MAB298–Elements of Topology

September 4, 2023

Contents

Chapter 1

Point set topology

1.1 Topological spaces and open sets

Definition 1. Let X be a non-empty set. A collection τ of subsets of X is said to be a *topology* on X if

(i) X and the empty set, $\emptyset$, belong to τ,

(ii) the union of any (finite, countably infinite, uncountably infinite,...) number of sets in τ belongs to τ, and

(iii) the intersection of any two sets in τ belong to τ.

The pair (X, τ) is called a *topological space*. The elements of τ are called *open* sets. These open sets allow us to significantly generalise many ideas from analysis.

Example 1. Let $X = \{a, b, c, d, e\}$ and

$$\tau_1 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}.$$

Then τ_1 is a topology on X as it satisfies conditions (i), (ii), (iii) of Definition 1.

Example 2. Let $X = \{a, b, c, d, e\}$ and

$$\tau_2 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, e\}, \{b, c, d\}\}.$$

Then τ_2 is not a topology on X as the intersection

$$\{a, c, e\} \cap \{b, c, d\} = \{c\}$$

of two members of τ_2 does not belong to τ_2; that is, τ_2 does not satisfy condition (iii) of Definition 1.

Example 3. Let $\mathbb{Z}$ be the set of all integer numbers and let τ consists of $\mathbb{Z}$, $\emptyset$, and all finite subsets of $\mathbb{Z}$. Then τ is not a topology on $\mathbb{Z}$, since the infinite union

$$\{1\} \cup \{2\} \cup \{3\} \cup \ldots \{n\} \cup \cdots = \mathbb{N}$$

of members of τ does not belong to τ; that is τ does not have property (ii) of Definition 1.

Any set X can be endowed with at least two different topologies in the following natural way.

Definition 2. Let X be a non-empty set and let τ be the collection of all subsets of X.[1] Then τ is called the *discrete topology* on the set X. The topological space (X, τ) is called a *discrete space*. In other words, X is discrete iff every $A \subset X$ is open.

We note that τ in Definition 2 does satisfy the conditions of Definition 1 and so is indeed a topology.

Definition 3. Let X be any non-empty set and let $\tau = \{X, \emptyset\}$. Then τ is called the *indiscrete topology* on the set X and (X, τ) is said to be an *indiscrete space*. In other words, X is indiscrete iff the only open sets in X are the empty set and X itself.

Once again we have to check that τ satisfies the conditions of Definition 1 and so is indeed a topology.

Example 4. The *standard topology/Euclidean topology* on the real line $\mathbb{R}$ is defined in the following way: a non-empty subset $A \subset \mathbb{R}$ is open if and only if for any $x \in A$ there is $\delta > 0$ such that $(x - \delta, x + \delta) \subset A$. Similarly, one defines the topology in $\mathbb{R}^n$: a non-empty subset $A \subset \mathbb{R}^n$ is open if and only if for any $x \in A$ there is $\delta > 0$ such that $B(x, \delta) \subset A$, where

$$B(x, \delta) = \{y \in \mathbb{R}^n \mid |x - y| < \delta\}$$

is the open ball of radius δ centered at x. We also let $\emptyset$ be an open set.

Check that this is a topology:

- $\mathbb{R}^n$ is obviously open ($B(x, 1) \subset \mathbb{R}^n$ for all $x \in \mathbb{R}^n$). Moreover $\emptyset$ is open by definition.
- Let $\{A_i\}_{i \in I}$ be an arbitrary collection of open sets. Let $x \in \cup_{i \in I} A_i$. Then $x \in A_i$ for some $i \in I$, and since A_i is open there exists $\delta > 0$ such that $B(x, \delta) \subset A_i$. Now observe that $B(x, \delta) \subset A_i \subset \cup_{i \in I} A_i$. So the value of δ that worked for A_i also works for $\cup_{i \in I} A_i$. So $\cup_{i \in I} A_i$ is open and the second property is satisfied.

[1] τ is called the *power set* of X, denoted $\mathcal{P}(X)$ or 2^X.

- Let A_1, A_2 be two open sets. Let $x \in A_1 \cap A_2$. Since each A_i is open, for each i there exists $\delta_i > 0$ such that $B(x, \delta_i) \subset A_i$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $B(x, \delta) \subset A_i$ for both i, or equivalently $B(x, \delta) \subset A_1 \cap A_2$. So the third property of a topology is satisfied.

The definition of a topological space is very general. There are many examples of topological spaces with very different and sometimes exotic properties.

Example 5. On the real line $\mathbb{R}$, we define the topology τ by introducing the following open subsets: $\mathbb{R}$, empty set $\emptyset$, and infinite intervals of the form $(a, +\infty)$, where $a \in \mathbb{R}$. To prove that this collection of subsets satisfies all necessary properties it suffice to notice that
1) $(a_1, +\infty) \cap (a_2, +\infty) = (a, +\infty) \in \tau$, where $a = \max\{a_1, a_2\}$, and
2) $\cup_{\alpha \in I}(a_\alpha, +\infty) = (a, +\infty) \in \tau$, where $a = \inf_{\alpha \in I}\{a_\alpha\}$.

1.2 Topology on metric spaces, induced topology, closed sets

$\mathbb{R}$ and $\mathbb{R}^n$ are just two particular cases of the following general construction allowing us to introduce "topology" on any metric space (X, d).

Recall the following standard definition.

Definition 4. A metric space is a pair (X, d) where X is a set, and d is a metric on X, that is a function from $X \times X$ to $\mathbb{R}$ that satisfies the following properties for all $x, y, z \in X$:

- $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$ (symmetry), and
- $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

Example 6. 1. $\mathbb{R}^n$ equipped with the Euclidean metric $d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ is a metric space.

2. Let X be a non-empty set. We define a metric on X according to the rule:
$$d(x, y) = \left\{ \begin{array}{ll} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{array} \right\}$$

3. Let $C[a, b]$ denote the set of all continuous functions from $[a, b]$ to $\mathbb{R}$. Define a metric on $C[a, b]$ by
$$d(f, g) = \max_{x \in [a,b]} |f(x) - g(x)|.$$

Clearly $d(f,g) = d(g,f)$ so the second property is satisfied. Clearly $d(f,g) \geq 0$ since $|f(x) - g(x)| \geq 0$ for all x. Notice that $d(f,g) = 0$ if and only if $f(x) = g(x)$ for all $x \in [a,b]$. Therefore $d(f,g) = 0$ if and only if $f = g$ so the first property is satisfied. For the final property notice that

$$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$$

for any $x \in [a,b]$ and $f, g, h \in C([a,b])$. Taking maximums we now observe that

$$\begin{aligned} d(f,g) &= \max_{x \in [a,b]} |f(x) - g(x)| \\ &\leq \max_{x \in [a,b]} (|f(x) - h(x)| + |h(x) - g(x)|) \\ &\leq \max_{x \in [a,b]} |f(x) - h(x)| + \max_{x \in [a,b]} |h(x) - g(x)| \\ &= d(f,h) + d(h,g). \end{aligned}$$

So we satisfy the final property.

4. Similarly, let $C^1[a,b]$ denote the set of all continuously differentiable functions from $[a,b]$ to $\mathbb{R}$. We can a metric on $C^1[a,b]$ by

$$d(f,g) = \max_{x \in [a,b]} |f(x) - g(x)| + \max_{x \in [a,b]} |f'(x) - g'(x)|.$$

A metric space (X,d) possesses the following natural topology.

Definition 5. A set U in a metric space (X,d) is open if and only if for any $x \in U$ there exists $\delta > 0$ such that $B(x,\delta) \subset U$, where

$$B(x,\delta) = \{y \in X \ : \ d(y,x) < \delta\}$$

is the open ball of radius δ centered at x.

Conclusion: With this definition of open every metric space is also a topological space.

Exercise. Generalise the arguments given in Example 4 to prove the conclusion above. Note that there are topological spaces that are not metric spaces. Can you think of one?

Obviously, the topology of $\mathbb{R}^n$ is a particular case of this construction: $d(x,y) = |x - y| = \sqrt{\sum (x_i - y_i)^2}$ for $x, y \in \mathbb{R}^n$.

It is very important that the above construction allows us to introduce topology on infinite-dimensional spaces. By the example above both

$C[a,b]$ and $C^1[a,b]$ can be made into topological spaces by equipping an appropriate metric.

Notice that different metrics d_1 and d_2 may define the same topology on X. For example, in $\mathbb{R}^n$ one can consider two different metrics:

- $d_1(x,y) = \sqrt{\sum_i (x_i - y_i)^2}$ (standard Euclidean metric) and
- $d_2(x,y) = \max_i |x_i - y_i|$

It is not hard to see that although d_1 and d_2 are different, in particular their open balls look different, the topologies defined by them on $\mathbb{R}^n$ are the same.

We will on occasion make use of the following complementary notion to open balls.

Definition 6. Let (X,d) be a metric space. Let $x \in X$ and $\delta > 0$. We let

$$\overline{B(x,\delta)} := \{y \in X : d(y,x) \leq \delta\}.$$

We call $\overline{B(x,\delta)}$ the closed ball of radius δ centred at x.

The following notion of convergence generalises that which we have seen for subsets of Euclidean space.

Definition 7. Let (X,d) be a metric spaces and $(x_n)_{n=1}^{\infty}$ be a sequence of elements of X. We say that $\lim_{n\to\infty} x_n = x$ for some $x \in X$ if $\lim_{n\to\infty} d(x_n,x) = 0$.

Another series of examples of topological spaces can be constructed by noticing that any subset Y of a topological space X is a topological space itself:

Definition 8. Given a topological space (X,τ) and $Y \subset X$, then the family $\tau_Y = \{B_Y = Y \cap B \mid B \in \tau\}$ is a topology for Y, called the subspace (or relative or induced) topology for Y. (Y,τ_y) is called a subspace of (X,τ).

`Closed sets`

Definition 9. A set A in a topological space X is called *closed*, if its complement $X \setminus A$ is open. (Recall that $X \setminus A = \{x \in X \ : \ x \notin A\}$).

`Set-Theoretic Digression: De Morgan Formulas`
Let Γ be an arbitrary collection of subsets of a set X. Then

$$X \setminus \bigcup_{A\in\Gamma} A = \bigcap_{A\in\Gamma} (X \setminus A),$$

and

$$X \setminus \bigcap_{A \in \Gamma} A = \bigcup_{A \in \Gamma} (X \setminus A),$$

Using these formulas, it is easy to derive:
`Main Properties of Closed Sets:`

Proposition 1. *(1) the intersection of any collection of closed sets is closed;*

(2) the union of any finite number of closed sets is closed;

(3) the empty set and the whole space X are closed.

In fact, these three properties can be viewed as the definition of a topological space in terms of closed subsets.

1.3 Interior, boundary and exterior of a set

This section is devoted to further expanding the vocabulary needed when we speak about phenomena in a topological space. Throughout this section, X denotes a topological space.

Definition 10. A neighborhood U of a point $x \in X$ is any open set containing this point.

Remark 1. If U is a neighborhood of x, then U is a neighborhood of any other point $y \in U$. More generally, an open set U is a neighborhood of any of its points.

Remark 2. A subset $A \subset X$ is open if and only if the following condition is satisfied:
For any $x \in A$ there exists a neighborhood $U = U_x$ such that $x \in U \subset A$ (in other words, each point $x \in A$ is contained in A together with a certain neighborhood U_x).

Remark 3. *As a rule, a point $x \in X$ admits many different neighborhoods.*

Example 7. 1. Let $X = \mathbb{R}$ with the standard topology. A typical neighborhood U of $x \in X$ is an interval $(x - \epsilon, x + \epsilon)$. However, according to the above definition, $(x - \epsilon, +\infty)$, as well as the whole line $\mathbb{R}$, are also considered as neighborhoods of x.

2. Let $X \subset \mathbb{R}^2$ be the (closed) half-plane $y \geq 0$. As a neighborhood of $x = (0, 0)$ in X we can take the semi-disc $U = \{x^2 + y^2 < \epsilon, y \geq 0\}$.

3. If X is a discrete space, then a neighborhood of x is just an arbitrary subset U which contains x, for example, this point itself $U = \{x\}$.

4. If X is an indiscrete space, then a point x admits the only neighborhood $U = X$.

`Interior, Exterior, and Boundary Points:`

Definition 11. Let X be a topological space, $A \subset X$ a subset, and $b \in X$ a point. The point b is

- an *interior point* of A if b has a neighborhood contained in A;
- an *exterior point* of A if b has a neighborhood disjoint with A;
- a *boundary point* of A if each neighborhood of b intersects both A and the complement of A.

It is not hard to see that for any set $A \subset X$ we have a natural partition of X into three *disjoint* subsets: 1) set of interior points of A, set of exterior points of A and 3) set of boundary points of A.

Definition 12.

- The set of all interior points of A is called the *interior* of A.

  Notation: $\operatorname{Int} A$.
- The set of all boundary points of A is called the *boundary* of A.

  Notation: ∂A.
- The set of all exterior points of A is called the *exterior* of A.

Example 8. Let $X = \mathbb{R}^2$ and $A = \{x^2 + y^2 < 1\}$ be an open disc. Then

- the interior points of A are those for which $x^2 + y^2 < 1$,
- the exterior points of A are those for which $x^2 + y^2 > 1$,
- the boundary points of A are those for which $x^2 + y^2 = 1$.

In particular, $\operatorname{Int} A = \{x^2 + y^2 < 1\}$ and $\partial A = \{x^2 + y^2 = 1\}$.

Example 9. Let $X = \mathbb{R}$, $A = [0, 1)$. Then the interior of $A = [0, 1)$ is $(0, 1)$, the exterior of A is $(-\infty, 0) \cap (1, +\infty)$, the boundary of A consists of two points 0 and 1.

Example 10. In a discrete space X, for every $A \subset X$ we have $\operatorname{Int} A = A$, $\partial A = \emptyset$ and the exterior of A is $X \setminus A$.

Less trivial examples:

Example 11. Let $X = \mathbb{R}$ and $A = \mathbb{Q} \subset \mathbb{R}$ (set of rational numbers). Then the interior and exterior of $\mathbb{Q}$ (in $\mathbb{R}$) are both empty, whereas $\partial\mathbb{Q} = \mathbb{R}$.

