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A study of the Greeks through the Black-Scholes Model

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Abstract

The Greek letters are quantities representing the sensitivity of the price of derivatives such as options to a change in underlying parameters on which the value of an instrument. Their crucial importance is represented by the hedging strategies. In this paper, we will discuss the behaviour of the Greeks with respect to volatility. We will then verify the presence of the so-called volatility smile, and its behaviour as a function of the time to maturity. Finally, we will study the behaviour of the call option price with the Black-Scholes model using the historical and implied volatility.

Keywords
Greeks, Options, Implied-Volatility, Russell-2000

1 Introduction

Option Greeks are financial measures of the sensitivity of an option's price to its underlying determining parameters, such as volatility or the price of the underlying asset. They are fundamental in the building of a hedging strategy, whose goal is to offset potential unfavourable moves in other investments, in particular by taking an opposite position in a related asset.

Here we do a brief recap of the Greek letters involved in this paper.

1.1 Delta

The *delta* (Δ) of an option is defined as the rate of change of the option price with respect to the price of the underlying asset. It is the slope of the curve that relates the option price to the underlying asset price. In general,

$$\Delta = \frac{\partial C}{\partial S} \tag{1}$$

where C is the price of the call option and S is the stock price.

If the *delta* of a portfolio is zero we say that it is *delta-neutral* [2]. The object of the *delta hedging* is to immunize this portfolio against small changes in the underlying asset price S, *i.e.* making it *delta-neutral*. The idea is to add a derivative to the portfolio. Since the price of a derivative is perfectly correlated with the underlying asset price, we should be able to balance the derivative against the portfolio in such a way that the adjusted portfolio becomes *delta-neutral* [1]. The *delta* is the number of units of the derivative which we will add to the a priori given portfolio.

Since the delta of an option does not remain constant the position remains *delta-hedged* for only a relatively short time interval: the hedge has to be adjusted periodically. This approach is called *discrete rebalanced delta hedge*, in contrast with *static hedging*, where a hedge is set up initially and never adjusted.

For a European call (put) option on a non-dividend-paying stock, it can be shown that

$$\Delta(call) = \Phi(d_1) \qquad \Delta(put) = \Phi(d_1) - 1 \tag{2}$$

where Φ is the cumulative distribution function for a standard normal distribution and $d_1 = \frac{\log S_0/K + (r+\sigma^2/2)T}{\sigma\sqrt{T}}$.

1.2 Gamma

The *gamma* (Γ) of a portfolio of options on an underlying asset is the rate of change of the portfolio's delta with respect to the price of the underlying asset. It is the second partial derivative of the portfolio with respect to asset price:

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2} \tag{3}$$

Gamma tells us how often we need to adjust our portfolio delta-neutral: if *gamma* is small, *delta* changes slowly, and adjustments to keep a portfolio delta-neutral do not need to be made too frequently, while if *gamma* is high in absolute value, *delta* is very sensitive to the price of the underlying asset, it is then quite risky to leave a *delta-neutral* portfolio unchanged for any length of time [2].
For a European option on a non-dividend-paying stock, the *gamma* is given by [2]

$$\Gamma = \frac{N(d_1)}{S_0 \sigma \sqrt{T}} \tag{4}$$

where N is the normal distribution.

To keep the *delta* constant, we want to have a position *gamma-neutral*. From the fact that the gamma of the underlying stock equals zero, it follows that we cannot use the stock itself to change the gamma of the portfolio. Since we want the adjusted portfolio to be both delta and gamma neutral, it is also obvious that we need two different derivatives in the hedge. The problem with this scheme is that the second derivative in general will destroy the delta-neutrality obtained by choosing the first derivative to be, together with the value of the portfolio, *delta-neutral* [1]. Thus to construct a *delta-gamma* neutral position we need another approach, following [1]:

1. Choose x_F, the number of options to be added, such that the portfolio consisting of Π and F is gamma neutral. This portfolio will generally not be delta-neutral.

2. Now add the underlying stock in order to make the portfolio delta-neutral.

Formally the value of the hedged portfolio will now be given by

$$V = \Pi + x_F \cdot F + x_S \cdot s \tag{5}$$

and knowing the delta and gamma of the underlying stock

$$\Delta_\Pi + x_F \cdot \Delta_F + x_S = 0 \tag{6}$$

$$\Gamma_P + x_F \cdot \Gamma_F = 0 \tag{7}$$

The solution is given by

$$x_F = -\frac{\Gamma_P}{\Gamma_F} \qquad x_S = \frac{\Delta_F \cdot \Gamma_\Pi}{\Gamma_F} - \Delta_\Pi \tag{8}$$

1.3 Theta

The *theta* (θ) of a portfolio of options is the rate of change of the value of the portfolio with respect to the passage of time. For a European call (put) option on a non-dividend-paying stock, it can be shown from the Black-Scholes formula that

$$\Theta(call) = -\frac{S_0 N(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}\Phi(d_2) \tag{9}$$

$$\Theta(put) = -\frac{S_0 N(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}\Phi(-d_2) \tag{10}$$

Theta is usually negative for an option since as time passes the option tends to become less valuable.
Theta is not the same type of hedge parameter as delta: there is uncertainty about the future stock price, not about the passage of time, so it doesn't make sense to hedge against the latter [2].

