21MAB143 Rings and Polynomials: Week 6

1 Resultants in many variables

This week we will extend the ideas of resultant and discriminant from Week 4 to the context of polynomials with more than 1 variable. In some cases this gives a good way of finding the common zeroes of two polynomials, or the multiple zeroes of a single polynomial.

The main idea is the following: suppose $f \in \mathbb{C}[x_1, \ldots, x_n]$ is a polynomial in n variables with coefficients in the complex numbers $\mathbb{C}$. Then we can also view f as a polynomial in just a single variable x_n, but now with coefficients in the ring $\mathbb{C}[x_1, \ldots, x_{n-1}]$. If we do this for two polynomials $f, g \in \mathbb{C}[x_1, \ldots, x_n]$ and compute the resultant using the Sylvester matrix as in Week 4 Defintiion 1.1, we end up with a polynomial in the remaining variables $\mathbb{C}[x_1, \ldots, x_{n-1}]$ that will tell us something about the common zeroes of f and g.

Now let's give the formal definition.

Definition 1.1. *Let $f, g \in \mathbb{C}[x_1, \ldots, x_n]$ be two polynomials in n variables. Write them both as polynomials in the variable x_n with coefficients in $\mathbb{C}[x_1, \ldots, x_{n-1}]$: they will take the form*

$$\begin{aligned} f &= a_d x_n^d + a_{d-1} x_n^{d-1} + \cdots + a_0 \\ g &= b_e x_n^e + b_{e-1} x_n^{e-1} + \cdots + b_0 \end{aligned}$$

where now each coefficient $a_0, \ldots, a_d, b_0, \ldots, b_e$ is a polynomial in $\mathbb{C}[x_1, \ldots, x_{n-1}]$.

Then we define the **resultant of f and g with respect to x_n**, *denoted $\mathrm{Res}_{x_n}(f, g)$, as*

$$\mathrm{Res}_{x_n}(f,g) = \det \begin{pmatrix} a_d & 0 & \cdots & 0 & b_e & 0 & \cdots & 0 \\ a_{d-1} & a_d & \cdots & 0 & b_{e-1} & b_e & \cdots & 0 \\ a_{d-2} & a_{d-1} & \ddots & 0 & b_{e-2} & b_{e-1} & \ddots & 0 \\ \vdots & \vdots & \ddots & a_d & \vdots & \vdots & \ddots & b_e \\ a_0 & a_1 & \cdots & \vdots & b_0 & b_1 & \cdots & b_{e-1} \\ 0 & a_0 & \ddots & \vdots & 0 & b_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_1 & \vdots & \vdots & \ddots & b_1 \\ 0 & 0 & \cdots & a_0 & 0 & 0 & \cdots & b_0 \end{pmatrix}$$

There are a few important remarks to make about this definition:

1. Although the matrix above looks identical to the Sylvester matrix introduced in Week 4, bear in mind that the entries of this matrix are **polynomials** in the variables $x_1, \ldots, x_{n-1}$ and therefore the resultant is also a polynomial in these variables.

2. There is nothing special about the variable x_n here: we could compute the resultant of f and g with respect to any of the variables $x_1, \ldots, x_n$, by expanding f and g in powers of the chosen variable.

3. Remember that if $n = 2$ then we usually call our variables x and y; in this case the resultant with respect to (say) the second variable is denoted $\mathrm{Res}_y(f, g)$.

Example: Let's find $\mathrm{Res}_y(f, g)$ when f and g are the following two polynomials:

$$f = y^2 - x^3 - 1$$
$$g = y - x - 1$$

The coefficient polynomials in this case are

$$a_2 = 1,\ a_1 = 0,\ a_0 = -x^3 - 1$$
$$b_1 = 1,\ b_0 = -x - 1$$

so the resultant is

$$\begin{aligned}
\mathrm{Res}_y(f, g) &= \det \begin{pmatrix} 1 & 1 & 0 \\ 0 & -x-1 & 1 \\ -x^3-1 & 0 & -x-1 \end{pmatrix} \\
&= (-x-1)^2 - (-(-x^3-1)) \\
&= -x^3 + x^2 + 2x \\
&= -x(x+1)(x-2).
\end{aligned}$$

In the one-variable case the resultant is a number which tells us whether or not two polynomials have a common roots. In the current context, the resultant is a polynomial. What does it tell us?

