23MAC260 Problem Sheet 9

Week 10 Lectures

Last updated: April 4, 2024

1. Let L be the lattice in the complex plane spanned by the two numbers

$$\omega_1 = \frac{3}{8} - \frac{3\sqrt{3}}{8} i$$
$$\omega_2 = -i$$

Find a complex number τ in the region

$$\mathcal{F} = \left\{ z \in \mathbb{C} : \mathrm{Im}(z) > 0,\ |\,\mathrm{Re}(z)| \leq \frac{1}{2},\ |z| \geq 1 \right\}$$

such that L is similar to the lattice

$$\mathbb{Z} \oplus \mathbb{Z} \cdot \tau.$$

2. Recall our definition of the action of $SL(2, \mathbb{Z})$ on the upper half-plane $\mathcal{H}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

Prove that if M_1 and M_2 are matrices in $SL(2, \mathbb{Z})$ then

$$M_1 \cdot (M_2 \cdot \tau) = (M_1 M_2) \cdot \tau$$

so we really do have a group action.

3. Show that if τ_1, τ_2 are two complex numbers in the upper half-plane H and M is a 2×2 matrix such that

$$M \cdot \tau_1 = \tau_2$$

then $\det M > 0$.

4. Show that for any $\tau \in \mathcal{H}$ we have
$$(-I_2) \cdot \tau = \tau.$$

5. Verify the following relations among the matrices
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}:$$

   (a) $S^2 = -I_2$.

   (b) $(ST)^3 = -I_2$.

   (Using the previous problem this shows that the transformations of $\mathcal{H}$ given by S and ST have order 2 and 3 respectively.)

6. Write the matrix
$$\begin{pmatrix} 11 & 14 \\ 7 & 9 \end{pmatrix} \in SL(2, \mathbb{Z})$$
   in terms of S and T.

7. In this problem we will prove Lemma 2.3 from Week 10, stating that $SL(2, \mathbb{Z})$ is generated by the two matrices S and T.

   (a) For a general matrix
$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$
   calculate SM and $T^n M$ for any integer n.

   (b) Use the previous part to show that, by multiplying M by suitable powers of S and T, we can reduce it to a matrix of the form
$$\widetilde{M} = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in SL(2, \mathbb{Z}).$$
   (Hint: Euclidean algorithm.)

   (c) Use the fact that $\widetilde{M} \in SL(2, \mathbb{Z})$ to conclude that $\alpha = \delta = \pm 1$. Hence write $\widetilde{M}$ in terms of S and T.

   (d) Conclude that M can be written in terms of S and T.