Elements of Topology A. Bolsinov (1)

Solutions (Summer 2017)

No. 1 (a) <u>Def.</u> Let X be a topological space, $A \subset X$ a subset and $b \in X$ a point.

(i) b is called an interior point of A if b has a neighborhood contained in A. [3] (bookwork)

(ii) b is called a boundary point of A if each neighborhood of b intersects both A and the complement of A. [3] (bookwork)

(iii) $\mathrm{Int}\,A$ is open.
To show this we use the following property of open sets:
$B \subset X$ is open if and only if for any point $b \in B$ there is a neighborhood $U(b)$ such that $U(b) \subset B$.

Let $b \in \mathrm{Int}\,A$. This means that there exists a neighborhood $U(b)$ such that $U(b) \subset A$. Let b' be any other point of $U(b)$. Clearly, $U(b)$ can be considered as a neighborhood of b' and therefore b' is an interior point of A.
In other words, $U(b)$ consists of interior points of A, i.e. $U(b) \subset \mathrm{Int}\,A$. Thus, every point $b \in \mathrm{Int}\,A$ is contained in $\mathrm{Int}\,A$ together with a certain neighborhood. Hence, $\mathrm{Int}\,A$ is open as required. [4] (bookwork)

(B) Describe the interior and boundary:

(i) $\mathbb{R} \setminus \mathbb{Z}$

interior $= \ldots \cup (-2,-1) \cup (-1,0) \cup (0,1) \cup (1,2) \cup (2,3) \cup \ldots =$

$$= \bigcup_{n=-\infty}^{+\infty} (n, n+1)$$

[3]

(standard problem)

boundary $= \mathbb{Z}$

(ii) $(2,3) \cup \{\frac{1}{n}, n \in \mathbb{N}\}$

[3]

(standard problem)

interior $= (2,3)$

boundary $= \{2\} \cup \{3\} \cup \{0\} \cup \{\frac{1}{n}, n \in \mathbb{N}\}$

(iii) $(0, +\infty) \cap \mathbb{Q}$

[3]

interior $= \emptyset$

boundary $= [0, +\infty)$

(Standard problem)

$\mathbb{R} \setminus \mathbb{Z}$ is open (as a union of open intervals).

[1]

The other two sets are not open.

(standard problem)

No 2.

(a) Def. X is a Hausdorff topological space, if for any two distinct points $x, y \in X$ there exist neighborhoods $\mathcal{U}(x)$ and $V(y)$ which are disjoint, i.e., $\mathcal{U}(x) \cap V(y) = \emptyset$.

[3] (bookwork)

(b) Prof. $\underset{\text{compact}}{X} \xrightarrow{f} \underset{\text{Hausdorff}}{Y}$: continuous bijection

We only need to prove that $f^{-1} : Y \to X$ is continuous.
The continuity of f^{-1} means that the preimage of any open set $A \subset X$ under f^{-1} is open. Using the duality between open and closed sets we can reformulate this condition as follows: f^{-1} is continuous if and only if the preimage of any closed set $C \subset X$ under f^{-1} is closed. Notice that the preimage of the inverse map f^{-1} is just the image of C under the direct map!
Thus, the statement ~~of the~~ is equivalent to the fact that the image $f(C)$ of any closed subset $C \subset X$ is closed as a subset of Y. But this fact is a combination of the three following statements:

1) C is compact as a closed subset of a compact space X.
2) $f(C)$ is compact as the image of a compact set under a continuous map.
3) $f(C)$ is closed, since $f(C)$ is a compact subset of the Hausdorff topological space Y.

[5] (bookwork)

(c) Compact or not?