Proposition 1.2. *Let* $f, g \in \mathbf{C}[x_1, \ldots, x_n]$. *A point* $(z_1, \ldots, z_{n-1}) \in \mathbf{C}^{n-1}$ *is a zero of* $\mathrm{Res}_{x_n}(f, g)$ *if and only if there is some* z_n *such that* $(z_1, \ldots, z_n) \in \mathbf{C}^n$ *is a common zero of* f *and* g, *meaning that*

$$f(z_1, \ldots, z_n) = g(z_1, \ldots, z_n) = 0.$$

Proof. Fix $(z_1, \ldots, z_{n-1}) \in \mathbf{C}^{n-1}$. Write $\widetilde{f}$ and $\widetilde{g}$ to denote the polynomials we get by substituting the numbers $z_1, \ldots, z_{n-1}$ in for the variables $x_1, \ldots, x_{n-1}$ in f and g respectively. Then $\widetilde{f}$ and $\widetilde{g}$ are polynomials of one variable, and we have

$$\mathrm{Res}(\widetilde{f}, \widetilde{g}) = (\mathrm{Res}_{x_n}(f, g))(z_1, \ldots, z_{n-1}).$$

In other words, we get the same result whether we form the resultant first and then plug in $z_1, \dots, z_{n-1}$ or vice-versa.

Now a number z_n such that $f(z_1, \dots, z_n) = g(z_1, \dots, z_n) = 0$ is exactly the same thing as a common zero of $\tilde{f}$ and $\tilde{g}$. By our results from Week 4, such a number exists if and only if $\mathrm{Res}(\tilde{f}, \tilde{g}) = 0$. By the equality above, this is the case if and only if $\mathrm{Res}_{x_n}(f, g)$ has a zero at $(z_1, \dots, z_{n-1})$, as claimed. $\square$

Figure 1 gives a geometric illustration of the proposition in the case $n = 2$ with the polynomials from our previous example:

$$f = y^2 - x^3 - 1$$
$$g = y - x - 1$$

We saw above that $\mathrm{Res}_y(f, g) = -x(x+1)(x-2)$, so it has roots at -1, 0, 2. According to the Proposition, this means that any common zeroes (z_1, z_2) of f and g must have $z_1 = -1$ or $z_1 = 0$ or $z_1 = 2$.

For example when $z_1 = 2$, the polynomials $\tilde{f}$ and $\tilde{g}$ defined in the proof are obtained by just looking at f and g on the vertical "slice" $z_1 = 2$: substituting we get

$$\tilde{f} = y^2 - 9$$
$$\tilde{g} = y - 3$$

These polynomials do indeed have a common root at 3, and so we get the point $(2, 3)$ as a common zero of f and g.

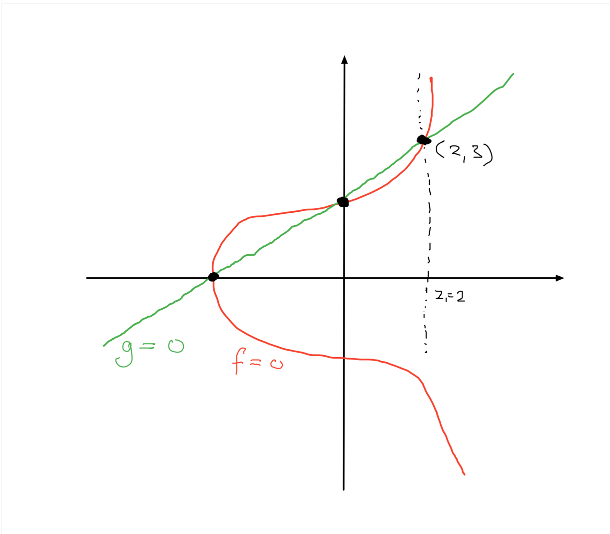

Figure 1: Zeroes of $\mathrm{Res}_y(f, g)$ correspond to common zeroes of f and g