Example 12. Let $A = \{b\}$ be a one-point subset in $\mathbb{R}$ with the topology from Example 5 (open sets are $\mathbb{R}$, $\emptyset$ and $(a, +\infty)$, $a \in \mathbb{R}$). Then $\text{Int } A = \emptyset$ (Why?), the boundary of $A = \{b\}$ is $(-\infty, b]$, and the exterior is $(b, +\infty)$.

Notice the following obvious facts:

Remark 4. 1) The exterior of A is the interior of $X \setminus A$;
2) $\partial A = \partial(X \setminus A)$.

Proposition 2. *For any $A \subset X$, its interior* Int A *is open. Moreover,* Int A *can be characterized as the largest (with respect to inclusion) open set in X contained in A, i.e., an open set that contains any other open subset of A.*

Proof. Let $x \in \text{Int } A$, then x is an interior point of A, i.e., by definition there exist a neighborhood U of x such that $U \subset A$. Let $y \in U$ be any other point. Then U is a neighborhood of y which is entirely contained in A. Therefore, y is an interior point of A. Thus, $U \subset \text{Int } A$. Conclusion: Every point $x \in \text{Int } A$ is contained in Int A together with a certain neighborhood, so Int A is open.

Now let U be an arbitrary open set in X contained in A. If $x \in U$, then $x \in \text{Int } A$. In the definition of interior we can take U as a neighbourhood of x contained in A. Therefore $U \subset \text{Int } A$. Since U was arbitrary we see that the second part of our proposition holds, i.e. Int A is the largest open set in X contained in A. $\square$

Corollary 1. *For any $A \subset X$, its boundary ∂A is closed.*

Proof. The space X can be presented as the disjoint union of Int A, Int $(X \setminus A)$ and ∂A. In particular, $\partial A = X \setminus (\text{Int } A \cup \text{Int } (X \setminus A))$. Since, Int A and Int $(X \setminus A)$ are both open, so is $\text{Int } A \cup \text{Int } (X \setminus A)$. Thus ∂A is closed as the complement to the open set $\text{Int } A \cup \text{Int } (X \setminus A)$. $\square$

Corollary 2. *The exterior of a set is the largest open set disjoint with A. It is obvious that the exterior of A is* Int $(X \setminus A)$.

Proposition 3. *A set is open if and only if it coincides with its interior.*

Proof. If A is open, then each point $x \in A$ is in Int A (according to Remark 2). Thus, $A = \text{Int } A$.

Conversely, If $A = \text{Int } A$, then A is open since so is Int A (Proposition 2). $\square$

1.4 Closure, adherent, limit and isolated points

Definition 13. If (X, τ) is a topological space and $A \subset X$, then the closure of A, denoted by $\bar{A}$, is the smallest closed subset (with respect to inclusion) of X which contains A.

Note that A is closed if and only if $A = \bar{A}$.

Every subset of topological space has closure. It is the intersection of all closed sets containing this set:

$$\bar{A} = \bigcap \{C \subset X \ : \ A \subset C \text{ and } C \text{ is closed } \}.$$

Proposition 4. *If (X, τ) is a topological space with $A, B \subset X$, then*
(i) $\bar{\emptyset} = \emptyset$
(ii) $A \subset \bar{A}$
(iii) $\bar{\bar{A}} = \bar{A}$
(iv) $\overline{A \cup B} = \bar{A} \cup \bar{B}$
(v) *if $B \subset A$ then $\bar{B} \subset \bar{A}$.*

Proof: Exercise!

Theorem 1. *Given a topological space with $A \subset X$, then $x \in \bar{A}$ iff for each neighborhood U of x, $U \cap A \neq \emptyset$.*

Proof. $\Rightarrow$: Let $x \in \bar{A}$ and let U be an (open) neighborhood of x. By contradiction, assume $U \cap A = \emptyset$, then $A \subset X \setminus U$. Since $X \setminus U$ is closed, we conclude that $\bar{A} \subset X \setminus U$ and, consequently, $x \in X \setminus U$, thereby contradicting the assumption that $x \in U$. Thus, $U \cap A \neq \emptyset$

$\Leftarrow$: Assume that for each neighborhood U of x, we have $U \cap A \neq \emptyset$. Let (by contradiction) $x \notin \bar{A}$, i.e. $x \in X \setminus \bar{A}$. Since $X \setminus \bar{A}$ is open, $X \setminus \bar{A}$ can be treated as a neighborhood of x so that, by hypothesis, $(X \setminus \bar{A}) \cap A \neq \emptyset$, which is a contradiction (with the fact that $A \subset \bar{A}$). Thus, $x \in \bar{A}$. □

`Adherent, limit and isolated points`

Definition 14. A point $b \in X$ is an *adherent point* for a set A if each neighborhood of b intersects A.

A point $b \in X$ is a *limit point* of a set A, if each neighborhood of b intersects $A \setminus b$.

It is easy to see that every limit point of a set is its adherent point, but there are adherent points which are not limit ones. Example: Let $X = \mathbb{R}$ (with the standard topology) and $A = (0, 1) \cup \{2\}$. Then 2 is an adherent point for A, but not a limit one.

Definition 15. A point b is an isolated point of a set A if $b \in A$ and b has a neighborhood disjoint with $A \setminus b$.

The following statement is just the reformulation of Theorem 1.

Proposition 5. *The closure of a set A is the set of the adherent points of A.*

Here is a simple corollary of this fact.

Proposition 6. *A set A is closed if and only if A contains all of its limit points.* (*Similarly, A is closed if and only if A contains all of its adherent points.*)

Proof. Assume that A is closed. So $A = \bar{A}$. Proposition 5 together with the fact $A = \bar{A}$ implies that the set of adherent points equals A. Since every limit point of A is also an adherent point of A, it follows that A contains all of its limit points.

Now suppose that A contains all of its limit points. We will show that A^c is open and therefore conclude that A is closed. Let us suppose that A^c is not open. Then there exists $x \in A^c$ such that every open neighbourhood of x intersects A. Since $x \notin A$, it follows that every neighbourhood of x intersects $A \setminus \{x\}$. Therefore x is a limit point of A. However, this contradicts the fact that A contains all of its limit points. Therefore A^c is open and A must be closed.

□

Using Theorem 1, we now describe the relationship between the closure, interior and boundary of A.

First of all notice that any boundary point $x \in \partial A$ belongs to the closure $\bar{A}$. Indeed, any neighborhood of x intersects A (see Definition), so $x \in \bar{A}$ by Theorem 1 and therefore, $\partial A \subset \bar{A}$ and, in particular, $A \cup \partial A \subset \bar{A}$.

Next notice that $A \cup \partial A$ is closed as the complement of the open set $\mathrm{Int}\,(X \setminus A)$. Therefore, $\bar{A} \subset A \cup \partial A$.

Thus, we obtain

Proposition 7. $\bar{A} = A \cup \partial A$.

Equivalently, the closure of a set A is the complement of the exterior of A:

$$\bar{A} = X \setminus \mathrm{Int}\,(X \setminus A)$$

and also

$$\bar{A} = \mathrm{Int}\,A \cup \partial A$$

It is very important to note that all the above arguments and constructions apply to topological spaces of any nature.

Corollary 3. (1) *The boundary of a set A is the set $\bar{A} \setminus \operatorname{Int} A$.*

(2) *A set A is closed if and only if $\partial A \subset A$.*

(3) *The boundary of a set A equals the intersection of the closure of A and the closure of the complement of A: $\partial A = \bar{A} \cap \overline{(X \setminus A)}$.*

1.5 Everywhere dense and nowhere dense sets

`Everywhere Dense Sets`

Definition 16. A set $A \subset X$ is called *everywhere dense* if $\bar{A} = X$.

Proposition 8. *A set is everywhere dense if and only if it intersects any nonempty open set.*

Example 13. The set $\mathbb{Q}$ is everywhere dense in $\mathbb{R}$.

`Nowhere Dense Sets`

Definition 17. A set $A \subset X$ is nowhere dense if its exterior is everywhere dense.

Proposition 9. *A set A is nowhere dense in X if and only if each neighborhood of each point $x \in X$ contains a point y such that the complement of A contains y together with a neighborhood of y.*

Example 14. (1) $\mathbb{R}$ is nowhere dense in $\mathbb{R}^2$; (2) $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$.

1.6 Continuous maps

`Set-Theoretic Digression: Maps and Main Classes of Maps`

A map f of a set X to a set Y is a triple consisting of X, Y, and a rule, which assigns to every element of X exactly one element of Y. There are other words with the same meaning: mapping, function, etc. If f is a map of X to Y, then we write $f : X \to Y$, or $X \xrightarrow{f} Y$.

The element $b \in Y$ assigned by f to an element a of X is denoted by $f(a)$ and called the image of a under f. We write $b = f(a)$, or $a \overset{f}{\mapsto} b$, or $f : a \mapsto b$.

A map $f : X \to Y$ is

- a *surjective map*, or just a *surjection* if every element of Y is the image of at least one element of X.

- an *injective map*, *injection*, or one-to-one map if every element of Y is the image of at most one element of X.

- a *bijective map*, *bijection*, or *invertible map* if it is both surjective and injective.

`Image and Preimage`

The image of a set $A \subset X$ under a map $f : X \to Y$ is the set of images of all points of A. It is denoted by $f(A)$. Thus $f(A) = \{f(x) \mid x \in A\}$.

The image of the entire set X (i.e., the set $f(X)$) is the image of f, it is often denoted by $\operatorname{Im} f$.

The preimage of a set $B \subset Y$ under a map $f : X \to Y$ is the set of elements of X with images in B. It is denoted by $f^{-1}(B)$. Thus $f^{-1}(B) = \{a \in X \mid f(a) \in B\}$.

Be careful with these terms: their etymology can be misleading. For example, the image of the preimage of a set B can differ from B. And even if it does not differ, it may happen that the preimage is not the only set with this property. Hence, the preimage cannot be defined as a set whose image is the given set.

`Composition`

The *composition* of maps $f : X \to Y$ and $g : Y \to Z$ is the map $g \circ f : X \to Z$: $x \mapsto g\big(f(x)\big)$.

`Inverse and Invertible Map`

A map $g : Y \to X$ is *inverse* to a map $f : X \to Y$ if $g \circ f = \operatorname{id} X$ and $f \circ g = \operatorname{id} Y$. A map having an inverse map is (called) invertible.

A map f is invertible iff it is a bijection. Indeed, if $f : X \to Y$ is a bijection, then every $y \in Y$ has exactly one preimage $x \in X$ (i.e., such that $f(x) = y$) and to define the inverse map $f^{-1} : Y \to X$ we simply put $f^{-1}(y) = x$. If an inverse map exists, then it is unique.

`Continuous maps and continuity`

Continuity is the central concept of topology. Essentially, topological spaces have the minimum necessary structure to allow a definition of continuity. Continuity in almost any other context can be reduced to this definition by an appropriate choice of topology.

There are two equivalent ways to define the notion of a continuous map.

Definition 18. A map $f : X \to Y$ is continuous at $x \in X$ if for any open neighborhood B of $f(x)$, there is a neighborhood A of x such that $f(A) \subset B$.

The map $f : X \to Y$ itself is continuous if and only if it is continuous at all points $x \in X$.

Definition 19. A map $f : X \to Y$ is continuous if and only if for every open set B in Y, its inverse $f^{-1}(B)$ is also an open set.

Proposition 10. *These definitions are equivalent.*

Proof. Definition 18 $\Rightarrow$ Definition 19:

Assume that $f : X \to Y$ is continuous at every point $x \in X$. Let B be an open set in Y and $A = f^{-1}(B)$ be its preimage in X. Take any point $x \in A = f^{-1}(B)$, and let $y = f(x) \in B$. Since B is open, it can be treated as a neighborhood of y.

As f is continuous at x, there is a neighborhood U of x such that $f(U) \subset B$, hence $U \subset f^{-1}(B) = A$. Thus A contains a neighborhood U of x. Since x was an arbitrary point of A, it follows that A is open and hence f is continuous in the sense of Definition 19.

Definition 19 $\Rightarrow$ Definition 18:

Assume that $f : X \to Y$ is continuous in the sense of Definition 19.

Consider an arbitrary point $x \in X$, its image $y = f(x) \in Y$ and an arbitrary neighborhood B of y. Since f is continuous, we conclude that $f^{-1}(B)$ is open in X and hence a neighborhood of x. Since $f(f^{-1}(B)) \subset B$, it follows that f is continuous at x in the sense of Definition 18. $\square$

Example 15. 1) Let $X = \mathbb{R}$ with the standard topology. Then the continuous maps $f : X \to X$ are just usual continuous maps (in the sense of Calculus).

2) Let X have the discrete topology. Then a map $f : X \to Y$ is continuous for any topology on Y.

3) Let Y have the trivial (indiscrete) topology. Then a map $f : X \to Y$ is continuous for any topology on X.

4) Any constant map $f : X \to Y$, $f(x) = y_0 \in Y$, where y_0 is a fixed element, is continuous for any topologies on X and Y.

5) The identity map $f : X \to X$, $f(x) = x$, is continuous for any topology on X.

6) Note that in the latter example we consider the same topology on the both X's. If we consider $f : (X, \tau_1) \to (X, \tau_2)$, $f(x) = x$, then f may appear to be discontinuous, if $\tau_1 \neq \tau_2$. Indeed, let $X_1 = \mathbb{R}$ with the standard topology and $X_2 = \mathbb{R}$ with the discrete topology. Then the "identity" map $f : X_1 \to X_2$ is not continuous (a single point $x_0 \in X_2$ is open, but its preimage $f^{-1}(\{x_0\}) = \{x_0\}$ is not open in X_1).

`Simplest properties of continuous maps`

Proposition 11. *If two maps $f : X \to Y$ and $g : Y \to Z$ are continuous, then their composition $g \circ f : X \to Z$ is continuous.*

Proof. Let B be an open set in Z. Then its preimage $g^{-1}(B)$ is open in Y (since g is continuous) and the preimage of this preimage $f^{-1}(g^{-1}(B))$ is open in X (since f is continuous). It remains to notice that $f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B)$. Thus, the preimage of B under the composition $g \circ f$ is open and, therefore, $g \circ f$ is continuous by Definition 19. $\square$

Proposition 12. *1) Let A be a subspace of X. The inclusion $i : A \to X$ is continuous.*

2) Let $f : X \to Y$ be a continuous map and $A \subset X$. The restriction $f|_A : A \to Y$ is continuous.

