MAB298-Elements of Topology: Solution Sheet 5

Connectedness

1. Let X be a topological space, A and B connected (pathwise connected) subsets of X. Let $A \cap B \neq \emptyset$. Prove that $A \cup B$ is connected (resp. pathwise connected).

   Let A and B be connected and $A \cap B \neq \emptyset$. Assume, by contrary, that $A \cup B$ is disconnected, i.e. there are disjoint open sets C and D such that $A \cup B = C \cup D$ and $(A \cup B) \cap C \neq \emptyset$, $(A \cup B) \cap D \neq \emptyset$. Consider the intersections $A_1 = A \cap C$ and $A_2 = A \cap D$. Clearly $A = A_1 \cup A_2$ is a partition of A into two open disjoint subsets. Since A is connected, one of these subsets has to be empty. Without loss of generality we may assume that $A_1 = A \cap C = \emptyset$, and A is contained entirely in D, i.e., $A \subset D$. Similarly, it is easy to see that either $B \cap C$ or $B \cap D$ is empty. Consider these two situations:

   If $B \cap C$ is empty, then (taking into account that $A \cap C$ is empty too) we conclude $(A \cup B) \cap C = \emptyset$, which contradicts our assumption.

   If $B \cap D$ is empty, then B is contained entirely in C and we have: $A \subset D$, $B \subset C$. Since C and D are disjoint, we get $A \cap B = \emptyset$. Contradiction.

   In the case of path connectedness, the proof is easier. We only need to show that each point $y \in A$ can be connected with each point $z \in B$ by a connected path.

   Take an arbitrary point x of the intersection $A \cap B$. This point can be connected by a continuous path with $y \in A$, and with $z \in B$ (by another path). This obviously means that $y \in A$ and $z \in B$ can be connected by the path that consists of two parts: first we go from y to x and then from x to z.

2. Let A be a connected subset of a topological space X. Prove that its closure $\bar{A}$ is also connected.

   Let A be a connected subspace of X. By contradiction, assume that its closure $\bar{A}$ is disconnected, i.e., there are disjoint open sets C and D such that $\bar{A} \subset C \cup D$ and neither $\bar{A} \cap C$ nor $\bar{A} \cap D$ is empty.

Since A is connected, it has to be contained entirely either in C, or in D. Assume for definiteness that $A \subset C$ and $A \cap D = \emptyset$. Take an arbitrary point $x \in \bar{A} \cap D$. Since $x \in \bar{A}$, this point is an adherent for A. By definition, any neighborhood of x (in particular, D) contains some points of A. In other words, $A \cap D \neq \emptyset$. Contradiction.

3. Does the connectedness of $A \cup B$ and $A \cap B$ imply that of A and B?

   No. As an example consider $A = (-1, 0] \cup (1, 2)$ and $B = (-2, -1] \cup [0, 1]$. The union of A and B is $A \cup B = (-2, 2)$, the intersection consists of just one point $A \cap B = \{0\}$. Both the interval $(-2, 2)$ and one-point set $\{0\}$ are connected, whereas A and B are not.

4. Prove that if A is a proper nonempty subset of a connected space X, then $\partial A \neq \emptyset$.

   Let $A \subset X$ be a proper nonempty subset and X be connected. As we know, X admits a natural partition into three disjoint subsets: the interior, boundary and exterior of A. The interior and exterior of A are open. If we assume that the boundary $\mathrm{Fr}\, A$ is empty, then we get the partition of X into two open nonempty disjoint subsets $\mathrm{Int}\, A$ and $\mathrm{Int}\,(X \setminus A)$, which is impossible since X is connected.

5. Prove that X is disconnected if and only if there is a continuous surjection $f : X \to S^0$, where S^0 is a discrete two-point space.

   Suppose that X is disconnected. Then $X = A \cup B$ where A and B are disjoint open non-empty sets. If $S^0 = \{1, -1\}$ is a discrete topological space that consists of two points, then we can define a continuous map $f : X \to S^0$ by putting $f(A) = 1$, $f(B) = -1$. The continuity of f immediately follows from the fact that both A and B are open.

   Conversely, if we assume that there is a continuous surjective map $f : X \to S^0$, then we can introduce two natural subsets

$A, B \subset X$ by putting $A = f^{-1}(1)$, $B = f^{-1}(-1)$. Since f is continuous and S^0 is discrete, A and B are both open. Since f is a surjection, neither A, nor B is empty. Thus, we have a partition of X into two open disjoint nonempty subsets: $X = A \cup B$, so that X is disconnected.

6. Which of the following spaces are connected?

   1) $[0, 1]$ with discrete topology;

   Disconnected because any discrete space (which contains at least 2 points) is disconnected.

   2) $[0, 1]$ with indiscrete topology;

   Connected because any indiscrete space is connected.

   3) $\mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}, (a, +\infty), a \in \mathbb{R}\}$;

   Connected, because any two nonempty proper open subsets, i.e. subsets of the form $(a_1, +\infty)$, $(a_2, +\infty)$, have non-trivial intersection:

$$(a_1, +\infty) \cap (a_2, +\infty) = (a, +\infty), \quad \text{where} \quad a = \max(a_1, a_2).$$

   4) the set of all $n \times n$-matrices;

   Connected. The set of all $n \times n$ matrices is a vector space of dimension $m = n^2$. Any vector space $\mathbb{R}^m$ is path connected and therefore connected. To see that $\mathbb{R}^m$ is path connected it suffices to notice that any two points $u, v \in \mathbb{R}^m$ can be joined by the continuous path $x(t) = u + t(v - u)$, $t \in [0, 1]$.

