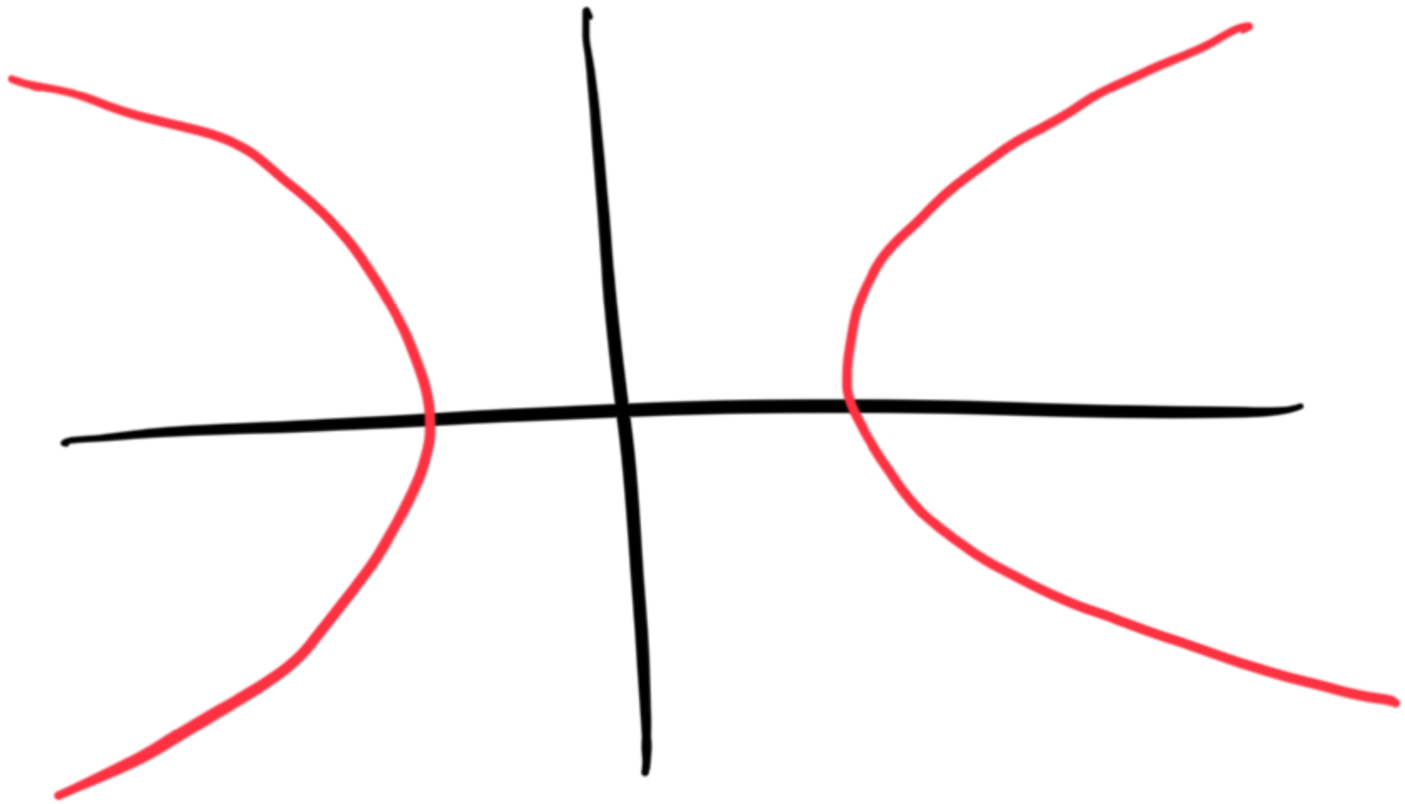
(c) (i) $A = \left\{ \frac{1}{n^2+1} : n \in \mathbb{N} \right\} \subset \mathbb{R}$



Is A compact? No, A is not compact because A is not closed (we are using the Heine-Borel theorem:

$A \subset \mathbb{R}^n$ compact $\Longleftrightarrow$ A is closed & bounded)

(ii) $A = \{ x^2 - 2y^2 = 15 \}$



A is closed, but not bounded, therefore not compact.

e.g. $\left\{ \left((15+2n^2)^{\frac{1}{2}}, n \right) : n \in \mathbb{N} \right\} \subset A$

unbounded

(iii) $A = \{x^4 + y^4 + \sin x = 10\}$

A is closed and bounded $\Longrightarrow$ A is compact

closed

$A = f^{-1}(\{10\})$

continuous

$f(x,y) = x^4 + y^4 + \sin x$

$|x|, |y| < 10$

otherwise:

$f(x,y) \geq 10^4 - 1 > 10$

(iv) $A = [0,1) \subset \mathbb{R}$ with the topology

$\tau = \{\mathbb{R}, \emptyset, (a, +\infty)\}$

$\mathbb{R}$

Compact or not?

