Precalculus: Functions and Their Graphs

Ryan Lee

July 13, 2024

1 Lines in the Plane

Lines are fundamental objects in precalculus and can be represented in various forms.

1.1 Slope-Intercept Form

The slope-intercept form of a line is $y = mx + b$, where:

- m is the slope (rate of change)
- b is the y-intercept (where the line crosses the y-axis)

1.2 Point-Slope Form

The point-slope form is $y - y_1 = m(x - x_1)$, where (x_1, y_1) is a point on the line and m is the slope.

1.3 Slope Formula

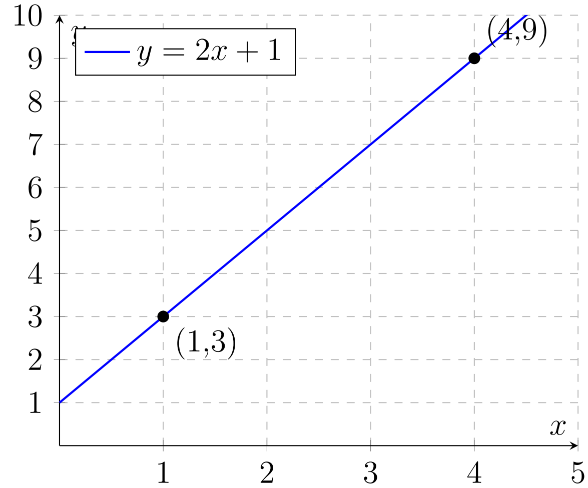
For two points (x_1, y_1) and (x_2, y_2), the slope is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

1.4 Example 1: Finding the Equation of a Line

Find the equation of a line passing through points (1, 3) and (4, 9).

Solution: 1) Calculate the slope: $m = \frac{9-3}{4-1} = \frac{6}{3} = 2$
2) Use point-slope form with (1, 3): $y - 3 = 2(x - 1)$
3) Simplify to slope-intercept form: $y = 2x + 1$



2 Functions

A function is a rule that assigns each element of one set (domain) to exactly one element of another set (range).

2.1 Key Concepts

- Domain: Set of all possible input values (x-values)
- Range: Set of all possible output values (y-values)
- Independent variable: Input variable (usually x)
- Dependent variable: Output variable (usually y)

2.2 Function Notation

We write $f(x) = y$ to denote a function f that takes an input x and produces an output y.

2.3 Evaluating Functions

To evaluate a function, substitute the given x-value into the function.

2.4 Example 2: Evaluating a Function

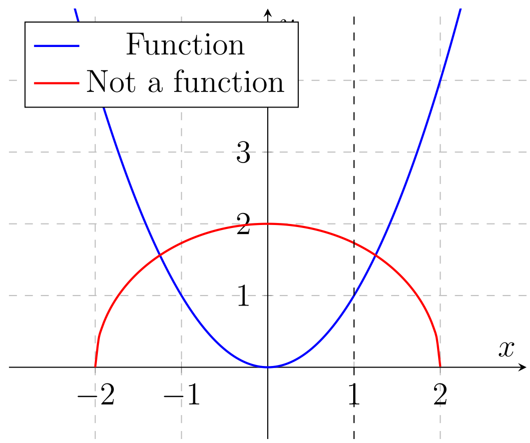
If $f(x) = x^2 - 3x + 2$, find $f(4)$ and $f(-1)$.

Solution: For $f(4)$: $f(4) = (4)^2 - 3(4) + 2 = 16 - 12 + 2 = 6$

For $f(-1)$: $f(-1) = (-1)^2 - 3(-1) + 2 = 1 + 3 + 2 = 6$

2.5 Vertical Line Test

A graph represents a function if and only if no vertical line intersects the graph more than once.



3 Graphs of Functions

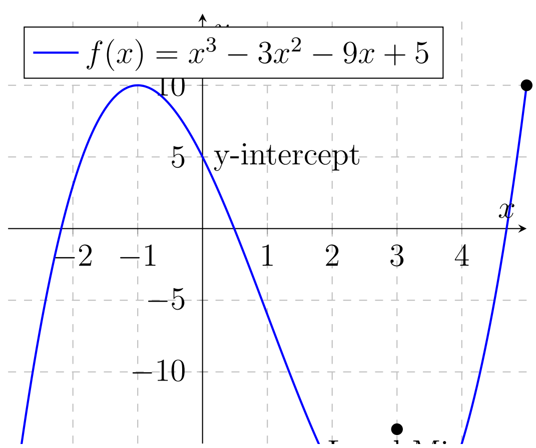
The graph of a function is the set of all points (x, y) where $y = f(x)$.