1.1 Intersections of plane curves

We just saw how the resultant helps us to find common zeroes of multivariable polynomials. In this section we focus on the case $n = 2$ and on polynomials with **real** coeffcients. Polynomials in $\mathbf{R}[x, y]$ correspond to curves in the plane, and their common zeroes are exactly the intersection points of those curves. In this context we have a well-defined procedure to find the intersection points using the resultant. We can describe it as follows:

Step 1 Compute $\mathrm{Res}_y(f, g)$. This will be a polynomial in x only: let's call it $R(x)$.

Step 2 Find the real roots of $R(x)$.

Step 3 For each real root r_1 of $R(x)$, find all the points $(r_1, r_2) \in \mathbf{R}^2$ which have r_1 as first coordinate and which are zeroes of both f and g.

Let's do an example to see how it works in practice.

Example: Consider the following two polynomials in $\mathbf{R}[x, y]$:

$$f = x^2 - 2y^2 - 1$$
$$g = 2y^2 - xy - 1$$

Let's find the intersection points in $\mathbf{R}^2$ of the two curves:

$$C_1 = V(\langle f \rangle) = \left\{(r_1, r_2) \in \mathbf{R}^2 \mid f(r_1, r_2) = 0\right\}$$
$$C_2 = V(\langle g \rangle) = \left\{(r_1, r_2) \in \mathbf{R}^2 \mid g(r_1, r_2) = 0\right\}$$

Step 1: First we compute $\mathrm{Res}_y(f, g)$.
We have

$$f = -2y^2 + x^2 - 1 \quad \text{so}$$
$$a_2 = -2,\ a_1 = 0,\ a_0 = x^2 - 1$$
$$g = 2y^2 - xy - 1 \quad \text{so}$$
$$b_2 = 2,\ b_1 = -x, b_0 = -1$$

Inserting these into the Sylvester matrix we get

$$\mathrm{Res}_y(f, g) = \det \begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & -x & 2 \\ x^2 - 1 & 0 & -1 & -x \\ 0 & x^2 - 1 & 0 & -1 \end{pmatrix}$$

We can compute this for example by expanding along the first row: we get

$$\mathrm{Res}_y(f,g) = -2 \cdot \det \begin{pmatrix} -2 & -x & 2 \\ 0 & -1 & -x \\ x^2-1 & 0 & -1 \end{pmatrix} + 2 \cdot \det \begin{pmatrix} 0 & -2 & 2 \\ x^2-1 & 0 & -x \\ 0 & x^2-1 & -1 \end{pmatrix}$$

which simplifies to give

$$\mathrm{Res}_y(f,g) = 2x^4 - 14x^2 + 16.$$

Step 2: Next we need to find the real roots of $\mathrm{Res}_y(f,g)$. Here we will use the fact that our polynomial from Step 1 has only even powers of x, so we can view it as a polynomial in x^2.

Substituting $u = x^2$ in the polynomial above, we get

$$2u^2 - 14u + 16$$

The quadratic formula shows that this polynomial has roots

$$u = \frac{7 \pm \sqrt{17}}{2}$$

and so since $u = x^2$ the 4 roots of the resultant are

$$r_1 = \pm\sqrt{\frac{7 \pm \sqrt{17}}{2}}.$$

Notice that since $\sqrt{17} < 7$ all 4 roots are real.