Claim: A is compact

Pf: Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of A. Since $0 \in A$, there is $U_{\alpha_0} \in \mathcal{U}$ s.t. $0 \in U_{\alpha_0}$. Since U_{α_0} is open, $U_{\alpha_0} = (a, +\infty)$ for some $a < 0$. Then $A = [0,1) \subset U_{\alpha_0}$ and $\{U_{\alpha_0}\}$ can be taken as a finite subcover of A. Hence A is compact. $\blacksquare$

③ (a) <u>Claim</u>: If $f: X \to Y$ is continuous and X is connected, then $f(X)$ is connected.

<u>Pf</u>: By contradiction assume that $f(X)$ is disconnected, i.e. $f(X) = A_1 \cup B_1$ where A_1, B_1 are disjoint, nonempty open sets (in the induced topology). This means, in particular, that there are sets

in particular, that there are open sets $A, B \subset Y$ s.t. $A_1 = A \cap f(X)$ and $B_1 = B \cap f(X)$. Since f is continuous, the preimages $f^{-1}(A)$ and $f^{-1}(B)$ are open in X. Notice that

$$f^{-1}(A) = f^{-1}(A_1)$$

$$f^{-1}(B) = f^{-1}(B_1)$$

Thus, we have a partition of X into non-empty disjoint open sets :

$$X = f^{-1}(A) \cup f^{-1}(B).$$

This contradicts the fact that X is connected. Conclusion: $f(X)$ is connected. $\square$

(b) $\mathbb{R}$ with $\tau = \{\mathbb{R}, \emptyset, (a, +\infty)\}$.

$$a \in \mathbb{R}$$

Connected or not?

<u>Answer</u>: Yes.

<u>Proof</u>: By contradiction, assume

that there is a partition into disjoint non-empty open sets:

$$\mathbb{R} = A \cup B.$$

By def. of τ, $A = (a, +\infty)$ for some $a \in \mathbb{R}$. But then

$$B = \mathbb{R} \setminus A = (-\infty, a]$$

is not open. The contradiction shows that $(\mathbb{R}, \tau)$ is connected. $\square$

(c) $X = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ where

$$Y = \{ x^2 + y^2 = 1 \} \subset \mathbb{R}^2 \quad \text{circle}$$

Are they homeomorphic or not?

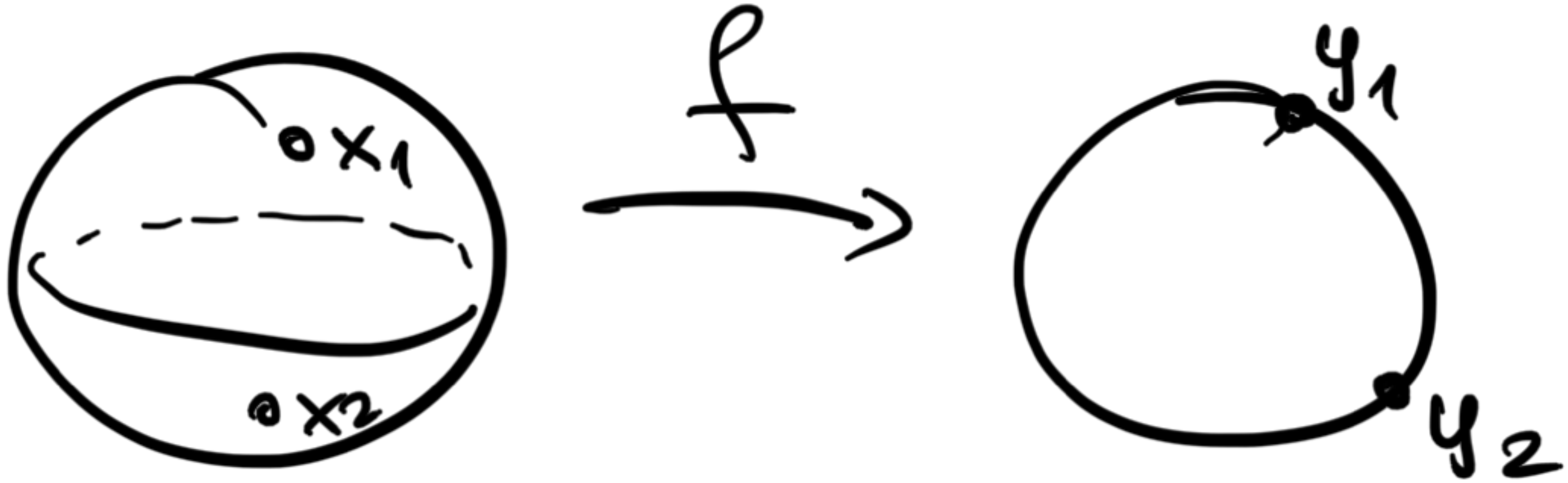
Answer: X and Y are not homeomorphic.

