MAB298-Elements of Topology: Problem Sheet 4

Compactness

1. Let the topology on $\mathbb{R}$ be defined by the collection of subsets $\{\emptyset,\ \mathbb{R},\ (a, +\infty), a \in \mathbb{R}\}$ (see Example 5 in Lecture Notes). Is $\mathbb{R}$ with this topology compact?

2. Prove by definition that $[0, 1)$ is not compact

3. Prove by definition that $\mathbb{Z}$ as a subset of $\mathbb{R}$ is not compact.

4. Let X be compact and C_i, $i \in \mathbb{N}$ be a collection of closed sets such that any finite intersection $\cap_{i=1}^{N} C_i$ is not empty. Prove that $\cap_{i=1}^{\infty} C_i \neq \emptyset$.

5. Let A and B be two compact subsets of a space X. Does it follow that $A \cup B$ is compact?

6. Which of the topological spaces listed below are compact:

   1) $[0, 1]$ with the discrete topology

   2) $\mathbb{Z}$ with the discrete topology

   3) $\mathbb{R}$ with the indiscrete topology

   4) closed half plane $\{(x, y)\ :\ y \geq 0\}$

   5) sphere $\{x^2 + y^2 + z^2 = 1\}$

   6) open disc $\{x^2 + y^2 < 1\}$ in $\mathbb{R}^2$

   7) annulus $\{1 < x^2 + y^2 < 4\}$

   8) punctured sphere $\{x^2 + y^2 + z^2 = 1\} \setminus \{(0, 0, 1)\}$

   9) $\mathbb{Q}$ as a subset in $\mathbb{R}$

   10) $\{(x^4 + y^4)(1 + x^2 + y^2) = 10\}$ in $\mathbb{R}^2$

   11) $\{\frac{x^4+y^4}{1+x^2+y^2} \leq 10\}$ in $\mathbb{R}^2$

   12) $\{\sin^4 x + \cos^4 y = 1\}$ in $\mathbb{R}^2$

   13) $\{(x + \sin y)^2 + (y + \sin x)^2 = 100\}$ in $\mathbb{R}^2$

   14) $[0, 1] \cap \mathbb{Q}$

7. Prove that if $f : [0,1] \to \mathbb{R}$ is a continuous function, then $f([0,1])$ is a segment (i.e., closed interval). More generally, if $X \subset \mathbb{R}^n$ is compact and connected and $f : X \to \mathbb{R}$ is a continuous function, then $f(X)$ is a segment.

8. Let A be a subset of $\mathbb{R}^n$. Prove that A is compact iff each continuous numerical function on A is bounded.

9. In the space $C^0([0,1])$ of continuous functions on $[0,1]$ with the standard distance $d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$, consider the closed ball B of radius 1 centered at zero (i.e., at the zero function):
$$B = \{f \in C^0([0,1]) \ : \quad |f(x)| \leq 1\}$$
Is B compact?