Loughborough University

ELEMENTS OF TOPOLOGY
(22MAB298)

Semester 2 22/23 — In-Person Exam paper

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

Help during the exam

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

You may **not** use a calculator for this exam.

Write your answer for every question in the appropriate space. Indicate within that space if you give additional parts of your answer elsewhere.

Answer all 3 questions.

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1. (a) Give the definition of a topology. [4]

Solution: Let X be a non-empty set. A collection τ of subsets of X is said to be a topology on X if: $X, \emptyset \in \tau$, the union of any number of sets in τ belongs to τ, the intersection of any two sets in τ belongs to τ. (It is fine to say finite intersections. They do not need to introduce X formally, or say it is non-empty. Mark dropped if they do not make clear that an arbitrary union is in τ.)

(b) Let $X = \{1, 2, 3, 4\}$ and consider the collection of subsets

$$\tau = \{\{1, 2, 3, 4\}, \{1\}, \{2, 3\}, \{2, 4\}, \{2\}\}.$$

Does τ define a topology for X? Explain your answer. [3]

Solution: τ does not contain $\emptyset$ so it is not a topology.

(c) Consider $\mathbb{R}$ equipped with the usual topology coming from the Euclidean metric. State whether the following sets are open with respect to this topology:

(i) $\mathbb{R} \setminus (\mathbb{Q} \cap (0, \infty))$. [2]

(ii) $(-1, 0) \cup (0, 1)$. [2]

(iii) $\{x : \sin(x) > 0\}$. [2]

You do not need to explain your answers.

Solution: $\mathbb{R} \setminus (\mathbb{Q} \cap (0, \infty))$ is not open. $(-1, 0) \cup (0, 1)$ is open. $\{x : \sin(x) > 0\}$ is open.

(d) Let (X, τ) be a topological space and $Y \subset X$. Define the subspace topology for Y and prove that it actually defines a topology. [7]

Solution: The subspace topology for Y is given by $\tau_Y = \{A = Y \cap B : B \in \tau\}$. $Y, \emptyset \in \tau_Y$ because $Y = X \cap Y$ and $\emptyset = \emptyset \cap Y$. Let $\{A_i\}_{i \in I}$ be a collection of sets each belonging to τ_Y. Then for each i there exists $B_i \in \tau$ such that $A_i = Y \cap B_i$. Now notice that $\cup_{i \in I} A_i = \cup_{i \in I} Y \cap B_i = Y \cap (\cup_{i \in I} B_i)$. $\cup B_i \in \tau$ therefore $\cup A_i \in \tau_Y$. Let $A_1, A_2 \in \tau_Y$. Then there exists $B_1, B_2 \in \tau$ such that $A_1 = Y \cap B_1$ and $A_2 = Y \cap B_2$. Then $A_1 \cap A_2 = (Y \cap B_1) \cap (Y \cap B_2) = Y \cap (B_1 \cap B_2)$. Since $B_1 \cap B_2 \in \tau$ we have $A_1 \cap A_2 \in \tau_Y$. (Three marks for the definition of the subspace topology. Four marks for explaining why it is is a topology.)

(e) Let $X = \{1, 2, 3\}$ and

$$\tau = \{\{1, 2, 3\}, \emptyset, \{1\}, \{2\}\}.$$

Prove that there exists no metric d for X such that τ is the topology generated by d. Hint: Consider open balls centred at 3. [5]

Solution: Suppose such a metric d existed. Let $r = \min\{d(1,3), d(2,3)\}$. $r > 0$ by the definition of a metric. Then $B_{r/2}(3) = \{3\}$. So $\{3\}$ is an open set with respect to this metric. $\{3\} \notin \tau$. Therefore no such metric exists. (Three marks awarded for explaining that $\{3\}$ would be an open set if such a metric existed.)

2. (a) Let $X = \mathbb{R}$ and $Y = \mathbb{Q}$. Calculate the interior, boundary, and exterior of the subset $Y \subset X$ with respect to the following topologies:

   (i) The topology given by the standard Euclidean metric. [4]

   (ii) The discrete topology. [4]

   (iii) The indiscrete topology. [4]

   Explain your solutions.

Solution:

i. $Int\mathbb{Q} = \emptyset$, $\partial\mathbb{Q} = \mathbb{R}$, $Ext\mathbb{Q} = \emptyset$. This is because for any $x \in \mathbb{R}$ and $r > 0$, the ball $B(x,r)$ contains elements of $\mathbb{Q}$ and $R \setminus \mathbb{Q}$.

ii. $Int\mathbb{Q} = \mathbb{Q}$. $\partial\mathbb{Q} = \emptyset$, $Ext\mathbb{Q} = \mathbb{R} \setminus \mathbb{Q}$. This follow because in the discrete topology $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both open.

iii. $Int\mathbb{Q} = \emptyset$, $\partial\mathbb{Q} = \mathbb{R}$, $Ext\mathbb{Q} = \emptyset$. This is because for the indiscrete topology, for any $x \in \mathbb{Q}$ the only open set containing x is $\mathbb{R}$. Since $\mathbb{R}$ has non empty intersection with $\mathbb{Q}$ and $R \setminus \mathbb{Q}$ it follows that every x is in the boundary of $\mathbb{Q}$.

