Loughborough University

ELEMENTS OF TOPOLOGY
(16MAC298)

Summer 2017 — 2 hours

Answer **THREE** questions.
Any calculator from the University's approved list may be used.

1. (a) Let A be a subset of a topological space X.
   (i) Give the definition of an interior point of A. [3]
   (ii) Give the definition of a boundary point of A. [3]
   (iii) Prove that the interior $\mathrm{Int}\, A$ of A is open. [4]

   (b) Describe the interior and the boundary of the following subsets of $\mathbb{R}$:
   (i) $\mathbb{R} \setminus \mathbb{Z}$; [3]
   (ii) $(2,3) \cup \{\frac{1}{n}, n \in \mathbb{N}\}$; [3]
   (iii) $(0, +\infty) \cap \mathbb{Q}$. [3]
   Which of these subsets are open? Justify your answers. [1]

2. (a) Give the definition of a Hausdorff topological space. [3]

   (b) Let $f : X \to Y$ be a continuous bijection between a compact topological space X and a Hausdorff topological space Y. Prove that f is a homeomorphism. [5]

   (c) Which of the following spaces are compact (justify your answer):
   (i) $[0,1]$ with the discrete topology; [2]
   (ii) $[0,1]$ as a subset of $\mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}, (a, +\infty), a \in \mathbb{R}\}$; [2]
   (iii) $\{\frac{1}{n}, n \in \mathbb{N}\} \cup \{0\}$ as a subset in $\mathbb{R}$ (with the standard topology); [2]
   (iv) the rectangle $\{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 2,\ 0 \le y \le 1\}$; [2]
   (v) the set $\{(x,y) \in \mathbb{R}^2 \mid \sin^4 x + \sin^4 y \le 1\}$; [2]
   (vi) the curve in $\mathbb{R}^2$ given by the equation $x^4 + y^4 = 1$. [2]

3. (a) Are X and Y homeomorphic? Justify your answer:
   (i) $X = (0,1),\ \ Y = (-\infty, +\infty)$; [3]
   (ii) $X = \{x^2 + y^2 + z^2 \leq 1\},\ \ Y = \{x^2 + y^2 + z^2 \geq 1\}$; [3]
   (iii) $X = [0,1),\ \ Y = (1,3]$; [3]
   (iv) $X = S^1$ (the circle), $\ Y = [0,1]$. [3]

   (b) Give the definition of a pathwise connected topological space. [4]

   (c) Let X be the set of orthogonal 2×2 matrices viewed as a subset of the four-dimensional vector space of real 2×2 matrices:

$$X = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad AA^\top = \mathrm{Id} \right\}.$$

   Is X pathwise connected? Justify your answer. [4]

4. (a) Give the definition of a manifold. [5]

   (b) Is the subset $X \subset \mathbb{R}^n$ given by the equation(s) below a manifold? Justify your answer.
   (i) $X = \{\ x\,(x^2 + y^2 - 1) = 0\ \} \subset \mathbb{R}^2$; [5]
   (ii) $X = \{x - y - z = 1,\ x^3 + y^3 + z^3 = 0\} \subset \mathbb{R}^3$. [5]

   (c) Consider the surface M obtained from the fundamental polygon

$$ABD^{-1}CDB^{-1}CA^{-1}$$

   by pairwise identification of its edges. Find the Euler characteristic of M and determine its topological type. [5]