1.4 Vega

This parameter is necessary because in the real-world volatility of the asset underlying a derivative is not constant: this means that the value of a derivative is liable to change because of movements in volatility. The *vega* of a portfolio of derivatives is the rate of change of the value of the portfolio with respect to the volatility of the underlying asset

$$\mathcal{V} = \frac{\partial \Pi}{\partial \sigma} \tag{11}$$

Vega represents the sensibility of the portfolio's value to the changes in volatility. While a position in the underlying has zero vega, the vega of the portfolio can be changed by adding a position in a traded option. If $\mathcal{V}$ is the vega of the portfolio and $\mathcal{V}_T$ is the vega of a traded option, a position of $\mathcal{V}/\mathcal{V}_T$ in the traded option makes the portfolio vega neutral [2]. Unfortunately building a position $\mathcal{V}$-neutral does not correspond in general to a position Δ-neutral and vice-versa.

1.5 Implied volatility

The one parameter in the Black-Scholes pricing formulas that cannot be directly observed is the volatility of the stock price. We know that it can be estimated from the history of the stock price but the latter is backwards-looking. The implied volatility, on the other hand, is the volatility implied by options prices observed in the market [2]. Unfortunately, the Black-Scholes pricing formula cannot be inverted to find σ as a function of S_0, K, r, T and c, but we can retrieve it with an iterative approach, using several values of σ to find more and more correct values of the price, until the correct value of the call option price and then the iterations can stop.

Graphing implied volatilities against strike prices for a given expiry produces a skewed "smile" instead of the expected flat surface. The pattern differs across various markets. It is believed that investor reassessments of the probabilities of fat-tail have led to higher prices for out-of-the-money options. This anomaly implies deficiencies in the standard Black–Scholes option pricing model which assumes constant volatility and log-normal distributions of underlying asset returns. Empirical asset returns distributions, however, tend to exhibit fat-tails (kurtosis) and skew. Modelling the volatility smile is an active area of research in quantitative finance, and better pricing models such as the stochastic volatility model partially address this issue [5].

2 Materials & Methods

In this paper, we study the greeks' superficies as a function of the Spot price S and the time to maturity, keeping fixed the strike price, the interest rate and the volatility. In particular, we want to observe the behaviour of these superficies with a shock of volatility.
In the second part, we want to compute those quantities and check the implied volatility smile for a real-world case. The asset we choose as underlying is the index Russell-2000 (^RUT in the New York Stock Exchange - Nasdaq). The choice of an index as underlying is to satisfy the need for European options which is typical of indexes. The Russell 2000 Index is a small-cap stock market index that makes up the smallest 2,000 stocks in the Russell 3000 Index. The Russell 2000 is managed by FTSE Russell and is widely regarded as a bellwether of the U.S. economy because of its focus on smaller companies that focus on the U.S. market.

Because it tracks the performance of small-cap stocks, the Russell 2000 serves as a very different benchmark than other major indexes, like the S&P 500 or the Dow Jones Industrial Average (DJIA), which focus on much larger companies.

The strong focus on small-cap companies means the Russell 2000 may show more volatility than these indices because smaller companies have more limited financial resources than big companies and are less equipped to weather negative changes in the overall economy than their larger counterparts. However, with that greater potential for risk comes built-in greater potential to grow exponentially.

Like the S&P 500, many economists consider the Russell 2000 a reasonably accurate barometer of the U.S. economy, particu-