3.1 Key Features of Graphs

- x-intercepts: Points where the graph crosses the x-axis ($f(x) = 0$)
- y-intercept: Point where the graph crosses the y-axis ($x = 0$)
- Increasing/Decreasing: Intervals where the function values increase or decrease
- Maximum/Minimum points: Highest and lowest points on the graph

3.2 Example 3: Analyzing a Graph

Consider the graph of $f(x) = x^3 - 3x^2 - 9x + 5$:



Observations:

- y-intercept: (0, 5)
- x-intercepts: approximately at x = -1.5, x = 1, and x = 3.5

- Local maximum at approximately (-1, 18)
- Local minimum at approximately (3, -14)
- Increasing on intervals (-, -1) and (3,)
- Decreasing on interval (-1, 3)

4 Transformations of Functions

Transformations allow us to modify the graph of a function in predictable ways.

4.1 Shifting Graphs

4.1.1 Vertical Shifts

For a function $f(x)$:

- $f(x) + k$ shifts the graph k units up
- $f(x) - k$ shifts the graph k units down

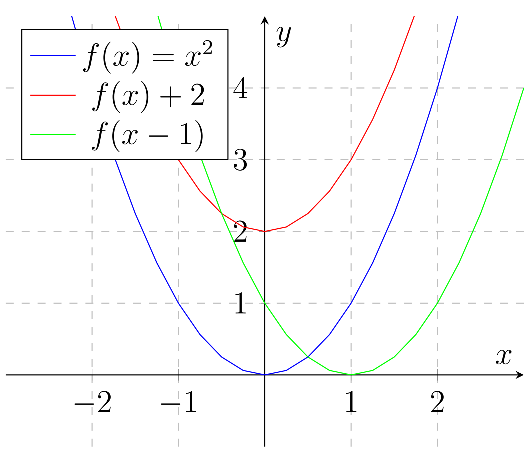
4.1.2 Horizontal Shifts

For a function $f(x)$:

- $f(x - h)$ shifts the graph h units right
- $f(x + h)$ shifts the graph h units left

4.2 Example 4: Shifting Graphs

Consider the function $f(x) = x^2$. Graph $f(x)$, $f(x) + 2$, and $f(x - 1)$.

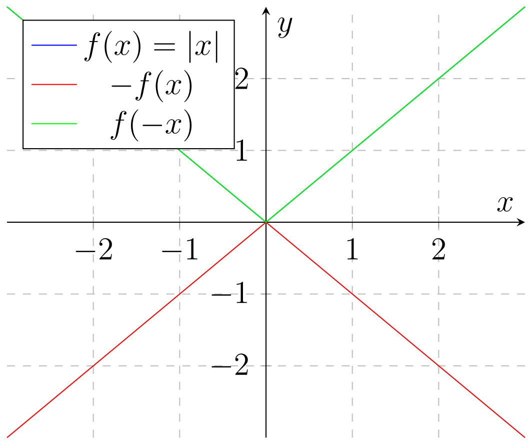


4.3 Reflecting Graphs

- $-f(x)$ reflects the graph of $f(x)$ over the x-axis
- $f(-x)$ reflects the graph of $f(x)$ over the y-axis

4.4 Example 5: Reflecting Graphs

Graph $f(x) = |x|$, $-f(x)$, and $f(-x)$.