Proof. Exercise!

Proposition 13. *Let $f : X \to Y$ be continuous map and $B \subset Y$ be closed. Then its preimage $f^{-1}(B)$ is closed in X.*

Proof. Since B is closed its complement is open. Therefore by continuity $f^{-1}(B^c)$ is open. The result now follows upon observing $f^{-1}(B)^c = f^{-1}(B^c)$.

Example 16. Is it true that the image of any open (resp. closed) subset under a continuous map is necessarily open (resp. closed)? The answer is negative.

Consider the map $\sin : \mathbb{R} \to \mathbb{R}$. The image of an open interval $(-2\pi, 2\pi)$ is $[-1, 1]$ (which is not open).

Consider the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{1+x^2}$. Then the image of the closed set $[-1, +\infty)$ is $(0, 1]$ (which is not closed).

Proposition 14. *The image of an everywhere dense set under a surjective continuous map is everywhere dense.*

Proof. Let $A \subset X$ be everywhere dense and $f : X \to Y$ be a surjective continuous map. We are going to prove that $f(A)$ is everywhere dense in Y. Assume the contrary, i.e. that $f(A)$ is not everywhere dense. Then there exist a non-empty open set $V \subset Y$ such that, $V \cap f(A) = \emptyset$. Consider the preimage $f^{-1}(V)$, it is a non-empty open subset in X (since f is continuous and surjective). Because A is everywhere dense, there exists a point $x \in A \cap f^{-1}(V)$. Then $f(x) \in f(A) \cap V = \emptyset$. Contradiction. $\square$

Example 17. *Is it true that the image of nowhere dense set under a continuous map is nowhere dense? The answer is negative. Consider, for example, the map $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x$ (in other words, f is the projection of $\mathbb{R}^2$ onto the horizontal line). Take $A = \{y = 0\} \subset \mathbb{R}^2$. Obviously, A is nowhere dense as a subset of $\mathbb{R}^2$. But its image $f(A)$ is the whole line $\mathbb{R}$ (which is, of course, everywhere dense in itself).*

`Reformulation for metric spaces`

Let X and Y be two metric spaces, $a \in X$. A map $f : X \to Y$ is continuous at a iff for every ball with center at $f(a)$ there exists a ball with center at a whose image is contained in the first ball.

Let X and Y be two metric spaces. A map $f : X \to Y$ is continuous at a point $a \in X$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every point $x \in X$ the inequality $d(x, a) < \delta$ implies $d(f(x), f(a)) < \varepsilon$.

This means that the definition of continuity usually studied in Calculus, when applicable, is equivalent to the above definition stated in terms of topological structures.

`Properties of Continuous Functions`

Let $f, g : X \to \mathbb{R}$ be continuous. The maps $X \to \mathbb{R}$ defined by formulas
$x \mapsto f(x) + g(x)$,
$x \mapsto f(x)g(x)$,
$x \mapsto f(x) - g(x)$,
$x \mapsto |f(x)|$,
$x \mapsto \max\{f(x), g(x)\}$,
$x \mapsto \min\{f(x), g(x)\}$
are continuous.

In addition, if $0 \notin g(X)$, then the map $X \to \mathbb{R} : x \mapsto f(x)/g(x)$ is continuous.

`Continuity of Distances`

For every point x_0 of a metric space X, the function $X \to \mathbb{R} : x \mapsto d(x, x_0)$ is continuous.

The same is true if we replace x_0 by a subset $A \subset X$. In this case the distance $d(x, A)$ between x and A is defined by $d(x, A) = \inf_{y \in A} d(x, y)$.

1.7 Homeomorphism, topological properties

Definition 20. Let X and Y be topological spaces. A map $f : X \to Y$ is called a *homeomorphism* if it satisfies the following three properties:

- f is bijective,
- f is continuous,
- f^{-1} is continuous.

Definition 21. Topological spaces X and Y are called *homeomorphic* if there exists a homeomorphism $f : X \to Y$.

`Simplest properties of homeomorphisms.`

Proposition 15. • *The identity map of a topological space* id $: X \to X$ *is a homeomorphism;*

- *The inverse map of a homeomorphism is a homeomorphism (i.e., if $f : X \to Y$ is homeomorphism, then $f^{-1} : Y \to X$ is a homeomorphism);*

- *The composition of homeomorphisms is a homeomorphism (i.e., if $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then $g \circ f : X \to Z$ is a homeomorphism).*

Corollary 4. *"Being homeomorphic" is an equivalence relation. In other words:*

1) X is homeomorphic to itself;

2) if X is homeomorphic to Y, then Y is homeomorphic to X;

3) if X is homeomorphic to Y and Y is homeomorphic to Z, then X is homeomorphic to Z.

Example 18. An open interval (a, b) is homeomorphic to $\mathbb{R}$. As an example of a homeomorphism between them we can take

$$f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}, \qquad f(x) = \tan x = \frac{\sin x}{\cos x}.$$

Obviously, f is a bijection and the inverse map is $f^{-1} = \tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$. Both f and f^{-1} are continuous.

Example 19. There are continuous bijections $f : X \to Y$ with the discontinuous inverse $f^{-1} : Y \to X$. As an example consider

$$f : [0, 1) \to S^1 \text{ (circle)}, \qquad f(x) = e^{2\pi i x},$$

Here we consider S^1 as the unit circle in $\mathbb{C}$: $S^1 = \{z \in \mathbb{C} \ : \ |z| = 1\}$. This f is a continuous bijection, but not a homeomorphism.

A homeomorphism $f : X \to Y$ satisfies the following fundamental property: if $A \subset X$ is open, then $f(A) \subset Y$ is open and vice versa if $B \subset Y$ is open, then $f^{-1}(B)$ is open in X. This means that f establishes a bijection not only between X and Y but also between the topological structures on X and Y. In other words, homeomorphic space are topologically isomorphic. In particular, the following (obvious) statement holds.

Proposition 16. *Let $f : X \to Y$ be a homeomorphism. Then for every $A \subset X$:*

- *A is closed iff $f(A)$ is closed,*

- $f(\bar{A}) = \overline{f(A)}$,
- $f(\text{Int } A) = \text{Int } f(A)$,
- $f(\partial A) = \partial f(A)$,
- *A is a neighborhood of $x \in X$ iff $f(A)$ is a neighborhood of $f(x) \in Y$,*
- *etc*

`Homeomorphism Problem and Topological Properties`

One of the classical problems in topology is the *homeomorphism problem*: to find out whether two given topological spaces are homeomorphic. In each special case, the character of solution depends mainly on the answer. In order to prove that two spaces are homeomorphic, it suffices to present a homeomorphism between them. Essentially this is what one usually does in this case. To prove that two spaces are not homeomorphic, it does not suffice to consider any special map, and usually it is impossible to review all the maps. Therefore, for proving the nonexistence of a homeomorphism one uses indirect arguments. In particular, we can find a property or a characteristic shared by homeomorphic spaces and such that one of the spaces has it, while the other does not. Properties and characteristics that are shared by homeomorphic spaces are called *topological properties and invariants*. Obvious examples are the cardinality (i.e., the number of elements) of the set of points and the set of open sets. Less obvious properties are the main object of the next lectures.

1.8 Compactness

This section is devoted to a topological property playing a very special role in topology and its applications. It is a sort of topological counterpart for the property of being finite in the context of set theory.

We all recall the important and useful theorem from calculus, that functions which are continuous on a closed and bounded interval take on a maximum and minimum value on that interval. The classic theorem of Heine-Borel asserts that every covering of such an interval by open sets has a finite subcover. In this section, we use this feature of closed and bounded subsets to define the corresponding notion, compactness, in a general topological space.

Definition 22. Let X be a topological space. A cover for X is a family of subsets $U = \{U_\alpha \ : \ \alpha \in I\}$ of X such that $X = \cup_{\alpha \in I} U_\alpha$. The case when all U_α are open is very important. In this situation, we say that U is an open cover. A subfamily $V \subset U$ is called a *subcover* (or, *subcovering*) if V still covers X, i.e. $X = \cup_{U_{\alpha_i} \in V} U_{\alpha_i}$.

Similarly, if $A \subset X$ is a subset then a cover for A is a family of subsets $U = \{U_\alpha \ : \ \alpha \in I\}$ of X such that $A \subset \cup_{\alpha \in I} U_\alpha$; and $V \subset U$ is a subcover for A if $A \subset \cup_{U_{\alpha_i} \in V} U_{\alpha_i}$.

Definition 23. A topological space X is *compact* if each open cover of X contains a finite subset that also covers X (i.e., a finite subcovering).

Example 20. 1. Any finite space is compact.

2. An indiscrete space is compact.

3. The line $\mathbb{R}$ is not compact. An open cover which does not admit any finite subcover is, for example, $U = \{(-n, n) \ : \ n \in \mathbb{N}\}$

4. The open interval $(0, 1)$ is not compact. Indeed, the infinite family of open intervals $(\frac{1}{n}, 1)$ covers $(0, 1)$, i.e. $(0, 1) = \cup_{n=1}^{\infty} (\frac{1}{n}, 1)$, but we can't choose any finite subcover.

Theorem 2. (Heine–Borel) *A closed interval* $[a, b]$ *is compact.*

Proof. Consider an open cover $U = \{U_\alpha, \alpha \in I\}$ and define a subset $C \subset [a, b]$:

$$C = \{x \in [a, b] \ : \ [a, x] \text{ admits a finite subcover}\}$$

Obviously, C is not empty since $a \in C$ (the point a is covered by a certain $U_\alpha \in U$) and C is bounded. Consider the least upper bound of C:

$$y = \sup C$$

We are going to show that $y \in C$ and, in fact, $y = b$ so that $[a, b]$ admits a finite subcover.

There is an element U_β which covers y. Since U_β is open there exists $\epsilon > 0$ such that $(y - \epsilon, y + \epsilon) \subset U_\beta$. Since $y = \sup C$, there is $x \in C$ such that $y - \epsilon < x \leq y$. The interval $[a, x]$ admits a finite subcovering $U_{\alpha_1}, \ldots, U_{\alpha_n}$. Then by adding U_β to this collection $U_{\alpha_1}, \ldots, U_{\alpha_n}$, we get a finite subcover for $[a, y]$ so that $y \in C$. Moreover, if $y < b$, then in fact $U_\beta, U_{\alpha_1}, \ldots, U_{\alpha_n}$ is a finite subcover for $[a, y + \epsilon/2]$ so that $y + \epsilon/2 \in C$ and y cannot be $\sup C$. Thus, the only possibility is $y = b$, i.e. $[a, b]$ admits a finite subcover and $[a, b]$ is compact as was to be proved. □

1.9 Properties of compact spaces

Proposition 17. *If a subset X of a Euclidean space $\mathbb{R}^n$ (equipped with the usual topology) is compact, then X is bounded and closed.*

Proof. Consider the cover of $\mathbb{R}^n$ by the balls $B_n = \{x \in \mathbb{R}^n \mid |x| < n, n \in \mathbb{Z}$. Since X is compact, we can find a finite subcover so that $X \subset \cup_{n=1}^{N} B_n = B_N$ and therefore X is bounded.

To prove that X is closed, assume the contrary, i.e. X is not closed. This means that there exists a limit point $a \in \mathbb{R}^n$ of X such that $a \notin X$. For any point $x \in X$ we can find disjoint neighborhoods U_x of x and V_x of a. The neighborhoods U_x (x is varying throughout X) form an open cover of X. Since X is compact, we can choose a finite subcover $U_{x_1}, \dots, U_{x_N}$ so that $X \subset U = U_{x_i} \cup \dots \cup U_{x_N}$. Consider $V = V_{x_1} \cap \dots \cap V_{x_N}$. Obviously, V is a neighborhood of a. Since $U_{x_i} \cap V_{x_i}$ do not intersect, we conclude that $U \cap V = \emptyset$. Therefore $X \cap V = \emptyset$ and a cannot be a limit point of X (since a admits a neighborhood which does not intersect X). □

Exercise. A compact subset of a metric space is bounded.

It turns out that the converse is also true: the above two conditions, i.e. being bounded and closed, are sufficient for $X \subset \mathbb{R}^n$ to be compact.

Theorem 3. *A subset X of a Euclidean space $\mathbb{R}^n$ (equipped with the usual topology) is compact if and only if it is closed and bounded.*

Proposition 18. *A closed subset A of a compact space X is compact.*

Proof. Let U be any open cover of A, then $U' = \{U, X \setminus A\}$ is an open cover for X. Since X is compact, we can choose a finite subcover for X. Obviously, the same finite subcover (the set $X \setminus A$ should be excluded) can be considered as a finite subcover for A. So A is compact.

□

Theorem 4. *A continuous image of a compact space is compact.* (*In other words, if X is a compact space and $f : X \to Y$ is a continuous map, then $f(X)$ is compact.*)

Proof. Consider an arbitrary open cover $U = \{U_\alpha, \alpha \in I\}$ of $f(X) \subset Y$. Then its "preimage" $V = \{V_\alpha = f^{-1}(U_\alpha), \alpha \in I\}$ is an open cover of X. Since X is compact we can choose a finite subcover $V_{\alpha_1}, \dots, V_{\alpha_n}$:

$$X \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$$

Then

$$f(X) \subset f(V_{\alpha_1}) \cup \dots \cup f(V_{\alpha_n}) \subset U_{\alpha_1} \cup \dots \cup U_{\alpha_n},$$

i.e., $U_{\alpha_1}, \dots, U_{\alpha_n}$ is a finite subcover for $f(X)$ and, therefore, $f(X)$ is compact, as needed. □

Theorem 5. *A continuous real valued function on a compact space is bounded and attains its maximal and minimal values.* (*In other words, if X is a compact space and $f : X \to \mathbb{R}$ is a continuous function, then there exist $x_1, x_2 \in X$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for every $x \in X$, i.e. $a = f(x_1) = \min_{x \in X} f(x)$, $b = f(x_2) = \max_{x \in X} f(x)$.*)

Proof. Consider the image $f(X) \subset \mathbb{R}$ of X under f. By Theorem 4, $f(X)$ is compact and, consequently, bounded and closed (Theorem 3). Thus, f is bounded.