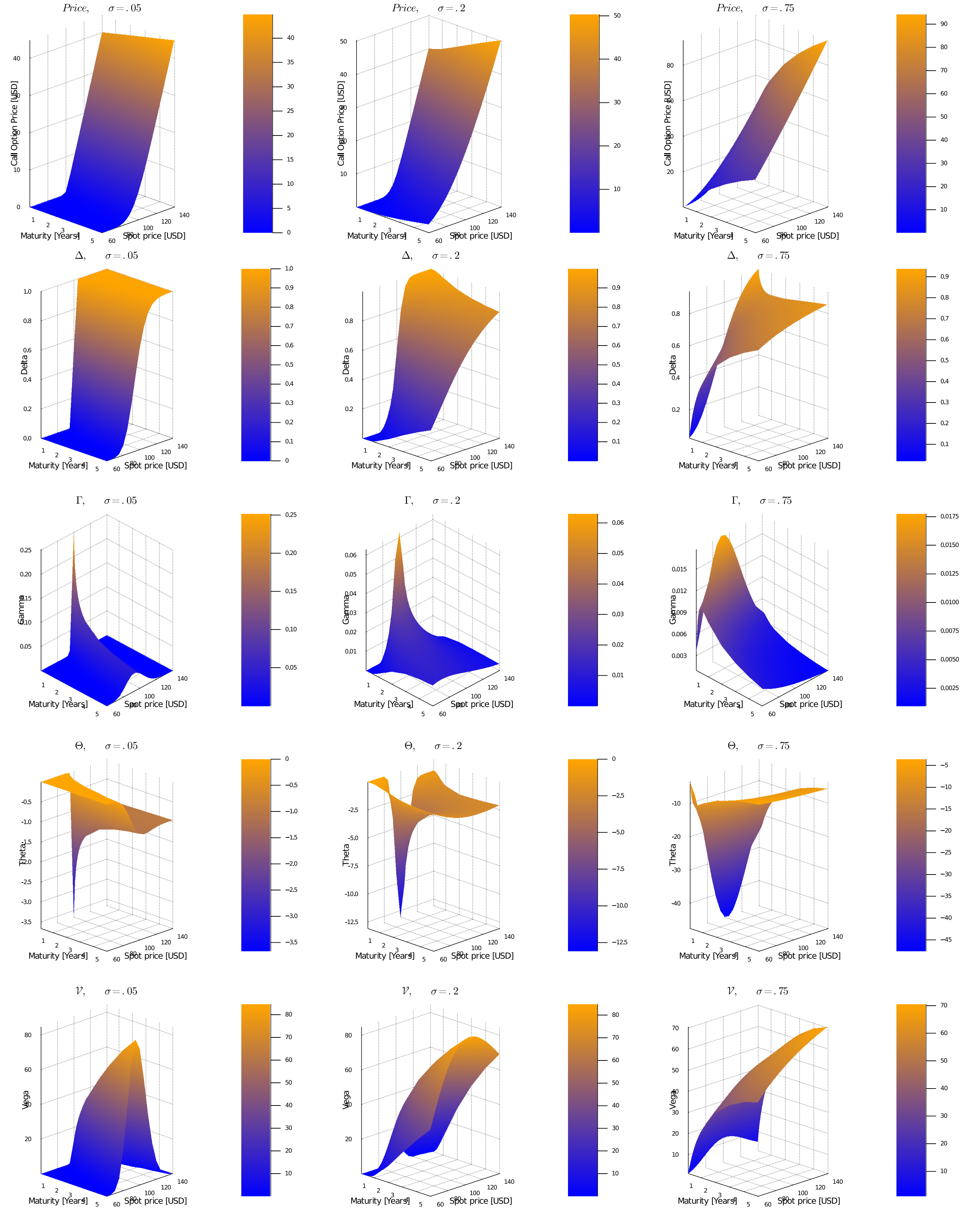

Figure 1: Call option price and Greeks (Delta, Gamma, Theta and Vega) surfaces for the different values of volatility.

larly as it applies to smaller companies. What's more, the Russell 2000 is frequently viewed as a bellwether for the economy as the smaller businesses it follows can be the engines of job growth, and tracking them can help economists forecast where the U.S. economy is going [4] [3].

In a last part we want to verify the different behaviour of the historical volatility and the implied volatility for options ATM and verify the different price prediction for an option for the different volatilities for each maturity.

The data used are taken from the Yahoo Finance site and which was consulted in the day 29 march.

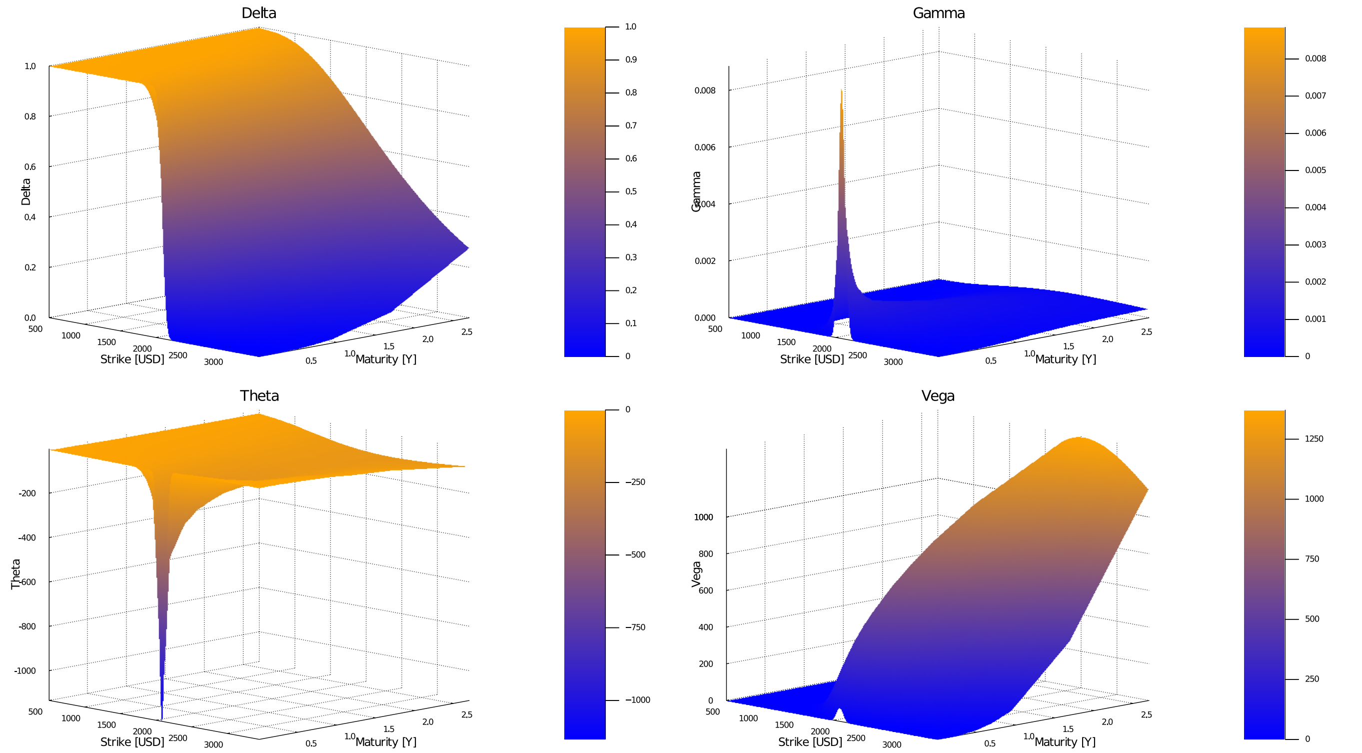

Figure 2: Greeks (Delta, Gamma, Theta and Vega) surfaces for the RUT index.

3 Results & Discussion

Here we present the main results of the paper. This section will be divided in three different subsection: in the first we discuss from a theoretical point of view the behaviour of the greeks surfaces as a function of the volatility. Then we show the greeks for a real-world case, using the Russell 2000 index, and the implied volatility smile, other than a study of the difference between the call option price predicted by the Black-Scholes model, using either the historical volatility or the implied volatility.

3.1 Greeks

We consider in this first theoretical part a fictional asset, fixing the parameters reported in Table 1 and varying the Spot price (in $[60, 65, ..., 135, 140]$) and the time to maturity (in $[0.1, 0.2, ..., 0.9, 1, 2, ..., 5]$). Given this grid of parameters, we

Table 1: Fixed parameters for the calculations.

Strike price [USD]	Interest rate	Annualized Vol.
100	.01	.05; .20; .75

compute the greeks according to the methods defined earlier in this paper. In Figure 1 we report a grid for the surfaces obtained for the call option price and the greeks we are considering for the different values of volatility considered. The choice of those values of volatility is due to the need of stressing the different behaviour and it is not so realistic[1].

Considering first the call option price, we observe the increase of price, both with respect to the spot price and the time to maturity, being more and more stiff as the volatility grows. This is because a larger value of volatility means a greater value of instability and thus a greater level of risk. This becomes, in terms of options, an higher price.

Delta represents the first derivative of the call option price with respect to the price of the underlying: a higher volatility increases the delta for OTM options. The more volatility, the less OTM an OTM option really is. A higher volatility decreases the delta for ITM options. The more volatility, the less ITM an ITM option really is. So, more volatile stocks therefore have a less pronounced delta, as we see in the plots.

The behaviour of Gamma, Theta and Vega are, on the other hand, similar, with the obvious adjustments. For the three of them, as the volatility decreases the surface becomes more and more peaked, while for larger values of volatility the surfaces are smoother and more dispersed.

3.2 Volatility smile

For this case we need to use a real asset, since of course it would not make sense computing the implied volatility for a option whose price is given by the Black-Scholes model: the implied volatility would be flat, as the historical volatility. The asset we use is the Russell2000 index. In Figure 3 (a) we show the trend of the Stock price in the last years. In Figure 3 (b) is reported the call option price surface according to the Black-Scholes model using the historical volatility, computed as the standard deviation of the returns of a time range corresponding to the time to maturity. The interest rate is retrieved from the

[1] The VIX index, measuring the market expectation of the implied volatility of the S&P index, reached during the COVID-19 crisis the record value of ~ 80 and RVX, which measures the expected volatility of RUT ~ 85, just for the 1 month maturity and for a very small time range.

