Normal subgroups + Quotients, 1

We talked last week about left cosets of H in G:

$$gH = \{gh \mid h \in H\}.$$

Can also consider right cosets

$$Hg = \{hg \mid h \in H\}.$$

In general, the left and right cosets can differ; only the same for special subgroups:

Definition: Let H be a subgroup of G. H is normal if left + right cosets coincide:

$$\boxed{gH = Hg} \quad \forall g \in G.$$

Equivalently, H is normal if

$$\boxed{ghg^{-1} \in H} \quad \forall g \in G, \forall h \in H.$$

Examples

(i) $\{e\} \subset G$ and $H = G$ are normal.

(ii) If G is commutative then $ghg^{-1} = h$ so any $H \subset G$ is normal.

(iii) $A_n \subset S_n$ is normal: $\forall\, h \in A_n,\ g \in S_n$

$$\mathrm{sgn}(ghg^{-1}) = \mathrm{sgn}(g)\,\mathrm{sgn}(h)\,\underbrace{\mathrm{sgn}(g^{-1})}_{=\,\mathrm{sgn}(g)}$$

$$= \mathrm{sgn}(g)^2\,\underbrace{\mathrm{sgn}(h)}_{=\,1} = 1$$

so $ghg^{-1} \in A_n$.

On the other hand e.g. $\langle (12) \rangle \subset S_3$ is not normal: e.g. $(13)(12)(13)^{-1}$

$$= (13)(12)(13) = (23)$$

which is no longer in $\langle (12) \rangle$.

Quotient Groups

Theorem: Let $H \subset G$ be a normal subgroup. Define an operation on G/H by:

$$(g_1H)(g_2H) = g_1g_2H \qquad (*)$$

Then $(*)$ makes G/H into a group, called the quotient of G by H.

Proof: Hardest part is to check that (*) is a well-defined operation. Need to show that if $g_1H = \gamma_1 H$ and $g_2 H = \gamma_2 H$, then

$$g_1 g_2 H = \gamma_1 \gamma_2 H.$$

To see this:

$$g_1 H = \gamma_1 H \Rightarrow g_1 \gamma_1^{-1} \in H \quad (1)$$
$$g_2 H = \gamma_2 H \Rightarrow g_2 \gamma_2^{-1} \in H \quad (2)$$

Now multiply (2) by γ_1 on the left and γ_1^{-1} on the right:

$$\gamma_1 g_2 \gamma_2^{-1} \gamma_1^{-1} \in H \quad (3)$$

Multiply (1) + (3): $(g_1 \gamma_1^{-1})(\gamma_1 g_2 \gamma_2^{-1} \gamma_1^{-1}) \in H$

$$\Rightarrow g_1 g_2 \underbrace{\gamma_2^{-1} \gamma_1^{-1}}_{= (\gamma_1 \gamma_2)^{-1}} \in H$$

$$\Rightarrow (g_1 g_2)(\gamma_1 \gamma_2)^{-1} \in H$$

$$\Rightarrow (g_1 g_2) H = (\gamma_1 \gamma_2) H$$

as required.

So (*) is a well-defined operation on the set of cosets G/H.

Now checking the group axioms is easy:

- Associativity follows from associativity in G.
- Identity element $eH = H$
- Inverses $(gH)^{-1} = g^{-1}H$.

$\blacksquare$

<u>Example:</u> Dihedral group D_n.

The subgroup $H = \{e, r, \ldots, r^{n-1}\}$ r = rotation

is normal: left cosets are

any reflection

H and $D_n \setminus H = \{s_1, \ldots, s_n\} = sH$

right cosets are

H and $D_n \setminus H = \{s_1, \ldots, s_n\} = Hs$

Products in D_n/H:

$H \cdot H = (eH)(eH) = eH = H$

$H \cdot (sH) = (sH)(H) = sH$

$(sH)(sH) = s^2 H = H$.

The map $D_n/H \longrightarrow \mathbb{Z}_2$

$$\begin{array}{ccc} D_n/H & \longrightarrow & \mathbb{Z}_2 \\ \| & & \| \\ \{H, sH\} & & \{0,1\} \end{array}$$

$$H \longmapsto 0$$
$$sH \longmapsto 1$$

is an isomorphism.

Homomorphisms and Quotients

Let G and G' be groups.

Remember that a homomorphism

$$\varphi: G \longrightarrow G'$$

means a map such that

$$\varphi(xy) = \varphi(x)\varphi(y) \quad \forall x, y \in G$$

(Similar to isomorphism, but not necessarily bijective.)

Next time: We'll see that quotient groups of G are "the same as" homomorphisms from G onto other groups.