MAB298-Elements of Topology: Solution Sheet 4

Compactness

1. Let the topology on $\mathbb{R}$ be defined by the collection of subsets $\{\emptyset,\ \mathbb{R},\ (a, +\infty), a \in \mathbb{R}\}$ (see Example 5 in Lecture Notes). Is $\mathbb{R}$ with this topology compact?

   No. To show this we need an example of an open cover which does not admit any finite subcover. Consider the cover which consists of the following open subsets:

   $$(0, +\infty), (-1, +\infty), (-2, +\infty), \dots, (-n, +\infty), \dots$$

   Obviously, $\mathbb{R} = \cup_{n=0}^{\infty}(-n, +\infty)$ so that this is indeed an open cover of $\mathbb{R}$, but we can't choose any finite subcover.

2. Prove by definition that $[0, 1)$ is not compact.

   The proof is similar. The collection of open sets:

   $$\left\{\left(-\epsilon, 1 - \frac{1}{n}\right)\right\}_{n=1,2,3,\dots}$$

   is an open cover for $[0, 1)$ which admits no finite subcover.

3. Prove by definition that $\mathbb{Z}$ as a subset of $\mathbb{R}$ is not compact.

   The proof is similar. The collection of open sets:

   $$\{(n - \epsilon, n + \epsilon)\}_{n\in\mathbb{Z}}$$

   is an open cover for $\mathbb{Z}$ which admits no finite subcover.

4. Let X be compact and C_i, $i \in \mathbb{N}$ be a collection of closed sets such that any finite intersection $\cap_{i=1}^{N} C_i$ is not empty. Prove that $\cap_{i=1}^{\infty} C_i \neq \emptyset$.

   By contradiction, assume that $\cap_{i=1}^{\infty} C_i = \emptyset$. Consider the collection of open sets

   $$A_i = X \setminus C_i, \quad i = 1, 2, 3, \dots$$

   (Each A_i is open since C_i is closed.)

We have

$$\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty}(X \setminus C_i) = X \setminus (\cap_{i=1}^{\infty} C_i) = X \setminus \emptyset = X.$$

Thus, the collection of A_i is an open cover of X. Since X is compact, we can choose a finite subcover:

$$A_{i_1}, A_{i_2}, \ldots, A_{i_n}$$

so that $X = A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_n} = X \setminus (C_{i_1} \cap C_{i_2} \cap \cdots \cup C_{i_n})$. The obtained equality

$$X = X \setminus (C_{i_1} \cap C_{i_2} \cap \cdots \cup C_{i_n})$$

means that $C_{i_1} \cap C_{i_2} \cap \cdots \cup C_{i_n} = \emptyset$ (otherwise, the right hand side would be "strictly less" than X). But this conclusion is in contradiction with the assumption that any finite intersection $\cap_{i=1}^{N} C_i$ is not empty.

5. Let A and B be two compact subsets of a space X. Does it follow that $A \cup B$ is compact?

   $A \cup B$ is compact. Use the fact that an open cover for $A \cup B$ is also an open cover of A, and an open cover of B. Then apply the definition.

6. Which of the topological spaces listed below are compact:

   1) $[0, 1]$ with the discrete topology

   No, because X is an infinite discrete space.

   2) $\mathbb{Z}$ with the discrete topology

   No, because X is an infinite discrete space.

   3) $\mathbb{R}$ with the indiscrete topology

   Yes, because any indiscrete space is compact.

   4) closed half plane $\{(x, y) \ : \ y \geq 0\}$

   Here and below we use the following compactness criterion: *a subset X of $\mathbb{R}^n$ is compact iff X is both bounded and closed.* The closed half plane is not compact, because it is not bounded.

5) sphere $\{x^2+y^2+z^2=1\}$

Yes, because the sphere is bounded and closed.

6) open disc $\{x^2+y^2<1\}$ in $\mathbb{R}^2$

No, because the open disc X is not closed (the boundary points do not belong to X).