   5) the set $O(2)$ of orthogonal 2×2-matrices:

$$O(2) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : A^{-1} = A^{\top} \right\},$$

   (the same question for $O(n)$, the set of orthogonal $n \times n$-matrices);

   Disconnected. $O(2)$ consists of two components A and B: orthogonal matrices with $\det = 1$ and orthogonal matrices with $\det = -1$.

   It is not hard to see that

$$A = \left\{ \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}, \quad \phi \in [0, 2\pi] \right\},$$

$$B = \left\{ \left(\begin{array}{cc} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{array} \right), \quad \phi \in [0, 2\pi] \right\},$$

To explain why A and B are open subsets in $O(2)$, it suffices to notice that $A = f^{-1}(0, +\infty)$ and $B = f^{-1}(-\infty, 0)$ where $f : O(2) \to \mathbb{R}$ is a continuous function defined by $f(X) = \det X$.

6) the set of all real triangular 2×2 matrices

$$\left\{ A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\};$$

Connected. The set of all real triangular 2×2 matrices is a vector space of dimension 3, and we know that $\mathbb{R}^3$ is obviously connected.

7) the set of all real triangular 2×2 matrices with positive determinant

$$\left\{ A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \ : \ \det A > 0 \right\};$$

Disconnected. This set X can be represented as the union of two open nonempty disjoint subsets

$$X_1 = \left\{ A = \left(\begin{array}{cc} a & b \\ 0 & c \end{array} \right), \quad \det A > 0, \ a > 0 \right\}$$

and

$$X_2 = \left\{ A = \left(\begin{array}{cc} a & b \\ 0 & c \end{array} \right), \quad \det A > 0, \ a < 0 \right\}$$

The fact that X_1 and X_2 are open follows immediately from the following representation:

$$X_1 = f^{-1}(0, +\infty), \quad X_1 = f^{-1}(-\infty, 0),$$

where $f : X \to \mathbb{R}$ is defined by $f(A) = a$ (the first diagonal element).

8) $\{x^2 + y^2 - z^2 = a\}$ in $\mathbb{R}^3$ (consider 3 cases $a = -1, 0, 1$);

Connected for $a > 0$ (one sheet hyperboloid) and $a = 0$ (cone), and disconnected for $a < 0$ (two sheet hyperboloid).

9) $(x^2 + y^2 - 1)(x^2 + y^2 - 4) = 0$ in $\mathbb{R}^2$;

Disconnected. This set consists of two components $x^2+y^2-1 = 0$ and $x^2 + y^2 - 4 = 0$. Each of them is represented as a circle (of radius 1 and 2 respectively). These circles do not intersect, i.e. are disjoint.

10) $(x^2 + y^2 - 4)((x - 1)^2 - y^2 - 1) = 0$ in $\mathbb{R}^2$.

Connected. This set represents the union of two circles $x^2+y^2-4 = 0$ and $(x - 1)^2 + y^2 - 1 = 0$. Each of them is obviously connected. Since these circles intersect (the intersection point is $(2, 0)$), then their union is connected (see Question 1).

7. Using the connectedness concept, prove that $[0, 1)$, $[0, 1]$, $\mathbb{R}$, S^1 (circle), S^2 (sphere) are pairwise nonhomeomorphic.

   Each of the spaces $[0, 1)$, $[0, 1]$, $\mathbb{R}$ satisfy the following property: they becomes disconnected after removing one appropriate point. The circle S^1 and sphere S^2 do not have such a point: they remain connected if we remove **any** (just one!) of their points.

   This shows that $[0, 1)$, $[0, 1]$, $\mathbb{R}$ are topologically different from (i.e. non-homeomorphic to) S^1 and S^2.

   To distinguish S^1 and S^2 it suffices to notice that after removing any two points, S^1 becomes disconnected, whereas S^2 remains connected.

   To distinguish $\mathbb{R}$ from $[0, 1)$ and $[0, 1]$ we notice that after removing **any** point $x \in \mathbb{R}$, the real line $\mathbb{R}$ splits into two components, whereas $[0, 1)$ and $[0, 1]$ both remain connected after removing the point 0.

   Finally, to prove that $[0, 1)$ and $[0, 1]$ are not homeomorphic, we may notice that $[0, 1]$ remains connected after removing two (special!) points, namely 0 and 1. This does not hold for $[0, 1]$: removing any two points from $[0, 1)$ makes this space disconnected.

8. Prove that a circle is not homeomorphic to any subset of $\mathbb{R}$.

   The circle S^1 is not homeomorphic to any subset X of $\mathbb{R}$. Indeed, if X contains at least 3 points, we can find a point $x \in X$ such that $X_1 = X \cap (x, +\infty)$ and $X_2 = X \cap (-\infty, x)$ are not empty. Then the partition $X \setminus \{x\} = X_1 \cup X_2$ shows that $X \setminus \{x\}$ is disconnected. The situation with the circle S^1 is different: $S^1 \setminus \{a\}$ is connected for any point $a \in S^1$.

9. Prove that the set $\{xy = 0\} \subset \mathbb{R}^2$ is not homeomorphic to $\mathbb{R}$.

   If we remove the point $(0, 0)$ from $\{xy = 0\}$, then this space splits into four components, whereas after removing any point from $\mathbb{R}$ we get always two components only.