<u>Proof</u>: By contradiction, assume that there is a homeomorphism

$$f : X \to Y.$$

Take two points $x_1, x_2 \in X$ and

and two points $x_1, x_2 \in X$ and
let $y_1 = f(x_1)$, & $y_2 = f(x_2)$.



Notice: $Y \setminus \{y_1, y_2\}$ is disconnected, whereas $X \setminus \{x_1, x_2\}$ is still connected. Thus the restriction

$$f|_{X \setminus \{x_1, x_2\}} : X \setminus \{x_1, x_2\} \longrightarrow Y \setminus \{y_1, y_2\}$$

is a continuous map such that the image of a connected space is disconnected, which is impossible. This contradiction shows that S^2 and S^1 are not homeomorphic. $\blacksquare$

(d) $X = \left\{ A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : \begin{array}{l} \det A \neq 0, \\ a, b, c \in \mathbb{R} \end{array} \right\}$

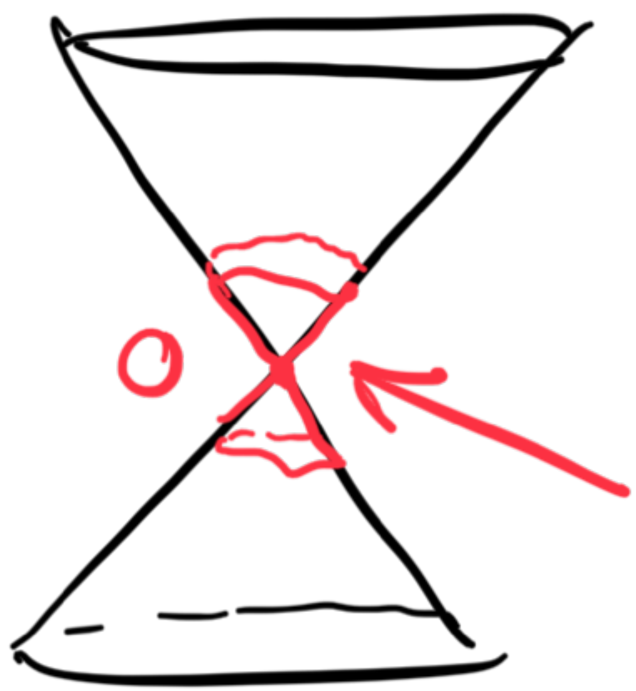
Is X connected or not?

Answer: Not connected

(See solution of 2019 exam)

4 (a) $X = \{x^2+y^2-z^2=0\} \subset \mathbb{R}^3$

Is X a manifold or not?



X is a cone in $\mathbb{R}^3$

U neighborhood of the origin

This neighborhood cannot be homeomorphic to an interval $B^1=(-r,r)$.

Indeed, if we remove a point from $\mathcal{U}$ (not the origin), then $\mathcal{U}$ remains connected, whereas removing any point from B^1 leads to splitting into two components.

On the other hand, $\mathcal{U}$ cannot be homeomorphic to a disk $B^2 = \{x^2+y^2<r^2\}$ either. Indeed, if we remove the origin from $\mathcal{U}$, then it splits into two components, whereas $B^2 \setminus \{\text{point}\}$ is still connected for

any point in B^2. The same argument shows that $\mathcal{U}$ cannot be homeomorphic to any $B^n = \{x_1^2 + x_2^2 + \dots + x_n^2 < r^2\}$ for any $n > 2$.

