MAB298-Elements of Topology: Solution Sheet 2
Closure of a set, interior, exterior and boundary points

1. What is the closure of $\mathbb{Q}$ in $\mathbb{R}$?

   The closure of $\mathbb{Q}$ in $\mathbb{R}$ is $\mathbb{R}$, because any point $a \in \mathbb{R}$ is adherent for $\mathbb{Q}$ (recall: this means that in any neighborhood $U(a)$ there are rational points, i.e. elements of $\mathbb{Q}$).

2. Is it true that $\overline{A \cup B} = \bar{A} \cup \bar{B}$?

   Yes, it is. By definition, $\overline{A \cup B}$ is the smallest closed set that contains both A and B. Since $\bar{A}$ is the smallest closed set that contains A, we have $\bar{A} \subset \overline{A \cup B}$. Similarly, $\bar{B} \subset \overline{A \cup B}$. Therefore, $\bar{B} \cup \bar{A} \subset \overline{A \cup B}$. Conversely, $\bar{A} \cup \bar{B}$ is a closed set that contains both A and B, but $\overline{A \cup B}$ is the smallest set with this property so that $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$. Thus, $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

3. Find the closure of $\{a\}$ in $\mathbb{R}$ with the topology as in Problem Sheet 1, 3(b).

   The closed sets in this topology are $\emptyset$, $\mathbb{R}$ and $(-\infty, c]$, $c \in \mathbb{R}$. The smallest of them which contains a is obviously $(-\infty, a]$. Thus $\overline{\{a\}} = (-\infty, a]$.

4. Find the boundary of (a, b) in $\mathbb{R}$ with the topology as Problem Sheet 1, 3(b).

   Let us consider an arbitrary point $c \in \mathbb{R}$ and determine its type (boundary, interior or exterior).

   Case 1: $c \leq b$, or equivalently $c \in (-\infty, b]$. A neighborhood of c is $(c - \varepsilon, +\infty)$. Obviously, this neighborhood contains both points which belong to (a, b) and points which don't. Thus, c is a boundary point.

   Case 2: $c > b$. Then there exists $\varepsilon > 0$ such that $c - \varepsilon > b$. Then the neighborhood $(c - \varepsilon, +\infty)$ of c does not intersect the interval (a, b). This means that c is an exterior point of (a, b).

   Conclusion: the boundary of (a, b) in this topological space is $(-\infty, b]$.

5. Prove that a set A is closed if and only if $\partial A \subset A$.

   A is closed if and only if it coincides with its closure $\bar{A}$. Using Proposition 7 in Lecture Notes which states that $\bar{A} = A \cup \partial A$, we conclude that A is closed if and only if $A = A \cup \partial A$. The latter relation obviously means that $\partial A \subset A$, as required.

   Another explanation. We know that the ambient topological space X can be presented as the disjoint union of three sets $\mathrm{Int}\, A$, ∂A and $\mathrm{Int}\,(X \setminus A)$ (exterior of A). Moreover the interior $\mathrm{Int}\, A$ and exterior $\mathrm{Int}\,(X \setminus A)$ are open, whereas the boundary ∂A is closed.

   If $\partial A \subset A$, then $A = \mathrm{Int}\, A \cup \partial A$ and A is closed as the complement to the open set $\mathrm{Int}\,(X \setminus A)$. Conversely, if A is closed then its complement $X \setminus A$ is open and, therefore, coincides with its interior: $X \setminus A = \mathrm{Int}\,(X \setminus A)$. It follows from this that $A = X \setminus \mathrm{Int}\,(X \setminus A) = \mathrm{Int}\, A \cup \partial A$ and, consequently, A contains ∂A.

6. Find limit and isolated points of the sets $(0,1] \cup \{2\}$ and $\{1/n \mid n \in \mathbb{N}\}$ in $\mathbb{R}$.

   For the set $(0,1] \cup \{2\}$, the limit points are those from $[0,1]$, the isolated point is 2.

   For the set $\{1/n \mid n \in \mathbb{N}\}$, the limit point is 0, the isolated points are $1/n$, $n \in \mathbb{N}$.

7. Let $X \subset \mathbb{R}$ be the subset obtained from the closed interval $[0,1]$ by removing all the points of the form $\dfrac{1}{n}$, $n = 1, 2, 3, \ldots$, i.e.:

$$X = [0,1] \setminus \left\{ \frac{1}{n},\ n \in \mathbb{N} \right\}$$

   Is X open? closed? Describe the interior, boundary, closure, and adherent points of X?

