Loughborough University

ELEMENTS OF TOPOLOGY
(17MAC298)

Summer 2018 | 2 hours

Answer **THREE** questions.

Any calculator from the University's approved list may be used.

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1. (a) Let A be a subset of a topological space X.

   (i) Give the definition of an adherent point of A. [3]

   (ii) Prove that $x \in \overline{A}$ if and only if x is an adherent point of A. [5]

   (iii) Let $X = \mathbb{R}$ with the discrete topology and $A = (0, 1)$. What is the closure of A? [2]

   (b) Sketch the following subsets A and describe their closures $\overline{A}$ in $\mathbb{R}^2(x, y)$ with the standard topology:

   (i) $A = \{x^2 + y^2 < 1\}$; [2]

   (ii) $A = \{x \leq 0,\ y \leq 0\}$; [2]

   (iii) $A = \{x + y = 1,\ x > 0\}$; [2]

   (iv) $A = \{(x, y) = (m, n),$ where $m, n \in \mathbb{Z}\}$. [2]

   Which of these subsets A are closed? Justify your answers. [2]

2. (a) Give the definition of a Hausdorff topological space. Give an example of a topological space X which is not Hausdorff. [5]

   (b) Prove that a compact subset of a Hausdorff topological space is closed. [5]

   (c) Which of the following sets are compact? Justify your answer:

   (i) $\mathbb{R}$ with the topology $\tau = \{\emptyset,\ \mathbb{R}$ and $(a, +\infty),$ where $a \in \mathbb{R}\}$; [2]

   (ii) $\{1 - \frac{1}{n}, n \in \mathbb{N}\}$ as a subset of $\mathbb{R}$ (with the standard topology); [2]

   (iii) the subset of $\mathbb{R}^3$ given by the inequality $x^2 + y^2 + z^2 \geq 1$; [2]

   (iv) the subset of $\mathbb{R}^2$ given by the equation $x^4 + \sin^4 y = 1$; [2]

   (v) the square $\{(x, y) \in \mathbb{R}^2 \ :\ x \in [0, 1], y \in [0, 1]\}$. [2]

3. (a) Give the definition of a continuous map. [3]

   (b) Prove that the image of a connected topological space under a continuous map is connected. [5]

   (c) Let $X = \mathbb{R}$ with the discrete topology, $Y = \mathbb{R}$ with the standard topology, and $f : X \to Y$, $f(x) = x$.

      (i) Is X connected? [2]

      (ii) Is Y connected? [2]

      (iii) Is f continuous? [2]

      (iv) Is f a homeomorphism? [2]

      (v) Are X and Y homeomorphic? [2]

      (vi) Give an example of a continuous map $Y \to X$. [2]

      Justify your answers.

4. (a) Is the subset $X \subset \mathbb{R}^2$ a manifold? Justify your answer.

      (i) $X = \{xy + y - x - 1 = 0\}$; [3]

      (ii) $X = \{\cosh x + \cosh y = 10\}$; [3]

      (iii) $X = \{x = 0,\ y \geq 0\}$. [4]

   (b) State the Classification Theorem for closed surfaces. [5]

   (c) Consider the surface M obtained from the fundamental polygon

$$ABC^{-1}BD^{-1}A^{-1}CD$$

   by pairwise identification of its edges. Find the Euler characteristic of M and determine its topological type. [5]