MAB298-Elements of Topology: Solution Sheet 3
Continuous maps and homeomorphisms

1. Let $f : X \to Y$ be a continuous map between two metric spaces, and (x_n) be a convergent sequence. Is it true that the sequence $y_n = f(x_n)$ converges?

   Yes, $y_n = f(x_n)$ converges to $y = f(x)$, where $x = \lim_{n\to\infty} x_n$. Indeed, consider an arbitrary neighborhood $V(y)$ of y. Then, since f is continuous, there exists a neighborhood $U(x)$ such that $f(U(x)) \subset V(y)$. The convergence of x_n to x implies that starting from a certain natural number $N \in \mathbb{N}$ we have $x_n \in U(x)$ for all $n > N$. Then $y_n = f(x_n) \in V(y)$, $n > N$. But this exactly means that y_n converges to $y = f(x)$.

2. Prove that the image of an everywhere dense set under a surjective continuous map is everywhere dense.

   Let $f : X \to Y$ be a continuous surjective map and $A \subset X$ be everywhere dense in X. To prove that $B = f(A)$ is everywhere dense in Y we need to show that $\bar{B} = Y$, i.e. each point $y \in Y$ is adherent for B. In other words, we need to show that in any neighborhood $V(y)$ there are points of B.

   Let $V(y)$ be an arbitrary neighborhood of $y \in Y$. Since f is surjective, there is $x \in X$ such that $f(x) = y$, and since f is continuous, there is a neighborhood $U(x)$ such that $f(U(x)) \subset V(y)$. Now we use the fact that A is everywhere dense in X: in $U(x)$ there are points of A. Take such a point $a \in A \cap U(x)$. Then $b = f(a) \in V(y)$, i.e. $V(y) \cap B \neq \emptyset$. Thus, $B = f(A)$ is everywhere dense in Y, as needed to be proved.

3. Is it true that the image of nowhere dense set under a continuous map is nowhere dense?

   No. Example: Take the vertical projection $p : \mathbb{R}^2 \to \mathbb{R}$, $p(x, y) = x$ and consider the line $A = \{y = x\}$ in $\mathbb{R}^2$. This line A is nowhere dense in $\mathbb{R}^2$, however its image $p(A)$ coincides with $\mathbb{R}$ and is obviously everywhere dense in $\mathbb{R}$.

4. Let $X = C^1[0, 1]$ be the space of continuously differentiable functions on $[0, 1]$ with the metric $d(f, g) = \max_{x\in[0,1]} |f(x) - g(x)|$.

Let $F_1 : X \to \mathbb{R}$ be defined by $F_1(f) = \int_0^1 f(x)dx$. Is F_1 continuous?

Let $F_2 : X \to \mathbb{R}$ be defined by $F_2(f) = f'(0)$. Is F_2 continuous?

The map F_1 (integration) is continuous. To see this, consider an arbitrary function $f \in C^1[0,1]$ and let $a = F_1(f) = \int_0^1 f(x)dx$. To check that F_1 is continuous at f, we need to show that for any $\varepsilon > 0$ one can find $\delta > 0$ such that $d(f,g) < \delta$ implies $|F_1(f) - F_1(g)| < \varepsilon$.

In our case this can easily be done. It suffices to put $\delta = \varepsilon$. Then

$$|F_1(f) - F_1(g)| = \left| \int_0^1 (f(x) - g(x))\, dx \right| \le \int_0^1 |f(x) - g(x)| dx \le$$

$$\le \int_0^1 \max_{x\in[0,1]} |f(x) - g(x)| dx = \int_0^1 d(f,g)\, dx < \int_0^1 \varepsilon\, dx = \varepsilon.$$

Roughly speaking, the continuity of the integration means the following:

if two functions are sufficiently close in the sense of the above metric d, then their integrals are close too.

The map F_2 (differentiation at 0) is not continuous. This means, in fact, that

even if two functions are very close in the sense of the metric d, their derivatives may be substantially different.

More rigorously we will show that F_2 is not continuous at the point $f = 0$. By definition, this means that there exists $\varepsilon > 0$ such that for any $\delta > 0$ we can find a function g such that $\max_{x\in[0,1]} |g(x)| < \delta$, but $g'(0) > \varepsilon$.

Put $\varepsilon = 1/2$. Then for any fixed $\delta > 0$ we can take the function $g(x) = \frac{\delta}{2} \sin \frac{2x}{\delta}$ for which we have:

$$\max_{x\in[0,1]} |g(x)| = \max_{x\in[0,1]} \frac{\delta}{2} \sin \frac{2x}{\delta} \le \frac{\delta}{2} < \delta,$$

but

$$F_2(g) = g'(0) = \cos \frac{2x}{\delta} \Big|_{x=0} = 1 > \frac{1}{2}.$$

This shows that F_2 is discontinuous at $f = 0$.

