23MAC260 Problem Sheet 9: Solutions

Week 10 Lectures

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1. Let L be the lattice in the complex plane spanned by the two numbers

$$\omega_1 = \frac{3}{8} - \frac{3\sqrt{3}}{8}\,i$$
$$\omega_2 = -i$$

Find a complex number τ in the region

$$\mathcal{F} = \left\{ z \in \mathbb{C} : \mathrm{Im}(z) > 0,\, |\mathrm{Re}(z)| \leq \frac{1}{2},\, |z| \geq 1 \right\}$$

such that L is similar to the lattice

$$\mathbb{Z} \oplus \mathbb{Z} \cdot \tau.$$

Solution: We follow the strategy described in the Example in Section 3 of the Week 10 lecture notes.

Step 1: First we know that L is similar to the lattice spanned by $\{1, \omega\}$ where

$$\begin{aligned}
\omega &= \omega_1/\omega_2 \\
&= \frac{1}{-i}\left(\frac{3}{8} - \frac{3\sqrt{3}}{8}\,i\right) \\
&= \frac{3\sqrt{3}}{8} + \frac{3}{8}\,i \\
&= \frac{3}{4}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\,i\right) \\
&= \frac{3}{4}\exp(\pi i/6).
\end{aligned}$$

The last line shows that $|\omega| = \frac{3}{4}$ so ω is not in the region $\mathcal{F}$.

Step 2: Now we apply the transformations S and T as appropriate to move ω into the region $\mathcal{F}$.

Since $|\omega| < 1$ we start by applying the transformation

$$S : z \mapsto -\frac{1}{z}$$

to get

$$\begin{aligned} S(\omega) &= -\frac{1}{\omega} \\ &= -\frac{4}{3}\exp(-\pi i/6) \\ &= \frac{4}{3}\exp(5\pi i/6) \end{aligned}$$

where in the last line we used the fact that $-\exp(i\theta) = \exp(i(\theta + \pi))$ for any θ.

Now we compute that

$$\begin{aligned} \mathrm{Re}(S(\omega)) &= \frac{4}{3}\cos(5\pi/6) \\ &\approx -1.15 \end{aligned}$$

so $S(\omega)$ is not in the region $\mathcal{F}$ either.

Step 3: Next we apply the transformation

$$T = z \mapsto z + 1$$

to get

$$\begin{aligned} TS(\omega) &= S(\omega) + 1 \\ &= \frac{4}{3}\exp(5\pi i/6) + 1. \end{aligned}$$

This number has real part

$$\begin{aligned} \mathrm{Re}(TS(\omega)) &= \frac{4}{3}\cos(5\pi i/6) + 1 \\ &\approx -0.15 \end{aligned}$$

and imaginary part

$$\begin{aligned} \mathrm{Im}(TS(\omega)) &= \frac{4}{3}\sin(5\pi/6) \\ &= \frac{2}{3} \end{aligned}$$

so we compute that

$$
\begin{aligned}
|\mathrm{TS}(\omega)| &\approx \sqrt{(-0.15)^2 + (2/3)^2} \\
&< 1
\end{aligned}
$$

hence $\mathrm{TS}(\omega)$ is (still!) not in the region $\mathcal{F}$.

Step 4: Finally, we apply the transformation S again to get

$$
\begin{aligned}
\mathrm{STS}(\omega) &= S\left(\frac{4}{3}\exp(5\pi i/6) + 1\right) \\
&= -\left(\frac{1}{\frac{4}{3}\exp(5\pi i/6) + 1}\right).
\end{aligned}
$$

Call this number τ; then one can check that

$$\tau \approx 0.33 + 1.42\, i$$

So $|\operatorname{Re}(\tau)| \leq \frac{1}{2}$ and $|\tau| \geq 1$, hence τ is indeed in the region $\mathcal{F}$.

So $\tau = \mathrm{STS}(\omega)$ is the required number.

2. Recall our definition of the action of $\mathrm{SL}(2, \mathbb{Z})$ on the upper half-plane $\mathcal{H}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

Prove that if M_1 and M_2 are matrices in $\mathrm{SL}(2, \mathbb{Z})$ then

$$M_1 \cdot (M_2 \cdot \tau) = (M_1 M_2) \cdot \tau$$

so we really do have a group action.