Consider $a = \inf f(X)$ and $b = \sup f(X)$. It is easy to see that a and b are adherent points of $f(X)$. Indeed, by definition, for any $\epsilon > 0$ there exists $y \in f(X)$ such that $y \in (b - \epsilon, b]$ (so that every neighborhood of b contains points of $f(X)$, i.e., b is adherent for $f(X)$). Since $f(X)$ is closed, it contains all of its adherent points. In particular, $a, b \in f(X)$, i.e., there exist $x_1, x_2 \in X$ such that $a = f(x_1)$ and $b = f(x_2)$. Hence,

$$f(x_1) \leq f(x) \leq f(x_2) \quad \text{for every} \quad x \in X,$$

as was to be proved. $\square$

Notice that for non-compact topological spaces, the statement of Theorem 5 is, in general, false.

Example 21. Let $X = (0, 1)$ and $f : (0, 1) \to \mathbb{R}$ is defined by $f(x) = \frac{1}{1-x}$. This function is continuous on $(0, 1)$, but not bounded.

Another example: $f : (0, 1) \to \mathbb{R}$ is defined by $f(x) = x + 5$. Then $\inf_{x \in (0,1)} f(x) = 5$ and $\sup_{x \in (0,1)} f(x) = 6$ but none of them is attained on $(0, 1)$ (since $0 = f^{-1}(5) \notin (0, 1)$, $1 = f^{-1}(6) \notin (0, 1)$.

1.10 Compactness for metric spaces

The notion of compactness we have seen for topological spaces is defined using open covers. As we will see the following notion is equivalent for metric spaces and in certain cases is more useful.

Definition 24. Let (X, d) be a metric space. We say that (X, d) is sequentially compact if every sequence admits a convergent subsequence, i.e. for every $(x_n)_{n=1}^{\infty}$ there exists $(x_{n_k})_{k=1}^{\infty}$ and x such that $\lim_{k \to \infty} d(x_{n_k}, x) = 0$.

Proposition 19. *Let (X, d) be a metric space. Then (X, d) is compact if and only if (X, d) is sequentially compact.*

Proof. We will split the proof of this theorem into two parts.

Compactness implies sequential compactness. Assume that (X, d) is compact. Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence with no convergence subsequence. Then for any $x \in X$ there exists $\epsilon_x > 0$ such that $B(x, \epsilon_x)$ contains finitely many terms in $(x_n)_{n=1}^{\infty}$. Otherwise for any $\epsilon > 0$ there exists arbitrarily large n such that $x_n \in B(x, \epsilon)$. This fact can be used to construct a convergent subsequence of $(x_n)_{n=1}^{\infty}$ that converges to x. The collection $\{B(x, \epsilon_x)\}_{x \in X}$ form an open cover of X and therefore admits a finite subcover. However, since $(x_n)_{n=1}^{\infty}$ contains infinitely many terms in X there must be an element of this subcover which contains infinitely many terms. This contradicts the definition of ϵ_x. Therefore $(x_n)_{n=1}^{\infty}$ must have a convergent subsequence.

Sequential compactness implies compactness. We start by proving the following claim: If (X, d) is sequentially compact and $\{U_i\}$ is an open cover of X, then there exists $\epsilon > 0$ such that for any $x \in X$ the open ball $B(x, \epsilon)$ is contained in U_i for some i.

Suppose this claim is false, then for any $n \in \mathbb{N}$ we can $x_n \in X$ such that $B(x_n, 1/n)$ is not contained in any U_i. By sequential compactness we can find $x \in X$ and a subsequence (x_{n_k}) such that $\lim_{k\to\infty} x_{n_k} = x$. Replacing (x_n) with (x_{n_k}) we can assume that $\lim_{n\to\infty} x_n = x$ and $B(x_n, 1/n)$ is not contained in a U_i for any $n \in \mathbb{N}$. The x we have constructed lies in some U_i and U_i is open. Therefore there exists $m \in \mathbb{N}$ such that $B(x, 1/m) \subset U_i$. Since $\lim_{n\to\infty} x_n = x$ there exists N such that for all $n \geq N$ we have $d(x_n, x) < 1/2m$. Therefore if we pick $n \geq \max\{N, 2m\}$ we have

$$B(x_n, 1/n) \subseteq B(x_n, 1/2m) \subseteq B(x, 1/m) \subset U_i.$$

This contradicts the definition of (x_n) so the claim holds.

Let $\epsilon > 0$ be in the claim above. We now claim that we can cover X with finitely many balls of radius ϵ. If this is not possible then define the sequence $(x_n)_{n=1}^{\infty}$ recursively as follows. Let x_1 be arbitrary. Suppose we have defined $\{x_1, \ldots, x_n\}$ for some $n \in \mathbb{N}$ and $d(x_l, x_m) \geq \epsilon$ for all $1 \leq l < m \leq n$. Since $\cup_{n=1}^{\infty} B(x_n, \epsilon)$ does not cover x, there exists $x_{n+1} \in X$ such that $d(x_{n+1}, x_m) \geq \epsilon$ for all $1 \leq l \leq n$. Clearly we can repeat this process indefinitely and in doing so define sequence $(x_n)_{n=1}^{\infty}$. However, this (x_n) has the property that $d(x_n, x_m) \geq \epsilon$ for distinct $n, m \in \mathbb{N}$. Therefore this sequence has no convergent subsequence. This contradicts our sequential compactness assumption and completes our proof of the claim.

Now let $\{U_i\}$ be an arbitrary open cover of X and less $\epsilon > 0$ be as in the first claim. By the second claim we can cover X with finitely many balls of radius ϵ. By the first claim each of these is contained in U_i for some i and so X can be covered by finitely many elements of $\{U_i\}$. $\square$

The parameter $\epsilon > 0$ introduced in the first claim in the proof above is known as the Lebesgue number of a cover.

With Proposition (19) in mind, when speaking about metric spaces we will use the terms compact and sequentially compact interchangeably.

1.11 Connectedness

Speaking informally, a topological space X is connected if it "consists of just one piece".

The precise definition is as follows.

Definition 25. A topological space X is called *connected* if X has no partition into two disjoint non-empty open subsets. (In other words, X cannot be presented as $X = A \cup B$ where A, B are non-empty open subsets and $A \cap B = \emptyset$).

X is said to be *disconnected*, if there exist open non-empty disjoint sets A and B such that $X = A \cup B$.

Notice that, since these sets A and B are complements of one another, they are both closed as well as both open. This observation immediately implies the following reformulation of the definition.

Proposition 20. *A topological space X is connected iff X and $\emptyset$ are the only subsets which are clopen (i.e., both closed and open).*

Example 22. (1) An indiscrete space is connected.

(2) A discrete space X cannot be connected unless it consists of one element only.

(3) The subspace $\mathbb{Q}$ of $\mathbb{R}$ is not connected because $\mathbb{Q} = A \cup B$, where $A = (\infty, \sqrt{2}) \cap \mathbb{Q}$, $B = (\sqrt{2}, +\infty) \cap \mathbb{Q}$, is a partition.

(4) The set $X = [0, 1] \cup (2, 3) \subset \mathbb{R}$ (as a topological space with the induced topology) is disconnected. Indeed, in the induced topology $[0, 1]$ and $(2, 3)$ are both open and give a partition of X.

Theorem 6. $\mathbb{R}$ *is connected.*

Proof. (by contradiction) Assume that $\mathbb{R}$ admits a partition into two disjoint open non-empty sets $\mathbb{R} = A \cup B$. Let $a \in A$ and $b \in B$. For definiteness, we assume that $a < b$. Consider the set

$$C = \{x \in \mathbb{R} \mid [a, x] \subset A\}.$$

This set is not empty (since $a \in C$) and bounded from above (since $x < b$ for all $x \in C$).

Let $y = \sup C$. Since A and B give a partition of $\mathbb{R}$, this point y belongs to either A or B. Consider these two possibilities.

Assume $y \in A$. Since A is open, there is a neighborhood $U_y = (y - \epsilon, y + \epsilon)$ such that $U_y \subset A$. Then obviously $x = y + \epsilon/2 \in C$ so that y cannot be $\sup C$. Thus, $y \notin A$.

Assume $y \in B$. Since B is open, there is a neighborhood $U_y = (y - \epsilon, y + \epsilon)$, such that $U_y \subset B$. This neighborhood contains no points of C. In other words, C is located on the left of U, i.e. $x \leq y - \epsilon$ for all $x \in C$ and, in particular, $\sup C \leq y - \epsilon$ so that $y \neq \sup C$. Thus, $y \notin B$.

We see that $y = \sup C$ belongs to neither A, nor B. This contradicts the fact that $\mathbb{R} = A \cup B$. □

Exercise. Prove that $A \subset \mathbb{R}$ is connected iff A is an interval.

Note. By an interval in $\mathbb{R}$, we mean any subset I such that whenever $a < b < c$ and whenever $a \in I$ and $c \in I$ then $b \in I$. It is routine to check that the only ones are

$$(a,b), [a,b], [a,b), (a,b], [a,\infty), (a,\infty), (-\infty,b), (\infty,b], (-\infty,\infty) = \mathbb{R}$$

and $\{a\}$ for real $a, b, a < b$ where appropriate. It turns out that these are exactly the connected subsets of $\mathbb{R}$.

Theorem 7. *Connectedness is preserved by continuous maps. In other words, if X is connected and $f : X \to Y$ is a continuous map, then the image $f(X)$ is connected.*

Proof. (by contradiction) Assume that $f(X)$ is disconnected, i.e. that $f(X) = A_1 \cup B_1$ where A_1 and B_1 are disjoint non-empty and open (in the induced topology). This means, in particular, that there are open sets $A, B \subset Y$ such that $A_1 = A \cap f(X)$, $B_1 = B \cap f(X)$. Since f is continuous, the preimages $f^{-1}(A)$ and $f^{-1}(B)$ are open in X. Thus, we have a partition $X = f^{-1}(A) \cup f^{-1}(B)$ of X into non-empty disjoint open sets. This contradicts to the fact that X is connected. □

Corollary 5. (Intermediate Value Theorem) *Let $f : X \to \mathbb{R}$ be a continuous function on a connected space X and $a = f(x_1), b = f(x_2)$, $a < b$, $x_1, x_2 \in X$. Then for any $c \in (a,b)$ there exists $x_3 \in X$ such that $f(x_3) = c$. In other words, if a and b are values of a continuous function f, then f takes all intermediate values between a and b.*

Proof. (by contradiction) If $c \notin f(X)$, then the sets $A = f^{-1}(-\infty, c)$ and $B = f^{-1}(c, +\infty)$ form a partition of X into non-empty disjoint open sets. This contradicts the connectedness of X. $\square$

Corollary 6. (Fixed point theorem) *If $f : [0,1] \to [0,1]$ is continuous, then it has a fixed point, i.e. there exists some $x \in [0,1]$ such that $f(x) = x$.*

Proof Consider $g(x) = f(x) - x$. Then $g : [0,1] \to \mathbb{R}$ is continuous. Further, $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$ so that 0 is intermediate between $g(0)$ and $g(1)$. Thus, by the Intermediate Value Theorem, there exists $x \in [0,1]$ such that $0 = g(x) = f(x) - x$ i.e. such that $f(x) = x$.

Note: Given continuous $h : [a,b] \to [a,b]$, it follows that h has a fixed point since $[a,b] \simeq [0,1]$ (are homeomorphic) and "every continuous function has a fixed point" is a topological property.

Exercise. Let X be disconnected and $X = A \cup B$ be a partition of X. If Y is any connected subset of X, then $Y \subset A$ or $Y \subset B$.

1.12 Pathwise Connectedness

Definition 26. A topological space (X, τ) is *pathwise connected* iff for any $x, y \in X$, there exists a continuous map $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. Such a map f is called a *path* from x to y.

Example 23. (1) An indiscrete space is pathwise connected (because any map $f : [0,1] \to X$ is continuous).

(2) A discrete space is not pathwise connected (unless it consists of a single point).

(3) $\mathbb{R}$ is pathwise connected.

(4) Every convex set $X \subset \mathbb{R}^n$ is pathwise connected.

Exercise. Let X be a discrete space. Prove that a continuous map $f : [0,1] \to X$ is constant, i.e. the interval $[0,1]$ is mapped entirely to a certain point $x_0 \in X$: $f([0,1]) = x_0 \in X$.

Theorem 8. *Every pathwise connected space is connected.*

Proof. (by contradiction) Assume that X is pathwise connected, but disconnected. Consider a partition of X into two disjoint non-empty open subsets: $X = A \cup B$. Take $x \in A$ and $y \in B$ and a continuous path $f : [0,1] \to X$ from x to y. Then $[0,1] = f^{-1}(A) \cup f^{-1}(B)$ is a partition of $[0.1]$ into two disjoint non-empty open sets, which is impossible since $[0,1]$ is connected. $\square$

Example 24. The converse is false: there are connected spaces which are not pathwise connected. Consider the following example, the topologist's sine curve:

$$X = X_1 \cup X_2 = \{y = \sin\frac{1}{x}\ :\ x > 0\} \cup \{(0, y)\ :\ y \in [-1, 1]\}.$$

X is a connected space, but no path can be found from $(0, 0)$ to any point on the graph $\{y = \sin\frac{1}{x}, x > 0\}$.

Notice that the natural partition $X = X_1 \cup X_2$ does not satisfy the conditions from the definition of a connected space: X_2 is not open. To see this, notice that X_1 is not closed! Indeed, any point $p \in X_2$ is a limit point for X_1: in every neighborhood of such a point (i.e., in a two-dimensional disc $B(p, \epsilon)$) there are points of X_1.

Theorem 9. *Pathwise connectedness is preserved by continuous maps. In other words, if X is pathwise connected and $f : X \to Y$ is a continuous map, then the image $f(X)$ is pathwise connected.*

Proof. Let $y_1, y_2 \in f(X)$. Take $x_1, x_2 \in X$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since X is pathwise connected, there is a continuous path $\gamma : [0, 1] \to X$ from x_1 to x_2. Consider the composition $f \circ \gamma : [0, 1] \to Y$. Obviously, $f \circ \gamma$ is a continuous path between y_1 and y_2 in the image $f(X)$. Thus, $f(X)$ is pathwise connected. □

The next example shows how the concepts of connectedness and compactness can be used to distinguish non-homeomorphic spaces.