5. Prove that the following plane figures are homeomorphic to each other.

   (a) A half-plane: $\{x \geq 0\}$;

   (b) a quadrant: $\{x, y \geq 0\}$;

   (c) an angle: $\{x \geq y \geq 0\}$;

   (d) a square without two sides: $\{0 \leq x, y < 1\}$;

   (e) a disk without a boundary point: $\{x^2 + y^2 \leq 1, y \neq 1\}$;

   (f) a half-disk without the diameter: $\{x^2 + y^2 \leq 1, y > 0\}$;

   Figure 1 shows how these figures can be continuously deformed one to another in such a way that the resulting map is a homeomorphism. In topology, indicating such a deformation is usually considered as a rigorous proof of the fact that two given spaces are homeomorphic (although this kind of proof is rather informal). If the topological objets under consideration are complicated, such an explanation is sometimes the only thing we can do.

   However, for the above domains in $\mathbb{R}^2$, homeomorphisms between them can be described by explicit formulas.

   Let us denote the above domains (a),..., (f) by $X_1, \ldots, X_6$ respectively.

   By (r, ϕ) we denote the standard polar coordinates in $\mathbb{R}^2$. Recall that $x = r \cos \phi$, $y = r \sin \phi$.

   A homeomorphism between X_2 and X_1 can be given by:

   $$f_{21} : X_2 \to X_1, \qquad f_{21}(r, \phi) = \left(r,\ 2\phi - \frac{\pi}{2}\right).$$

   It is easy to see that f_{21} is a continuous bijection. The inverse map

   $$f_{12} : X_2 \to X_1, \qquad f_{12}(r, \phi) = \left(r,\ \frac{1}{2}(\phi + \frac{\pi}{2})\right)$$

is continuous too.

A homeomorphism between X_3 and X_2 can be given by

$$f_{32}: X_3 \to X_2, \qquad f_{32}(r,\phi) = (r,\ 2\phi)\,.$$

A homeomorphism between X_4 and X_2 can be given by

$$f_{42}: X_4 \to X_2, \qquad f_{42}(x,y) = \left(\tan\frac{\pi x}{2}, \tan\frac{\pi y}{2}\right).$$

A homeomorphism between X_5 and X_1 can be constructed geometrically in the following way. Instead of X_1, it will be more convenient to take the half plane $\{y \geq 0\}$. We shall still denote it by X_1.

To find the image of a point $P \in X_5$ in X_1, we first consider the ray starting at $S = (0,1)$ (the removed boundary point) and passing through P (see Fig. 2). This ray intersects the boundary circle $\{x^2 + y^2 = 1\}$ at a certain point Q, and the horizontal line $y = 0$ at a point R. By definition, we set the image of P to be the point $f(P)$ on the vertical ray starting at R such that the distance between $f(P)$ and R is the ratio $\dfrac{PQ}{PS}$. Geometrically this means that the segment SQ (the endpoint S is excluded) is mapped onto this vertical ray in such a way that Q is mapped to R and when $P \in SQ$ moves from Q to S, its image $f(P)$ moves on this ray in the vertical direction approaching the infinity as P tends to S. Obviously, f is continuous and admits a well-defined inverse map $f^{-1}: X_1 \to X_5$ which can be easily described in geometric terms and is continuous.

If we denote the coordinates of P by (x,y), then it is not so hard to express the equation of the ray and the coordinates of all the points Q, R and $f(P)$ in terms of (x,y).

To prove that X_6 and X_1 are homeomorphic, it will be convenient to replace X_6 by the half disc of the same size but located in a bit different way: $x^2 + (y-1)^2 \leq 1, y < 1$ (see. Fig 2). We leave the same notation X_6 for this half disk.

Now a homeomorphism between X_6 and $X_1 = \{y \geq 0\}$ can be constructed just in the same same way as before. To find the image of a point $P \in X_6$ in X_1, we first consider the ray starting at $S = (0, 1)$ (the center of the disk) and passing through P (see Fig. 3). This ray intersects the boundary circle $\{x^2 + (y-1)^2 = 1\}$ at a certain point Q, and the horizontal line $y = 0$ at a point R. By definition, we set the image of P to be the point $f(P)$ on the vertical ray starting at R such that the distance between $f(P)$ and R is the ratio $\dfrac{PQ}{PS}$. Geometrically this means that the segment SQ (the endpoint S is excluded) is mapped onto this vertical ray in such a way that Q is mapped to R and when $P \in SQ$ moves from Q to S, its image $f(P)$ moves on this ray in the vertical direction approaching the infinity as P tends to S. Obviously, f is continuous and admits a well-defined inverse map $f^{-1} : X_1 \to X_6$ which can be easily described in geometric terms and is continuous.

