MAB298-Elements of Topology: Problem Sheet 1

Topological spaces, open and closed sets

1. Let X consist of four elements: $X = \{a, b, c, d\}$. Which of the following collections of its subsets generate a topology on X:

   (a) $\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}, \{a, b\}$;

   (b) $\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}$;

   (c) $\emptyset, X, \{a, c, d\}, \{b, c, d\}$?

2. Let τ be a topology in $\mathbb{R}$ such that the intervals $[a, b]$ are open for all $a < b$. Prove that this topology is discrete.

3. Consider the collection of subsets of $\mathbb{R}$ that consists of:

   (a) $\mathbb{R}$, $\emptyset$ and all infinite "closed" intervals $[a, +\infty)$, $a \in \mathbb{R}$;

   (b) $\mathbb{R}$, $\emptyset$ and all infinite "open" intervals $(a, +\infty)$, $a \in \mathbb{R}$.

   Is this topology or not?

4. Let X be a plane. Let τ consist of $\emptyset$, X, and all open disks with center at the origin. Do X and τ define a topological space?

5. Let X be $\mathbb{R}$, and let τ consist of the empty set and all infinite subsets of $\mathbb{R}$. Do X and τ define a topological space?

6. List all topologies in a two-element set, say, in $\{0, 1\}$.

7. Let (X, τ) be a discrete topological space. Define a metric d on X such that the corresponding (metric) topology coincides with τ.

8. Let (X, τ) be an indiscrete topological space which contains at least two elements. Prove that there is no metric d on X such that the corresponding (metric) topology coincides with τ.

9. Find examples of sets that are

   (a) both open and closed simultaneously (open-closed);

   (b) neither open, nor closed.

10. Give an explicit description of closed sets in

    (a) a discrete space;

    (b) an indiscrete space;

    (c) $\mathbb{R}$ with topology as in 3(b) (open sets are $\mathbb{R}$, $\emptyset$ and the infinite intervals $(a, +\infty)$).

11. Is a "closed" segment $[a, b]$ closed in

    (a) $\mathbb{R}$ with the usual topology?

    (b) $\mathbb{R}$ with the topology defined in Ex. 3(b)?

12. Prove that the half-open interval $[0, 1)$ is neither open nor closed in $\mathbb{R}$, but is both a union of closed sets and an intersection of open sets.