   Answer:

   X is not open, as $0 \in X$ is not an interior point.

   X is not closed, as $1/2$ is an adherent point of X which does not belong to X.

$\text{Int}\, X = \cup_{n \in \mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n} \right)$.

$\partial X = \left\{ 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$.

$\bar{X} = [0, 1]$.

The set of adherent points of X is $\bar{X} = [0, 1]$.

8. Consider $A = [0, 1)$ as a subset of $\mathbb{R}$ with

   (a) the standard topology,

   (b) the discrete topology,

   (c) the indiscrete topology.

   Describe the interior, boundary, closure, and adherent points of A?

   Answer:

   (a) $\text{Int}\, A = (0, 1), \quad \bar{A} = [0, 1], \quad \partial A = \{0, 1\}$;

   (b) $\text{Int}\, A = A = [0, 1), \quad \bar{A} = A = [0, 1), \quad \partial A = \emptyset$;

   (c) $\text{Int}\, A = \emptyset, \quad \bar{A} = \mathbb{R}, \quad \partial A = \mathbb{R}$.

   The set of adherent points of A always coincides with $\bar{A}$.

9. Prove that $\partial A = \partial(X \setminus A)$.

   It follows immediately from the definition of a boundary point: c is a boundary point iff each neighborhood of c contains points from both A and $X \setminus A$. This definition does not change if we interchange A and $X \setminus A$.

10. Do the following equalities hold true for any sets A and B:

    (a) $\text{Int}\,(A \cup B) = \text{Int}\, A \cup \text{Int}\, B$,

    (b) $\text{Int}\,(A \cap B) = \text{Int}\, A \cap \text{Int}\, B$?

    (a) is wrong.

    Example: $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$. Then $\text{Int}\,(A \cup B) = \text{Int}\, \mathbb{R} = \mathbb{R}$, whereas $\text{Int}\, A \cup \text{Int}\, B = \emptyset \cup \emptyset = \emptyset$.

    (b) is true.

If x is an interior point of $A \cap B$, then there is a neighborhood $U(x)$ which entirely belongs to $A \cap B$. In particular, $U(x) \subset A$ and $U(x) \subset B$. This means that x is an interior point for both A and B and we conclude that $\text{Int } (A \cap B) \subset \text{Int } A \cap \text{Int } B$.

Conversely, if $x \in \text{Int } A \cap \text{Int } B$, i.e. x is an interior point for both A and B, then there exist neighborhoods $U_1(x)$ and $U_2(x)$ such that $U_1(x) \subset A$ and $U_2(x) \subset B$. Take the intersection $U_3(x) = U_1(x) \cap U_2(x)$. Obviously, $U_3(x)$ is a neighborhood of x and moreover $U_3(x) \subset A \cap B$. This means that x is an interior point of $A \cap B$ and we conclude that $\text{Int } A \cap \text{Int } B \subset \text{Int } (A \cap B)$.

Conclusion: $\text{Int } A \cap \text{Int } B = \text{Int } (A \cap B)$.

11. Let X be a metric space and $x \in X$. Consider the open ball of radius r

$$B(x, r) = \{y \in X \mid d(x, y) < r\}$$

and the sphere of the same radius and with the same center:

$$S_r = \{y \in X \mid d(x, y) = r\}$$

Is it true that $S_r = \partial B(x, r)$? Is it true that the closure of $B(x, r)$ equals $\{y \in X \mid d(x, y) \leq r\}$?

In $\mathbb{R}^n$ with the standard metric, these statements are true. But in general, this is false.

The simplest example counterexample is a discrete space. Consider, for instance, $\mathbb{Z}$ as a subset of $\mathbb{R}$ with the induced distance function.

The open ball of radius 1 centered at 0 (in $\mathbb{Z}$) consists of the only point 0:

$$B_1(0) = \{x \in \mathbb{Z} \ : \ |x| < 1\} = \{0\}$$

The corresponding sphere

$$S_1(0) = \{x \in \mathbb{Z} \ : \ |x| = 1\}$$

consists of two points 1 and -1.

Obviously, $B_1(0)$ is closed, so its closure is just $\{0\}$, but not $\{x \in \mathbb{Z} : d(0, x) \leq 1\} = \{-1, 0, 1\}$. The boundary of $B_1(0)$ in this case case is empty.