Step 3: Finally, for each root r_1 found in Step 2, we want to find all values of r_2 such that

$$f(r_1, r_2) = g(r_1.r_2) = 0. \tag{$*$}$$

We know from Proposition 1.2 that for each r_1 which is a root of the resultant, there is at least one r_2 such that the equations $(*)$ are satisfied. On the other hand, if $f(r_1, r_2) = g(r_1, r_2) = 0$, then $(f+g)(r_1, r_2) = 0$ also. In this case

$$f + g = x^2 - xy - 2$$

so if $f(r_1, r_2) = 0$ then we get

$$r_1^2 - r_1 r_2 - 2 = 0$$

which can be rearranged to give

$$r_2 = r_1 - \frac{2}{r_1}.$$

So for each root r_1 of the resultant, we get **exactly** one r_2 such that $f(r_1, r_2) = g(r_1, r_2) = 0$.

For example, if

$$r_1 = \sqrt{\frac{7\sqrt{17}}{2}} \approx 2.4$$

then we compute

$$\begin{aligned} r_2 &= r_1 - \frac{2}{r_1} \\ &= \sqrt{\frac{7\sqrt{17}}{2}} - \frac{2}{\sqrt{\frac{7\sqrt{17}}{2}}} \\ &\approx 1.5 \end{aligned}$$

In this way, we get 4 intersection points in $C_1 \cap C_2$.

2 Discriminants in many variables

As in the case of one variable, we can use the resultant to define the **discriminant**. In one variable this was a number that told us whether a polynomial had a multiple root; in the case of many variables, the discriminant will be a polynomial which tells us the location (if any) of multiple zeroes or "singular points" of the given polynomial.

Definition 2.1. *Let $f \in \mathbf{C}[x_1, \dots, x_n]$ be a polynomial in n variables. Write it as a polynomial in the variable x_n with coefficients in $\mathbf{C}[x_1, \dots, x_{n-1}]$: say*

$$f = a_d x_n^d + a_{d-1} x_n^{d-1} + \cdots + a_0$$

Let f_n denote the partial derivative of f with respect to x_n: so

$$f_n = d a_d x_n^{d-1} + \cdots + a_1$$

Then we define the **discriminant of f with respect to the variable x_n**, *denoted by $\mathrm{Disc}_{x_n}(f)$, to be*

$$\mathrm{Disc}_{x_n}(f) = \left(\frac{(-1)^{d(d-1)/2}}{a_d} \right) \mathrm{Res}_{x_n}(f, f_n).$$

Just as with the definition of resultant, this defintion of discriminant looks formally identical to the one from Week 4, but it means something different: in this case, $\mathrm{Disc}_{x_n}(f)$ is a polynomial in the variables $x_1, \dots, x_{n-1}$.

Remark: In the definition above we divide by a_d. In general a_d is a (nonconstant) polynomial in the variables $x_1, \dots, x_{n-1}$, so it has no multiplicative inverse: why is it OK to divide by it? The answer is that in the Sylvester matrix computing $\mathrm{Res}_{x_n}(f, f_n)$, the only nonzero entries in the top row will be a_d and $d a_d$. By the usual properties of determinants, this means that the polynomial $\mathrm{Res}_{x_n}(f, f_n)$ is always divisible by a_d, so the discriminant as defined above really is a polynomial in $x_1, \dots, x_{n-1}$.

Proposition 1.2 above told us that roots of the resultant correspond to common roots of the two polynomials. This immediately implies the corresponding fact for the discriminant:

Proposition 2.2. *Let $f \in \mathbf{C}[x_1, \dots, x_n]$ be a polynomial in n variables. A point $(z_1, \dots, z_{n-1}) \in \mathbf{C}^{n-1}$ is a zero of $\mathrm{Disc}_{x_n}(f)$ if and only if there is some z_n such that $(z_1, \dots, z_n) \in \mathbf{C}^n$ is a common zero of f and its partial derivative f_n.*

Proof. Put $g = f_n$ in Proposition 1.2. □

2.1 Singular points of plane curves

Again we switch to the case $n = 2$ and focus on real polynomials. Given a polynomial $f \in \mathbf{R}[x, y]$ which defines a plane curve

$$C = \{(r_1, r_2) \in \mathbf{R}^2 \mid f(r_1, r_2) = 0\}$$

we would like to understand the geometric meaning of roots of $\mathrm{Disc}_y(f)$.