Thus, we have described homeomorphisms $X_2 \to X_1$, $X_3 \to X_2$, $X_4 \to X_2$, $X_5 \to X_1$ and $X_6 \to X_1$. Since being homeomorphic is an equivalence relation, we conclude that all the domains $X_1, \dots, X_6$ are pairwise homeomorphic.

6. Prove that the following plane domains are homeomorphic to each other:

   (a) punctured plane $\mathbb{R}^2 \setminus (0, 0)$;

   (b) punctured open disk $B^2 \setminus (0, 0) = \{0 < x^2 + y^2 < 1\}$;

   (c) annulus $\{a < x^2 + y^2 < b\}$, where $0 < a < b$;

   (d) plane without a disk: $\mathbb{R}^2 \setminus D^2$;

   (e) plane without a segment: $\mathbb{R}^2 \setminus [0, 1]$;

   See Fig. 4 which illustrates the deformations between these domains.

   More formally, homeomorphisms between these domains (which we denote by Y_1, Y_2, Y_3, Y_4, Y_5 respectively) can be defined as follows.

   As a homeomorphism between Y_2 (punctured disk) and Y_1 we can take the map

$$f_{21} : Y_2 \to Y_1, \quad f_{21}(r, \phi) = \left(\tan \frac{\pi r}{2}, \phi\right) \qquad \text{(in polar coordinates)}.$$

As a homeomorphism between Y_3 (annulus) and X_1 we can take

$$f_{31} : Y_3 \to Y_1, \quad f_{31}(r, \phi) = (h(r), \phi) \qquad \text{(in polar coordinates)},$$

where $h(r)$ is a numerical function which establishes a homeomorphism between the interval (a, b) and the half line $\mathbb{R}^+ = (0, +\infty)$, for example, $h(r) = \dfrac{r-a}{b-r}$.

As a homeomorphism between $Y_4 = \{x^2+y^2 > 1\}$ (plane without a disk) and Y_1 we can take

$$f_{41} : Y_4 \to Y_1, \quad f_{41}(r, \phi) = (r-1, \phi) \qquad \text{(in polar coordinates)}.$$

A homeomorphism between Y_4 (plane without a disc) and Y_5 (plane without a segment) can be constructed in the following way. For simplicity, we assume that the removed segment is $y = 0, x \in [-1, 1]$.

Let $P \in Y_4$ and the coordinates of P be (x, y). If $|x| \geq 1$, then we do not change the position of P, i.e., we put $f(P) = P$. Let $|x| < 1$. This condition means that P is located either over or under the removed disk. For definiteness we assume that P is in the upper half plane. Then we take the vertical ray starting from the boundary circle $x^2 + y^2 = 1$ (i.e., from the point $S = (x, \sqrt{1-x^2})$) and passing through P and move it down until its origin touches the horizontal line $y = 0$ (see Fig. 5). In coordinates, this means that

$$f(x, y) = (x, y - \sqrt{1-x^2}).$$

If P is located in the lower half plane, we consider a similar ray but directed downwards and move it up until its origin touches the horizontal line (see Fig. 5). In coordinates:

$$f(x, y) = (x, y + \sqrt{1-x^2}).$$

As a result of this mapping, the plane without the disk will be transformed to the plane without the segment $[-1, 1]$ (the diameter of this disc). This map can be globally defined by the following formula which involves all possible situations:

$$f_{45}(x, y) = \left(x, y - \operatorname{sgn}(y)\sqrt{\max\{0, 1-x^2\}}\right),$$

where $\operatorname{sgn}(y) = -1, 0, 1$ depending on whether y is negative, zero or positive respectively.

We have constructed homeomorphisms $Y_2 \to Y_1$, $Y_3 \to Y_1$, $Y_4 \to Y_1$ and $Y_4 \to Y_5$. Taking into account that "being homeomorphic" is an equivalence relation, we may conclude that Y_1, Y_2, Y_3, Y_4, Y_5 are all pairwise homeomorphic.

7. Let $K = \{a_1, \ldots, a_n\} \subset \mathbb{R}^2$ be a finite set. The complement $\mathbb{R}^2$. Convince yourself that any two planes with n punctures are homeomorphic, i.e., the position of $a_1, \ldots, a_n$ in $\mathbb{R}^2$ does not affect the topological type of $\mathbb{R}^2 \setminus \{a_1, \ldots, a_n\}$.

8. Let $D_1, \ldots, D_n \subset \mathbb{R}^2$ be pairwise disjoint closed disks. The complement of the union of its interiors is said to be plane with n holes. Convince yourself that any two planes with n holes are homeomorphic, i.e., the location of disks $D_1, \ldots, D_n$ does not affect the topological type of $\mathbb{R}^2 \setminus \bigcup_{i=1}^n \operatorname{Int} D_i$.

9. Prove that a mug (with a handle) is homeomorphic to a doughnut.

   See a nice video clip presented in Wikipedia:

   http://en.wikipedia.org/wiki/Homeomorphism