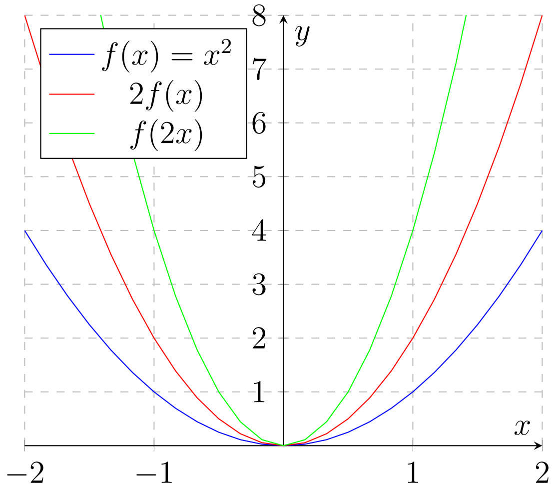
4.5 Stretching and Compressing Graphs

For a constant $a > 0$:

- $af(x)$ stretches the graph vertically by a factor of a if $a > 1$, or compresses it if $0 < a < 1$
- $f(ax)$ compresses the graph horizontally by a factor of a if $a > 1$, or stretches it if $0 < a < 1$

4.6 Example 6: Stretching and Compressing

Graph $f(x) = x^2$, $2f(x)$, and $f(2x)$.



5 Combinations of Functions

We can combine functions to create new functions using basic arithmetic operations.

5.1 Sum and Difference

For functions $f(x)$ and $g(x)$:

- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$

5.2 Product and Quotient

For functions $f(x)$ and $g(x)$:

- $(f \cdot g)(x) = f(x) \cdot g(x)$
- $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$

5.3 Composition

The composition of f and g is denoted as $(f \circ g)(x) = f(g(x))$.

5.4 Example 7: Function Combinations

Let $f(x) = x^2$ and $g(x) = x + 1$. Find and evaluate at $x = 2$:

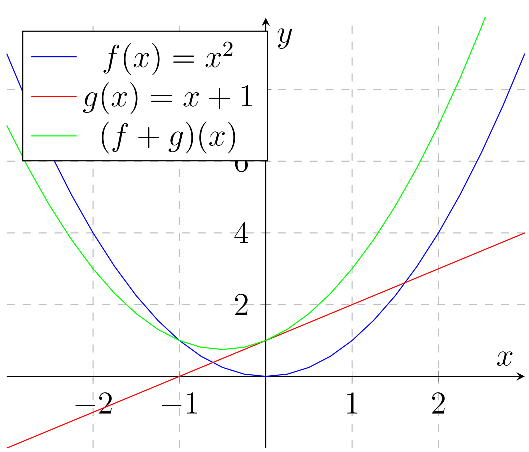
- $(f+g)(x)$
- $(f \cdot g)(x)$
- $(f \circ g)(x)$

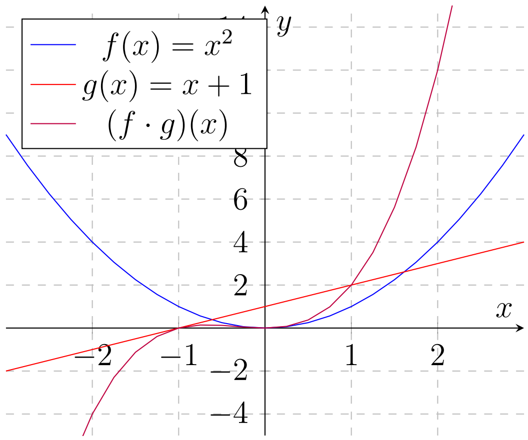
Solution:

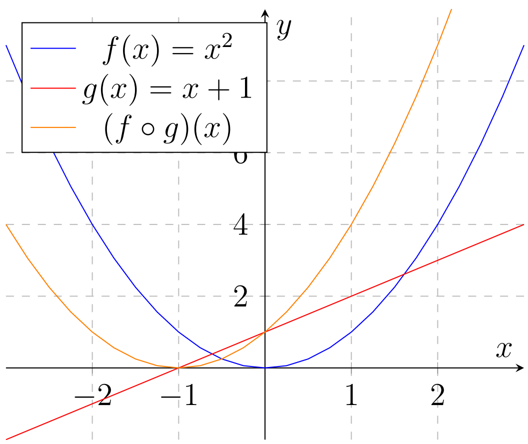
- $(f+g)(x) = x^2 + (x+1)$ $(f+g)(2) = 2^2 + (2+1) = 4 + 3 = 7$
- $(f \cdot g)(x) = x^2(x+1)$ $(f \cdot g)(2) = 2^2(2+1) = 4 \cdot 3 = 12$
- $(f \circ g)(x) = f(g(x)) = f(x+1) = (x+1)^2$ $(f \circ g)(2) = (2+1)^2 = 3^2 = 9$

5.5 Graphical Representation of Function Combinations

Let's visualize the combinations of $f(x) = x^2$ and $g(x) = x + 1$.