Definition 2.3. *Let* C *be a plane curve defined by a polynomial* $f \in \mathbf{R}[x, y]$*: that is,*

$$C = \{(r_1, r_2) \in \mathbf{R}^2 \mid f(r_1, r_2) = 0\}$$

A point $(r_1, r_2) \in C$ *is called a* **singular point** *of* C *if both partial derivatives of* f *are zero at that point: so*

$$f(r_1, r_2) = f_x(r_1, r_2) = f_y(r_1, r_2) = 0.$$

The definition says that singular points of C are those where the gradient vector of f vanishes; therefore C does not have a well-defined tangent line at a singular point.

For a given plane curve C, we would like to locate the singular points of C in $\mathbf{R}^2$, if any. By Proposition 2.2 we know that if (r_1, r_2) is a singular point, then r_1 must be among the roots of $\mathrm{Disc}_y(f)$.

So we can proceed as follows: given a root r_1 of $\mathrm{Disc}_y(f)$, we can find all the points $(r_1, r_2) \in C$ which have r_1 as first coordinate. For each such point p we can evaluate the partial derivatives $f_x(r_1, r_2)$ and $f_y(r_1, r_2)$: if both equal zero, then p is a singular point of C.

Figure 2.1 gives an illustration of various possibilites. If the curve C is defined by a polynomial f, then $\mathrm{Disc}_y(f)$ will have roots at r_1 and s_1 (and perhaps elsewhere too). The point (r_1, r_2) is a singular point of the curve C, whereas (s_1, s_2) is not. The green line is the tangent line to the curve C at (s_1, s_2): this is a vertical line, indicating that $f_y(s_1, s_2) = 0$, and this explains why $\mathrm{Disc}_y(f)$ has a root at s_1.

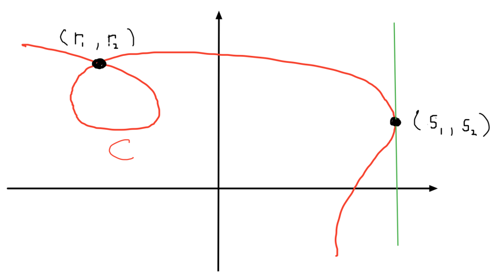

Figure 2: Different kinds of zeroes of $\mathrm{Disc}_y(f)$

We can summarise the procedure for finding singular points of a plane curve C as follows:

Step 1: Compute the discriminant $\mathrm{Disc}_y(f)$. This is a polynomial in the variable x: call it $D(x)$.

Step 2: Find all real roots of $D(x)$.

Step 3: Find all points $(r_1, r_2) \in C$ where r_1 is a root of $D(x)$ and $f_x(r_1, r_2) = f_y(r_1, r_2) = 0$.

Example: Let C be the ellipse defined by the equation

$$2x^2 + 3y^2 - 1 = 0$$

An ellipse has a well-defined tangent direction at each point, so it should not have any singular points. Let us check this is what our computations show. We have

$$\begin{aligned} f &= 3y^2 + 2x^2 - 1 \\ f_y &= 6y \end{aligned}$$

so the coefficient polynomials in this case are

$$\begin{aligned} &a_2 = 3,\ a_1 = 0,\ a_0 = 2x^2 - 1 \\ &b_1 = 6,\ b_0 = 0 \end{aligned}$$

and hence we compute the discriminant as

$$\begin{aligned} \mathrm{Disc}_y(f) &= \left(\frac{(-1)^{2\cdot 1/2}}{3}\right) \mathrm{Res}_y(f, f_y) \\ &= -\frac{1}{3} \det \begin{pmatrix} 3 & 6 & 0 \\ 0 & 0 & 6 \\ 2x^2 - 1 & 0 & 0 \end{pmatrix} \\ &= -24x^2 + 12 \end{aligned}$$