Proposition 21. *The intervals (of three different types) $X_1 = (0, 1)$, $X_2 = [0, 1)$ and $X_3 = [0, 1]$ are not homeomorphic.*

Proof. $X_3 = [0, 1]$ is compact whereas $X_1 = (0, 1)$, $X_2 = [0, 1)$ are not (X_1 and X_2 are not closed as subsets of $\mathbb{R}$). Since "compactness" is a topological property, we conclude that X_3 is homeomorphic neither to X_1, nor X_2.

To show that $[0, 1)$ is not homeomorphic to $(0, 1)$, (by contradiction) we assume that a homeomorphism $f : [0, 1) \to (0, 1)$ exists. Consider the image of 0. This is a certain point $a = f(0) \in (0, 1)$. Since f is a bijection, we have

$$f(0) = a \qquad \text{and} \qquad f\big(X_2 \setminus \{0\}\big) = X_1 \setminus \{a\}$$

But $X_2 \setminus \{0\} = (0, 1)$ is connected, whereas $X_1 \setminus \{a\} = (0, a) \cup (a, 1)$ is disconnected. This contradicts to the fact that connectedness is preserved under continuous maps. □

Comment. In this proof we used the following obvious general principle: if $f : X \to Y$ is a homeomorphism and $A \subset X$, then A is homeomorphic to $f(A)$ and $X \setminus A$ is homeomorphic to $Y \setminus f(A)$.

1.13 Hausdorff spaces

Let X be a topological space.

Definition 27. X is a *Hausdorff* topological space, if for any two distinct points $x, y \in X$ there exist neighborhoods $U(x)$ and $V(y)$ which are disjoint, i.e., $U(x) \cap V(y) = \emptyset$.

Examples of Hausdorff topological spaces are:

- Real line $\mathbb{R}$ with the standard topology; indeed if $x, y \in \mathbb{R}$ and $x \neq y$, then the neighborhoods $U_{\epsilon/3}(x) = (x - \epsilon/3, x + \epsilon/3)$ and $V_{\epsilon/3}(y) = (y - \epsilon/3, y + \epsilon/3)$ are disjoint, if we take $\epsilon = |x - y| = \mathrm{dist}(x, y)$.
- Euclidean space $\mathbb{R}^n$
- Any metric space (X, d). The reason is just the same as for the real line $\mathbb{R}$. If $x, y \in X$, $x \neq y$ and $\epsilon = d(x, y) > 0$, then the open balls

  $$\begin{aligned} U_{\epsilon/3}(x) &= \{z \in X \ : \ d(x, z) < \epsilon/3\}, \\ V_{\epsilon/3}(y) &= \{z \in X \ : \ d(y, z) < \epsilon/3\} \end{aligned}$$

  represent disjoint neighborhoods of x and y (due to the triangle inequality $d(x, y) < d(x, z) + d(y, z)$).
- Any subset of $\mathbb{R}^n$. In particular, almost all spaces encountered in analysis, as a rule, satisfy the Hausdorff property.
- Any subset of any metric space.
- More generally, if X is a Hausdorff topological space and $Y \subset X$, then Y is Hausdorff too.

Examples of non-Hausdorff topological spaces:

- Any indiscrete topological space (which contains at least two points).
- Real line $\mathbb{R}$ with the topology $\tau = \{\mathbb{R}, \emptyset, (a, +\infty), a \in \mathbb{R}\}$. This space is not Hausdorff because any two non-empty open subsets of this space have a non-trivial intersection.
- The most famous example of a non-Hausdorff space is the line with two origins, or bug-eyed line. This space X is obtained from two copies $\mathbb{R}_1$ and $\mathbb{R}_2$ of the real line $\mathbb{R}$ by identifying all pairs of corresponding points $x \in \mathbb{R}_1$ and $x \in \mathbb{R}_2$ except for the origin 0. As a result, we obtain a line with two distinct origins $0_1 \in \mathbb{R}_1$ and $0_2 \in \mathbb{R}_2$.

This identification can be considered as a map $p : \mathbb{R}_1 \cup \mathbb{R}_2 \to X$. Then the subset $U \subset X$ is open (by definition) if and only if its preimage under the identification map $p^{-1}(U)$, is open as a subset of the disjoint union $\mathbb{R}_1 \cup \mathbb{R}_2$ (see the next section for more explanations on the disjoint union).

The Hausdorff property fails for these two origins 0_1 and 0_2, their neighborhoods cannot be disjoint.

Hausdorff topological spaces satisfy many important properties. In particular:

Proposition 22. *Let X be a Hausdorff topological space. Let $\{x\} \subset X$ be a subset that consists of a single point $x \in X$. Then $\{x\}$ is closed. Shortly: each point of a Hausdorff topological space is closed.*

Proof. We need to verify that $X \setminus \{x\}$ is open. Let $y \in X \setminus \{x\}$, $y \neq x$. Then x and y have disjoint neighborhoods $U(x)$ and $V(y)$. Since $x \notin V(y)$, we have $V(y) \subset X \setminus \{x\}$. In other words, each point y is contained in $X \setminus \{x\}$ together with a certain neighborhood. Thus, $X \setminus \{x\}$ is open and, therefore, $\{x\}$ is closed. $\square$

Proposition 23. *Let X be a Hausdorff topological space and $Y \subset X$ is compact. Then Y is closed.*

Proof. Let us prove that Y contains all of its limit points. By contradiction, assume that y is a limit point of Y, but $y \notin Y$.

For any point $x \in Y$ and y we can find disjoint neighborhoods $U(x)$ and $V_x(y)$. (We denote the neighborhood of y by $V_x(y)$ to emphasize thatthis neighborhood depends on $x \in Y$).

Obviously, the neighborhoods $U(x)$, $x \in Y$, all together form an open covering of Y. Since Y is compact we may choose a finite subcover $\{U(x_1), \dots, U(x_k)\}$ so that $Y \subset U(x_1) \cup \dots \cup U(x_k)$. Consider the corresponding neighborhoods $V_{x_1}(y), \dots, V_{x_k}(y)$ and take the intersection $V(y) = V_{x_1}(y) \cap \dots \cap V_{x_k}(y)$. Obviously, $V(y)$ is disjoint with each of U_{x_i} and, therefore, we have $V(y) \cap (U(x_1) \cup \dots \cup U(x_k)) = \emptyset$. Hence $V(y) \cap Y = \emptyset$, since Y is covered by $U(x_1), \dots, U(x_k)$. Thus, we have found a neighborhood of y which contains no points of Y. This contradicts the fact that y is a limit point of Y.

$\square$

The following statement is very often applied to recognize if two given topological spaces are homeomorphic or not.

Theorem 10. *Let X be compact, Y be Hausdorff and $f : X \to Y$ be continuous bijection. Then f is a homeomorphism.*

Proof. We only need to prove that $f^{-1} : Y \to X$ is continuous. The continuity of f^{-1} means that the preimage of any open set $A \subset X$ under f^{-1} is open. Using the duality between open and closed sets we can reformulate this condition as follows: f^{-1} is continuous iff the preimage of any closed set $C \subset X$ under f^{-1} is closed. Notice that the preimage of a set A under the inverse map f^{-1} is just the image of A under the direct map f. Thus, the statement of the theorem is equivalent to the fact that the image $f(C)$ of any closed subset $C \subset X$ is closed as a subset of Y.

But this fact is a simple combination of the three following statements:

1) C is compact as a closed subset of a compact space (Proposition 18).

2) $f(C)$ is compact as the image of a compact set under a continuous map (Theorem 4).

3) $f(C)$ is closed, since $f(C)$ is a compact subset of the Hausdorff topological space Y (Proposition 23). □

1.14 New spaces from old ones

Disjoint union

One can form many "new" topological spaces out of "old" ones. The simplest way to do so is to take the *disjoint union* of two topological spaces, where, while keeping the spaces separate, we think of them as a single space. For example, $\{-1, 1\}$ is the disjoint union of two one-pint spaces $\{-1\}$ and $\{1\}$. In general, if we have two topological spaces X_1, X_2, we define the disjoint union $X_1 \sqcup X_2$ as the set of ordered pairs (x, i) such that $x \in X_i$ and $i \in \{1, 2\}$. The open sets in $X_1 \sqcup X_2$ are just the disjoint unions of an open set in X_1 with an open set in X_2. For example, an open set in $\mathbb{R} \sqcup \mathbb{R}$ would be $(0, 1) \sqcup (0, 1)$, $(0, 1) \sqcup (2, 3)$ or $\emptyset \sqcup (0, 1)$.

Other examples:

- $S^0 \simeq \{-1\} \sqcup \{1\}$ (here "$\simeq$" means homeomorphic to.)
- $O(3) \simeq SO(3) \sqcup SO(3)$, where we identify the first copy of $SO(3)$ with orthogonal matrices of determinant 1 and the second copy with orthogonal matrices of determinant -1.

Properties of disjoint unions:

1. Disjoint unions preserve compactness (if X_1, X_2 are compact then so is $X_1 \cup X_2$).

2. Disjoint unions preserve the Hausdorff property (if X_1, X_2 are Hausdorff then so is $X_1 \sqcup X_2$).

3. Disjoint unions are always disconnected. Conversely, every disconnected space is the disjoint union of two non-empty spaces.

4. Continuous functions $f : X_1 \sqcup X_2 \to \mathbb{R}$ correspond to coordinate-wise continuous functions $f_1 : X_1 \to \mathbb{R}$ and $f_2 : X_2 \to \mathbb{R}$.

Product spaces

The next simplest construction are products, e.g. $\mathbb{R} \times \mathbb{R} \simeq \mathbb{R}^2$. More generally, as a set, $X_1 \times X_2$ consists of pairs (x_1, x_2) where $x_1 \in X_1$, $x_2 \in X_2$. If $A_1 \subset X_1$, $A_2 \subset X_2$ are open sets, then we define $A_1 \times A_2 \subset X_1 \times X_2$ to be open. These sets are called "open rectangles". But these cannot be all open sets in $X_1 \times X_2$ since the topology has to be closed under unions, and the union of two rectangles need not be a rectangle (in contrast to the intersection). So we add these unions to our collection of open sets.

Examples:

1. $S^1 \times [0, 1]$ is a cylinder in $\mathbb{R}^2$.

2. $S^1 \times S^1$ is the two-dimensional torus.

3. $S^1 \times (0, \infty) \simeq \mathbb{R}^2 \setminus \{0\}$. This can be seen by using "polar coordinates" $(x, y) = (r \cos\theta, r \sin\theta)$.

We only discuss finite products here, but one can also consider infinite products.

Properties of finite products:

1. Products preserve compactness (i.e. if X_1, X_2 are compact, then so is $X_1 \times X_2$). This is Tychonov's theorem.

2. Products preserve connectedness (i.e. if X_1, X_2 are connected, then so is $X_1 \times X_2$).

3. Products preserve the Hausdorff property (i.e. if X_1, X_2 are Hausdorff, then so is $X_1 \times X_2$).

Quotient spaces

Recall: $\sim$ is called an *equivalence relation* on a set X if it is:

1. reflexive: $x \sim x$;

2. symmetric $x \sim y \iff y \sim x$;
3. transitive: $x \sim y$, $y \sim z \implies x \sim z$.

We denote the set of *equivalence classes* of $x \in X$ by $[x]$. Then the *quotient set* $X/\sim$ is defined as the set of all equivalence classes of X. The natural map $p : X \to X/\sim$, $p(x) := [x]$, is surjective. If X is a topological space, we define a topology on $X/\sim$ by declaring $A \subset X/\sim$ to be open iff $p^{-1}(A)$ is open in X. We call $X/\sim$ a quotient space. An important special case of quotient spaces is the following: If X is a topological space and $A \subset X$ is a subspace, we define an equivalence relation by identifying all points in A (i.e. all points in A belong to the same equivalence class), while leaving all the other points equivalent only to themselves. This quotient space is denoted by X/A.

There are many examples of quotient spaces:

1. $[0,1]/\{0,1\} \simeq S^1$ (we say that the points 0 and 1 are "glued" together);
2. $[0,1]^2/\partial[0,1]^2 \simeq S^2$; similarly $D^2/\partial D^2 \simeq S^2$ (here $D^2 = \{x^2+y^2 \leq 1\}$ is the unit ball in $\mathbb{R}^2$);
3. $\mathbb{R}/\mathbb{Z} \simeq S^1$ (The standard meaning of $\mathbb{R}/\mathbb{Z}$ is different from the construction explained above. Here $\mathbb{Z}$ is considered as a group acting on $\mathbb{R}$ and two points in $\mathbb{R}/\mathbb{Z}$ are considered equivalent if they differ by an integer. This is an example of a so-called orbit space.)
4. On $\mathbb{R}^n \setminus \{0\}$ define $a \sim b \iff a = \lambda b$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Then $\mathbb{R}P^{n-1} := (\mathbb{R}^n \setminus \{0\})/\sim$ is called *real projective space*. This is a "space of lines through the origin". It can also be viewed as the $n-1$-dimensional unit sphere S^{n-1} with antipodal points identified. The *projective line* $\mathbb{R}P^1$ is homeomorphic to the circle S^1.

1.15 Complete metric spaces and the Baire Category theorem

The purpose of this section is to state and prove the Baire Category theorem. This is an important result that can be applied in lots of different parts of mathematics to prove the existence of exotic objects. We begin by introducing the notion of completeness for a metric space.

Definition 28. Let (X, d) be a metric space. A sequence $(x_n)_{n=1}^\infty$ is a Cauchy sequence if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Definition 29. A metric space (X, d) is complete if every Cauchy sequence converges to a point in X.

Example 25. 1. $\mathbb{R}$ is complete.

2. $\mathbb{Q}$ is not complete (There is Cauchy sequence of rational numbers converging to $\sqrt{2}$.)

3. The space $C[a, b]$ of real valued functions on $[a, b]$ equipped with the metric $d(f, g) = \max_{x \in [a,b]} |f(x) - g(x)|$ is complete (exercise).

Proposition 24. *Let (X, d) be a compact metric space. Then (X, d) is complete.*

Proof. Suppose that (X, d) is compact and that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in X. By compactness, $(x_n)_{n=1}^{\infty}$ has a subsequence, say $(x_{n_k})_{k=1}^{\infty}$ converging to a point x in X. We now show that our original sequence also converges to this x.