6 Inverse Functions

An inverse function "undoes" what the original function does.

6.1 Definition

For a function f, its inverse function f^{-1} satisfies:

- $f(f^{-1}(x)) = x$
- $f^{-1}(f(x)) = x$

6.2 Finding Inverse Functions

To find the inverse of $y = f(x)$: 1. Replace $f(x)$ with y 2. Interchange x and y 3. Solve for y 4. Replace y with $f^{-1}(x)$

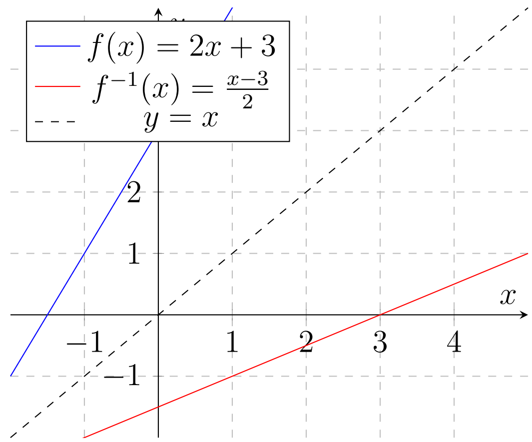
6.3 Example 8: Finding an Inverse Function

Find the inverse of $f(x) = 2x + 3$.

Solution: 1. $y = 2x + 3$ 2. $x = 2y + 3$ 3. $x - 3 = 2y$ $\frac{x-3}{2} = y$ 4. $f^{-1}(x) = \frac{x-3}{2}$

6.4 Graphical Relationship

The graph of $f^{-1}(x)$ is the reflection of $f(x)$ over the line $y = x$.

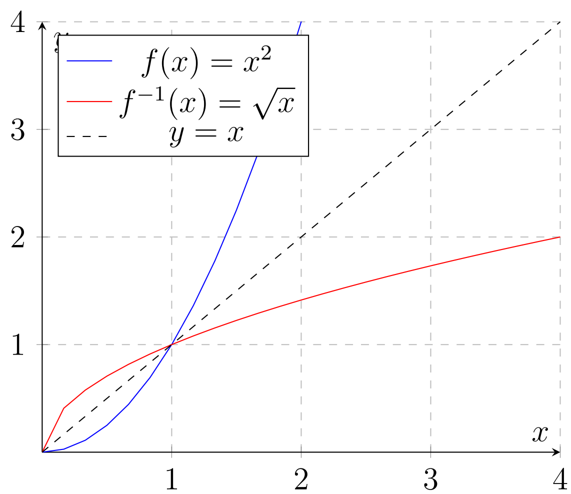


6.5 More Examples of Inverse Functions

Let's look at some other common functions and their inverses.

6.5.1 Square Function and Square Root Function

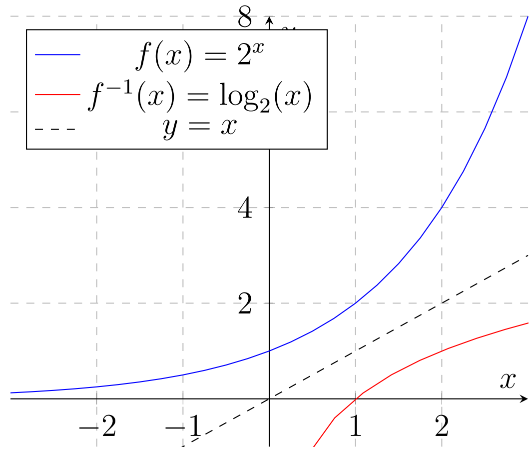
$f(x) = x^2$ (for $x \geq 0$) and its inverse $f^{-1}(x) = \sqrt{x}$



Note: We restrict the domain of $f(x) = x^2$ to non-negative numbers to ensure it has an inverse.

