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A comparison between the convergence rate of the Cox-Ross-Rubenstein and the Leisen-Reimer binomial trees to the Black-Scholes model

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Attar Aidin[1]

[1] Dipartimento di Fisica e Astronomia "Galileo Galilei", Università degli Studi di Padova

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Abstract

The binomial tree model is a simple option-pricing toy model based on some assumptions. In this paper, we will inspect the convergence of two discrete binomial models, the Cox-Ross-Rubinstein and the Leisen-Reimer, to the continuous Black-Scholes model. We will then implement the three models with a real-world case, using an underlying asset which is the Teva Pharmaceutical Industries Limited stocks, to continue the pharmaceutical trend of this series of papers. The results obtained will confirm the theoretical results already given in the Leisen-Reimer paper.

Keywords
Cox-Ross-Rubinstein, Leisen-Reimer, Black-Scholes, Option pricing

1 Introduction

The binomial option pricing model, introduced by Cox, Ross and Rubinstein (CRR) in 1979, is a widely used method to value options. The mathematical passages to price an option with such a model have been already explained in previous work [1]. Here we want to focus on the convergence of this model to the Black-Scholes option pricing formula. Indeed, as explained in [2], one way of deriving the famous Black-Scholes-Merton result for valuing a European option on a non-dividend-paying stock is by allowing the number of time steps in a binomial tree to approach infinity.

Without reporting the whole procedure we just report here the CRR formula

$$f = (1 + r\Delta t)^{-1} \sum_{j=0}^{n} f^{u^j d^{n-j}} \binom{n}{j} q^j (1-q)^{n-j} \tag{1}$$

and the Black-Scholes formula

$$C = S_0 N(d_1) - K e^{rT} N(d_2) \tag{2}$$

with

$$d_1 = \frac{logS_0/K + (r + \sigma^2/2)T}{\sigma T} \qquad d_2 = d_1 - \sigma\sqrt{T} \tag{3}$$

Furthermore, as explained in [3] the order of this convergence is 1, meaning that, if $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n; if positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha} = \lambda}$$

then $\{p_n\}_{n=0}^{\infty}$ converges to p with order α, with asymptotic error constant λ.

In the paper [3] in 1995 by Dietmar Leisen and Matthias Reimer, a new binomial model is defined, the nowadays so-called Leisen-Reimer binomial tree, where the calculated option prices converge smoothly to the Black-Scholes solution and remarkably, they even achieve an order of convergence two with much smaller initial error. This can be useful in practical use, where a higher number of steps required more computing resources, desired qualities of a binomial model are for the convergence to be fast and smooth.

In the Leisen-Reimer model, probabilities must be calculated before moving sizes. First, we need to calculate d_1 and d_2 which are the same defined for the Black-Scoles in Equation (3). Now we can use d_2 to compute the probability of an up move in a Leisen-Reimer tree: $p = h^{-1}(d_2)$, with $h^{-1}(z)$ the Peizer-Pratt inversion function, which provides discrete binomial estimates for the cumulative normal distribution function. The function we use for this inversion is the following:

$$h^{-1}(z) = \frac{1}{2} + \frac{sign(z)}{2} \times \\ \times \sqrt{1 - exp\left(-\left(\frac{z}{n + \frac{1}{3} + \frac{.1}{(n+1)}}\right)^2 \left(n + \frac{1}{6}\right)\right)} \tag{4}$$

At this point, we can evaluate the up and down move size formulas as

$$u = e^{r\Delta t} \frac{p'}{p} \tag{5}$$

and

$$u = e^{r\Delta t} \frac{1 - p'}{1 - p} \tag{6}$$

with $p' = h^{-1}(d_1)$.

It is worth saying that this method works only with an odd number of steps, indeed, while an even number of steps does not break the calculations, it leads to imprecise option results.

