Problem Sheet 9

Q1: L : $\omega_1 = \frac{3}{8} - \frac{3\sqrt{3}}{8} i$

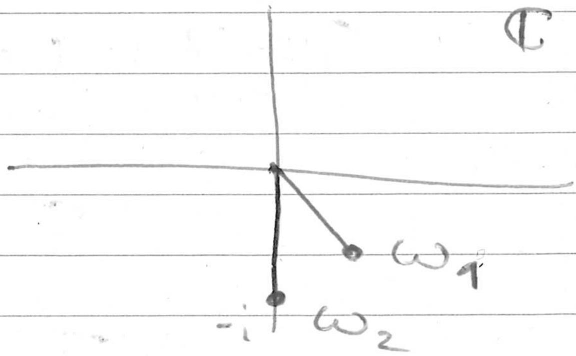
$\omega_2 = -i$.

Step 1: We want to find a lattice similar to L of the form

$\mathbb{Z} \oplus \mathbb{Z} \cdot \omega$ with $\omega \in \mathfrak{H}$

To do this, ω should be one of ω_2/ω_1 or ω_1/ω_2. Which one?

The picture shows we should take



$$\omega = \omega_1/\omega_2$$

$$= \frac{1}{-i}\left(\frac{3}{8} - \frac{3\sqrt{3}}{8} i\right)$$

$\frac{1}{-i} = i$

$$= i\left(3/8 - \frac{3\sqrt{3}}{8} i\right)$$

$$= \frac{3\sqrt{3}}{8} + 3/8\, i \quad \in \mathfrak{H}$$

Note :

$$\omega = \frac{3\sqrt{3}}{8} + \frac{3}{8} i$$

$$= \frac{3}{4}\Big(\underbrace{\frac{\sqrt{3}}{2}}_{\cos(\pi/6)} + \underbrace{\tfrac{1}{2}}_{\sin(\pi/6)} i\Big)$$

$$= \frac{3}{4}\left(\exp(i\pi/6)\right)$$

So $|\omega| = 3/4$.

Hence $\omega \notin \mathcal{F}$.

$$\mathcal{F} = \left\{ \begin{array}{l} |\mathrm{Re}(z)| \le 1/2, \\ |z| \geq 1 \end{array} \right\}$$

Step 2 : Use transformations S & T

$$S : z \longmapsto -1/z$$

$$T : z \longmapsto z + 1$$

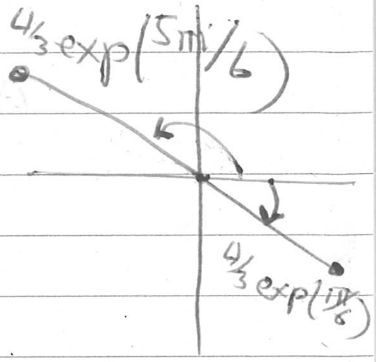
to move ω into $\mathcal{F}$

Since $|\omega| < 1$ let's start by using S :

$$S\omega = -1/\omega$$

$$= \frac{-1}{\frac{3}{4}\exp(i\pi/6)}$$

$$= -\frac{4}{3}\exp(-i\pi/6)$$



$$= 4/3\exp(5\pi i/6)$$

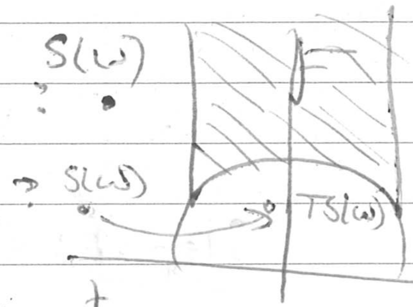
Is this number in $\mathcal{F}$?

Compute:

$$\mathrm{Re}(Sw) = 4/3\cos(5\pi/6)$$

$$\approx -1.15 < -0.5.$$

So $Sw \notin \mathcal{F}$.



<u>Step 3</u>: Next apply transformation

$$T : z \longmapsto z+1.$$

Get $TS(w) = 4/3\exp(5\pi i/6) + 1.$

Is it in $\mathcal{F}$?

Its real part is $4/3\cos(5\pi/6) + 1 \approx -0.15$

So $|\mathrm{Re}(TS(w))| \leq 1/2$ ✓

Is $|TS(\omega)| \geq 1$ or not?

Know $\operatorname{Re} TS(\omega) \approx -0.15$

Also $\operatorname{Im} TS(\omega) = \operatorname{Im} S(\omega)$

$$= \tfrac{4}{3} \sin\left(\tfrac{5\pi}{6}\right)$$

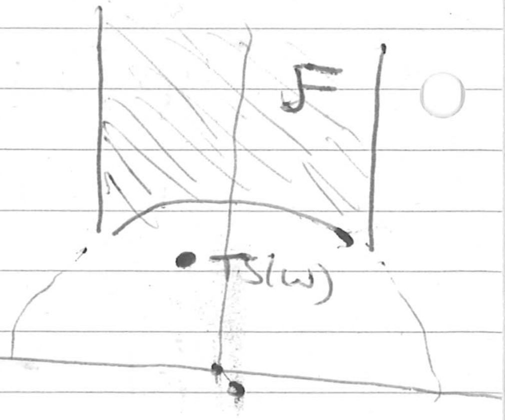
$$= \tfrac{2}{3}$$

So $|TS(\omega)| = \sqrt{\operatorname{Re}(TS(\omega))^2 + \operatorname{Im}(TS(\omega))^2}$

$$\approx \sqrt{(-0.15)^2 + (2/3)^2} < 1$$

So $TS(\omega) \notin \mathcal{F}$

<u>Step 4:</u> Apply S again:



$STS(\omega)$

$$= S\left(\tfrac{4}{3}\exp\left(5\pi i/6\right) + 1\right)$$

$$= -\left(\frac{1}{\tfrac{4}{3}\exp\left(5\pi i/6\right) + 1}\right)$$

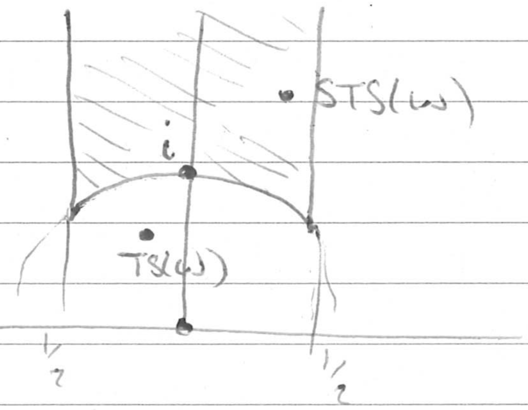
$$= -\left(\frac{1}{\left(\tfrac{4}{3}\cos\left(5\pi/6\right) + 1\right) + \left(\tfrac{4}{3}\sin 5\pi/6\right) i}\right)$$

$$= -\left(\frac{1}{\left(\frac{4}{3}\left(-\frac{\sqrt{3}}{2}+1\right)\right)+\left(\frac{4}{3}\cdot\frac{1}{2}\right)i}\right)$$

$$= -\left(\frac{1}{\left(-\frac{2}{\sqrt{3}}+\frac{4}{3}\right)+\left(\frac{2}{3}\right)i}\right)$$

$$\approx 0.33 + 1.42\, i$$

Now:



$|\mathrm{Re}(STS\omega)| \approx 0.33 < \frac{1}{2}$

$|STS(\omega)| \geq 1.42 > 1$

$\therefore STS(\omega) \in \mathcal{F}$ as reqd. $\square$