MAB298-Elements of Topology: Solution Sheet 7 Manifolds and Implicit Function Theorem

1. Describe an atlas for the sphere S^2 which consists of two charts. Generalize this example to the case of the n-dimensional sphere $\{x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$.

   Consider the sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$ and two points $S = (0,0,1)$ and $N = (0,0,-1)$, the south and north poles of the sphere. Let $U_1 = S^2 \setminus S$ and $U_2 = S^2 \setminus N$. It is easy to see that U_1 and U_2 covers the sphere and each of them is homeomorphic to an open two-dimensional disc B. Thus, $\{U_1, U_2\}$ can be viewed as an atlas consisting of two charts. To be exact, we also need to specify homeomorphisms $\phi_i : U_i \to B$. For this particular atlas, instead of a disc B one usually considers the whole plane $\mathbb{R}^2$ (such a replacement is allowed because B and $\mathbb{R}^2$ are homeomorphic). As a standard homeomorphism $\phi_1 : U_1 \to \mathbb{R}^2$ one can take the so-called stereographic projection which is defined as follows (Figure 1). Let P be an arbitrary point of the sphere different from S. We consider the (unique) ray with the origin S and passing through P. This ray intersects the horizontal coordinate plane $\mathbb{R}^2 = Oxy$ at a certain point Q. By definition we put $\phi_1(P) = Q$. It is easy to see that this rule defines a natural bijection between $U_1 = S^2 \setminus S$ and $\mathbb{R}^2$ which is obviously continuous in both directions. The homeomorphism $\phi_2 : U_2 \to \mathbb{R}^2$ is obtained by replacing S with N.

   The construction for S^n is absolutely similar.

2. Describe an atlas for the (2-dimensional) torus which consists of 4 charts.

   The atlas for the torus T^2 consisting of four charts can be constructed as follows. We know that T^2 can be obtained from a square by pairwise identification of its edges. As the first chart we can take the interior of the square. This chart covers almost the whole torus except for the set which can be represented as two circles glued at one point (try to sketch this situation). Before gluing, this point was 4 vertices of the square. Consider a small neighborhood of this point. This is another chart. The part of the torus which is still uncovered is now the disjoint union of two intervals. For each of these intervals we can find a neighborhood homeomorphic to a disc. We take them as the third and forth charts. As a result, we have an atlas consisting of 4 charts.

3. Consider the subset X in $\mathbb{R}^3$ given by the following equation:

$$x^3 + 3xy^2 + 3xz^2 + 2y^3 + 5yz^2 + z^3 = 1.$$

Using the Implicit Function Theorem, verify that X is a manifold.

$F(x, y, z) = x^3 + 3xy^2 + 3xz^2 + 2y^3 + 5yz^2 + z^3 = 1$. Consider the differential of F:

$$dF = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = (3x^2+3y^2+3z^2, 6xy+6y^2+5z^2, 6xz+10yz+3z^2).$$

It is easy to see that $dF = 0$ implies that $3x^2 + 3y^2 + 3z^2 = 0$ and, therefore, $x = y = z = 0$. Hence, the only point where the differential dF vanishes is $(0, 0, 0)$. But this point does not belong to the set $X = \{F(x, y, z) = 1\}$ so that dF does not vanish on X, and according to the Implicit Function Theorem, X is a manifold.

4. Consider the subset X of $\mathbb{R}^2$ given by the equation

$$(x^2 + y^2 - 1)(x^2 - 2x + y^2) = 0.$$

Is X a manifold?

$(x^2 + y^2 - 1)(x^2 - 2x + y^2) = 0$

This equation means that X is the union of two sets given by the equations $x^2 + y^2 - 1 = 0$ and $x^2 - 2x + y^2 = 0$ respectively.

