23MAC260 Problem Sheet 3

Week 3

Last updated February 8, 2024

1. Put each of the following curves in Weierstrass form:

   (a) $y^2 = -x^3 + 2x^2 + 3x - 1$

   (b) $y^2 = 8x^3 - 4x^2 - 2x + 1$

   (c) $y^2 = \omega x^3 + \omega^2 x^2 + x$ (where $\omega = \exp(2\pi i/3)$).

   Which of these equations define elliptic curves?

2. We defined the discriminant Δ of a cubic polynomial

$$f(x) = x^3 + ax + b$$

   in Weierstrass form to be

$$\Delta = -4a^3 - 27b^2.$$

   Now suppose that f has roots x_0, x_1, x_2.

   (a) Write a and b in terms of x_0, x_1, and x_2. What is $x_0 + x_1 + x_2$?

   (b) Prove that

$$\Delta = (x_0 - x_1)^2(x_0 - x_2)^2(x_1 - x_2)^2.$$

   (This gives a different proof that $\Delta = 0$ if and only if f has a multiple root.)

3. (Week 3 notes, p.8) For the elliptic curve

$$y^2 = x^3 + 73$$

and the point $P = (2, 9)$, use the formula from lectures to compute the points

$$2P,\ 3P,\ 4P.$$

(A calculator will be useful here.)

Do you think there is an integer n such that $nP = O$? We will provide a definite answer later in the module.

4. Consider the curve C_t given by the equation

$$Y^2Z = tX^3 - XZ^2 - Z^3$$

where $t \in \mathbb{C}$ is a parameter. As we vary the parameter t, we get a family of cubic curves in $\mathbb{P}^2$.

   (a) Find all values of t such that C_t is not an elliptic curve. (Hint: use the discriminant Δ.)

   (b) What does the curve C_0 (obtained by setting $t = 0$ in the equation above) look like?