MILESTONE 1

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1. Solve the following linear system by using Gaussian Elimination Approach.

a. $x_1 + 2x_2 + 3x_3 + 4x_4 = 13$
$2x_1 - x_2 + x_3 = 8$
$3x_1 - 2x_2 + x_3 + 2x_4 = 13$

Form matrix:

R_1: $x_1 + 2x_2 + 3x_3 + 4x_4 = 13$

R_2: $2x_1 - x_2 + x_3 = 8$

R_3: $3x_1 - 2x_2 + x_3 + 2x_4 = 13$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 2 & -1 & 1 & 0 & 8 \\ 3 & -2 & 1 & 2 & 13 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 2 & -1 & 1 & 0 & 8 \\ 3 & -2 & 1 & 2 & 13 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & -5 & -5 & -8 & -18 \\ 3 & -2 & 1 & 2 & 13 \end{array}\right]$$

$R_2 - 2R_1$:
$2 - 2(1) = 0$
$-1 - 2(2) = -5$
$1 - 2(3) = -5$
$0 - 2(4) = -8$
$8 - 2(13) = -18$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & -5 & -5 & -8 & -18 \\ 3 & -2 & 1 & 2 & 13 \end{array}\right] \xrightarrow{R_3 \to R_3 - 3R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & -5 & -5 & -8 & -18 \\ 0 & -8 & -8 & -10 & -26 \end{array}\right]$$

$R_3 - 3R_1$:
$3 - 3(1) = 0$
$-2 - 3(2) = 8$
$1 - 3(3) = -8$
$2 - 3(4) = -10$
$13 - 3(13) = -26$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & -5 & -5 & -8 & -18 \\ 0 & -8 & -8 & -10 & -26 \end{array}\right] \xrightarrow{R_3 - \frac{8}{5}R_2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & -5 & -5 & -8 & -18 \\ 0 & 0 & 0 & 14/5 & 14/5 \end{array}\right]$$

$R_3 - \frac{8}{5}R_2$:
$0 - \frac{8}{5}(0) = 0$
$-8 - \frac{8}{5}(-5) = 0$
$-8 - \frac{8}{5}(-5) = 0$
$-10 - \frac{8}{5}(-8) = \frac{14}{5}$
$-26 - \frac{8}{5}(-18) = \frac{14}{5}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & -5 & -5 & -8 & -18 \\ 0 & 0 & 0 & 14/5 & 14/5 \end{array}\right] \xrightarrow{\frac{5}{14}R_3} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & -5 & -5 & -8 & -18 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & -5 & -5 & -8 & -18 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{array}{rr} x_1 + 2x_2 + 3x_3 + 4x_4 = 13 & (1) \\ -5x_2 - 5x_3 - 8x_4 = -18 & (2) \\ x_4 = 1 & (3) \end{array}$$

sub (3) into (2)
$-5x_2 - 5x_3 - 8(1) = -18$
$-5x_2 - 5x_3 = -10$
$x_2 + x_3 = 2$
$x_2 = 2 - x_3$ (4)

sub (3) into (1)
$x_1 + 2x_2 + 3x_3 + 4(1) = 13$
$x_1 + 2x_2 + 3x_3 = 9$
sub (4)
$x_1 + 2(2 - x_3) + 3x_3 = 9$
$x_1 + 4 - 2x_3 + 3x_3 = 9$
$x_1 + x_3 = 5$
$x_1 = 5 - x_3$

From the calculation above,
$x_1 = 5 - x_3$
$x_2 = 2 - x_3$
$x_3 = x_3$
$x_4 = 1$

let $x_3 = t$ as free parameter

$x_1 = 5 - t$
$x_2 = 2 - t$
$x_3 = t$
$x_4 = 1$

$\therefore x_1 = 5 - t,$
$x_2 = 2 - t,$
$x_3 = t,$
$x_4 = 1,$
where t is a free parameter #

b. $$\begin{aligned} x_1 + x_2 - x_3 - x_4 &= 1 \\ 2x_1 + 5x_2 - 7x_3 - 5x_4 &= -2 \\ 2x_1 - x_2 + x_3 + 3x_4 &= 4 \\ 5x_1 + 2x_2 - 4x_3 + 2x_4 &= 6 \end{aligned}$$

Form matrix:

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & -3 & 3 & 5 & 2 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & -3 & 3 & 5 & 2 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right] \xrightarrow[R_4 - 5R_1]{R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right] \xrightarrow{R_4 + R_2} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & -4 & 4 & -3 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & -4 & 4 & -3 \end{array}\right] \xrightarrow{R_4 - 2R_3} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow$$

$x_1 + x_2 - x_3 - x_4 = 1$ (1)

$3x_2 - 5x_3 - 3x_4 = -4$ (2)

