ELEMENTS OF TOPOLOGY
(22MAB298)

Semester 2 22/23 In-Person Exam paper

Please fill in:

ID number:		Desk number:	

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

Help during the exam

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

You may **not** use a calculator for this exam.

Write your answer for every question in the appropriate space. Indicate within that space if you give additional parts of your answer elsewhere.

Answer all 3 questions.

1. (a) Give the definition of a topology. [4]

Continue at the end of the booklet (if necessary).

(b) Let $X = \{1, 2, 3, 4\}$ and consider the collection of subsets

$$\tau = \{\{1, 2, 3, 4\}, \{1\}, \{2, 3\}, \{2, 4\}, \{2\}\}.$$

Does τ define a topology for X? Explain your answer. [3]

Continue at the end of the booklet (if necessary).

(c) Consider $\mathbb{R}$ equipped with the usual topology coming from the Euclidean metric. State whether the following sets are open with respect to this topology:

(i) $\mathbb{R} \setminus (\mathbb{Q} \cap (0, \infty))$. [2]

(ii) $(-1, 0) \cup (0, 1)$. [2]

(iii) $\{x : \sin(x) > 0\}$. [2]

You do not need to explain your answers.

Continue at the end of the booklet (if necessary).

(d) Let (X, τ) be a topological space and $Y \subset X$. Define the subspace topology for Y and prove that it actually defines a topology. [7]

Continue at the end of the booklet (if necessary).

(e) Let $X = \{1, 2, 3\}$ and

$$\tau = \{\{1, 2, 3\}, \emptyset, \{1\}, \{2\}\}.$$

Prove that there exists no metric d for X such that τ is the topology generated by d. Hint: Consider open balls centred at 3. [5]

Continue at the end of the booklet (if necessary).

2. (a) Let $X = \mathbb{R}$ and $Y = \mathbb{Q}$. Calculate the interior, boundary, and exterior of the subset $Y \subset X$ with respect to the following topologies:

   (i) The topology given by the standard Euclidean metric. [4]

   (ii) The discrete topology. [4]

   (iii) The indiscrete topology. [4]

   Explain your solutions.

Continue at the end of the booklet (if necessary).

(b) Let (X, τ) and (Y, τ') be topological spaces. Define what it means for a function $f : X \to Y$ to be continuous. [4]

Continue at the end of the booklet (if necessary).

(c) Let $X = \mathbb{R}$ be equipped with the indiscrete topology. Let (Y, d) be a metric space. Prove that if $f : X \to Y$ is continuous then f is a constant function. [9]

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Continue at the end of the booklet (if necessary).

3. (a) Define what it means for a topological space to be connected. [4]

Continue at the end of the booklet (if necessary).

(b) Let $X = (-1, 0) \cup (0, 1)$.

(i) Equip X with a topology τ such that X is not connected with respect to this topology. [3]

(ii) Equip X with a topology τ' such that X is connected with respect to this topology. [3]

Continue at the end of the booklet (if necessary).

(c) Let (X, τ) be a topological space. Suppose that there exists a continuous and surjective function $f : X \to \{-1, 1\}$ when $\{-1, 1\}$ is equipped with the discrete topology. Prove that X is not connected. [5]

Continue at the end of the booklet (if necessary).

(d) Define what it means for a topological space (X, τ) to be pathwise connected. [4]

Continue at the end of the booklet (if necessary).

(e) Let (X, τ) and (Y, τ') be topological spaces. Assume that (X, τ) is pathwise connected and $f : X \to Y$ is continuous and surjective. Prove that (Y, τ') is pathwise connected. [6]

Continue at the end of the booklet (if necessary).

Extra space for answers to any questions: