Elements of Topology

Exam 2021 June

No 1. (a) See end of notes

(b) $A = [0,1]$ (standard question)

(i) $A \subset X_1 = \mathbb{R}$ with the discrete topology

A is closed as every subset of a discrete topological space is closed [2]

(ii) $A \subset X_2 = \mathbb{R}$ with the indiscrete topology

A is not closed as the only closed subsets in this case are ϕ and $\mathbb{R}$. [2]

(iii) $A \subset X_3 = \mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, (a,+\infty)_{a \in \mathbb{R}}\}$

A is not closed as the closed subsets in this case are ϕ, $\mathbb{R}$, $(-\infty, a]$, $a \in \mathbb{R}$ [2]

For (ii): $\bar{A} = \mathbb{R}$ as $\mathbb{R}$ is the smallest closed subset that contains A.

For (iii): $\bar{A} = (-\infty, 1]$ as this interval is the smallest closed subset that contains A.

[3]

(c) see end of notes

(d) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function.
Then $A = \{(x,y) \in \mathbb{R}^2 : -1 \leq f(x,y) \leq 1\}$ can be understood as $A = f^{-1}([-1,1])$

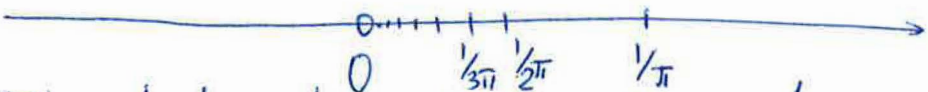
By definition, the preimage of any open subset $B \subset \mathbb{R}$ under f is open. Since $f^{-1}(\mathbb{R} \setminus B) = \mathbb{R}^2 \setminus f^{-1}(B)$, we see that the preimage of any closed set $C = \mathbb{R} \setminus B$ is closed (being the complement to the open set $f^{-1}(B)$).
Since $[-1,1]$ is a closed subset of $\mathbb{R}$, $A = f^{-1}([-1,1])$ is closed too.

[3]
(unseen)

(e) Notice that $\sin \frac{1}{x}$ is not continuous at $x = 0$, so the statement from (d) cannot be applied.

$$\sin \frac{1}{x} = 0 \Rightarrow \frac{1}{x} = \pi k, \; k \in \mathbb{Z} \Rightarrow x = \frac{1}{\pi k}, \; k \in \mathbb{Z}, \; k \neq 0$$

[3]
(unseen)

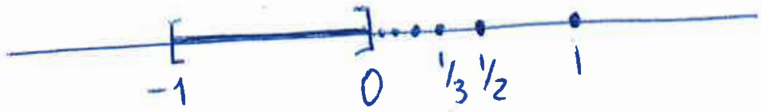


The set A contains the sequence $\frac{1}{\pi k}$, $k \in \mathbb{N}$ convergent to 0.
Hence, 0 is a limit point of A which does not belong to A. $\Rightarrow A$ is not closed

No. 2 (a), (b) see end of notes

(c) Compact or not? (standard question)

(i)



bounded and closed $\Rightarrow$ compact [2]

(ii) $\{x^4+y^4-x^2-y^2 \le 1/2\} \subset \mathbb{R}^2$

closed subset [2]

$x^4+y^4-x^2-y^2 \le 1/2$

$(x^2-\frac{1}{2})^2+(y^2-\frac{1}{2})^2 \le 1/2+1/4+1/4=1$ bounded and closed $\Rightarrow$ compact

$\Rightarrow$ $x^2-\frac{1}{2} \le 1$ and $y^2-\frac{1}{2} \le 1$ $\Rightarrow$ bounded

(iii) $\{\sin(x+y)=\cos(x-y)\}=A$ [2]

closed but not bounded $\Rightarrow$ not compact

as $(\frac{\pi}{4}+2\pi k, \frac{\pi}{4}) \in A$

$k \to +\infty$ [3]

(iv) $(0,1) \subset X=\mathbb{R}$ with indiscrete topology

compact as every subset of an indiscrete topological space is indiscrete and therefore compact.

(v) $(0,1) \subset X=\mathbb{R}$ with $\tau=\{\mathbb{R}, \emptyset, (a,+\infty), a\in\mathbb{R}\}$

Not compact. $\{U_k=(\frac{1}{k},+\infty), k\in\mathbb{N}\}$ is an open covering that does not admit any finite subcovering.

No 3 (a) see end of notes

(b) Consider $f : X \to Y$ continuous

X pathwise connected

Let $y_1, y_2 \in f(X)$. Take $x_1, x_2 \in X$ such that $y_1 = f(x_1)$, $y_2 = f(x_2)$. Since X is pathwise connected, there is a continuous path $\gamma : [0,1] \to X$ from x_1 to x_2. Consider the composition $f \circ \gamma : [0,1] \to Y$. Obviously, $f \circ \gamma$ is a continuous path between y_1 and y_2 in the image $f(X)$. Thus, $f(X)$ is pathwise connected, as required.

[4]
(bookwork)

(c) $X = \mathbb{Z}$ with the indiscrete topology.

Yes, there is a continuous path from 0 to 1.

For example
$$f(t) = \begin{cases} 0, & \text{if } t \in [0, 1/2) \\ 1, & \text{if } t \in [1/2, 1] \end{cases}$$

[4]
(unseen)

This map $f : [0,1] \to \mathbb{Z}$ is continuous as any map to an indiscrete topological space is continuous.

In a similar way, one can construct a continuous path from k to m, $k, m \in \mathbb{Z}$. Hence, X is pathwise connected.

(d)

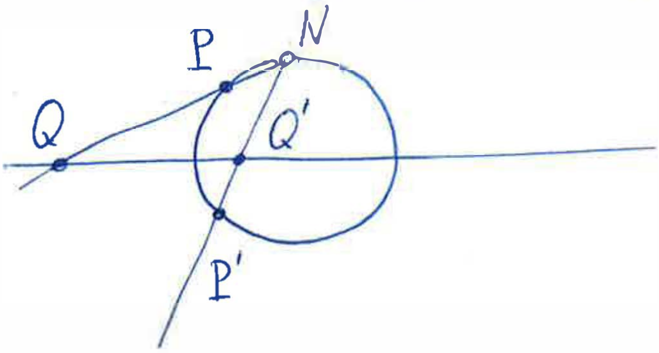
) see end of notes

(standard question)

(e) (i) No, because X is connected, whereas Y is not. [2]

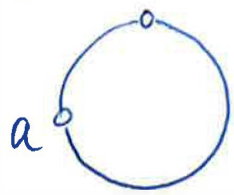
(ii) Yes, the stereographic projection

$f : X \to \mathbb{R}$, $\quad f: P \mapsto Q$
$\qquad\qquad\qquad\quad P' \mapsto Q'$ [2]



is a homeomorphism between X and $\mathbb{R}$

(iii) No, indeed if we remove a point from X, then X splits into two connected components.



$X \setminus \{a\}$

On the other hand, if we remove a point from $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus \{p\}$ remains connected

. [2]

No 4.

(a) See end of notes

(b) $X = \{ e^{x+y} - 2x - 2y + xz + yz = 0 \} \subset \mathbb{R}^3$.

We first find the singular points of $F(x,y,z) = e^{x+y} - 2x - 2y + xz + yz$

$$dF = 0 \iff \begin{aligned} & e^{x+y} - 2 + z = 0 && (1) \\ & e^{x+y} - 2 + z = 0 && (2) \\ & x + y = 0 && (3) \end{aligned}$$

[4]
(standard question)

Substituting $x+y=0$ into (1) and (2) gives

$$1 - 2 + z = 0 \implies z = 1$$

Thus, the singular points of F are the points satisfying

$$x + y = 0, \ z = 1$$

Substituting these values into $F(x,y,z)$ gives

$$F(x,y,z) = e^0 - 2 \cdot 0 + 1 \cdot 0 = 1 \neq 0.$$

Thus, X does not contain any singular points.
Therefore, by the Implicit Function Theorem,
X is a manifold of dimension 2.

(c) $X = \{ (x+y)(x^2+y^2) = 0 \} \subset \mathbb{R}^2$ Straightforward computation ~~The implicit function theorem~~ shows that $(0,0)$ is a singular point. And $(0,0) \in X$.

However, $(x+y)(x^2+y^2) = 0$ means that

either $x+y=0$

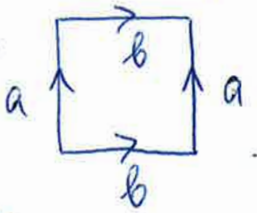
or $x^2+y^2=0$, i.e. $x=y=0$

[4] (unseen)

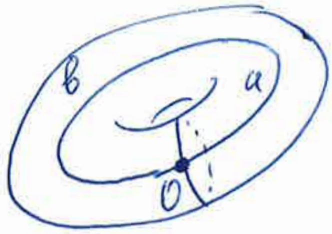
Thus, X is the union of the line $x+y=0$ and the point $(0,0)$ that belongs to this line, i.e. X is simply the line $x+y=0$ and, therefore, is a manifold of dimension 1.

[4] (bookwork)

(d) The torus can be defined as the surface obtained from the square



by pairwise identification of the edges as shown.



This representation defines an admissible partition of the torus that contains

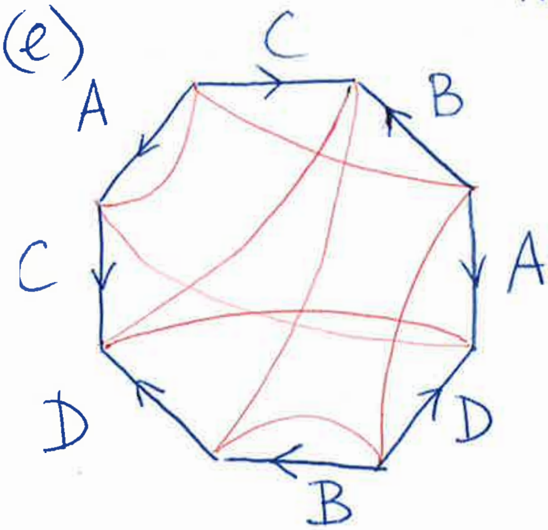
one face $F=1$

two edges $E=2$ and

one vertex $V=1$

Thus, the Euler characteristic is $\chi = 1-2+1 = 0$.

(e)



The surface is orientable, as each edge appears in combination $\ldots A \ldots A^{-1} \ldots$

The Euler characteristic is

$\chi = F - E + V = 1 - 4 + 1 = -2$

[4] (standard question)

Orientable with $\chi = -2 \Rightarrow$

M is the sphere with two handles.

② (b) (i) X, Y Hausdorff $\Rightarrow X \times Y$ Hausdorff

<u>Proof</u>: Let $(x_1, y_1), (x_2, y_2) \in X \times Y$ be distinct points, i.e. either $x_1 \neq x_2$ or $y_1 \neq y_2$. If $x_1 \neq x_2$, let

$$\underbrace{U(x_1)}_{\text{nhood of } x_1} \cap \underbrace{U(x_2)}_{\text{nhood of } x_2} = \emptyset$$

(possible since X is Hausdorff)

Then $\underbrace{U(x_1) \times Y}_{\text{nhood of } (x_1, y_1)} \cap \underbrace{U(x_2) \times Y}_{\text{nhood of } (x_2, y_2)} = \emptyset$.

