23MAC260 Problem Sheet 8: Solutions

Week 8 Lectures
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1. Let $f(z)$ be the meromorphic function

$$f(z) = \frac{3\cos^2(z)}{\sin^2(2z)}.$$

(a) Compute the order $\mathrm{ord}(f, 0)$ of f at 0.

Solution: We use the series expansions for $\cos$ and $\sin$ around 0:

$$\cos(z) = 1 - \frac{1}{2}z^2 + \cdots$$
$$\sin(z) = z - \frac{1}{6}z^3 + \cdots$$

to obtain

$$3\cos^2(z) = 3 - 3z^2 + \cdots$$
$$\sin^2(2z) = 4z^2 - \frac{16}{3}z^4 + \cdots$$

Putting these together, we see that that $f(z)$ can be written in the form

$$f(z) = \frac{1}{z^2}\left(\frac{3 - 3z^2 + \cdots}{4 - \frac{16}{3}z^2 + \cdots}\right)$$

Denote the expression in parentheses by $h(z)$. Then plugging in $z = 0$ we see that $h(0) = \frac{3}{4}$, so if we expand h in a power series about $z = 0$ we will get something of the form

$$h(z) = \frac{3}{4} + a_1 z + a_2 z^2 + \cdots$$

(In fact since the numerator and denominator of h only involve **even** powers of z, in our series expansion only even powers of z will have nonzero coefficients.)

Combining this with our previous equation gives

$$f(z) = \frac{3}{4}z^{-2} + \cdots$$

So $\operatorname{ord}(f, 0) = -2$.

(b) Compute the residue $\operatorname{Res}(f, 0)$ of f at 0.

Solution: Recall that $\operatorname{Res}(f, 0)$ means the coefficient of z^{-1} in the series expansion of f around 0. As remarked above, in the series expansion of the function h around 0, only even powers of z have nonzero coefficients, and since $f(z) = z^{-2}h(z)$, the same thing is true for f. Hence

$$\operatorname{Res}(f, 0) = 0.$$

(c) Compute the integral

$$\int_\gamma f(z)\, dz$$

where γ is the ellipse in the complex plane defined by the equation

$$17\operatorname{Re}(z)^2 + 23\operatorname{Im}(z)^2 = 13.$$

Solution: As you might guess, the solution has nothing to do with the form of the ellipse. We will use the Reside Theorem which says that

$$\int_\gamma f(z)\, dz = 2\pi\, i \sum_{i=1}^{n} \operatorname{Res}(f, z_i)$$

where $z_1, \ldots, z_n$ are the poles of f that lie inside γ.

Now the only possible poles of f are those points where its denominator $\sin^2(2z)$ equals 0. We have

$$\begin{aligned} \sin^2(2z) &= 0 \\ \Leftrightarrow \sin(2z) &= 0 \\ \Leftrightarrow z &= \frac{k}{2}\pi, \quad k \in \mathbb{Z}. \end{aligned}$$

Now if $k \neq 0$ then $|\frac{k}{2}\pi| > 1$ and so $17(\frac{k}{2}\pi)^2 > 13$, so these (possible) poles of f are all outside the ellipse γ. Hence there is only one possible pole of f inside γ, namely $z = 0$. Considering the Laurent expansion of f around 0, we can see that $z = 0$ is indeed a pole, since the expansion starts with a negative power of z. Therefore the Residue Theorem says that

$$\begin{aligned} \int_\gamma f(z)\, dz &= 2\pi\, i \operatorname{Res}(f, 0) \\ &= 0. \end{aligned}$$

Remark: Using the trigonometric identity

$$\sin(2z) = 2\sin(z)\cos(z)$$

we can rewrite f as

$$f(z) = \frac{3}{4}\left(\frac{1}{\sin^2(z)}\right)$$

This expression for f shows that in fact it has poles only at $z = k\pi$ for $k \in \mathbb{Z}$.

2. Show that if $f(z)$ is meromorphic at a point $z_0 \in \mathbb{C}$, then for the function $g(z) = f'(z)/f(z)$ we have

$$\mathrm{Res}(g, z_0) = \mathrm{ord}(f, z_0).$$

(This was used in the proof of the Equivalence Theorem in Week 9.)