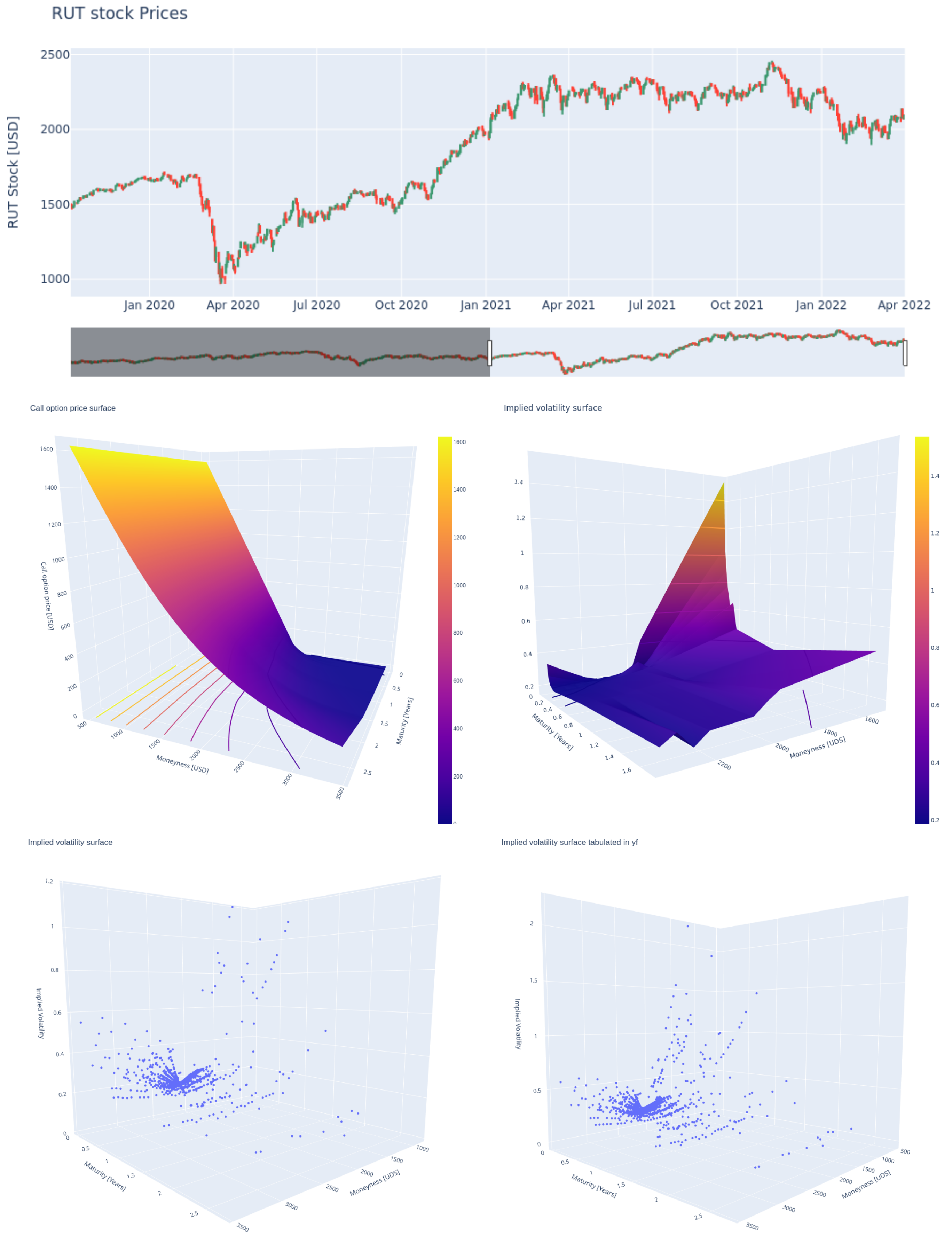

Figure 3: Top (a): Stock price trend for the last 2 years. The boxes represent the spread between the open and close values and the lines represent the spread between the low and high values. Increasing candles are drawn in green whereas decreasing are drawn in red. (Centre-left (b): Call option price surface, according to the Black-Scholes model, for the RUT index options. Centre-right (c): Implied volatility surface. Bottom left (d) and right (e): scatter plot for the implied volatility surface, for the computed values and the tabulated ones.

