MAB298-Elements of Topology: Problem Sheet 7
Manifolds and Implicit Function Theorem

1. Describe an atlas for the sphere S^2 which consists of two charts. Generalize this example to the case of the n-dimensional sphere $\{x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$.

2. Describe an atlas for the (2-dimensional) torus which consists of 4 charts.

3. Consider the subset X in $\mathbb{R}^3$ given by the following equation:

$$x^3 + 3xy^2 + 3xz^2 + 2y^3 + 5yz^2 + z^3 = 1.$$

Using the Implicit Function Theorem, verify that X is a manifold.

4. Consider the subset X of $\mathbb{R}^2$ given by the equation

$$(x^2 + y^2 - 1)(x^2 - 2x + y^2) = 0.$$

Is X a manifold?

5. Prove that the set $X \subset \mathbb{R}^4$ given by two equations:

$$x^2 + y^2 + z^2 + u^2 = 1, \qquad x^2 + y^2 - z^2 - u^2 = 0,$$

is a two dimensional compact connected manifold. (This manifold is homeomorphic to a torus).

6. Prove that the set of 2-dimensional orthogonal matrices

$$O(2) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad : \quad AA^\top = \mathrm{Id} \right\}$$

is a manifold. What is the dimension of $O(2)$?

7. Prove that the set $GL(n, \mathbb{R})$ of all invertible $n \times n$ matrices is a manifold.

8. Prove that every connected manifold X is path connected.