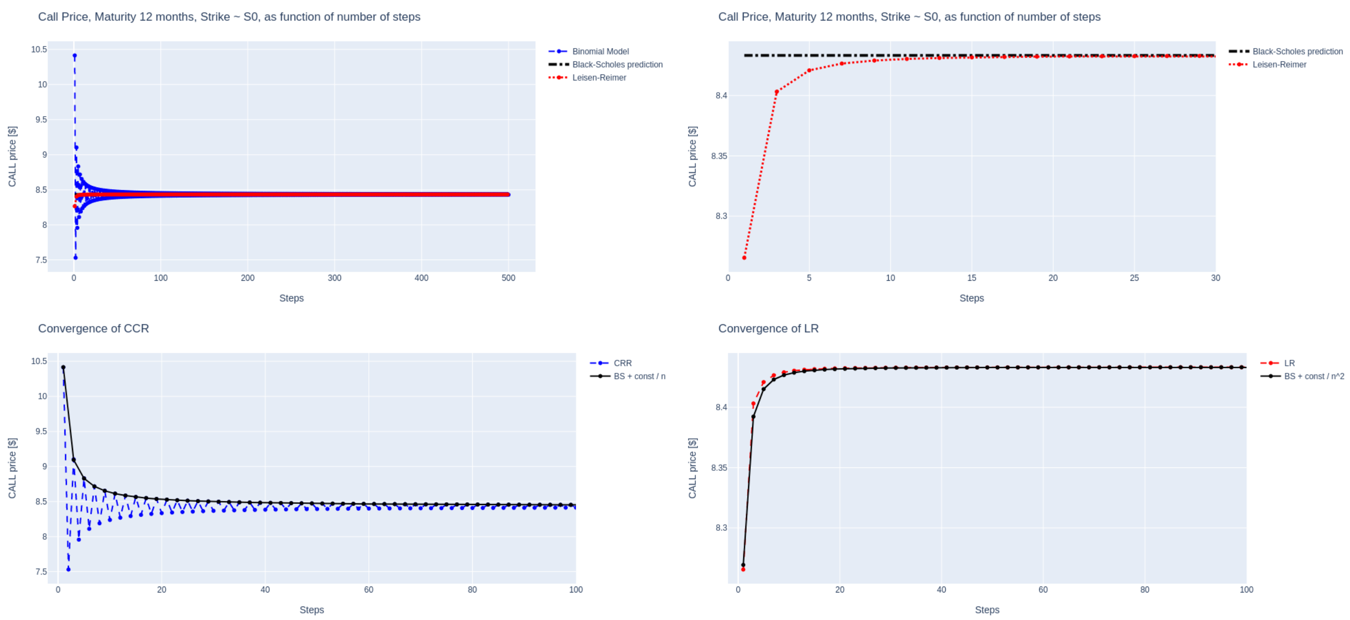

Figure 1: Top right (a): Call option price for a European option using three different models as a function of the number of steps. Top left (b): Convergence of the LR model to Black-Scholes. Bottom left (c) Convergence of CRR model and fit of the top points with a $1/n$ function with offset given by the Black-Scholes price. Bottom right (d) Convergence of LR model and fit with a $1/n^2$ function with offset given by the Black-Scholes price.

2 Materials & Methods

In this paper, we try to verify computationally the convergence of the call option price predicted by the Cox-Ross-Rubinstein and the Leisen-Reimer binomial trees to the Black-Scholes price. We use in the first part some fixed values for the parameters we need, these values are reported in Table 1.

Table 1: Parameters used to investigate the convergence.

S	r	σ	K	T
100	.01	.2	100	$1y$

In the second part, we use a real underlying: the asset chosen is Teva Pharmaceutical Industries Limited (TEVA in NYSE). Teva Pharmaceutical Industries Limited, a pharmaceutical company, develops, manufactures, markets, and distributes generic medicines, speciality medicines, and biopharmaceutical products in North America, Europe, and internationally. The company offers sterile products, hormones, high-potency drugs, and cytotoxic substances in various dosage forms, including tablets, capsules, injectables, inhalants, liquids, transdermal patches, ointments, and creams. It also develops, manufactures, and sells active pharmaceutical ingredients. In addition, it focuses on the central nervous system, pain, respiratory, and oncology areas. Its products in the central nervous system include Copaxone for the treatment of relapsing forms of multiple sclerosis; AJOVY for the preventive treatment of migraine; and AUSTEDO for the treatment of tardive dyskinesia and chorea associated with Huntington's disease. The company's products in the respiratory market comprise ProAir, QVAR, ProAir Digihaler, AirDuo Digihaler, ArmonAir Digihaler, BRALTUS, CINQAIR/CINQAERO, DuoResp Spiromax, and AirDuo RespiClick/ArmonAir RespiClick for the treatment of asthma and chronic obstructive pulmonary disease. Its products in the oncology market include Bendeka, Treanda, Granix, Trisenox, Lonquex, and Tevagrastim/Ratiograstim. Teva Pharmaceutical Industries Limited has a collaboration MedinCell for the development and commercialization of multiple long-acting injectable products, including a risperidone suspension for the treatment of patients with schizophrenia. The company was founded in 1901 and is headquartered in Tel Aviv-Yafo, Israel [4].