(Four marks for each part. Two marks for correctly determining all sets. One mark awarded for correctly determining one set. Two marks for an appropriate explanation. They may use the fact the interior, boundary, and exterior partition $\mathbb{R}$.)

(b) Let (X, τ) and (Y, τ') be topological spaces. Define what it means for a function $f : X \to Y$ to be continuous. [4]

Solution: A map $f : X \to Y$ is continuous if for every open set $B \in \tau'$, $f^{-1}(B) \in \tau$. (Full marks also awarded for saying the following: A map $f : X \to Y$ is continuous at $x \in X$ if for any open neighbourhood B of $f(x)$, there is a neighbourhood A of x such that $f(A) \subset B$. f is continuous if it is continuous at all points $x \in X$.)

(c) Let $X = \mathbb{R}$ be equipped with the indiscrete topology. Let (Y, d) be a metric space. Prove that if $f : X \to Y$ is continuous then f is a constant function. [9]

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Solution: Suppose that f is not constant. Then there exists distinct $y_1, y_2 \in Y$ and $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. There exists $r > 0$ such that $B_r(y_1) \cap B_r(y_2) = \emptyset$ by properties of a metric. Therefore $f^{-1}(B_r(y_1)) \cap f^{-1}(B_r(y_2)) = \emptyset$. Both $f^{-1}(B_r(y_1))$ and $f^{-1}(B_r(y_2))$ are non-empty and open by continuity. Since X is equipped with the indiscrete topology we must then have $X = f^{-1}(B_r(y_1)) = f^{-1}(B_r(y_2))$. But this contradicts $f^{-1}(B_r(y_1)) \cap f^{-1}(B_r(y_2)) = \emptyset$. (Partial marks awarded for using properties of the metric to create disjoint open balls around y_1 and y_2. Further marks awarded for stating that the preimages of these balls are open.)

3. (a) Define what it means for a topological space to be connected. [4]

Solution: X is said to be connected if there exists no disjoint partition of X into non-empty open sets. (At most two marks awarded if disjointness is not mentioned. One mark dropped if it is not made clear that the open sets are non-empty.)

(b) Let $X = (-1, 0) \cup (0, 1)$.

(i) Equip X with a topology τ such that X is not connected with respect to this topology. [3]

(ii) Equip X with a topology τ' such that X is connected with respect to this topology. [3]

Solution: Equip X with the topology coming from the Euclidean metric. Then $(-1, 0)$ and $(0, 1)$ are both non-empty open sets. Therefore X is not open with respect to this topology. Now equip X with the indiscrete topology. Then the only non-empty open set is X. Therefore X is connected. (Three marks for each topology. Marks dropped if there is insufficient explanation accompanying the topology.)

(c) Let (X, τ) be a topological space. Suppose that there exists a continuous and surjective function $f : X \to \{-1, 1\}$ when $\{-1, 1\}$ is equipped with the discrete topology. Prove that X is not connected. [5]

Solution: By assumption $f^{-1}(\{-1\})$ is open and $f^{-1}(\{1\})$ is open. These sets are disjoint. They are non-empty because of surjectivity. Moreover we have $X = f^{-1}(\{1\}) \cup f^{-1}(\{-1\})$ and so X is not connected. (Marks dropped if the application of surjectivity is not made explicit.)

(d) Define what it means for a topological space (X, τ) to be pathwise connected. [4]

Solution: A topological space (X, τ) is pathwise connected if for any $x, y \in X$, there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

(Marks dropped if continuity of f is not stated. Marks dropped if not made clear that this holds for all x and y.)

(e) Let (X, τ) and (Y, τ') be topological spaces. Assume that (X, τ) is pathwise connected and $f : X \to Y$ is continuous and surjective. Prove that Y is pathwise connected. [6]

Solution: Let $y_1, y_2 \in Y$ be arbitrary. By surjectivity, there exists $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since (X, τ) is pathwise connected there exists a continuous function $\gamma : [0, 1] \to X$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Now observe that $f \circ \gamma : [0, 1] \to Y$ is continuous, because compositions of continuous functions are continuous. Moreover $f(\gamma(0)) = f(x_1) = y_1$ and $f(\gamma(1)) = f(x_2) = y_2$. Since y_1 and y_2 are arbitrary we deduce that (Y, τ') is pathwise connected. (Marks dropped if application of surjectivity is not made clear. Marks dropped if continuity of $f \circ \gamma$ is not made explicit.)