7) annulus $\{1<x^2+y^2<4\}$

No, because this annulus X is not closed (the points lying on the circle $x^2+y^2=1$ are boundary points for X but do not belong to X).

8) punctured sphere $\{x^2+y^2+z^2=1\}\setminus\{(0,0,1)\}$

No, because the punctured sphere X is not closed (the point $(0,0,1)$ is a limit point for X, but does not belong to X).

9) $\mathbb{Q}$ as a subset in $\mathbb{R}$

No, because $\mathbb{Q}$ is not closed in $\mathbb{R}$ (the closure of $\mathbb{Q}$ is $\mathbb{R}$, i.e. $\bar{\mathbb{Q}}\neq\mathbb{Q}$). (Notice that $\mathbb{Q}$ is not bounded either.)

10) $\{(x^4+y^4)(1+x^2+y^2)=10\}$ in $\mathbb{R}^2$

Yes, because this set X is bounded and closed. To show that X is closed we may use the following general result: *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a continuous function, then each level set of f, i.e. the set*

$$X_a=\{(x_1,\dots,x_n)\in\mathbb{R}^n \quad : \quad f(x_1,\dots,x_n)=a\}$$

is closed.

The proof is obvious: $X_a=f^{-1}(a)$ and it is closed as a preimage of a closed set under a continuous map. Here we consider $a\in\mathbb{R}$ as a one-point set which is obviously closed in the standard topology.

To show that X is bounded we need to do some estimates. Notice that $(x^4+y^4)(1+x^2+y^2)=10$ implies

$$x^4+y^4=\frac{10}{1+x^2+y^2}\leq 10$$

and therefore

$$x^4 \leq 10 \qquad \text{and} \qquad y^4 \leq 10.$$

Hence

$$|x| \leq \sqrt[4]{10} \qquad \text{and} \qquad |y| \leq \sqrt[4]{10},$$

which means that our set X is bounded.

11) $\{\frac{x^4+y^4}{1+x^2+y^2} \leq 10\}$ in $\mathbb{R}^2$

This set X is closed because it can be represented as $f^{-1}(-\infty, 10]$, where $f : \mathbb{R}^2 \to R$ is the (continuous) function defined by $f(x,y) = \frac{x^4+y^4}{1+x^2+y^2}$ and $(-\infty, 10]$ is closed.

Let us try to check whether X is bounded or not.

It is not hard to see that if either $|x|$ or $|y|$ tends to infinity, then $f(x,y)$ tends to $+\infty$ too. This observation can be reformulated in the following way: if we consider an unbounded subset $A \subset \mathbb{R}$, then the function f cannot be bounded on A from above.

However, on our set X, the function f is bounded from above, namely $f(x,y) \leq 10$. So we conclude that X must be bounded.

Thus, being both bounded and closed, X is compact.

12) $\{\sin^4 x + \cos^4 y = 1\}$ in $\mathbb{R}^2$

No, because this set is not bounded (since it is periodic). Indeed, the points $(0, 2\pi k)$, $k \in \mathbb{Z}$ belong to X and form an unbounded sequence.

13) $\{(x + \sin y)^2 + (y + \sin x)^2 = 100\}$ in $\mathbb{R}^2$

Yes, this set X is compact. Indeed, it is closed as a level set of a continuous function. And it is bounded. To see this, notice that $(x + \sin y)^2 + (y + \sin x)^2 = 100$ implies

$$(x + \sin y)^2 \leq 100 \qquad \text{and} \qquad (y + \sin x)^2 \leq 100,$$

$$|x + \sin y| \leq 10 \qquad \text{and} \qquad |y + \sin x| \leq 10,$$

Taking into account that $-1 \leq \sin y \leq 1$ and $-1 \leq \sin x \leq 1$, it is easy to see that

$$|x| \leq 11 \qquad \text{and} \qquad |y| \leq 11,$$

i.e., X is bounded.

