23MAC260 Problem Sheet 1

Lectures 1–3

1. Recall that a field K is *algebraically closed* if every nonconstant polynomial $f \in K[x]$ has a root in K. Let K be an algebraically closed field, and let $F(X, Y, Z)$ be a nonconstant homogeneous polynomial with coefficients in K. Show that the set

$$V(F) = \left\{[a, b, c] \in \mathbb{P}^2_K \mid F(a, b, c) = 0\right\}$$

is infinite.

2. Let p be a prime number. Show that the equation

$$X^3 + pY^3 + p^2Z^3 = 0$$

has no solutions in $\mathbb{Q}^3 \setminus \{(0, 0, 0\}$.

3. Let $f(x, y)$ be any polynomial in 2 variables. Show that $(f^h)_d = f$.

4. Let $F(X, Y, Z)$ be any homogeneous polynomial in 3 variables. Show that $(F_d)^h = F$ unless F is divisible by Z.

5. Let C be an elliptic curve: that is, a curve in $\mathbb{P}^2$ defined by the equation

$$Y^2Z = G(X, Z)$$

where the dehomogenisation $G_d(x)$ has 3 distinct roots.

Show that C intersects the line at infinity $\{Z = 0\}$ in a unique point $[0, 1, 0]$.

6. A **line** in $\mathbb{P}^2$ means a curve defined by a linear equation

$$aX + bY + cZ = 0$$

where a, b, c are not all 0.

Show that any line through the point $[0, 1, 0] \in \mathbb{P}^2$ is given by an equation of the form

$$aX + bZ = 0$$

for some a, b, not both 0.

7. Show that the curve C in $\mathbb{P}^2$ defined by the equation

$$Y^2Z = X^3 - 2X^2Z + XZ^2$$

is not an elliptic curve.

The following questions are optional and not examinable.

I. Let C be an ellipse given in the form

$$x^2 + \frac{y^2}{\alpha^2} = 1.$$

Show that the length $L(x_0)$ of the arc of C bounded by $x = -1$ and $x = x_0$ is given by

$$L(x_0) = \int_{-1}^{x_0} \frac{1 - \beta^2 x^2}{\sqrt{(1 - x^2)(1 - \beta^2 x^2)}} \, dx$$

where $\beta = 1 - \alpha^2$.

II. (For students who have taken MAC142 *Introduction to Algebraic Geometry*) In this question, you will prove that every nonsingular plane cubic can be put in the form of Equation (3) in the Week 1 notes (at least over $\mathbb{C}$). So let C be a curve in $\mathbb{P}^2$ defined by the equation

$$F(X, Y, Z) = 0$$

where F is a homogeneous cubic.

(a) Prove that C has at least 1 inflection point; that is, a point where the tangent line to C meets C to order 3. You may use the fact that inflection points of C are exactly the common zeroes of F and its *Hessian determinant*

$$H(F) = \det \begin{pmatrix} \frac{\partial^2 F}{\partial X^2} & \frac{\partial^2 F}{\partial X \partial Y} & \frac{\partial^2 F}{\partial X \partial Z} \\ \frac{\partial^2 F}{\partial X \partial Y} & \frac{\partial^2 F}{\partial Y^2} & \frac{\partial^2 F}{\partial Y \partial Z} \\ \frac{\partial^2 F}{\partial X \partial Z} & \frac{\partial^2 F}{\partial Y \partial Z} & \frac{\partial^2 F}{\partial Z^2} \end{pmatrix}$$

(b) Choose an inflection point $p \in C$. Show that there is a projective transformation φ of $\mathbb{P}^2$ which maps the point p to the point $[0, 1, 0]$ and maps the tangent line of C at p to the line defined by $Z = 0$. Deduce that the curve $C' = \varphi(C)$ is defined by an equation $F'(X, Y, Z) = 0$ where F' has no terms in Y^3, XY^2, or X^2Y,

(c) Finally, "complete the square" in Y to eliminate the YZ^2 and XYZ terms in $F'(X, Y, Z)$. Divide across by the coefficient of Y^2Z (which must be nonzero!) to get an equation in the form of Equation (3) in the Week 1 notes.