If $y_1 \neq y_2$, let

$$\underbrace{V(y_1)}_{\text{nhood of } y_1} \cap \underbrace{V(y_2)}_{\text{nhood of } y_2} = \emptyset$$

(Possible since Y is Hausdorff)

Then $\underbrace{X \times V(y_1)}_{\text{nhood of } (x_1, y_1)} \cap \underbrace{X \times V(y_2)}_{\text{nhood of } (x_2, y_2)} = \emptyset$

In both cases we have found disjoint nhoods of (x_1, y_1) and (x_2, y_2). $\square$

② (b) (ii) X, Y compact $\Rightarrow X \times Y$ compact

Proof: 1. We first prove for arbitrary $x_0 \in X$ that if N is open set of $X \times Y$ containing $x_0 \times Y$, then x_0 has a nhood W s.t. $N \supset W \times Y$.

Let $\{U_\alpha \times V_\alpha : \alpha \in I\}$ $\subset N$ be a cover of $x_0 \times Y$.

Since $x_0 \times Y$ is compact (being homeomorphic to Y) $\Rightarrow x_0 \times Y \subset U_1 \times V_1 \cup \ldots \cup U_n \times V_n$ (finitely many). Set $W := U_1 \cap \ldots \cap U_n$. This is an open set in X (intersection of finitely many open sets) and contains x_0. Then $\{U_i \times V_i : i = 1, \ldots, n\}$ covers $W \times Y$; indeed, if $(x, y) \in W \times Y$, then $(x_0, y) \in U_i \times V_i$ for some $i \in \{1, \ldots, n\} \Rightarrow$ $y \in V_i$. But $x \in U_i$ $\Rightarrow (x, y) \in U_i \times V_i$.

Since $U_i \times V_i \subset N$ and $\bigcup_{i=1}^{n} U_i \times V_i \supset W \times Y$

$\Rightarrow W \times Y \subset N$.

2. Let A be an open covering of $X \times Y$. For any $x_0 \in X$, $x_0 \times Y$ is compact and can

((2)(b) iii) cont'd)

thus be covered by finitely many $A_1, \dots, A_n \in \mathcal{A}$.

Then $N := A_1 \cup \dots \cup A_n$ is open and contains $x_0 \times Y$. By Step 1, N contains a set $W \times Y$ (W nhood of x_0). $\Rightarrow$ $W \times Y$ is covered by $A_1 \cup \dots \cup A_n$.

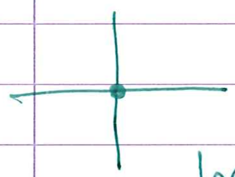
Thus, for each $x \in X$ $\exists$ nhood W_x of x s.t. $W_x \times Y$ is covered by finitely many elements of $\mathcal{A}$. Since $X = \bigcup_{x \in X} W_x \xRightarrow{X \text{ compact}} X = \bigcup_{j=1}^{k} W_j$ (finite subcover). Then

$$X \times Y = W_1 \times Y \cup \dots \cup W_k \times Y$$

and each $W_j \times Y$ may be covered by finitely many elements of $\mathcal{A}$. We have thus produced a finite subcover of $X \times Y$ $\Rightarrow$ $X \times Y$ is compact. $\square$

3 (d) The quotient map $\pi : X \to X/\sim$, $x \mapsto [x]$ is continuous & surjective. Hence, $X/\sim$ is the continuous image of a compact space and therefore compact (by Theorem 4).

(4) (a) $\{xy=0\}$ is <u>not</u> a manifold



since $(0,0) \in \{xy=0\}$ does not have a nbhood homeomorphic to a dish B^n. Indeed, for $n=1$, $B^n \setminus \{\text{point}\}$ has 2 connected components. For $n>1$, $B^n \setminus \{\text{point}\}$ is connected. However, $\{xy=0\} \setminus \{(0,0)\}$ has 4 connected components. The number of connected components is a topological invariant.