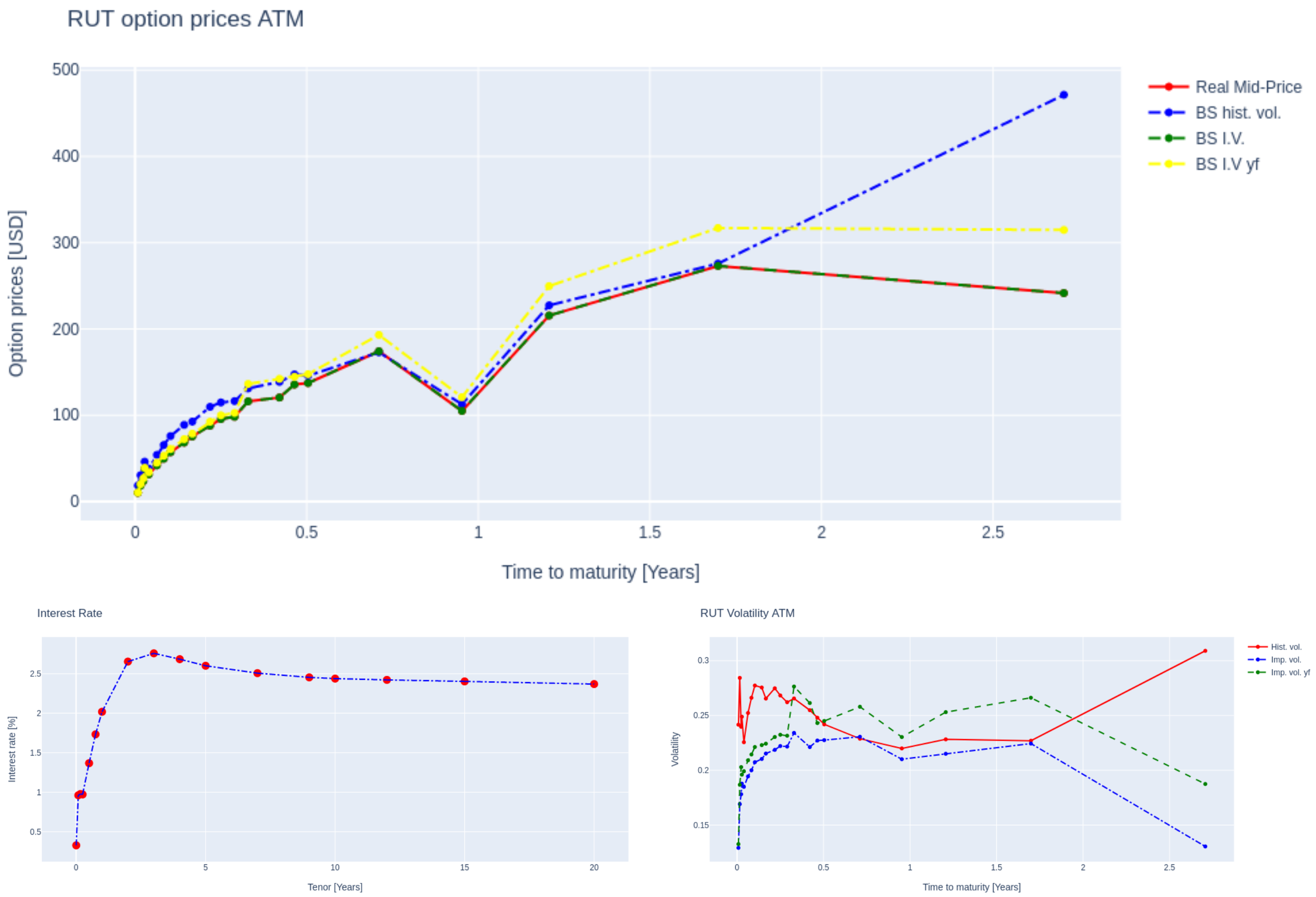

Figure 4: Top (a): Comparison between the real call option price and the expected one through the Black-Scholes formula using respectively the historical, the implied computed and the tabulated implied volatility. Bottom left (b): Interest rate Yield curve whit a linear interpolation between two points. Bottom right (c): Comparison between the historical volatility and the implied volatility, computed and tabulated

data provided, reported if Figure 4 (b), interpolating the middle values with a linear approach.

In Figure 3 (c), (d) and (e) are reported the implied volatility surfaces. In particular, in the first one a surface is reported: as we see the smile is emphasized in the first part, with closer maturities, while for longer maturities the surface flattens. Unfortunately, due to the not so cleanness of the data available in Yahoo Finance, we need to filter the couples (Maturity, Strike) to keep only the ones that have real values. The whole amount of points is reported in the figures at the bottom: from the scatter we see the smile for close maturities and the flattening for longer ones, but with a large amount of noise. We also report two versions of this scatter plot: the first one gives the implied volatility computed through the Black-Scholes formula (not inverting it, but iteratively giving values to σ until the correct option price is retrieved) and from the tabulated values in Yahoo Finance. We observe a not perfect correspondence: this is probably due to a different approach used by Yahoo Finance.

In Figure 2 we report the greeks computed for the Russell2000 index, using the methods already described. It is interesting to observe the different behaviour in the different parts of the charts. This is due to the fact that we used historical volatility, but a different value for each maturity we consider: from the stock price we can retrieve that longer maturities are affected by the COVID-19 crisis, therefore they have a greater volatility that will affect the greeks as explained in the previous subsection.

3.3 Historical and Implied Volatility for options ATM

We want now to compare the behaviour of the Black-Scholes formula using the historical volatility and the implied volatility for options at the money. In Figure 4 (a) is reported the call option price as a function of the time to maturity. In particular, out of the real mid-price, we report also the Black-Scholes predictions using either the historical volatility or the implied volatility. We observe that using the computed implied volatility, there is a perfect correspondence between the Black-Scholes prediction and the real Mid-Price. Of course, this is not a surprising result: we compute the implied volatility such that the two correspond. Thus at most this is proof that the computation of the implied volatility is successful.
On the other hand, we observe that there is not a perfect correspondence with the Black-Scholes model using the tabulated implied volatility: this disagreement is not so significant for closer maturities and grows as the tenor increases. Since we do not know exactly how Yahoo Finance computes the implied volatility, and how often, we cannot do a robust hypothesis. It can be probably due to an approach different from the Black-Scholes one or the not frequent enough update of this value.

