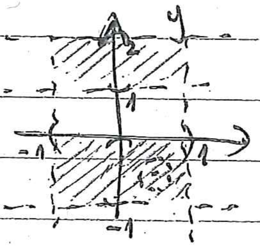
No 1 (a) Def. A set U in a metric space (X, d) is open if and only if for any $x \in U$ there exists $\delta > 0$ such that $B_\delta(x) \subset U$ where

$$B_\delta(x) = \{ y \in X : d(y,x) < \delta \}$$

is the ball of radius δ centered at x. bookwork [4]

1.(b)(i) $A_1 = \{ x \in (-1,1),\ y \in (-1,0) \cup (1,2) \}$

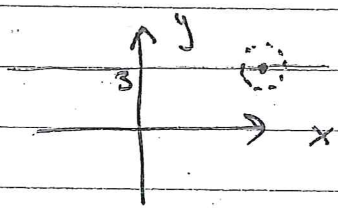


A_1 is open

Standard question

[2]

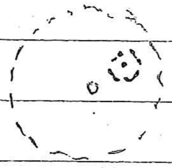
(ii) $A_2 = \{ y = 3,\ x \in (-\infty, \infty) \}$



A_2 is not open

[2]

(iii) $A_3 = \{ 0 < x^2 + y^2 < 1 \}$



A_3 is open

[2]

(c) Hausdorff : $\mathbb{R}$, $\mathbb{R}^n$, any metric space, ...

Non-Hausdorff : Indiscrete, Cofinite top., bug-eyed line, ... Standard question [4]

(d) (i) (X, d) is a metric space

Consider two points $x, y \in X$, $x \neq y$. Let $\varepsilon = d(x,y) > 0$,

then the open balls

$$U_{\varepsilon/3}(x) = \{ z \in X : d(x,z) < \varepsilon/3 \}$$

and

$$V_{\varepsilon/3}(y) = \{ z \in X : d(y,z) < \varepsilon/3 \}$$

represent disjoint neighborhoods of x and y

(due to the triangle inequality

$d(x,y) < d(x,z) + d(y,z)$ ..) bookwork [3]

(ii) Let $x \in X$. To prove that $\{x\}$ is closed, we need to verify that $X \setminus \{x\}$ is open.
Let $y \in X \setminus \{x\}$, i.e. $y \neq x$. Since X is Hausdorff, x and y have disjoint neighborhoods $U(x)$ and $V(y)$. Since $x \notin V(y)$, we have $V(y) \subset X \setminus \{x\}$. In other words, each point y is contained in $X \setminus \{x\}$ together with a certain neighborhood. Thus, $X \setminus \{x\}$ is open and, therefore, $\{x\}$ is closed.

bookwork
[3]

No 2 a) Def X is compact $:\Longleftrightarrow$ every open cover of X admits a finite subcover.

$\boxed{\text{bookwork } [3]}$

b) $\underset{\text{closed}}{A} \subset \underset{\text{compact}}{X}$. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of A $\Rightarrow$ $\mathcal{U}' = \{U_\alpha, X \setminus A\}$ is an open cover of $X \xRightarrow{X \text{ compact}} \exists$ finite subcover
$\Rightarrow$ same finite subcover (exclude $X \setminus A$) works for A.

$\boxed{\text{bookwork } [5]}$

standard question, but partially unseen

(c) (i) $[0,1] \subset \mathbb{R}$ compact (Heine - Borel) [2]

(ii) $(0,1) \subset (\mathbb{R}, \text{indiscrete})$ compact (every subset of an indiscrete space is compact) [2]

(iii) $[0,\infty) \subset (\mathbb{R}, \tau)$ compact [2]

If $\{U_\alpha\}$ open cover $\Rightarrow \exists \alpha_0$ s.t. $0 \in U_{\alpha_0}$

$\Rightarrow U_{\alpha_0}$ is a finite subcover since either

$U_{\alpha_0} = \mathbb{R}$ or $U_{\alpha_0} = (a,\infty)$ $(a<0)$.