Solution: Write out the entries of the matrices:

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

Then

$$M_1 M_2 = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}$$

On the other hand using the definition above we calculate:

$$\begin{aligned}
M_1 \cdot (M_2 \cdot \tau) &= M_1 \cdot \left(\frac{a_2\tau + b_2}{c_2\tau + d_2}\right) \\
&= \frac{a_1\left(\frac{a_2\tau+b_2}{c_2\tau+d_2}\right) + b_1}{c_1\left(\frac{a_2\tau+b_2}{c_2\tau+d_2}\right) + d_1} \\
&= \frac{a_1(a_2\tau + b_2) + b_1(c_2\tau + d_2)}{c_1(a_2\tau + b_2) + d_1(c_2\tau + d_2)} \\
&= \frac{(a_1a_2 + b_1c_2)\tau + (a_1b_2 + b_1d_2)}{(c_1a_2 + d_1c_2)\tau + (c_1b_2 + d_1d_2)} \\
&= (M_1M_2) \cdot \tau
\end{aligned}$$

3. Show that if τ_1, τ_2 are two complex numbers in the upper half-plane H and M is a 2×2 matrix such that

$$M \cdot \tau_1 = \tau_2$$

then $\det M > 0$.

Solution: Suppose that

$$\begin{aligned}
\tau_1 &= x_1 + i y_1 \\
\tau_2 &= x_2 + i y_2 \\
M &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\end{aligned}$$

Then

$$\begin{aligned}
M \cdot \tau_1 &= \frac{a\tau_1 + b}{c\tau_1 + d} \\
&= \frac{ax_1 + b + i\,ay_1}{cx_1 + d + i\,cy_1} \\
&= \frac{(ax_1 + b + i\,ay_1)(cx_1 + d - icy_1)}{(cx_1 + d)^2 + (cy_1)^2}
\end{aligned}$$

On one hand, by assumption $M \cdot \tau_1 = \tau_2$, so the above number equals τ_2. On the other hand, the imaginary part of this number is

$$\frac{(ad - bc)y_1}{(cx_1 + d)^2 + (cy_1)^2} = \det(M)\frac{y_1}{(cx_1 + d)^2 + (cy_1)^2}$$

Since $y_1 = \mathrm{Im}(\tau_1) > 0$, and denominator is a sum of squares of real numbers, this quantity is positive if and only if $\det(M) > 0$.

Putting these together, since we assume that $\tau_2 \in \mathcal{H}$, hence has positive imaginary part, we must have $\det(M) > 0$.

4. Show that for any $\tau \in \mathcal{H}$ we have

$$(-I_2) \cdot \tau = \tau.$$

Solution: This one is easy: putting $a = d = -1$ and $b = c = 0$ in the formula from Question 2, we get

$$\begin{aligned}(-I_2) \cdot \tau &= \frac{-\tau}{-1} \\ &= \tau.\end{aligned}$$

5. Verify the following relations among the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}:$$

(a) $S^2 = -I_2$.

(b) $(ST)^3 = -I_2$.

(Using the previous problem this shows that the transformations of $\mathcal{H}$ given by S and ST have order 2 and 3 respectively.)

Solution: This is just matrix multiplication. I will leave it to you.

6. Write the matrix

$$\begin{pmatrix} 11 & 14 \\ 7 & 9 \end{pmatrix} \in SL(2, \mathbb{Z})$$

in terms of S and T.

Solution: There is a general algorithm to solve this problem that we describe in the next question. In this question, we will first see how it works in this specific case.

Suppose that at a given step we have a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then

- if $|a| < |c|$ we multiply by S;
- if $|a| \geq |c|$ we mutliply by an appropriate power of T to reduce the maximum absolute value of the elements in the first column.