Let $\epsilon > 0$. Since $(x_n)_{n=1}^{\infty}$ is Cauchy there exists $N_1 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon/2$ for all $n, m \geq N_1$. Similarly since $(x_{n_k})_{k=1}^{\infty}$ converges to x there exists N_2 such that for all $n_k \geq N_2$ we have $d(x_{n_k}, x) < \epsilon/2$. Let $N = \max\{N_1, N_2\}$. Then for any $n \geq N$ and $n_k \geq N$, we have by the triangle inequality that

$$d(x, x_n) \leq d(x, x_{n_k}) + d(x_{n_k}, x_n) < \epsilon/2 + \epsilon/2.$$

□

Theorem 11 (Baire Category theorem)**.** *Let (X, d) be a complete metric space. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of open dense sets. Then $\cap_{n=1}^{\infty} A_n$ is dense in X.*

Proof. To prove our result it suffices to show that for any ball $B(x, r)$ we have $\cap_{n=1}^{\infty} A_n \cap B(x, r) \neq \emptyset$. Because A_1 is dense $A_1 \cap B(x, r) \neq \emptyset$ and is open, therefore there exists x_1 and $r_1 \in (0, 1)$ such that

$$\overline{B(x_1, r_1)} \subset A_1 \cap B(x, r).$$

Since each A_n is dense this process can be repeated. Therefore there exists a sequences (x_n) and $(r_n)_{n=1}^{\infty}$ satisfying $r_n \in (0, 1/n)$ and

$$\overline{B(x_n, r_n)} \subset A_n \cap B(x_{n-1}, r_{n-1})$$

for all $n \geq 2$. The sequence (x_n) is Cauchy because $x_n \subset B(x_m, r_m)$ for all $n > m$ and therefore $d(x_n, x_m) < 1/m$ by the definition of r_m. By completeness we know that there exists $y \in X$ such that $\lim_{n\to\infty} x_n = y$. Because each $\overline{B(x_n, r_n)}$ is closed and these balls satisfy $\overline{B}(x_n, r_n) \subset \overline{B}(x_m, r_m)$ for $m < n$, we may conclude that $y \in \overline{B(x_n, r_n)}$ for all $n \in \mathbb{N}$. Which by definition implies $y \in A_n$ for all n. Therefore $y \in \cap_{n=1}^{\infty} A_n \cap B(x, r)$. This completes our proof. $\square$

The Baire Category theorem naturally gives rise to the following notion.

Definition 30. Let (X, τ) be a topological space space. We call a set $A \subset X$ a G_δ set if it is a countable intersection of open sets.

Example 26.

- Any open set is a G_δ set.
- Any closed interval $[a, b] \subset \mathbb{R}$ is a G_δ set. ($[a, b] = \cap_{n=1}^{\infty}(a - 1/n, b + 1/n)$)
- A countable intersection of G_δ sets is a G_δ set.
- The union of finitely many G_δ sets is a G_δ set.

G_δ sets are important in several area of mathematics. For our purposes we are interested in dense G_δ sets. Such sets are useful because they provide a notion of typical for topological space, i.e. a property holds typically within a topological space if a dense G_δ set of points satisfy it. This property is well demonstrated by the following theorem.

Theorem 12. *Let (X, d) be a complete metric space. Let $X_1, X_2, X_3, \ldots$ be a countable collection of dense G_δ sets. Then $\cap_{i=1}^{\infty} X_i$ is a dense G_δ set.*

Proof. For each X_i there exists a collection of dense open sets $\{A_{i,n}\}_{n=1}^{\infty}$ such that $X_i = \cap_{n=1}^{\infty} A_{i,n}$. Consider the collection of all of the $A_{i,n}$, in particular consider $\{A_{i,n}\}_{i,n=1}^{\infty}$. Each of these sets is open and dense. Importantly this collection is countable. By the Baire Category theorem it follows that $\cap_{i,n=1}^{\infty} A_{i,n} = \cap_{i=1}^{\infty} X_i$ is a dense G_δ set. $\square$

Using this theorem we have the following result.

Theorem 13. *The irrational numbers are a dense G_δ set and the rational numbers do not contain a dense G_δ set.*

Proof. Notice that $\mathbb{R} \setminus \mathbb{Q} = \cap_{p/q \in \mathbb{Q}} \mathbb{R} \setminus \{p/q\}$. This intersection is over open and dense sets so $\mathbb{R} \setminus \mathbb{Q}$ is a dense G_δ set. Suppose the rational numbers contained a dense G_δ set. Then $(\mathbb{R} \setminus \mathbb{Q}) \cap \mathbb{Q}$ would be nonempty by our previous theorem. This is clearly not possible and we have a contradiction. $\square$

Exercise. Generalise the previous theorem to prove that if $X \subset \mathbb{R}$ is such that $\mathbb{R} \setminus X$ is countable, then X is a dense G_δ set.

We finish this section with an application of the above to a topic from Number Theory.

Definition 31. We say that a real number x is a Liouville number if for all $n \in \mathbb{N}$ there exists infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$

Liouville numbers can be thought of as numbers that are very well approximated by rational numbers. They play an important role in many parts of modern mathematics. Historically they are significant as the first examples of transcendental numbers. Those are numbers that do not satisfy a polynomial with integer coefficients. To show the existence of Liouville numbers we consider the following sets: For each $n \in \mathbb{N}$ let

$$L_n := \left\{ x : \left| x - \frac{p}{q} \right| < \frac{1}{q^n} \text{ for infinitely many } (p, q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

Notice that the Liouville numbers are precisely $\cap_{n=1}^{\infty} L_n$. We can write each L_n as follows

$$L_n := \bigcap_{N=1}^{\infty} \bigcup_{p \in \mathbb{Z}, q \geq N} (p/q - q^{-n}, p/q + q^n).$$

For each N the set $\bigcup_{p \in \mathbb{Z}, q \geq N} (p/q - q^{-n}, p/q + q^n)$ is a dense open set. Therefore L_n is a dense G_δ set. Moreover $\cap_{n=1}^{\infty} L_n$ is a dense G_δ set by the theorem above, i.e. the Liouville numbers are a dense G_δ-set. To prove the existence of transcendental numbers it is essential that the set of Liouville numbers does not contain just rational numbers. Using the fact that the irrational numbers are a dense G_δ set we have that the set of irrational Liouville numbers is a dense G_δ set.

Chapter 2

Manifolds

2.1 Manifolds

Definition 32. A topological space X is called a *manifold* of dimension n, if every point $x \in X$ possesses a neighborhood $U(x)$ homeomorphic to an n-dimensional open ball $B^n = \{x \in \mathbb{R}^n \ : \ |x| < r\} \subset \mathbb{R}^n$.

Examples:

- n-dimensional ball B^n;
- $\mathbb{R}^n$;
- any open subset of $\mathbb{R}^n$;
- sphere $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$;
- n-dimensional sphere $S^n = \{x_1^2 + x_2^2 + \cdots + x_3^2 = 1\} \subset \mathbb{R}^{n+1}$

Example of a topological space which is not a manifold:
$X = \{(x-y)(x+y) = 0\} \subset \mathbb{R}^2$, pair of intersecting lines. The intersection point $(0,0)$ does not admit a neighborhood homeomorphic to a ball B^n.

Indeed, a neighborhood U of this point in X is a ”cross”. If we remove this point from U, we obtain a disconnected set which consists of (at least) 4 connected components.

If we remove a point from a ball B^n, for $n \geq 2$, the ball remains connected. If $n = 1$, then B^1 is just an open interval $(-r, r)$. If we remove a point, we get a disconnected set, but the number of connected components is 2. Since the number of connected components is a topological invariant, we conclude that U cannot be homeomorphic to B^n whatever $n \in \mathbb{N}$ is.

Definition 33. *Let X be an n-dimensional manifold, $x \in X$ and $U = U(x)$ be a neighborhood of x which is homeomorphic to an n-dimensional ball $B \subset \mathbb{R}^n$ and let $\phi : U \to B$ denote the corresponding homeomorphism. Then the pair (U, ϕ) is called a chart (containing x).*

Definition 34. *A collection of charts which covers a manifold is called an atlas.*

An atlas is not unique as each manifold can be covered in multiple ways using different combinations of charts. This terminology is very natural: a geographical atlas (collection of charts representing the surface of the Earth) is an important particular case of topological atlases. The manifold in this case is the sphere = surface of the Earth.

Example 27. *A two-dimensional sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$ can naturally be covered by 6 charts:*

- *"upper" hemishpere: $U_1 = \{x^2 + y^2 + z^2 = 1, z > 0\}$,*
- *"lower" hemisphere: $U_2 = \{x^2 + y^2 + z^2 = 1, z < 0\}$,*
- *"right" hemisphere: $U_3 = \{x^2 + y^2 + z^2 = 1, y > 0\}$,*
- *"left" hemisphere: $U_4 = \{x^2 + y^2 + z^2 = 1, y < 0\}$,*
- *"front" hemisphere: $U_5 = \{x^2 + y^2 + z^2 = 1, x > 0\}$,*
- *"back" hemisphere: $U_6 = \{x^2 + y^2 + z^2 = 1, x < 0\}$.*

The maps $\phi_i : U_i \to B$ (where B is 2-dimensional disc) are defined to be natural projections to coordinate planes. For example, $\phi_1 : U_1 \to B$ is the projection of U_1 onto the open disc $B = \{x^2 + y^2 < 1\} \subset \mathbb{R}^2$ which sends the point $P = (x, y, z) \in U_1$ to the point $\phi_1(P) = (x, y) \in B$.

In Geography these hemispheres are known as Northern, Southern, Eastern, Western, Land and Water, see, for example,

http://en.wikipedia.org/wiki/Hemisphere.

Non-trivial examples of manifolds appear in physics as phase spaces of mechanical systems. Let us try to describe the pendulum phase space. The state of the pendulum is characterized by its position and velocity. See Fig. 1. The position is a point x on the circle (radius = length of the pendulum), the velocity is a vector v tangent to the circle at the position point. Thus, the phase space is the set of pairs (x, v), where $x \in S^1$ and $v \in$ tangent line to S^1 at x. It is not hard to see that from the topological point of view, this phase space can be regarded as a cylinder $S^1 \times \mathbb{R}^1$ (i.e., as a two-dimensional manifold).

A more complicated example is the phase space of the spherical pendulum (we allow it to swing in any direction). Then the position x of the pendulum is a point of the 2-dim sphere S^2, and the velocity $\dot{x}$ is a vector which belongs to the tangent plane to S^2 at the point x. This phase space is a four-dimensional manifold called the *tangent bundle* of the sphere.

Usually, speaking of manifolds one assumes two additional conditions:

1. X is a Hausdorff space;
2. X admits an atlas that consists of either countable or finite number of charts.

In the sequel, we shall follow this convention: our manifolds are always assumed to be Hausdorff spaces and with at most countable atlases (i.e., either finite, or countable).

2.2 Implicit Function Theorem

The Implicit Function Theorem is a powerful tool which allows us to recognize manifolds in various situations.

We start with an example which illustrates two essentially different possibilities.

Consider the subset M_a of $\mathbb{R}^3$ given by the following equation:

$$x^2 + y^2 - z^2 = a,$$

where a is a certain real number.

Is X_a a manifold? The answer depends on the value of the parameter a. There are three obvious cases:

- For $a > 0$, X_a is a hyperboloid of one sheet;
- For $a < 0$, X_a is a hyperboloid of two sheets;
- For $a = 0$, X_0 is a cone.

I refer again to Wikipedia, where you can find nice pictures:
http://en.wikipedia.org/wiki/Hyperboloid and
http://en.wikipedia.org/wiki/Quadric

It is easy to see that the first two cases give us manifolds, whereas the third one does not. The cone is not a manifold. The problem is the vertex of the cone, i.e., the point $(0,0,0)$ which has no neighborhood homeomorphic to a 2-disc (the proof is easy: if we remove this point from the cone, its neighborhood splits into two connected components, whereas the same operation applied to a 2-disc leaves it connected).

Such a point is usually called *singular* (all the others are *regular* in the sense that each of them possesses a nice neighborhood homeomorphic to a 2-disc). What is wrong with this point? What is the property which distinguish it from any other? The answer is that the differential of the function $F(x,y,z) = x^2 + y^2 - z^2$ that defines our surface vanishes at this point. Indeed,

$$dF = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = (2x, 2y, -2z)$$

vanishes exactly at the vertex of the cone, i.e., for $(x,y,z) = (0,0,0)$.

This simple observation is, in fact, the basement of the fundamental Implicit Function Theorem which, roughly speaking, says the following. Consider a smooth function F in three variables (x,y,z). Let $P_0 = (x_0, y_0, z_0)$ be one of the solutions of the equation $F(x,y,z) = a$, i.e., $F(x_0,y_0,z_0) = a$. Then in a small neighborhood we may linearize our equation and consider its linear approximation of the form:

$$\frac{\partial F}{\partial x}(P_0)(x - x_0) + \frac{\partial F}{\partial y}(P_0)(y - y_0) + \frac{\partial F}{\partial z}(P_0)(z - z_0) = 0$$

The idea is that the solutions of this linear equation in a neighborhood of P_0 are very close to the solutions to the original non-linear equation $F(x,y,z) = a$. In particular, the orthogonal projection of the set

$$\{F(x,y,z) = a\}$$

to the set

$$\frac{\partial F}{\partial x}(P_0)(x - x_0) + \frac{\partial F}{\partial y}(P_0)(y - y_0) + \frac{\partial F}{\partial z}(P_0)(z - z_0) = 0$$

is locally a homeomorphism. However, in order for this statement to hold we need to assume that $dF = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) \neq 0$ at this point P_0. It remains to notice that the solutions of the linearized equation form a two-dimensional vector space, i.e. $\mathbb{R}^2$.

Thus, the Implicit Function Theorem says that locally the space of solutions of a non-linear equation is similar (i.e., homeomorphic!) to the space of solutions of the corresponding linear equation. In fact, this theorem says much more, in particular it gives a local description of the space of solutions in very precise terms of differential calculus.

Here we confine ourselves with the topological aspects only. Here is a (weak) version of the Implicit Function Theorem which helps however to recognize manifolds.

Theorem 14. *Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, and $X = \{F(x, y, z) = a\}$ be one of its level sets. If $dF = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) \neq 0$ at any point $P \in X$, then X is a manifold of dimension 2.*

More generally, let $F : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, and $X = \{F(x_1, x_2, \dots, x_n) = a\}$ be one of its level sets. If

$$dF = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \dots, \frac{\partial F}{\partial x_n}\right) \neq 0 \quad \text{at any point } P \in X,$$

then X is a manifold of dimension $n - 1$.