The first equation defines the circle of radius 1 centered at the origin. The second can be rewritten as

$$x^2 - 2x + y^2 = (x - 1)^2 + y^2 - 1 = 0$$

and defines the circle of radius 1 centered at the point $(1, 0)$. These two circles intersects at two points P and Q. Although each of these circles is a manifold, their union X is not: $P, Q \in X$ are singular points, they do not admit any neighborhood $U \subset X$ homeomorphic to an interval (interval = 1-dim disc).

5. Prove that the set $X \subset \mathbb{R}^4$ given by two equations:

$$x^2 + y^2 + z^2 + u^2 = 1, \qquad x^2 + y^2 - z^2 - u^2 = 0,$$

is a two dimensional compact connected manifold. (This manifold is homeomorphic to a torus).

We have $x^2+y^2+z^2+u^2 = 1$, $x^2+y^2-z^2-u^2 = 0$. It is easy to see that this system can be rewritten in the following equivalent form

$$F_1(x, y, z, u) = x^2 + y^2 = 1/2$$
$$F_2(x, y, z, u) = z^2 + u^2 = 1/2$$

To prove that the set of solutions forms a manifold in $\mathbb{R}^4(x, y, z, u)$, we verify the condition of the Implicit Function Theorem. Consider the 2×4 matrix formed by the differentials dF_1 and dF_2:

$$\begin{pmatrix} 2x & 2y & 0 & 0 \\ 0 & 0 & 2z & 2u \end{pmatrix}$$

It is easy to see that the rows of this matrix are linearly independent (or, equivalently, its rank is 2) unless either $x = y = 0$, or $u = z = 0$. It is easy to see that neither $x = y = 0$, nor $u = z = 0$ may hold on X. Thus, the differentials of F_1 and F_2 are linearly independent everywhere on X so that X is a manifold by the Implicit Function Theorem.

The compactness of X easily follows from two observations: (i) X is closed as the intersection of two closed sets $\{F_1 = 1/2\}$ and $\{F_2 = 1/2\}$; (ii) X is bounded, because $F_1 = 1/2$ obviously implies $|x| < \sqrt{1/2}$ and $|y| < \sqrt{1/2}$ and, similarly, $F_2 = 1/2$ implies $|z| < \sqrt{1/2}$ and $|u| < \sqrt{1/2}$.

The path connectedness of X follows from the following argument. Let (x_0, y_0, z_0, u_0) and (x_1, y_1, z_1, u_1) belong to X. To construct a path which joins them, we notice that (x_0, y_0) and (x_1, y_1) can be considered as two points lying on the circle defined by $F_1 = 1/2$ and, therefore, can be joined by an arc of this circle. Formally, this means that there exists a continuous path $(x(t), y(t))$, $t \in [0,1])$, such that $(x(0), y(0)) = (x_0, y_0)$, $(x(1), y(1)) = (x_1, y_1)$, and any point $(x(t), y(t))$ satisfies $F_1 = 1/2$. Similarly, there exists a path $(z(t), u(t)$ that joins (z_0, u_0) and (z_1, u_1) and satisfies $F_2 = 1/2$ for each $t \in [0,1]$.

Now it remains to notice that the continuous path $(x(t), y(t), z(t), u(t))$ satisfies the both equations $F_1 = 1/2$ and $F_2 = 1/2$ (i.e., belongs to X) and joins the points (x_0, y_0, z_0, u_0) and (x_1, y_1, z_1, u_1). Thus, X is path connected and, therefore, connected.

6. Prove that the set of 2-dimensional orthogonal matrices

$$O(2) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad : \quad AA^\top = \mathrm{Id} \right\}$$

is a manifold. What is the dimension of $O(2)$?