The data used are taken from the Yahoo Finance site and the analysis involved the use of `Excel-VBA` and `Julia 1.7.0` using a `Jupyter` framework and can be found in the `GitHub` repository.

We consider as the price of the options the Last Price and not a mid-price between Ask and Bid values. This is due to the narrowness of the means used: Yahoo Finance does not provide those prices for the TEVA stocks.

On the other hand, the interest rates were taken from the `global-rates.com` site: since we deal with the American market we use the *USD LIBOR* interest rate. The values used are reported in Table 2

Table 2: Interest rates used

	Consultation date	Interest rate
1-Month	29-03-2022	0.44514%
3-Months	29-03-2022	0.98286%
6-Months	29-03-2022	1.45114%
12-Months	29-03-2022	2.08871%

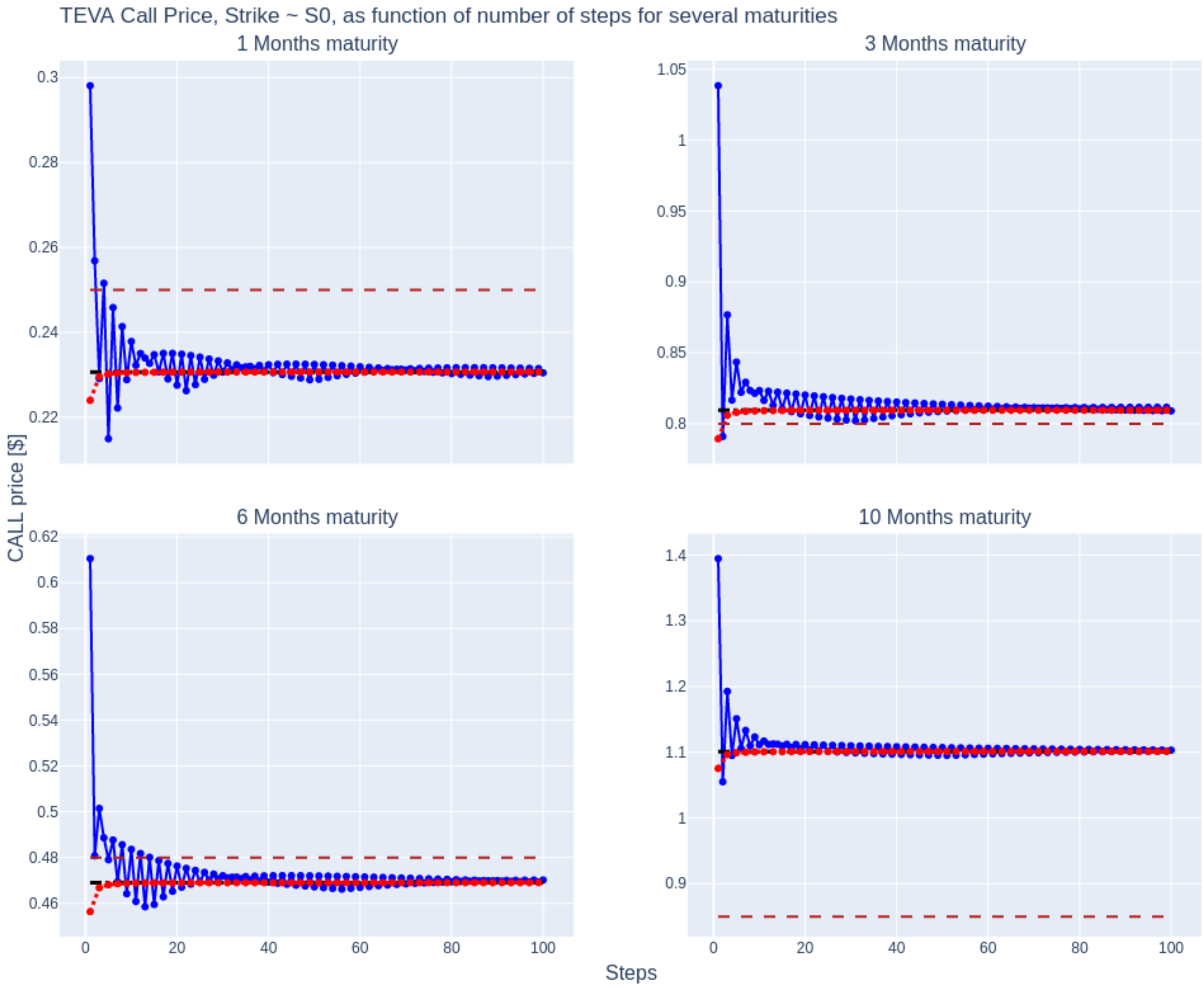

Figure 2: Top (a): Stock price trend for the last 5 years. The boxes represent the spread between the open and close values and the lines represent the spread between the low and high values. Increasing candles are drawn in green whereas decreasing are drawn in red. Bottom (b) Call option prices convergence for CRR (Blue) and LR (Red) binomial trees to BS (Black) as a function of the number of steps for different maturities and comparison with the real price (firebrick) of the option.

3 Results and Discussion

Here we describe the results of the task. We divide the section into the first part in which we describe the outcome for the fixed parameters in a more theoretical framework, while in the second part we price the options using real data.

3.1 Fixed parameters

We compute the results with the same parameters (Table 1) for the Cox-Ross-Rubinstein binomial tree, the Leisen-Reimer binomial tree and the Black-Scholes formula. In Figure 1 (a) and in Table 4 one can see the different behaviours together. We choose to stop after 500 steps which are already too much for both the binomial trees to converge. To have a better understanding of what happened with the LR model we report in Figure 1 (b) a zoom since in the previous Figure it is not clear for the very rapid convergence.

To check the order of convergence reported in [3] we try a fit for both the binomial models. In particular, for the CRR tree, we use the following function to fit

$$f_{CRR}(step) = Call_{Black\&Scholes} + \frac{p}{step}$$

where p is the parameter used to minimize the $\chi^{(}2)$ in the least-squares approach.

The same approach is used for the LR tree, as follows

$$f_{LR}(step) = Call_{Black\&Scholes} + \frac{p}{step^2}$$