$-2x_3 + 2x_4 = -2$ (3)

since R_4 is $0 = 1$, $\therefore$ no solution #

$R_2 - 2R_1$

$2 - 2(1) = 0$

$5 - 2(1) = 3$

$-7 - 2(-1) = -5$

$-5 - 2(-1) = -3$

$-2 - 2(1) = -4$

$R_3 - 2R_1$

$2 - 2(1) = 0$

$-1 - 2(1) = -3$

$1 - 2(-1) = 3$

$3 - 2(-1) = 5$

$4 - 2(1) = 2$

$R_4 - 5R_1$

$5 - 5(1) = 0$

$2 - 5(1) = -3$

$-4 - 5(-1) = 1$

$2 - 5(-1) = 7$

$6 - 5(1) = 1$

2. Develop an algorithm for Gaussian Approach so that it can be used to:	
a. solve all types of linear systems.	5M
b. find the inverse of the matrices.	5M

Algorithm for Gaussian Elimination

Input:
- A matrix \(A \) of size \(n \times n \)
- A vector \(b \) of size \(n \)

Output:
- A solution vector \(x \) of size \(n \) that satisfies \(Ax = b \)

Steps:
1. Augment the matrix \(A \) with the vector \(b \):
   - Create an augmented matrix \(M \) by appending \(b \) as the last column of \(A \).

2. Forward Elimination:
   - For each pivot row from 1 to \(n \):
   - Find the row with the largest absolute value in the pivot column below the pivot row and swap it with the pivot row.
   - For each target row below the pivot row:
   - Calculate the elimination factor as the ratio of the target row's pivot column element to the pivot row's pivot column element.
   - Subtract the elimination factor times the pivot row from the target row to eliminate the pivot column element in the target row.

3. Back Substitution:
   - Initialize the solution vector \(x \) with zeros.
   - For each row from \(n-1 \) down to 0:
   - Calculate the sum of the product of known solutions and corresponding coefficients.
   - Solve for the unknown in the current row by dividing the adjusted right-hand side value by the pivot element.

4. Return the solution vector \(x \).

Example of Pseudocode for Gaussian Elimination

```
Algorithm GaussianElimination(A, b):
    Input: A (n x n matrix), b (n x 1 vector)
    Output: x (solution vector)

    Augment A with b to form M
    for k = 0 to n-1 do
        Find the row with the largest element in column k and swap with row k
        for j = k+1 to n-1 do
            Compute elimination_factor = M[j][k] / M[k][k]
            for m = k to n do
                M[j][m] = M[j][m] - elimination_factor * M[k][m]

    x[n-1] = M[n-1][n] / M[n-1][n-1]
    for i = n-2 to 0 do
        sum_ax = 0
        for j = i+1 to n-1 do
            sum_ax = sum_ax + M[i][j] * x[j]
        x[i] = (M[i][n] - sum_ax) / M[i][i]

    return x
```

Algorithm for Finding the Inverse of a Matrix

Input:
- A matrix \(A \) of size \(n \times n \)

Output:
- An inverse matrix \(A^{-1} \) of size \(n \times n \)

Steps:

1. Augment the matrix \(A \) with the identity matrix \(I \):
   - Create an augmented matrix \(M \) by appending \(I \) to \(A \).

2. Forward Elimination:
   - For each pivot row from 1 to \(n \):
   - Find the row with the largest absolute value in the pivot column below the pivot row and swap it with the pivot row.
   - For each target row below the pivot row:
   - Calculate the elimination factor as the ratio of the target row's pivot column element to the pivot row's pivot column element.
   - Subtract the elimination factor times the pivot row from the target row to eliminate the pivot column element in the target row.

3. Back Substitution:
   - For each pivot row from \(n-1 \) down to 0:
   - For each row above the pivot row:
   - Calculate the elimination factor as the ratio of the row's pivot column element to the pivot row's pivot column element.
   - Subtract the elimination factor times the pivot row from the row to eliminate the pivot column element in the row.
   - Normalize the pivot row by dividing each element by the pivot element to make the pivot element equal to 1.

4. Extract the inverse matrix \(A^{-1} \) from the augmented matrix \(M \).

Example of Pseudocode for Finding the Inverse

```
Algorithm FindInverse(A):
    Input: A (n x n matrix)
    Output: A_inverse (n x n matrix)

    Augment A with the identity matrix I to form M
    for k = 0 to n-1 do
        Find the row with the largest element in column k and swap with row k
        for j = k+1 to n-1 do
            Compute elimination_factor = M[j][k] / M[k][k]
            for m = 0 to 2*n-1 do
                M[j][m] = M[j][m] - elimination_factor * M[k][m]

    for k = n-1 to 0 do
        for i = 0 to k-1 do
            Compute elimination_factor = M[i][k] / M[k][k]
            for m = 0 to 2*n-1 do
                M[i][m] = M[i][m] - elimination_factor * M[k][m]
        Normalize row k by dividing by M[k][k]

    Extract A_inverse from the augmented matrix M

    return A_inverse
```