This theorem has the following generalization to the case of several functions $F_1, \dots, F_k$.

Theorem 15. *Let $F_1, \dots, F_k : \mathbb{R}^n \to \mathbb{R}$ be smooth functions, $F_i = F_i(x_1, x_2, \dots, x_n)$, $i = 1, \dots, k < n$ and*

$$X = \left\{ \begin{array}{l} F_1(x_1, x_2, \dots, x_n) = a_1, \\ F_2(x_1, x_2, \dots, x_n) = a_2, \\ \dots \\ F_k(x_1, x_2, \dots, x_n) = a_k \end{array} \right\} \subset \mathbb{R}^n$$

be a common level set of these functions (i.e., the set of solutions of this system of functional equations). If the differentials

$$\begin{array}{l} dF_1 = \left(\frac{\partial F_1}{\partial x_1}, \frac{\partial F_1}{\partial x_2}, \dots, \frac{\partial F_1}{\partial x_n}\right) \\ dF_2 = \left(\frac{\partial F_2}{\partial x_1}, \frac{\partial F_2}{\partial x_2}, \dots, \frac{\partial F_2}{\partial x_n}\right) \\ \dots \\ dF_k = \left(\frac{\partial F_k}{\partial x_1}, \frac{\partial F_k}{\partial x_2}, \dots, \frac{\partial F_k}{\partial x_n}\right) \end{array}$$

are linearly independent at every point $P \in X$, then X is a manifold of dimension $n - k$.

Remark 5. *The "linear independence condition" in this theorem can obviously be replaced by the following:*

The Jacobi matrix of the functions $F_1, F_2, \dots, F_k$ has rank k:

$$\operatorname{rank}\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \dots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \dots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_k}{\partial x_1} & \frac{\partial F_k}{\partial x_2} & \dots & \frac{\partial F_k}{\partial x_n} \end{pmatrix} = k.$$

2.3 Surfaces as two-dimensional manifolds

Here we discuss one of the simplest but very important examples of manifolds, namely, manifolds of dimension two, or surfaces.

A surface M is characterized by the following property: each point $x \in M$ possesses a neighborhood $U(x)$ homeomorphic to a two-dimensional disc $B = \{x^2 + y^2 < r^2\} \subset \mathbb{R}^2$. In addition, we assume that:

1) M is a Hausdorff topological space;

2) M admits an atlas consisting of a finite or countable number of charts.

We start with examples:

- two-dimensional sphere $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$;
- hemisphere $S^+ = \{x^2 + y^2 + z^2 = 1,\ z > 0\} \subset \mathbb{R}^3$;
- two-dimensional plane $\mathbb{R}^2$;
- any open subset U of $\mathbb{R}^2$;
- graph of a smooth function $F(x, y)$ defined on a certain open domain $U \subset \mathbb{R}^2$, i.e.

  $$X = \{(x, y, z) \mid z = F(x, y),\ (x, y) \in U\} \subset \mathbb{R}^3.$$

  This easily follows from the fact that the projection $p : X \to U$, $p(x, y, z) = (x, y)$ establishes a natural homeomorphism between X and U which is, as we know, a two-dimensional manifold;
- level set of a smooth function of three variables:

  $$X = \{F(x, y, z) = a\} \subset R^3,$$

  provided the regularity condition hold:

  $$dF = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) \neq (0, 0, 0) \quad \text{for any } (x, y, x) \in X.$$

- torus T^2 (Fig. 2a), the surface obtained from a square by identifying its edges as shown in Fig 2b)
- Klein bottle K^2 (Fig. 3a), the surface obtained from a square by identifying its edges as shown in Fig 3b)
- Möbius strip (Fig. 4a), the surface obtained from a rectangle by identifying its edges as shown in Fig. 4b) (strictly speaking the Möbius strip is a surface in the above sense if we remove its boundary points, because a boundary point does not have any neighborhood homeomorphic to a disc)
- double torus, or pretzel (Fig. 5).

Our purpose is to distinguish various types of surfaces and to recognize which of them are homeomorphic and which are not.

First of all notice that homeomorphic surfaces may have quite different visual images. As an example consider the following surfaces:

1. one sheet hyperboloid $\{x^2 + y^2 - z^2 = 1\} \subset R^3$ (Fig. 6a);
2. infinite cylinder $\{x^2 + y^2 = 1\} \subset \mathbb{R}^3$ (Fig. 6b);
3. simple (cylindrical) strip (Fig. 6c) obtained from a rectangle by the "most natural" identification of two opposite sides (the boundary of this strip is supposed to be removed);

All these surfaces are homeomorphic to each other. So from the topological point of view we do not distinguish them.

Sometimes it is very useful to let surfaces have boundary points. By a boundary point $x \in M$ we mean a point which admits a neighborhood homeomorphic to a semidisc $B^+ = \{x^2 + y^2 < r^2,\ y \geq 0\} \subset R^2$.

Examples of such surfaces are:

- closed hemisphere $S^+ = \{x^2 + y^2 + z^2 = 1,\ z \geq 0\} \subset \mathbb{R}^3$;
- closed disc $\bar{B} = \{x^2 + y^2 \leq r^2\} \subset \mathbb{R}^2$;
- square, rectangle, triangle, n-gon (notice that vertices of these geometrical objects also have neighborhoods homeomorphic to a semidisc!)
- standard (cylindrical) strip and Möbius strip (the boundary points are now included)
- surfaces listed in our final page of figures.

Formally speaking, these objects are not *surfaces*, but *surfaces with boundary*!

Definition 35. A surface M is called *closed* if it is compact and has no boundary.

Among the surfaces considered above, the closed ones are the sphere, torus, Klein bottle and pretzel. All the others are either non-compact, or with boundary.

Another example of a closed surface is the so-called a projective plane which can be topologically defined in the following way.

Definition 36. A *projective plane* is the surface obtained from the sphere S^2 by pairwise identification of opposite points (x, y, z) and $(-x, -y, -z) \in S^2$. Notation: $\mathbb{R}P^2$.

This identification leads indeed to a certain surface. To see this, it is sufficient to notice that ”being a surface” is a ”local property”, which can be checked at each single point. Although the above identification is quite hard to imagine globally, the local identification in a neighborhood of a single point does not present any difficulty. If we take a small neighborhoods of (x, y, z) and $(-x, -y, -z)$ we can easily identify the pairs of corresponding points, and the result of this identification is a small piece (of a new surface!) homeomorphic to a disc. Thus, after identification each point still possesses a neighborhood homeomorphic to a disc so that the new topological space represent a certain surface. Moreover this surface is closed: it has no boundary and is compact as the image of the sphere (which is compact) under a continuous identification map.

Chapter 3

Further topological invariants

3.1 Euler characteristic.

In this section we discuss in loose terms an important quantity that we can associate to a closed surface. A polygon is a closed surface with flat, planar faces, e.g. a cube, a rectangle, a pyramid. A sphere is not a polygon.

The polygons are the building blocks of the sets we are interested in. We are actually interested in the sets one obtains by identifying polygons along their edges. To any such set we can associate the following three natural numbers

- number of faces F,
- number of edges E,
- number of vertices V.

Definition 37. Let M be a set obtained by identifying a finite number of polygons along their edges. The Euler characteristic of M is defined to be $\chi(M) = F - E + V$.

The following result shows how the Euler characteristic plays a role in studying arbitrary surfaces.

Proposition 25. *For any closed surface X there exists a finite collection of polygons with their edges identified, call it M, so that X is homeomorphic to M. Moreover, if M' is another set satisfying this property then $\chi(M') = \chi(M)$. Therefore we can define the Euler characteristic of X to be $\chi(M)$ for any such set.*

The value of the Euler characteristic is apparent in the following theorem.

Theorem 16. *Let X, X' be two closed surfaces satisfying $\chi(X) \neq \chi(X')$. Then X and X' are not homeomorphic.*

As an application of this theorem we will show that S^2 and the torus are not homeomorphic. It is simple to show that S^2 is homeomorphic to a cube (map a point x in the unit cube centred at the origin to $x/\|x\|$). The unit cube has six faces, 12 edges, and 8 vertices. Therefore $\chi(S^2) = 6-12+8 = 2$. It will be show in the lectures that the torus is homeomorphic to a polytope with Euler characteristic 0. Therefore S^2 and the torus are not homeomorphic.

3.2 The fundamental group

The last topic we will discuss in these notes is the beginnings of an important area known as algebraic topology. This topic attacks the problem of determining whether two topological spaces are homeomorphic by associating certain algebraic objects. The belief is then that if these algebraic objects are not isomorphic then the topological spaces are not homeomorphic. The idea behind this approach is that groups, rings, etc, are (in principal) easier to work with then topological spaces.

Definition 38. Two paths $f, f' : [0,1] \to X$ are called *path-homptopic* if they have the same initial point x_0 and end point x_1, and if there exists a continuous map $F : [0,1] \times [0,1] \to X$ such that

$$F(s,0) = f(s), \quad F(s,1) = f'(s)$$

for every $s \in [0,1]$, and

$$F(0,t) = x_0, \quad F(1,t) = x_1$$

for every $t \in [0,1]$. We call F a *path-homotopy* between f and f'.

We write $f \simeq_p f'$ if f and f' are path-homotopic.

Lemma 17. The relation $\simeq_p$ is an equivalence relation.

If f is a path we shall denote its equivalence class by $[f]$.

Definition 39. If f is a path from x_0 to x_1 and g is a path from x_1 to x_2, then the *product* $f * g$ is the path h from x_0 to x_2 given by

$$h(s) = \begin{cases} f(2s) & s \in [0,1/2], \\ g(2s-1) & s \in [1/2,1]. \end{cases}$$

The product of paths induces an operation on path-homotopy classes via

$$[f] * [g] = [f * g].$$

The operation $*$ on path-homotopy classes has almost all the properties of a group, except that $[f] * [g]$ is not defined for all paths, but only for those with $f(1) = g(0)$. But if we consider loops, i.e. paths with $x_0 = x_1$, then the set of path-homotopy classes with the operation $*$ indeed forms a group. It is called the **fundamental group** of X with base point x_0. It is denoted by $\pi_1(X, x_0)$.

Example 28. $\pi_1(\mathbb{R}^n, x_0)$ is the trivial group since any path f is nullhomotopic via the straight-path homotopy

$$F(s,t) = (1-t)e_{x_0}(s) + tf(s).$$

Here e_{x_0} denotes the constant path at x_0, i.e. $e_{x_0}(s) = x_0$.

The same argument shows that the fundamental group of any convex space X is trivial.

Does the fundamental group depend on the base point?

Theorem 18. *If X is path-connected, then for any $x_0, x_1 \in X$ the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.*

The fundamental group is a **topological invariant**, i.e. two homeomorphic spaces have the same fundamental group. It can be shown that $\pi(S^1, x_0) = \mathbb{Z}$ for any $x_0 \in S^1$. Consequently S^1 is not homeomorphic to any convex subset of $\mathbb{R}^n$.

3.3 Figures

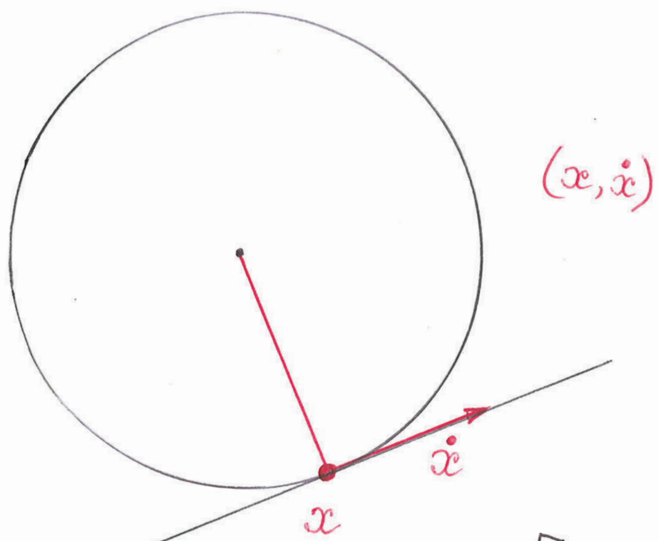

Figure 1.

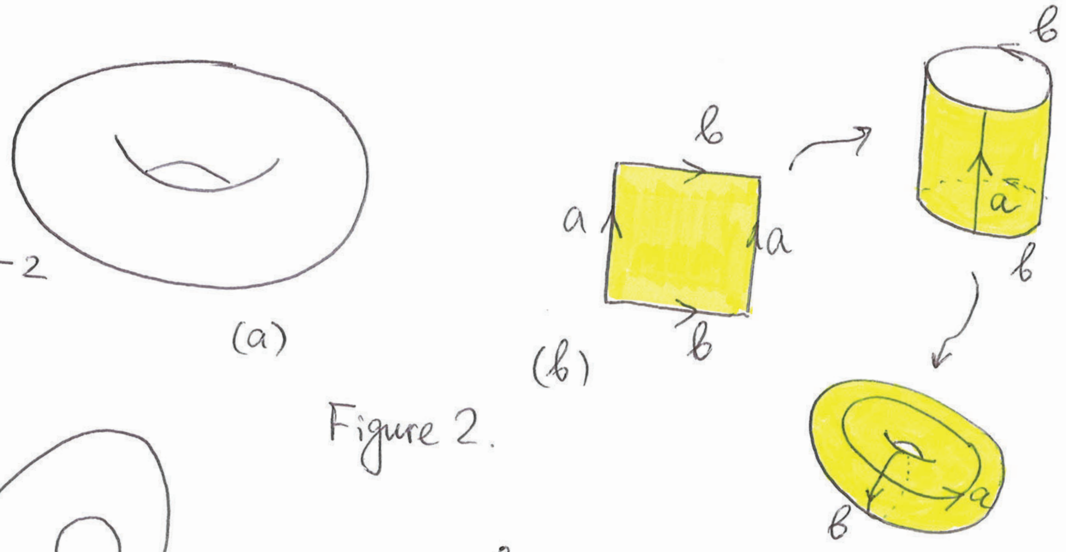

Figure 2.