(i) $[0,1]$ with the discrete topology (standard problem

is not compact

(any discrete infinite topological space is not compact) [2

(ii) $[0,1]$ as a subset of $\mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}, (a, +\infty)_{a \in \mathbb{R}}\}$

is compact. Indeed, let $\mathcal{U} = \{U_\alpha, \alpha \in I\}$ be an open cover of $[0,1]$. Let U_{α_0} be an element of this cover that contains the point 0. Obviously, $[0,1] \subset U_{\alpha_0}$ and therefore, $\{U_{\alpha_0}\}$ can be taken as a finite subcover. [2]

(iii) $\{\frac{1}{n}, n \in N\} \cup \{0\}$ as a subset of $\mathbb{R}$ (standard topology) is compact, because this subset is bounded and closed. [2]

(iv) the rectangle $\{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 2, 0 \le y \le 1\}$ is compact because this set is bounded and closed. [2]

(v) $\{(x,y) \in \mathbb{R}^2 \mid \sin^4 x + \sin^4 y \le 1\}$ is not compact, because this set is not bounded. Indeed, this set contains, for example, the unbounded sequence (x_n, y_n), $x_n = 2\pi n$, $y_n = 2\pi n$, [2]

(vi) the curve in $\mathbb{R}^2$ given by the equation $x^4 + y^4 = 1$ is compact, because it is bounded and closed.

[2]

No 3 (a) X, Y homeomorphic or not? (standard question)

(i) $X = (0,1)$, $Y = (-\infty, +\infty)$

$X \simeq Y$ homeomorphic

$f: X \to Y$, $f(x) = \tan\left(\pi\left(x - \frac{1}{2}\right)\right)$ is a homeomorphis

[3]

(page 5

(ii) $X = \{x^2+y^2+z^2 \leq 1\}$, $Y = \{x^2+y^2+z^2 \geq 1\}$

$X \not\simeq Y$ not homeomorphic, because [3]

X is compact, Y is not.

(iii) $X = [0,1)$, $Y = (1,3]$ [3]

$X \simeq Y$ homeomorphic

$f: X \to Y$, $f(x) = 3-2x$ is a homeomorphism

(iv) $X = S^1$, $Y = [0,1]$ [3]

$X \not\simeq Y$ not homeomorphic.

If we remove the point $\frac{1}{2}$ from Y and any point from X, then X remains connected, whereas Y becomes disconnected.

(b) Definition A topological space (X,τ) is pathwise connected if for any $x,y \in X$, there exists a continuous map $f: [0,1] \to X$ such that $f(0)=x$, $f(1)=y$. Such a map f is called a path from x to y. [4] (bookwork

(c) $X = \left\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, AA^T = Id\right\}$ is disconnected and therefore is not pathwise connected.

[4] (standard question)

$X = X_1 \sqcup X_2$

$X_1 = \{A \in X, \det A > 0\}$

$X_2 = \{A \in X, \det A < 0\}$

Since det is a continuous function,

X_1 and X_2 are open.

They are not empty, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in X_1$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in X_2$

They are disjoint $X_1 \cap X_2 = \emptyset$.

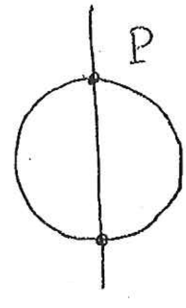
and any orthogonal matrix A belongs to either X_1, or X_2 because $\det A \neq 0$.

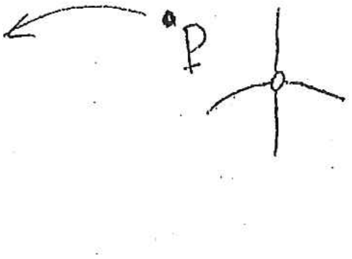
<u>No 4.</u> (a) Def. A topological space is called a manifold of dimension n, if every point $x \in X$ possesses a neighborhood $\mathcal{U}(x)$ homeomorphic to an n-dimensional open ball $B^n = \{x \in \mathbb{R}^n : |x| < r\} \subset \mathbb{R}^n$. [5] (bookwork)

(b) Manifold or not?

$$X = \{x(x^2+y^2-1) = 0\}$$

Geometrically X is the union of the circle $\{x^2+y^2=1\}$ and the line $\{x=0\}$.





X is not a manifold because $P=(1,0) \in X$ does not have any neighborhood homeomorphic to B^n.