The results of the fits are optimal and a visual representation is reported in Figures 1(c) and 1(d). They confirm the convergence order given since the CCR has an order of convergence one, while LR has an order two. We add that not only the convergence order is better, but there is a significant difference in the starting point as the parameters, reported in Table 3, which involves a systematically closer starting point for the LR model.

Table 3: Fit parameter found

Cox-Ross-Rubinstein	Leisen-Reimer
1.98	-0.16

Table 4: Distance between the binomial model and the Black-Scholes as a function of the number of steps made.

Step	CCR	LS
1	1.9814	-0.1679
11	0.1800	-0.0009
21	0.0939	-0.0002
31	0.0635	-0.0001
41	0.0480	-6.0197e-5
51	0.0385	-3.8862e-5
71	0.0277	-2.0027e-5
101	0.0194	-9.887e-6
151	0.0130	-4.420e-6
191	0.0103	-2.762e-6

3.2 Real Data

The historical prices of the stock TEVA described earlier, can be found in Figure 2(a). The procedure followed has been described in [1], and we briefly recap it here. We select a time range equal to the maturity we are considering and we compute the standard deviation of the returns computed as $\frac{S_{t+1}-S_t}{S_t}$ as the historical volatility. We are then ready to apply the CRR or the LR formula.

The results of the option price are reported in Figure 2(b), which confirms the convergence to the Black-Scholes price. We observe a significant problem in the 10-months maturity case: this is probably due to the fact there is not a value for the USD LIBOR at 10 months, thus we use the closer value for the 12 months, or also can be linked to an odd price since we are using the last price and not a mid-price.

Conclusions

We implemented a binomial recombining tree model using both a Cox-Rox-Rubinstein and a Leisen-Reimer approach. We compared the call option price using those methods and the Black-Scholes model and evaluated the order of convergence of those models to the latter. The results obtained are good and confirm the results of the Leisen-Reimer paper.

In the second part, we tried to use those methods to estimate the price of an option in a real-world approach. We used as underlying the stock TEVA, a pharmaceutical company. As we expect the convergence remains, but the comparison with the real prices is not always optimal, in particular for the 10-months maturity. This can be either due to the interest rate used or to the non-optimal value of the last price which can have fluctuations.

The future development of the work done should be the calculation of the other results of the models used such as the greeks, and the implementation of new models more complex, such as the Heston Model.
Another future work should be the implementation of those models, considering a dividend-paying asset.

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