This has roots at $\pm\frac{1}{\sqrt{2}}$.Substituting $x = \pm\frac{1}{\sqrt{2}}$ into the equation $f = 0$, we get $y = 0$. So there are two "candidate" singular points

$$p = \left(-\frac{1}{\sqrt{2}}, 0\right),\ q = \left(\frac{1}{\sqrt{2}}, 0\right).$$

But we calculate $f_x = 4x$, so $f_x(p) = -2\sqrt{2}$ and $f_x(q) = 2\sqrt{2}$. Neither one equals zero, so neither p nor q is a singular point of C, as expected.

Example: For a slightly more complicated example, let's try to find the singular points, if any, of the curve C defined by the equation

$$x^4 - x^2y + y^3 = 0.$$

Step 1: First we need to compute the discriminant $\mathrm{Disc}_y(f)$. We have

$$\begin{aligned} f &= y^3 - x^2y + x^4 \\ f_y &= 3y^2 - x^2 \end{aligned}$$

so the coefficients we insert into the Sylvester matrix are

$$a_3 = 1,\ a_2 = 0,,\ a_1 = -x^2,\ a_0 = x^4$$
$$b_2 = 3,\ b_1 = 0,\ b_0 = -x^2$$

This gives

$$\begin{aligned}
\mathrm{Disc}_y(f) &= \left(\frac{(-1)^{3\cdot 2/2}}{a_3}\right)\mathrm{Res}(f, f_y) \\
&= -\mathrm{Res}(f, f_y) \\
&= -\det\begin{pmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ -x^2 & 0 & -x^2 & 0 & 3 \\ x^4 & -x^2 & 0 & -x^2 & 0 \\ 0 & x^4 & 0 & 0 & -x^2 \end{pmatrix}
\end{aligned}$$

Computing this determinant we find

$$\begin{aligned}
\mathrm{Disc}_y(f) &= -27x^8 + 4x^6 \\
&= x^6(4 - 27x^2).
\end{aligned}$$

Step 2: Next we find the roots of $\mathrm{Disc}_y(f)$. From the previous expression we see that the roots of $\mathrm{Disc}_y(f)$ are

$$r_1 = 0 \text{ and } r_1 = \pm\sqrt{\frac{4}{27}}$$

Step 3: Now we have to decide, for each root r_1 of $\mathrm{Disc}_y(f)$, which points $(r_1, r_2) \in C$ have $f_x(r_1, r_2) = f_y(r_1, r_2) = 0$. We compute

$$\begin{aligned}
f_x &= 4x^3 - 2xy = 2x(2x^2 - y) \\
f_y &= 3y^2 - x^2
\end{aligned}$$

So if $f_x(r_1, r_2) = f_y(r_1, r_2) = 0$ then either $r_1 = r_2 = 0$ or $2r_1^2 - r_2 = 0$, that is

$$r_1^2 = \frac{1}{2}r_2. \tag{$**$}$$

Then $f_y = 0$ gives us $3r_2^2 - \frac{1}{2}r_2 = 0$, which we can solve to get $r_2 = 0$ or $r_2 = \frac{1}{6}$. Substituting back into $(**)$ then gives

$$r_1 = \pm\sqrt{\frac{1}{12}}.$$

Comparing this with Step 2, we see that the only value of r_1 which give both roots of $\mathrm{Disc}_y(f)$ and solutions of $f_x = f_y = 0$ is $r_1 = 0$.

The only point on the curve C with $r_1 = 0$ is the point $(0, 0)$. Since

$$f_x(0,0) = f_y(0,0) = 0$$

we see that the point $(0, 0)$ is indeed a singular point of C, and it is the only one.