Finally, the series corresponding to the historical volatility shows a significantly different behaviour out of some points. In particular, for the longest maturity considered there is a completely different behaviour: we said earlier that this last point is deeply influenced by the COVID crisis.

In Figure 4 (c) we observe directly the behaviour of the volatilities which generally confirms what we said earlier. The Yahoo Finance implied volatility seems to follow qualitatively the behaviour of the computed volatility, but there is generally an overestimation that grows with the time to maturity. On the other hand, the value of historical volatility does not follow in any way the behaviour of the implied volatility. We do not expect it to do that, since they are different quantities: while the first look backwards, aiming to behave as a realized volatility, the latter looks forward, behaving as an expectation of the future volatility.

Conclusions

We implemented the computation of the Greek Letters according to the Black-Scholes model and we observed the behaviour of the latter as a function of the spot price and the time to maturity. Then we observed the implied volatility smile surface and verified the flattening of it for longer maturities for a real-world asset. Finally, we computed the call option price at the money, according to the Black-Scholes model using the historical and the implied volatility and we observed the difference between the two.

The future development of the work done should be the implementation of the other Greeks, and the study of their behaviour. We also need to use cleaner datasets to avoid problems in the plot of the smile volatility surface.
Another future work should be the implementation of the methods we used, considering a dividend-paying asset.

References

[1] BJORK, T. *Arbitrage Theory in Continuous Time*, 3 ed. Oxford University Press, 2009.

[2] HULL, J. *Options, futures, and other derivatives*, 6. ed., pearson internat. ed ed. Pearson Prentice Hall, Upper Saddle River, NJ [u.a.], 2006.

[3] WALDEN SIEW, GORDON SCOTT, SUZANNE KVILHAUG. Russell 2000 index. https://www.investopedia.com/terms/r/russell2000.asp, 2022. [Online; accessed 3-April-2022].

[4] WIKIPEDIA CONTRIBUTORS. Russell 2000 index — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Russell_2000_Index&oldid=1073040097, 2022. [Online; accessed 4-April-2022].

[5] WIKIPEDIA CONTRIBUTORS. Volatility smile — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Volatility_smile&oldid=1080275543, 2022. [Online; accessed 4-April-2022].

A Appendix

A.1 Greeks and Implied volatility for a put option

Here we report without a deep analysis the results for a study equivalent to the one done in the paper, but for a put option. We already discussed the formulas for the greeks in case of a put option in the theoretical introduction.

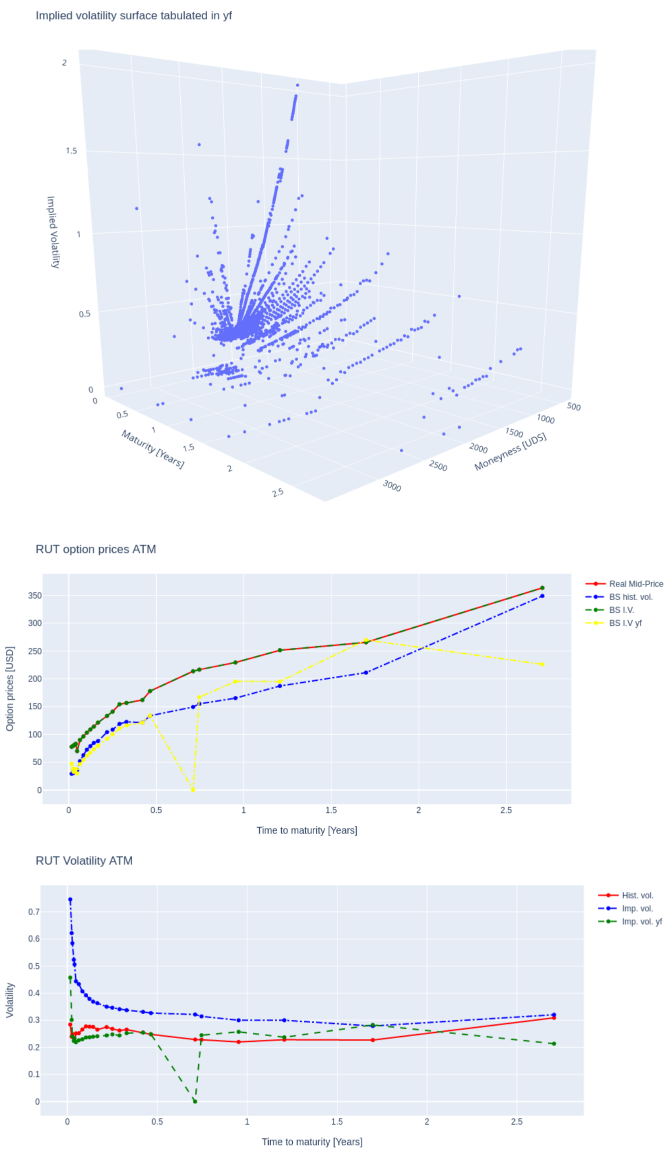

Figure 5: Top (a): Scatter plot for the implied volatility surface. As for the call option case we see an evident smile for closer maturities and a flattening for longer ones. The data reported are the ones tabulated in Yahoo Finance. We see also a very noisy pattern. Center (b): Option prices for an at the money put option. In the chart are reported the real mid-prices, the Black-Scholes price for historical volatility and for implied volatility, computed and tabulated. The pattern is similar as for the call option case, with the difference that there is not an evident difference between the real price and the BS one with historical volatility. Bottom (c): Comparison between the historical volatility and the implied volatility, computed and tabulated