Solution: Denote $\mathrm{ord}(f, z_0) = k$, so the series expansion of f around z_0 looks like:

$$f(z) = a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \cdots$$

where the coefficient a_k is nonzero. Differentiating this expresssion we get the series expansion for $f'(z)$ around z_0:

$$f'(z) = ka_k(z - z_0)^{k-1} + +(k + 1)(a_{k+1})(z - z_0)^k + \cdots$$

and hence

$$g(z) = \frac{ka_k(z - z_0)^{k-1} + (k + 1)(a_{k+1})(z - z_0)^k + \cdots}{a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \cdots}$$

We can divide top and bottom by $a_k(z - z_0)^k$ to rewrite this in the form

$$g(z) = \frac{k(z - z_0)^{-1} + \cdots}{1 + (z - z_0)\gamma(z - z_0)}$$

where $\gamma(z - z_0)$ is some power series with nonnegative powers of $z - z_0$ only. Using the series representation

$$\frac{1}{1 + x} = \sum_{n=0}^{\infty}(-1)^n x^n$$

applied with $x = (z - z_0)\gamma(z - z_0)$ we get

$$g(z) = k(z - z_0)^{-1} + \cdots$$

and hence $\mathrm{Res}(g, z_0) = k = \mathrm{ord}(f, z_0)$.

3. Let L be the lattice spanned by the complex numbers ω_1 ad ω_2. Let m, n, r, s be integers. Let L' be the lattice spanned by

$$\tau_1 = m\omega_1 + n\omega_2$$
$$\tau_2 = r\omega_1 + s\omega_2.$$

Show that $L = L'$ if and only if the matrix

$$M = \begin{pmatrix} m & n \\ r & s \end{pmatrix}$$

has determinant ± 1. (Equivalently, M is invertible and its inverse is also an integer matrix.)

Solution: First suppose that $L = L'$. That means in particular that ω_1 and ω_2 are elements of L'. Every element of L' is an integer linear combination of τ_1 and τ_2, so there are integers μ, ν, ρ, σ such that

$$\omega_1 = \mu\tau_1 + \nu\tau_2$$
$$\omega_2 = \rho\tau_1 + \sigma\tau_2$$

Substituting the original expressions for τ_1 and τ_2 into these new equations we get

$$\omega_1 = \mu(m\omega_1 + n\omega_2) + \nu(r\omega_1 + s\omega_2)$$
$$\omega_2 = \rho(m\omega_1 + n\omega_2) + \sigma(r\omega_1 + s\omega_2)$$

Simplify these gives

$$\omega_1 = (\mu m + \nu r)\omega_1 + (\mu n + \nu s)\omega_2$$
$$\omega_2 = (\rho m + \sigma_r)\omega_1 + (\rho n + \sigma s)\omega_2$$

We can write this as a matrix equation as follows:

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix} \cdot M \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

Now if we write $\omega_1 = x_1 + iy_1$ and $\omega_2 = x_2 + iy_2$ and consider the real and imaginary parts of the above equation we get

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix} \cdot M \cdot \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$$

Since ω_1 and ω_2 are not real multiples of each other, the matrix on the left-hand side is invertible. Multiplying on both sides by its inverse we get

$$I_2 = \begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix} \cdot M$$

So M is invertible with inverse an integer matrix, hence has determinant ± 1.

Conversely, if M has determinant ± 1 then there is an integer matrix

$$M^{-1} = \begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix}$$

Starting from the equation

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = M \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

we can multiply on both sides by M^{-1} to get

$$\begin{aligned} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} &= M^{-1} \cdot \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \\ &= \begin{pmatrix} \mu\tau_1 + \nu\tau_2 \\ \rho\tau_1 + \sigma\tau_2 \end{pmatrix} \end{aligned}$$

Since L is spanned by ω_1 and ω_2, this shows $L \subset L'$. But $L' \subset L$ since τ_1 and τ_2 are by definition linear combinations of ω_1 and ω_2. Hence $L = L'$.