(iv) $\{\frac{1}{4^n} : n \in \mathbb{N}\} \subset \mathbb{R}$ not compact [2]

$\{U_n = (\frac{1}{4^n} - \frac{1}{8^n}, \frac{1}{4^n} + \frac{1}{8^n}) : n \in \mathbb{N}\}$ is an open

cover which admits no finite subcover.

(v) $\{x^2 - y^2 = 1\} \subset \mathbb{R}$ not compact ~~since not~~ (Heine - Borel)

bounded; e.g. $x = n$, $y = \sqrt{n-1}$, $n \in \mathbb{N}$, is an

unbounded sequence in this set. [2]

(vi) $\{1 < x^4 + y^4 \leq 2\} \subset \mathbb{R}^2$ not compact since

not closed (Heine-Borel); e.g. $(1,0)$ is an adherent point which does not belong to the set. [2]

(a)

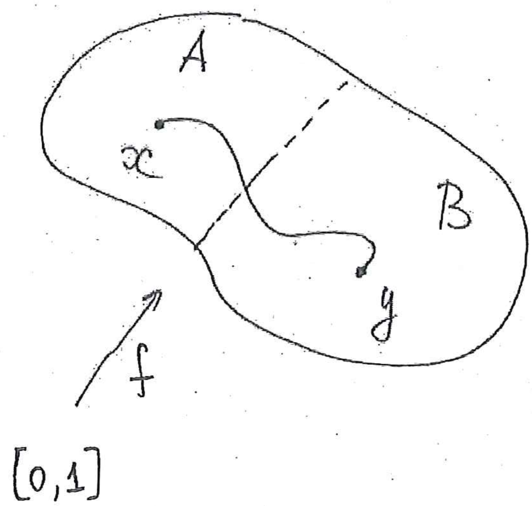
Def. A topological space (X,τ) is pathwise connected if and only if for any $x, y \in X$ there exists a continuous map $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. Such a map f is called a path from x to y.

bookwork [4]

(b) Thm. A pathwise connected topological space is connected.

Proof. By contradiction. Assume that X is pathwise connected but disconnected. Consider a partition of X into two disjoint non-empty open subsets: $X = A \cup B$. Take $x \in A$ and $y \in B$ and a continuous path $f : [0,1] \to X$ from x to y.

Then $[0,1] = f^{-1}(A) \cup f^{-1}(B)$ is a partition of $[0,1]$ into two disjoint non-empty open sets, which is impossible since $[0,1]$ is connected.



bookwork [4]

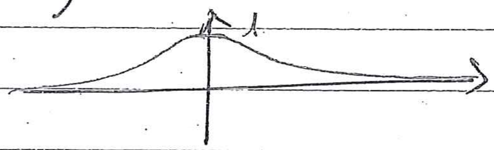
(c) $f: X \to Y$ is a homeomorphism $:\Leftrightarrow$

* f is bijective
* f is continuous
* f^{-1} is continuous

bookwork [3]

(d)(i) $f : \mathbb{R} \to (0,1]$, $f(x) = e^{-x^2}$

(unseen)



f is not injective

$(f(-x) = f(x) \ \forall x)$

[2]

$\Rightarrow f$ is not a homeo.

(ii) $f : [0, 2\pi) \to S^1$, $f(x) = e^{ix}$

not a homeo since S^1 is compact while $[0, 2\pi)$ is not.

[2]

(iii) $f : (0, \infty) \to (0, \infty)$, $f(x) = x^2$ is a homeo:

injective: $x^2 = y^2 \Rightarrow x = \pm y$ but only one of these can be > 0.

surjective: For $y > 0$, $f(\sqrt{y}) = y$ [2]

Continuous: polynomial ✓

f^{-1} continuous: $f^{-1}(y) = \sqrt{y}$ is continuous (Analysis I)

(e) $f : X \to Y$, $f(x) = x$

(unseen)

X ($\mathbb{R}$ disc), Y $\mathbb{R}$

[3]

* f cont. (every map from a discrete space to any other space is cont.)