Thus, the origin $O \in X$ does not admit any neighborhood homeomorphic to B^n, $n \geq 1 \Rightarrow$ <u>X is not a manifold.</u>

(b) $X = \{e^{x+y} - x - y + x^2 z + y^2 z = 2\}$

We apply the Implicit Function Theorem (IFT) to

$$F(x,y,z) = e^{x+y} - x - y + x^2 z + y^2 z$$

$$dF = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$$

$$= \left(e^{x+y} - 1 + 2xz,\ e^{x+y} - 1 + 2yz,\ x^2 + y^2\right)$$

Where is $dF(x,y,z) = (0,0,0)$?

$$\begin{cases} e^{x+y} - 1 + 2xz = 0 \\ e^{x+y} - 1 + 2yz = 0 \end{cases}$$

$$\left[\ \underbrace{x^2+y^2}_{\longrightarrow\ x=y=0} = 0\right.$$

Substituting $(0,0,z)$ into the equation defining X, we get

$$F(0,0,z) = \underbrace{e^0}_{=1} - 0 - 0 + 0\cdot z + 0\cdot z = 1 \neq 2$$

$\Rightarrow (0,0,z) \notin X$, therefore $dF \neq 0$ on X, and therefore X is a mani-fold of dimension $3-1=2$ (by the

|FT|).

(c) The Euler characteristic of the sphere is 2. Indeed, the sphere admits the following partition into cells (tetrahedron)

$\simeq$

$F = 4$

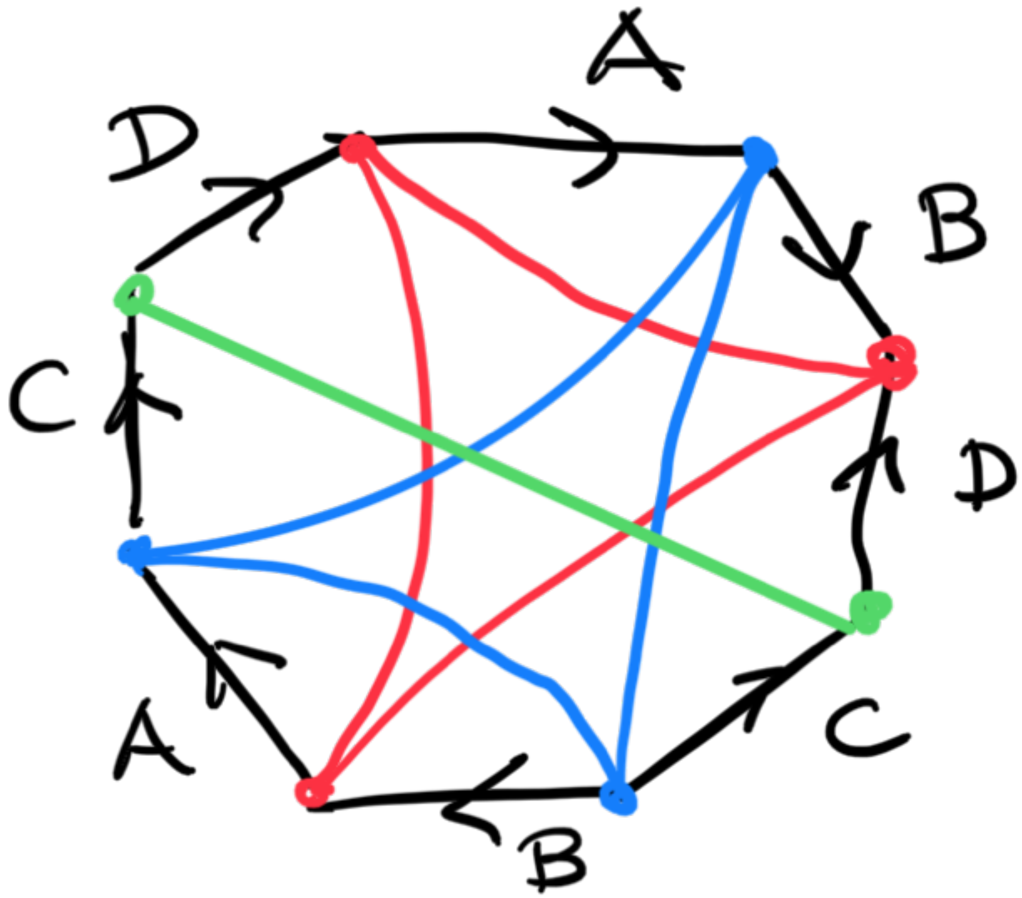
$E = 6$

$V = 4$

$\chi(S^2) = 4 - 6 + 4 = 2$

(d) $ABD^{-1}C^{-1}BACD$

What is the surface M?



$F = 1$

$E = 4$ (A, B, C, D)

$V = 3$

$$\chi(M) = 1 - 4 + 3 = 0$$

Orientable or not? No, since A appears in the form $--A--A\ldots.$

A non-orientable surface with zero Euler characteristic is homeomorphic to the <u>Klein bottle</u>.

$$\left(\begin{array}{l} M = S^2 + m \cdot \mu \\ \chi(M) = 2 - m = 0 \Rightarrow m = 2 \end{array} \right)$$