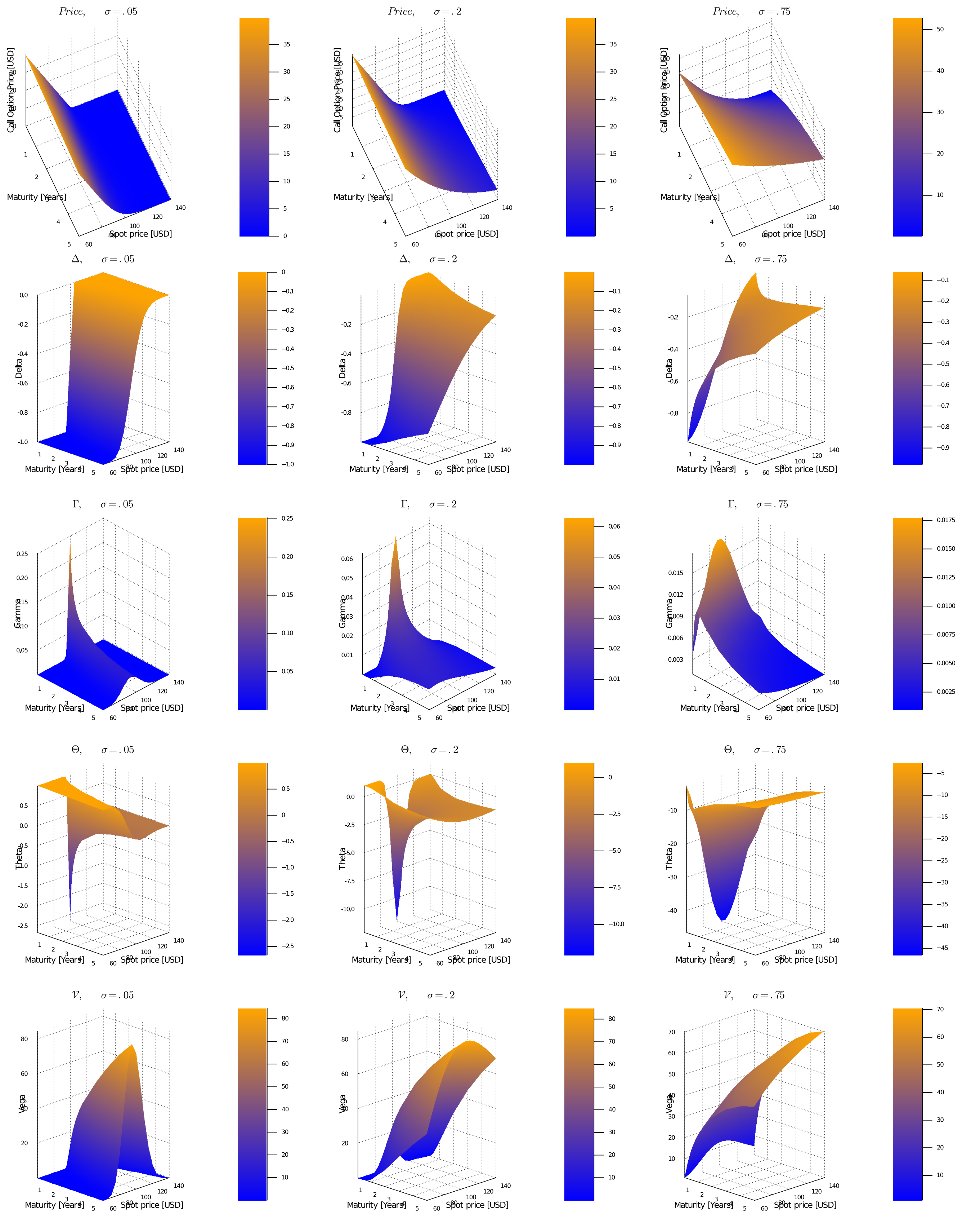

Figure 6: Prices and Greeks (Delta, Gamma, Theta and Vega) surfaces for the different values of volatility for a put option, with the parameters defined in Table 1. We observe for the prices the inverse behaviour with respect to the call option case, going from high prices for a low spot price to low ones for high spots. The behaviour on the maturity axes on the other hand is the same. The behaviour of the greeks surfaces is comparable with the call option case: we observe that only delta and theta changes.