Group Actions, Orbit Stabiliser Theorem

<u>Notation</u>: For sets X and Y, their <u>product</u> is

$$X \times Y = \{ (x,y) \mid x \in X,\ y \in Y \}.$$

<u>Definition</u>: An action of a group G on a set X means a map

$$\varphi : G \times X \longrightarrow X$$
$$(g, x) \longmapsto g(x)$$

such that

i) $(gh)(x) = g(h(x)) \ \forall g, h \in G, \ \forall x \in X$

ii) $e(x) = x \quad \forall x \in X$

<u>Remark</u>: For each $g \in G$ the map

$$\varphi_g : X \longrightarrow X \qquad \text{is a bijection.}$$
$$x \longmapsto g(x)$$

It's inverse is $\varphi_{g^{-1}} : x \longmapsto g^{-1}(x)$.

since $\varphi_{g^{-1}}(\varphi_g(x)) = g^{-1}(g(x)) \overset{(i)}{=} (g^{-1}g)(x) \overset{(ii)}{=} x$

⌈ Indeed, another way to define an action is:

a homomorphism $\varphi : G \longrightarrow \mathrm{Bij}(X)$ ⌋

↳ bijections $X \longrightarrow X$

Examples 1) In Lecture 18 we had groups of rotational symmetries:

T: tetrahedron
O: cube/octahedron
I: icosahedron/dodecahedron

For each solid we get 3 actions of the group:

on set of vertices, set of edges, and set of faces.

[Think about why properties (i) + (ii) are true].

2) Groups of matrices: the groups

$$GL(n,\mathbb{R}) = \{ A \in Mat_{n\times n}(\mathbb{R}) \mid \det(A) \neq 0 \}$$

$$O(n) = \{ A \in Mat_{n\times n}(\mathbb{R}) \mid A^T A = I_n \}$$

both act on $\mathbb{R}^n$:

$$(A, x) \longmapsto A(x) =: Ax \in \mathbb{R}^n.$$

i) $(AB)(x) = A(Bx) = A(B(x)) \quad \forall A, B, \ \forall x$

ii) $I(x) = Ix = x \quad \forall x.$ ∎

Orbits + Stabilisers

Now suppose group G acts on set X.

Definition: For $x \in X$, the orbit of x is

$$Orb(x) =: \{ g(x) \mid g \in G \} \subset X$$

The <u>stabiliser of x</u> is

$$\mathrm{Stab}(x) =: \{ g \in G \mid g(x) = x \} \subset G$$

<u>Example:</u> $SO(3) = \{ A \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) \mid A^TA = I, \det A = 1 \}$.

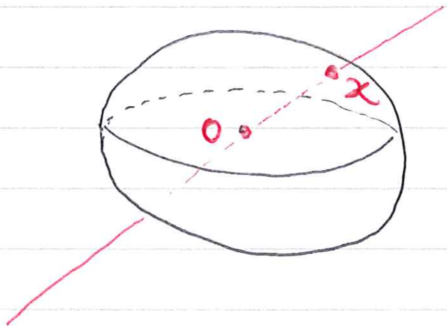
Let $x \neq 0$ be a vector in $\mathbb{R}^3$.

Then $\mathrm{Orb}(x) = \{ y \in \mathbb{R}^3 \mid y = Ax$ for some $A \in SO(3)\}$

orthogonal transformations preserve length!

$$= \{ y \in \mathbb{R}^3 \mid |y| = |x| \}$$

This is a sphere of radius $|x|$ centred at O.



$$\mathrm{Stab}(x) = \{ A \in SO(3) \mid Ax = x \}$$

This set consists of rotations around an axis through x, hence $\cong SO(2)$.

<u>Proposition:</u> $\mathrm{Stab}(x)$ is a subgroup of G.

<u>Proof:</u> Write $H = \mathrm{Stab}(x)$ for convenience.

i) If $h_1, h_2 \in H$ then $h_1(x) = h_2(x) = x$.

$$\Rightarrow (h_1 h_2)(x) \underset{\text{property (ii) in definition.}}{=} h_1(h_2(x)) = h_1(x) = x$$

So H is closed under multiplication

(ii) $e(x) = x$ so $e \in H$.

iii) If $h \in H$ then $h^{-1}(x) = h^{-1}(h(x)) \overset{\text{property (i)}}{=} (h^{-1}h)(x) \overset{(ii)}{=} x$

so $h^{-1} \in H$. So H is closed under inverses. $\blacksquare$

Theorem (Orbit-Stabiliser Theorem)

Let G act on X. For each $x \in X$ there is a bijection $\underbrace{\mathrm{Orb}(x)}_{\text{orbit of } x.} \longrightarrow \underbrace{G/\mathrm{Stab}(x)}_{\text{set of cosets of } \mathrm{Stab}(x) \text{ in } G}$

Proof: Write $H = \mathrm{Stab}(x)$.

Define a map $\Phi : \mathrm{Orb}(x) \longrightarrow G/H$

$$g(x) \longmapsto gH.$$

i) Φ is well-defined:

suppose $g_1(x) = g_2(x)$. Then $\underbrace{g_1^{-1}g_1(x)}_{= x} = g_1^{-1}g_2(x)$

So $g_1^{-1}g_2 \in \mathrm{Stab}(x) = H$.

$\Rightarrow g_2 = g_1 h$, some $h \in H$

$\Rightarrow g_2H = (g_1h)H = g_1(hH) = g_1H$

ii) Φ is surjective: check yourself.

iii) Φ is injective: if $g_1 H = g_2 H$

then $g_1^{-1} g_2 \in H$, so $g_1^{-1} g_2(x) = x$, hence
$\quad\quad\quad\quad\;\; \| \; stab(x)$

$$g_1(g_1^{-1} g_2)(x)) = g_1(x)$$
$$\iff \quad g_2(x) = g_1(x).$$

So Φ is a bijection. Therefore $|Orb(x)| = |G/H|$

We know $|G/H| = |G|/|H|$ (last lecture)

so $|G| = |H| \cdot |Orb(x)|$

$$= |Stab(x)| \cdot |Orb(x)|. \quad \blacksquare$$

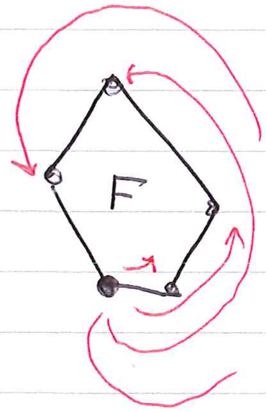
<u>Application</u>: Let's use this to count symmetries of the dodecahedron/icosahedron again.

Remember: I = group of rotational symmetries.

I acts on the set of faces of dodecahedron.

$|Faces| = 12.$

Let F be a face.



Then $\mathrm{Stab}(F) = \left\{ \begin{matrix} \text{rotational symmetries} \\ \text{sending } F \text{ to itself} \end{matrix} \right\}$

So $|\mathrm{Stab}(F)| = 5$.

On the other hand, we can find a rotation taking F to any chosen faces

So $\mathrm{Orb}(F) = \{\text{all faces}\}$

$\Rightarrow \quad |\mathrm{Orb}(F)| = 12$.

Hence $|I| = |\mathrm{Orb}(F)| \cdot |\mathrm{Stab}(F)| = 60$.

Remark: Could also use this to calculate the order of group of all symmetries $\tilde{I}$.

Let $\tilde{I}$ act on faces. Then for a face F we have $|\mathrm{Orb}(F)| = 12$ still, but

$\mathrm{Stab}(F) \cong D_5$ so $|\mathrm{Stab}(F)| = 10$

Hence $|\tilde{I}| = 120$.