Indeed, if we remove P from any of its sufficiently small neighborhoods, this neighborhood becomes disconnected and splits into at least 4 components.

If we remove a point from B^n, then $B^n \setminus \{x\}$ remains connected for $n \geq 2$ and $B^1 \setminus \{x\}$ splits into 2 components for $n=1$.

[5] (unseen)

(ii) $X = \{ x - y - z = 1 \; , \; x^3 + y^3 + z^3 = 0 \}$

We apply the Implicit Function Theorem

$$f_1(x) = x - y - z$$

$$f_2(x) = x^3 + y^3 + z^3$$

$$J = \begin{pmatrix} 1 & -1 & -1 \\ 3x^2 & 3y^2 & 3z^2 \end{pmatrix}$$

To find the rank of J we reduce J to the echelon form

$$\begin{pmatrix} 1 & -1 & -1 \\ 3x^2 & 3y^2 & 3z^2 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & -1 & -1 \\ 0 & 3y^2+3x^2 & 3z^2+3x^2 \end{pmatrix}$$

The rank is not maximal ($\Leftrightarrow$ i.e. <2) iff

$$\begin{matrix} 3y^2 + 3x^2 = 0 \\ 3z^2 + 3x^2 = 0 \end{matrix} \quad \Rightarrow \quad x = y = z = 0$$

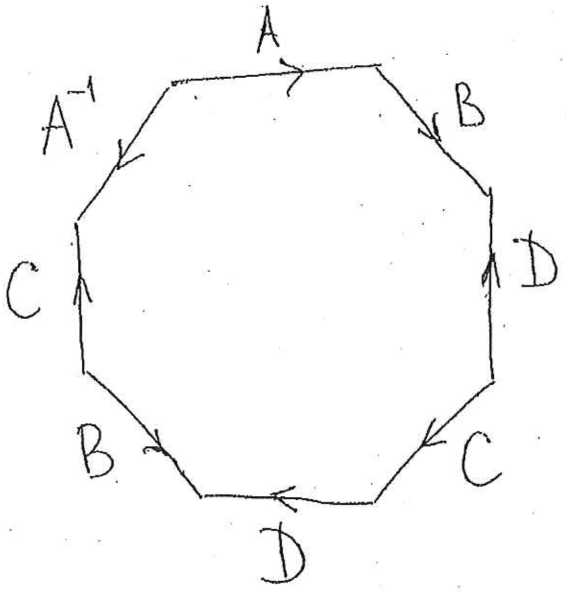
but this point $(0,0,0)$ does not belong to X (since $x - y - z = 1$)

Thus rank $J = 2$ on X, and therefore X is a smooth manifold of dimension $1 = 3 - 2$.

[5] (standard question)

(c) Fundamental polygon

$ABD^{-1}CDB^{-1}CA^{-1}$



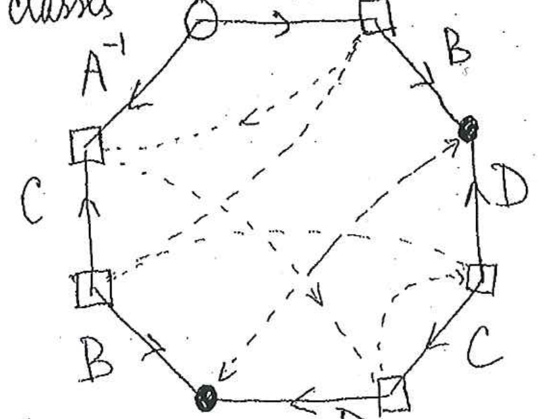
M is not orientable because the word $ABD^{-1}CDB^{-1}CA^{-1}$ contains combination ...C...C...

To determine the topological type of M we compute the Euler characteristic

$e(M) = V - E + F$

$F = 1, \ E = 4$

V (number of vertices) is computed by counting equivalence classes



○ □ ● 3 distinct classes, i.e. $V = 3$

$e(M) = 3 - 4 + 1 = 0.$

Thus M is a non-orientable compact surface of Euler charact. 0

Thus, M is a Klein bottle.

[5] standard question