* f invertible, $f^{-1}(y) = y$

* $f^{-1} : Y \to X$ not continuous

e.g. $\{0\} \subset X$ open, but $f^{-1}\{0\} = \{0\}$ not open in Y.

4. (a) Theorem (Implicit function theorem)

$F: \mathbb{R}^n \to \mathbb{R}$ smooth, $X = \{F = 0\}$. If

$dF \neq 0$ at every point of X, then X

is an $n-1$ dimensional manifold. bookwork [4]

(b) $X = \{F = 100\}$, $F = \frac{1}{4}x^4 + \frac{1}{2}y^2 x + xy$

$dF = \left(x^3 + \frac{1}{2}y^2 + y,\ yx + x\right)$ standard question [4]

$dF = 0 \Rightarrow x(y+1) = 0 \Rightarrow x = 0$ or $y = -1$

$F(0,y) = 0 \neq 100$

$F(x,-1) = \frac{1}{4}x^4 + \underbrace{\frac{1}{2}x - x}_{= -\frac{1}{2}x} \leq \frac{1}{4}x^4$

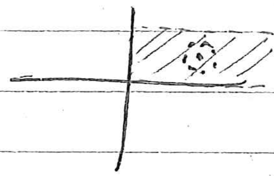
Hence, if $F(x,-1) = 100$, then $|x| \geq 400^{\frac{1}{4}}$.

But for $|x| \geq 400^{\frac{1}{4}}$, $y = -1$,

$$\left|x^3 + \tfrac{1}{2}y^2 + y\right| \geq 400^{\frac{3}{4}} - \tfrac{3}{2} \geq 400^{\frac{1}{2}} - \tfrac{3}{2} = 20 - \tfrac{3}{2} > 0$$

Thus, $dF \neq 0$ on $X \Rightarrow X$ is a one-dim. manifold.

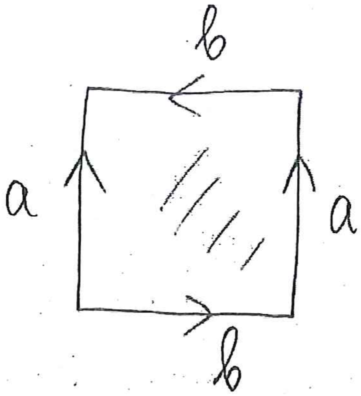
(c) $X = \{x > 0,\ 0 < y < 1\}$ unseen [4]



is an open subset of $\mathbb{R}^2$ and thus a manifold.

standard question [4]

(d)



Klein bottle can be obtained from a square by identification of its opposite edges as shown.

Thus $\chi(K^2) = V - E + F =$

$= 1 - 2 + 1 = 0$

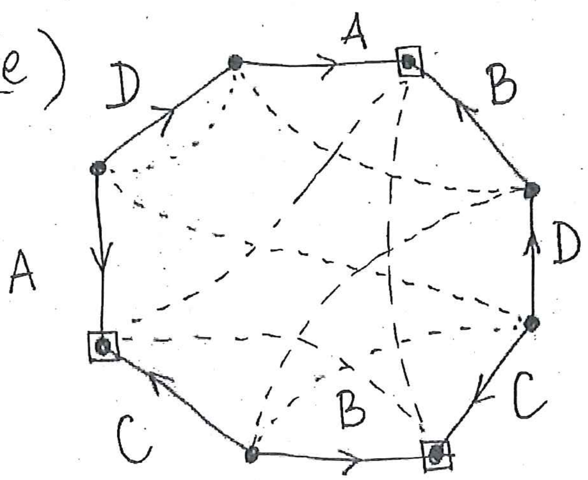
V = 1 one vertex after identification

E = 2 two edges a and b

F = 1 one face (the square)

standard question [4]

(e)



M is not orientable (as we have combination ... C ... C ...)

$\chi(M) = V - E + F = 2 - 4 + 1 = -1$

Conclusion: M is the sphere with 3 Möbius strips.

$\chi(S^2 + k \cdot \mu) = 2 - k$