The condition $AA^\top = \mathrm{Id}$ can be represented as three scalar equations

$$\begin{aligned} F_1(a,b,c,d) &= a^2 + b^2 = 1, \\ F_2(a,b,c,d) &= ac + bd = 0, \\ F_3(a,b,c,d) &= c^2 + d^2 = 1. \end{aligned}$$

The Jacobi matrix formed by their differentials $dF_i = \left(\frac{\partial F_i}{\partial a}, \frac{\partial F_i}{\partial b}, \frac{\partial F_i}{\partial c}, \frac{\partial F_i}{\partial d}\right)$ takes the form

$$\begin{pmatrix} 2a & 2b & 0 & 0 \\ c & d & a & b \\ 0 & 0 & 2c & 2d \end{pmatrix}$$

We need to verify that the rank of this matrix is equal to 3 for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(2)$

Consider the 3×3 determinants obtained from the Jacobi matrix by removing the fourth and third columns respectively:

$$\begin{vmatrix} 2a & 2b & 0 \\ c & d & a \\ 0 & 0 & 2c \end{vmatrix} = 4c(ad - bc), \qquad \begin{vmatrix} 2a & 2b & 0 \\ c & d & b \\ 0 & 0 & 2d \end{vmatrix} = 4d(ad - bc)$$

These determinants cannot vanish simultaneously on $O(2)$. Indeed, the matrices $A \in O(2)$ are all invertible so that $ad - bc \neq 0$.

Thus, we only need to verify that c and d cannot vanish simultaneously on $O(2)$. But it is evident, because $c = d = 0$ would means that the second row of $A \in O(2)$ vanishes, which is impossible since $\det A \neq 0$.

Thus, at least one of the above determinants does not vanish on $O(2)$, i.e., the rank of the Jacobi matrix is 3 for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(2)$

Thus the condition of the Implicit Function Theorem holds and therefore $O(2)$ is a manifold of dimension $4 - 3 = 1$.

7. Prove that the set $GL(n, \mathbb{R})$ of all invertible $n \times n$ matrices is a manifold.

   $GL(n, \mathbb{R})$ can be viewed as a subset of the n^2 dimensional vector space of all $n \times n$-matrices given by an inequality $\det A \neq 0$. Since det is a continuous function, then $GL(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n^2} = \{\text{all matrices}\}$. (Formal proof of this fact is that $GL(n, \mathbb{R})$ can be regarded as the preimage of the open set $(-\infty, 0) \cup (0, +\infty)$ under the continuous map $\det : \mathbb{R}^{n^2} \to \mathbb{R}$). An open set A of $\mathbb{R}^m$ is characterized by the fact that for each point $x \in A$ there is a ball B_x centered at x such that $B_x \subset A$. In particular, this means that each point possesses a neighborhod homeomorphic to a ball so that we have the following conclusion: *Any open subset of* $\mathbb{R}^m$ *is a manifold of dimension* m.

   In particular, $GL(n, \mathbb{R})$ is a manifold of dimension n^2 (as an open subset of $\mathbb{R}^{n^2} = \{\text{all matrices}\}$).

8. Prove that every connected manifold X is path connected.

   Assume that X is a connected manifold. Let us fix a point $x \in X$ and consider the set A of points which can be connected with x by a continuous path. (We need to show that this subset, in fact, coincides with the whole X).

   First of all notice that A is open. Indeed, if $y \in A$, then we can consider a neighborhood U of y homeomorphic to a disc. It is clear that every point $z \in U$ can be connected with $y \in U$ by a continuous path (because U is homeomorphic to a disc and, for a disc, this statement obviously holds). This shows that $z \in A$.

Indeed, z can be connected with x by a continuous path which consists of two parts: first we connect x with y and then y with z. Thus, $U \subset A$, i.e. each point $y \in A$ belongs to U together with some neighborhood. This means that A is open.

Now let us show that A is closed. Consider any limit point y of A. To prove that A is closed, we need to verify that $y \in A$ (i.e. A contains all of its limit points). Take a neighborhood U of y which is homeomorphic to disc. This neighborhood contains a point $z \in A$. Since x can be connected with z and z can be connected with y, we conclude that y can be connected with x so that $y \in A$.

Thus A is open and closed simultaneously. But X is connected and therefore the only non-empty set which is open and closed at the same time is X itself. Conclusion: $A = X$, i.e. any point $y \in X$ can be connected by a continuous path with the fixed point x. Thus, X is path connected.