23MAC260 Problem Sheet 4

Week 4 lectures

1. Let a and b be complex numbers such that $-4a^3 - 27b^2 \neq 0$. Let E and E' be the elliptic curves given the by two equations

$$\begin{aligned} E\colon\ y^2 &= x^3 + ax + b \\ E'\colon\ y^2 &= x^3 + ax - b. \end{aligned}$$

   (a) Show that $E \simeq E'$.

   (b) If a, $b \in \mathbb{R}$, show that E and E' are not isomorphic over $\mathbb{R}$ unless $b = 0$.

2. Consider the family of curves

$$E_t\colon \quad y^2 = x^3 + a(t)x + b(t)$$

   where $a(t)$ and $b(t)$ are polynomials in the parameter t. Suppose that

$$\Delta(t) = -4a(t)^3 - 27b(t)^2$$

   is not identically zero.

   (a) Show that there is a finite (possibly empty) set V of values for t such that E_t is an elliptic curve for all $t \in \mathbb{C} \setminus V$.

   (b) Suppose that neither of a and b is identically zero, that a and b have no common root, and that $3 \deg a \neq 2 \deg b$. Show that for every $c \neq 0$, -1728 there is an elliptic curve E_t in the family with $j(E_t) = c$.

3. *Legendre form.* A cubic is in **Legendre form** if it is given as

$$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

   for some number $\lambda \neq 0$, 1.

(a) Show that every cubic in Legendre form defines an elliptic curve.

(b) Transform the Legendre equation into Weierstrass form.

(c) Use the previous part to show that for every $j \neq 0, -1728$, there are exactly 6 values of λ such that $j(E_\lambda) = j$.

(d) Which values of λ give $j(E_\lambda) = 0$? Which give $j(E_\lambda) = -1728$?

4. Starting from the right-angled triangle with sides of length $(5, 12, 13)$, use the method described in the Week 4 lectures to produce another right-angled triangle with rational sides and area 30.