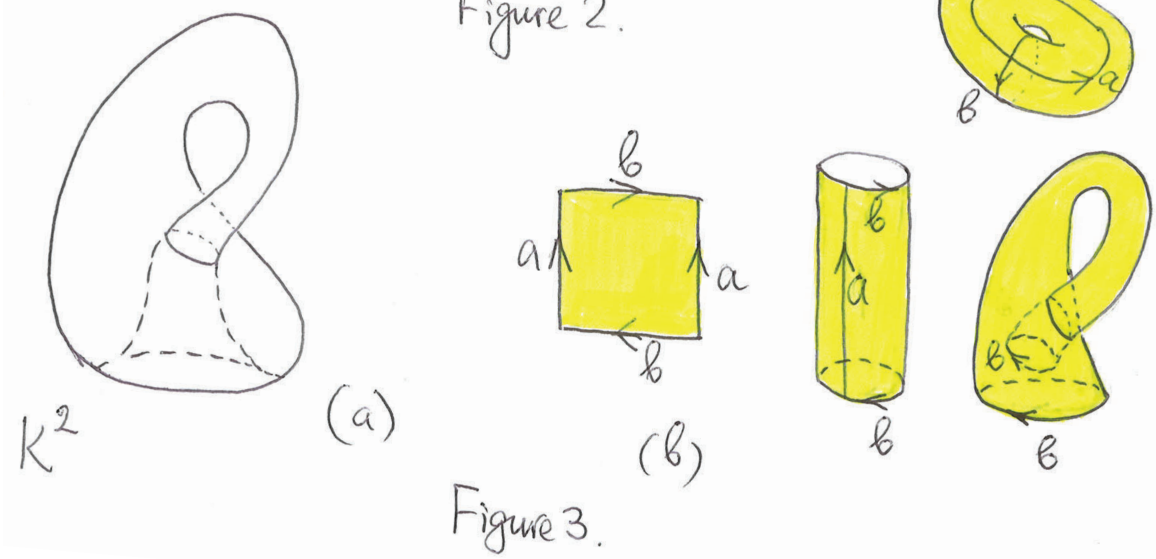

Figure 3.

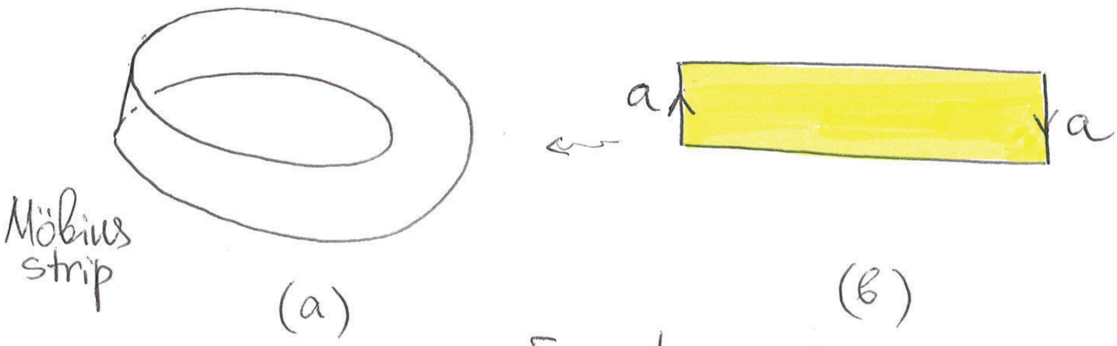

Figure 4.

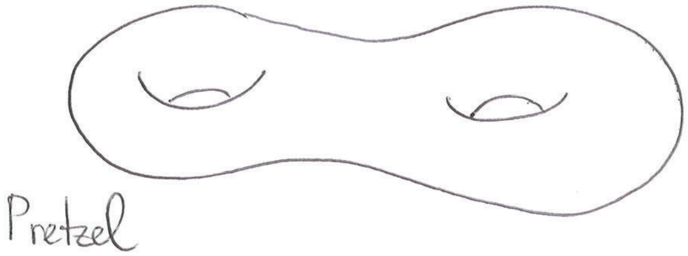

Figure 5.

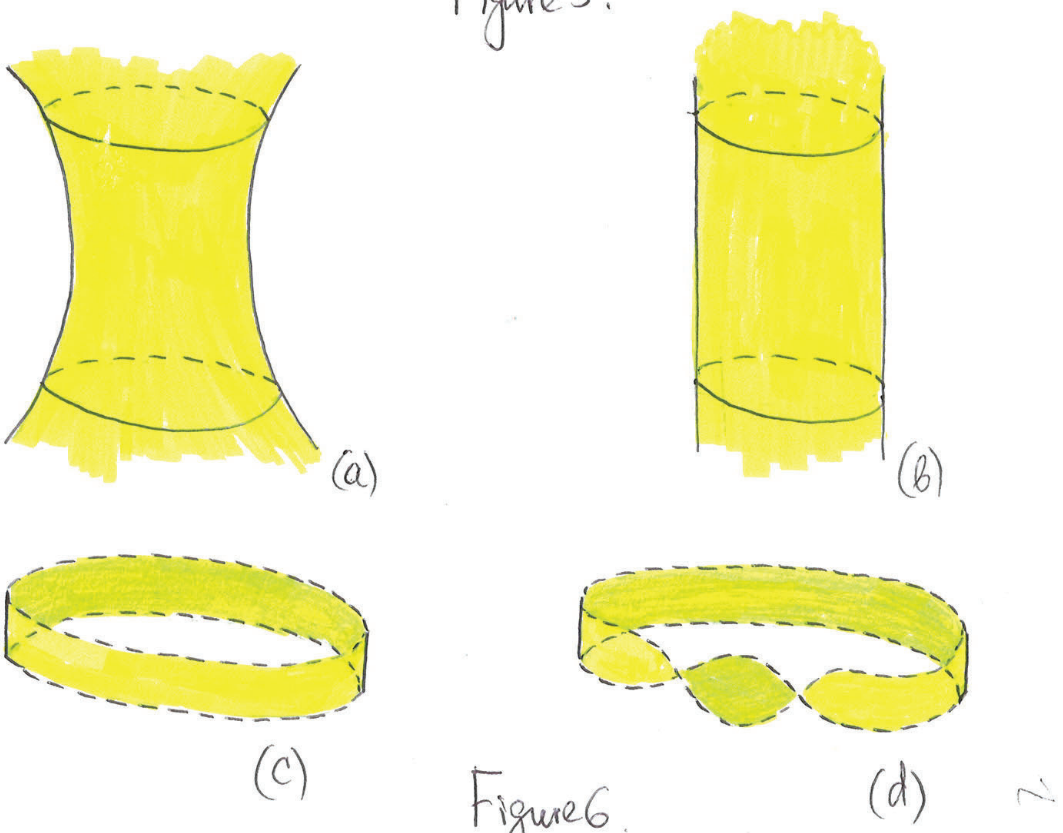

Figure 6.

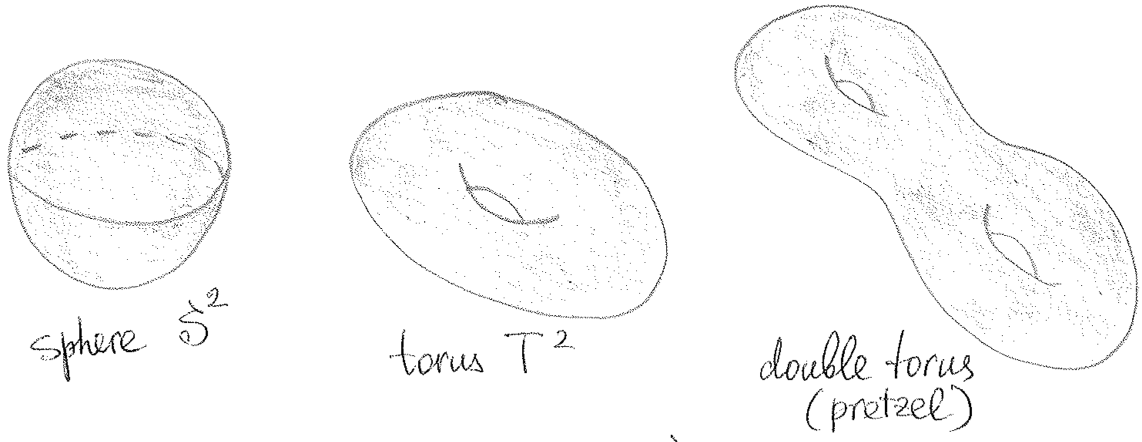

Fig. 1 (Closed surfaces)

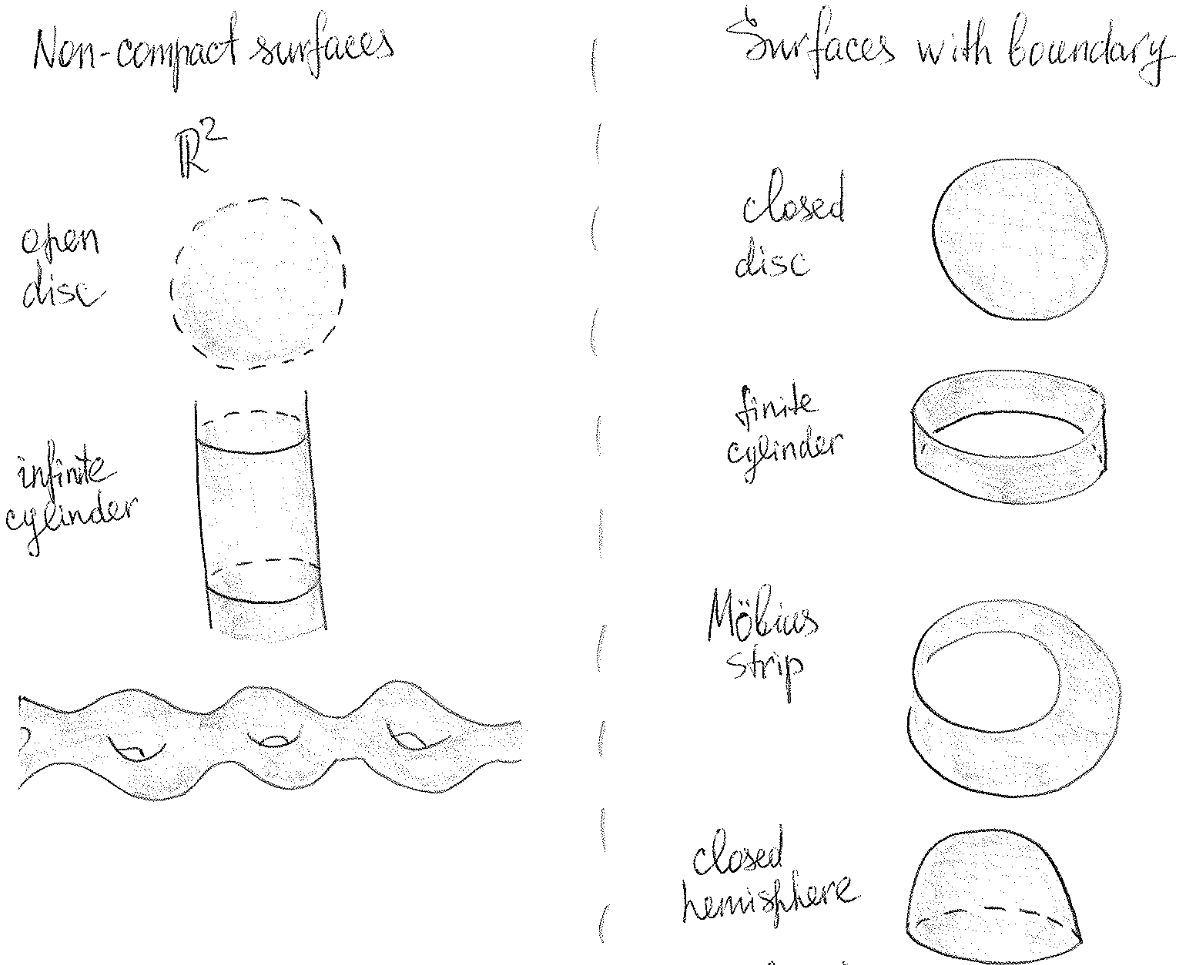

Fig. 2 (Non-closed surfaces)

Chapter 4

A topological proof of the infinitude of the primes

One of the most important and well known theorems in mathematics is the following statement.

Theorem 19. *There are infinitely many prime numbers.*

The most well known proof of this statement goes along the lines of assuming there are finitely many, then multiply them together, add one, and finally derive a contradiction. In this section we will give another proof of this statement. This proof uses the language of topology and was discovered by Hillel Furstenberg when he was an undergraduate student.

Proof. The first step is to define a topology on $\mathbb{Z}$. We say that a set $U \subset \mathbb{Z}$ is open if it is a union of sets of the form $S(a,b) = \{an+b : b \in \mathbb{Z}\} = a\mathbb{Z}+b$ or the empty set. We have to check this defines a topology:

- $\mathbb{Z}$ is open ($\mathbb{Z} = S(1,1)$).
- $\emptyset$ is open by definition.
- The union of open sets is open. Notice that the open sets are not those sets equal to $S(a,b)$, but the open sets are unions of sets of the form $S(a,b)$.
- The intersection of two open sets is open: Let U_1, U_2 be non-empty open sets. Let $x \in U_1 \cap U_2$. Then there exists a_1 such that $S(a_1,x) \subset U_1$ and a_2 such that $S(a_2,x) \subset U_2$. Then $x \in S(a_1a_2,x) \subset S(a_1,x) \subset U_1$ and $x \in S(a_1a_2,x) \subset S(a_2,x) \subset U_2$. So $S(a_1a_2,x) \subset U_1 \cap U_2$. Taking the union of these sets over $x \in U_1 \cap U_2$ yields that $U_1 \cap U_2$ is open.

Notice now that a finite non-empty set cannot be open (it cannot contain an $S(a,b)$). This is equivalent to the complement of a non-empty finite set cannot be closed. We also remark that each $S(a,b)$ is open and closed. See that $S(a,b) = \mathbb{Z} \setminus \cup_{j=1}^{a-1} S(a, b+j)$.

The following equality holds:

$$\mathbb{Z} \setminus \{-1, 1\} = \cup_{p\text{: p is prime}} S(p, 0).$$

The left hand side cannot be closed because it is the complement of a non-empty finite set. The right hand side is closed if there are finitely many primes. Here we are using that the union of finitely many closed sets is closed. Because a set cannot simultaneously be closed and not closed, we see that there must be infinitely many primes. $\square$