14) $[0,1] \cap \mathbb{Q}$

No, because this set is not closed ($\overline{[0,1] \cap \mathbb{Q}} = [0,1] \neq [0,1] \cap \mathbb{Q}$).

7. Prove that if $f : [0,1] \to \mathbb{R}$ is a continuous function, then $f([0,1])$ is a segment (i.e., closed interval). More generally, if $X \subset \mathbb{R}^n$ is compact and connected and $f : X \to \mathbb{R}$ is a continuous function, then $f(X)$ is a segment.

   Since X is connected, its image $f(X) \subset \mathbb{R}$ is connected too. As we know the connected subsets in $\mathbb{R}$ are intervals, i.e., subsets of the form

   $$(a,b), [a,b], (a,b], [a,b), \quad a \leq b$$

   (where we allow a and b to be $-\infty$ and $+\infty$ respectively).

   Since X is compact. its image $f(X)$ is compact too. It remains to notice that the only compact intervals are segments $[a,b]$, $a \leq b$; all the others are either unbounded or non-closed.

8. Let A be a subset of $\mathbb{R}^n$. Prove that A is compact iff each continuous numerical function on A is bounded.

   If A is compact, then $f(A)$ is a compact subset in $\mathbb{R}$ (as the image of a compact set under a continuous map). This implies that $f(A)$ is bounded. This is equivalent to saying that f is bounded on A.

   Let us prove the converse statement: *if each continuous function f is bounded on A, then A is compact.*

   By contradiction, assume that A is not compact. Then we have two possibilities: either A is not bounded, or A is not closed.

   Case 1: A is not bounded. This means, in particular, that X is not contained in any ball $B_R(0)$ of radius R centered at 0 (there are points $x \in A$ which are located outside of this ball). We can reformulate this condition saying that the function $f(x) = \sqrt{x_1^2 + \cdots + x_n^2}$ is not bounded. Thus, we have found an example of a continuous function which is not bounded on A. Contradiction.

Case 2: A is not closed. This means that there exist a limit point $y \in \mathbb{R}^n$ which does not belong to A. Consider the following function on A:

$$f : A \to \mathbb{R}, \qquad f(x) = \frac{1}{d(x,y)}$$

where $d(x,y)$ is the distance between $x \in X$ and $y \notin A$ (as usual, $d(x,y) = \sqrt{(x_1 - y_1)^2 + \ldots (x_n - y_n)^2}$).

Since $y \notin A$, then $d(x,y)$ does not vanish on X and consequently, f is continuous on X.

On the other hand, f is not bounded. The reason is as follows: y is a limit point of A, therefore for any $\epsilon < 0$ we can find $x \in A$ such that $d(x,y) < \epsilon$ and consequently

$$f(x) > \frac{1}{\epsilon}.$$

Since ϵ can be taken arbitrarily small, the value $f(x)$ can be arbitrarily large. In other words, f is not bounded. Contradiction.

9. In the space $C^0([0,1])$ of continuous functions on $[0,1]$ with the standard distance $d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$, consider the closed ball B of radius 1 centered at zero (i.e., at the zero function):

$$B = \{f \in C^0([0,1]) \ : \quad |f(x)| \leq 1\}$$

Is B compact?

No, this ball B is not compact (although B is closed and bounded!). It is more convenient to show that this set is not sequentially compact. Consider the sequence of continuous functions (f_n) given by

$$f_n(x) = \left\{ \begin{array}{ll} 1 - nx, & \text{for } 0 \leq x \leq 1/n \\ 0, & \text{for } 1/n < x \leq 1 \end{array} \right\}$$

This is a sequence of continuous functions. Importantly each function satisfies $f_n(0) = 1$ and $f_n(x) = 0$ for all $n > 1/x$. If this sequence has a subsequence that converges to a function f then f must satisfy $f(0) = 1$ and $f(x) = 0$ for all $x > 0$. However it is not possible for f to be continuous and satisfy these properties.