6.5.2 Exponential and Logarithmic Functions

$f(x) = 2^x$ and its inverse $f^{-1}(x) = \log_2(x)$

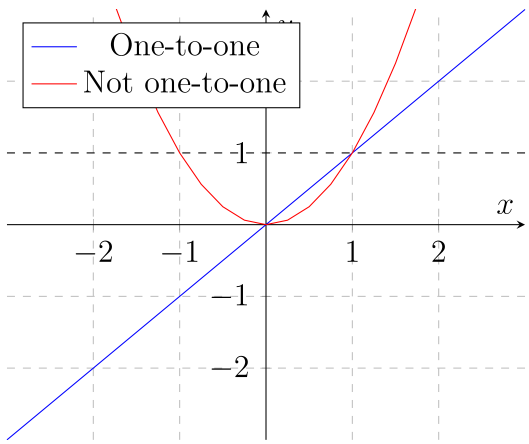

$f(x) = 2^x$
$f^{-1}(x) = \log_2(x)$
$y = x$
8
4
2
x
-2
-1
1
2

6.6 Properties of Inverse Function Graphs

1. The graphs of a function and its inverse are symmetric about the line $y = x$. 2. If (a, b) is a point on the graph of f, then (b, a) is a point on the graph of f^{-1}. 3. The domain of f becomes the range of f^{-1}, and vice versa. 4. Not all functions have inverses. To have an inverse, a function must be one-to-one (injective).

6.7 One-to-One Functions

A function is one-to-one if each element of the codomain is paired with at most one element of the domain. Graphically, this means that no horizontal line intersects the graph of the function more than once.



7 Practice Problems

7.1 Problem 1: Function Composition

Let $f(x) = x + 2$ and $g(x) = x^2$.

a) Find $(f \circ g)(x)$.
b) Find $(g \circ f)(x)$.
c) Evaluate $(f \circ g)(3)$.

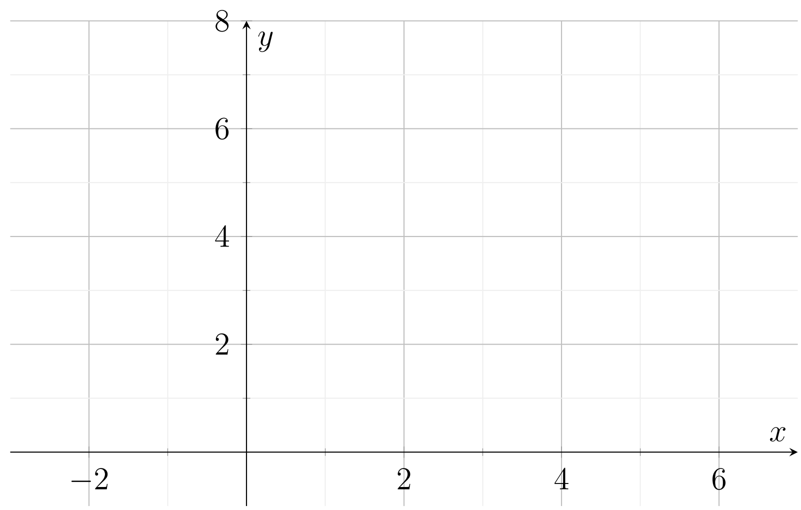
Answer:

7.2 Problem 2: Function Transformations

Starting with the function $f(x) = |x|$, graph $g(x) = |x - 2| + 3$.

Describe the transformations applied to $f(x)$ to get $g(x)$.

Answer:



7.3 Problem 3: Combining Functions

Let $f(x) = x + 1$ and $g(x) = x^2 - 4$.

a) Find an expression for $(f + g)(x)$.

b) Find an expression for $(f \cdot g)(x)$.

c) Evaluate $(f + g)(3)$ and $(f \cdot g)(3)$.

Answer:

7.4 Problem 4: Inverse Functions

Find the inverse function for $f(x) = 2x + 5$.

Then, find $f^{-1}(11)$.

Answer:

7.5 Problem 5: Graphing Functions

Graph the function $h(x) = x^2 - 4x + 3$.

