23MAC260 Problem Sheet 8

Week 8 Lectures

Last updated: April 4, 2024

1. Let $f(z)$ be the meromorphic function

$$f(z) = \frac{3\cos^2(z)}{\sin^2(2z)}.$$

   (a) Compute the order $\mathrm{ord}(f, 0)$ of f at 0.

   (b) Compute the residue $\mathrm{Res}(f, 0)$ of f at 0.

   (c) Compute the integral

$$\int_\gamma f(z)\,dz$$

   where γ is the ellipse in the complex plane defined by the equation

$$17\,\mathrm{Re}(z)^2 + 23\,\mathrm{Im}(z)^2 = 13.$$

2. Show that if $f(z)$ is meromorphic at a point $z_0 \in \mathbb{C}$, then for the function $g(z) = f'(z)/f(z)$ we have

$$\mathrm{Res}(g, z_0) = \mathrm{ord}(f, z_0).$$

3. Let L be the lattice spanned by the complex numbers ω_1 and ω_2. Let m, n, r, s be integers. Let L' be the lattice spanned by

$$\tau_1 = m\omega_1 + n\omega_2$$
$$\tau_2 = r\omega_1 + s\omega_2.$$

   Show that $L = L'$ if and only if the matrix

$$M = \begin{pmatrix} m & n \\ r & s \end{pmatrix}$$

   has determinant ± 1. (Equivalently, M is invertible and its inverse is also an integer matrix.)