The steps of the computation go as follows:

(a)

$$M = \begin{pmatrix} 11 & 14 \\ 7 & 9 \end{pmatrix}$$

(b)

$$T^{-1}M = \begin{pmatrix} 4 & 5 \\ 7 & 9 \end{pmatrix}$$

(c)

$$ST^{-1}M = \begin{pmatrix} -7 & -9 \\ 4 & 5 \end{pmatrix}$$

(d)

$$TST^{-1}M = \begin{pmatrix} -3 & -4 \\ 4 & 5 \end{pmatrix}$$

(e)

$$STST^{-1}M = \begin{pmatrix} -4 & -5 \\ -3 & -4 \end{pmatrix}$$

(f)

$$T^{-1}STST^{-1}M = \begin{pmatrix} -1 & -1 \\ -3 & -4 \end{pmatrix}$$

(g)

$$ST^{-1}STST^{-1}M = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$$

(h)

$$T^3ST^{-1}STST^{-1}M = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

(i)

$$\begin{aligned} ST^3ST^{-1}STST^{-1}M &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= T. \end{aligned}$$

Rearranging the last equation gives

$$\begin{aligned} M &= \left(ST^3ST^{-1}STST^{-1}\right)^{-1}T \\ &= TS^{-1}T^{-1}S^{-1}TS^{-1}T^{-3}S^{-1}T \end{aligned}$$

and using $S^{-1} = -S$ this can be written as

$$M = TST^{-1}STST^{-3}ST.$$

7. In this problem we will prove Lemma 2.3 in the Week 10 lecture notes, showing that $SL(2,\mathbb{Z})$ is generated by the two matrices S and T.

   (a) For a general matrix

   $$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

   calculate SM and $T^n M$ for any integer n.

   Solution: Again this is just matrix multiplication: we find

   $$SM = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$$

   and

   $$\begin{aligned} T^n M &= \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix} \end{aligned}$$

   (b) Use the previous part to show that, by multiplying M by suitable powers of S and T, we can reduce it to a matrix of the form

   $$\widetilde{M} = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in SL(2,\mathbb{Z}).$$

   (Hint: Euclidean algorithm.)

   Solution: First we can multiply M by S if necessary to assume without loss of generality that $|c| \leq |a|$. Now using the Division Algorithm we can write

   $$a = qc + r$$

   where the remainder r satisfies $r < |c|$. Rearranging this equation as

   $$r = a - qc$$

we then see from Part (a) that

$$T^{-q}M = \begin{pmatrix} r & b - qd \\ c & d \end{pmatrix}.$$

Now multiply by S to swap the rows (with a sign change):

$$ST^{-q}M = \begin{pmatrix} -c & -d \\ r & b - qd \end{pmatrix}.$$

Continuing in this way as in the Euclidean algorithm, we see that in the first column, we can make the absolute value of the smaller entry decrease at each step. Eventually we must reach a zero entry in the top-left corent, at which point we multiply by S one more time to switch the rows and get $\widetilde{M}$ in the form shown.

(c) Use the fact that $\widetilde{M} \in SL(2, \mathbb{Z})$ to conclude that $\alpha = \delta = \pm 1$. Hence write $\widetilde{M}$ in terms of S and T.

Solution: We have

$$\det(\widetilde{M}) = \alpha\delta.$$

Since $\widetilde{M} \in SL(2, \mathbb{Z})$ this means $\alpha\delta = 1$, hence $\alpha = \delta = \pm 1$.
If $\alpha = \delta = 1$ then

$$\begin{aligned} \widetilde{M} &= \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \\ &= T^{\beta}. \end{aligned}$$

If $\alpha = \delta = -1$ then

$$\begin{aligned} \widetilde{M} &= \begin{pmatrix} -1 & \beta \\ 0 & -1 \end{pmatrix} \\ &= -I_2 \cdot \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Frm Question 5 we know that $-I_2 = S^2$ so we can write this as

$$\widetilde{M} = S^2 T^{-\beta}.$$

(d) Conclude that M can be written in terms of S and T.

Solution: All the steps to transform M to $\widetilde{M}$ consisted of multiplication by S or powers of T, and we saw in the previous part that $\widetilde{M}$ is a product of powers of S and T. Putting these together, we can write M as a product of powers of S and T.