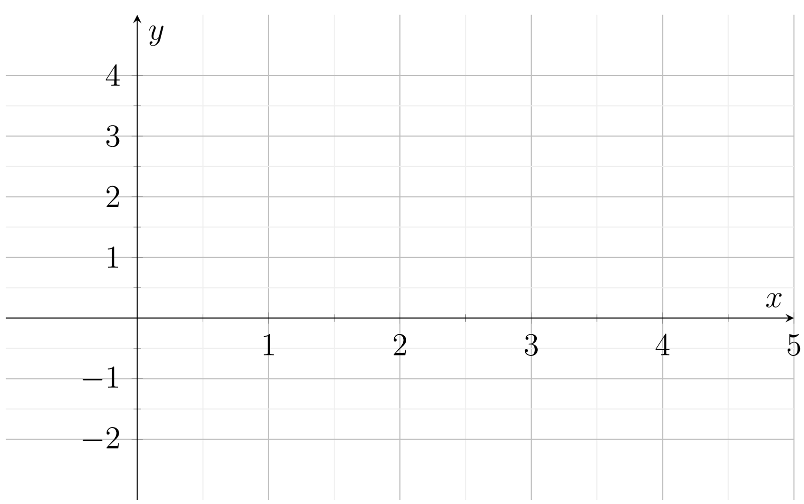
a) Find the y-intercept.

b) Find the x-intercepts.

c) Find the vertex of the parabola.

d) Is the parabola opening upward or downward?

Answer:



7.6 Problem 6: Function Domain and Range

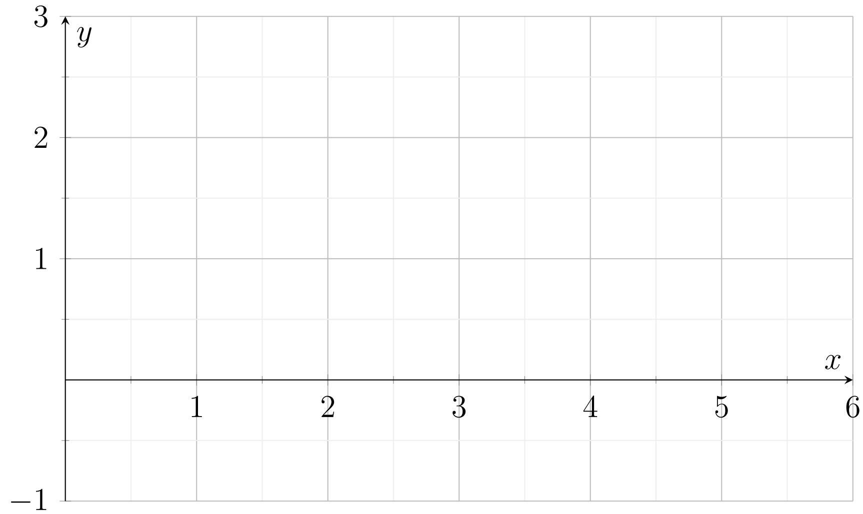
For the function $f(x) = \sqrt{x-1}$:

a) Find the domain of $f(x)$.

b) Find the range of $f(x)$.

c) Graph the function.

Answer:



8 Solutions to Practice Problems

8.1 Solution 1: Function Composition

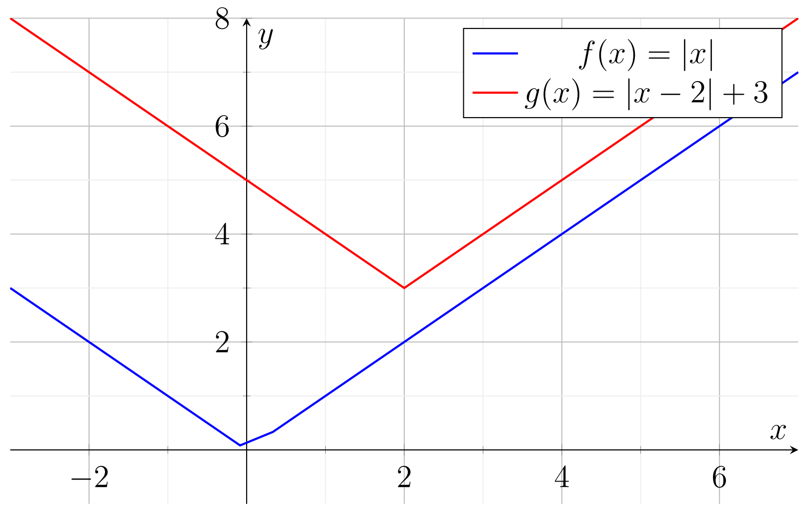
Let $f(x) = x + 2$ and $g(x) = x^2$.

