MAB298-Elements of Topology: Problem Sheet 5
Connectedness

1. Let X be a topological space, A and B connected (pathwise connected) subsets of X. Let $A \cap B \neq \emptyset$. Prove that $A \cup B$ is connected (resp. pathwise connected).

2. Let A be a connected subset of a topological space X. Prove that its closure $\bar{A}$ is also connected.

3. Does the connectedness of $A \cup B$ and $A \cap B$ imply that of A and B?

4. Prove that if A is a proper nonempty subset of a connected space X, then $\partial A \neq \emptyset$.

5. Prove that X is disconnected if and only if there is a continuous surjection $f : X \to S^0$, where S^0 is a discrete two-point space.

6. Which of the following spaces are connected?

   1) $[0, 1]$ with discrete topology;

   2) $[0, 1]$ with indiscrete topology;

   3) $\mathbb{R}$ the topology $\tau = \{\emptyset, \mathbb{R}, (a, +\infty), a \in \mathbb{R}\}$;

   4) the set of all $n \times n$-matrices;

   5) the set $O(2)$ of orthogonal 2×2-matrices:

   $$O(2) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : A^{-1} = A^{\top} \right\},$$

   6) the set of all real triangular 2×2 matrices

   $$\left\{ A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\};$$

   7) the set of all real triangular 2×2 matrices with positive determinant

   $$\left\{ A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \ : \ \det A > 0 \right\};$$

8) $\{x^2 + y^2 - z^2 = a\}$ in $\mathbb{R}^3$ (consider 3 cases $a = -1, 0, 1$);

9) $(x^2 + y^2 - 1)(x^2 + y^2 - 4) = 0$ in $\mathbb{R}^2$;

10) $(x^2 + y^2 - 4)((x - 1)^2 - y^2 - 1) = 0$ in $\mathbb{R}^2$.

7. Using the connectedness concept, prove that $[0, 1)$, $[0, 1]$, $\mathbb{R}$, S^1 (circle), S^2 (sphere) are pairwise nonhomeomorphic.

8. Prove that a circle is not homeomorphic to any subset of $\mathbb{R}$.

9. Prove that the set $\{xy = 0\} \subset \mathbb{R}^2$ is not homeomorphic to $\mathbb{R}$.

   If we remove the point $(0, 0)$ from $\{xy = 0\}$, then this space splits into four components, where after removing any point from $\mathbb{R}$ we get always two components only.