a) $(f \circ g)(x) = f(g(x)) = g(x) + 2 = x^2 + 2$
b) $(g \circ f)(x) = g(f(x)) = (x + 2)^2 = x^2 + 4x + 4$
c) $(f \circ g)(3) = f(g(3)) = f(3^2) = f(9) = 9 + 2 = 11$

8.2 Solution 2: Function Transformations

Starting with $f(x) = |x|$, to get $g(x) = |x - 2| + 3$:

1. Shift 2 units right: $|x - 2|$ 2. Shift 3 units up: $|x - 2| + 3$
Graph:



8.3 Solution 3: Combining Functions

Let $f(x) = x + 1$ and $g(x) = x^2 - 4$.

a) $(f + g)(x) = f(x) + g(x) = (x + 1) + (x^2 - 4) = x^2 + x - 3$
b) $(f \cdot g)(x) = f(x) \cdot g(x) = (x + 1)(x^2 - 4)$
c) $(f + g)(3) = 3^2 + 3 - 3 = 9 + 3 - 3 = 9$ $(f \cdot g)(3) = (3 + 1)(3^2 - 4) = 4(9 - 4) = 4(5) = 20$

8.4 Solution 4: Inverse Functions

For $f(x) = 2x + 5$:

1. Replace $f(x)$ with y: $y = 2x + 5$ 2. Swap x and y: $x = 2y + 5$ 3. Solve for y: $x - 5 = 2y$ $\frac{x-5}{2} = y$
Therefore, $f^{-1}(x) = \frac{x-5}{2}$
To find $f^{-1}(11)$: $f^{-1}(11) = \frac{11-5}{2} = \frac{6}{2} = 3$

8.5 Solution 5: Graphing Functions

For $h(x) = x^2 - 4x + 3$:

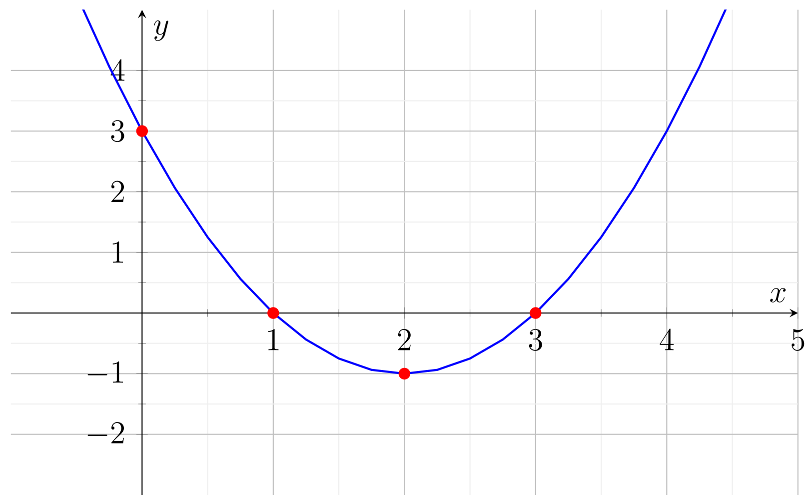
a) Y-intercept: When $x = 0$, $y = 0^2 - 4(0) + 3 = 3$. So, the y-intercept is (0, 3).

b) X-intercepts: Solve $x^2 - 4x + 3 = 0$ $(x - 1)(x - 3) = 0$ $x = 1$ or $x = 3$ X-intercepts are (1, 0) and (3, 0).

c) Vertex: $x = -b/(2a) = -(-4)/(2(1)) = 2$ $y = 2^2 - 4(2) + 3 = 4 - 8 + 3 = -1$ Vertex is (2, -1).

d) The parabola opens upward because the coefficient of x^2 is positive.

Graph:



8.6 Solution 6: Function Domain and Range

For $f(x) = \sqrt{x - 1}$:

a) Domain: The expression under the square root must be non-negative. $x - 1 \geq 0$ $x \geq 1$ Domain is $[1, \infty)$

b) Range: Since $\sqrt{x - 1} \geq 0$ for all x in the domain, and there's no upper limit, Range is $[0, \infty)$

c) Graph: