Università degli Studi di Padova
Dipartimento di Fisica e Astronomia "Galileo Galilei"
Master degree in Physics of Data
Course: Stochastic Methods for Finance

Pricing of a Call Option using the Binomial Model

GitHub Repo: https://github.com/aidinattar/Financial-Mathematics

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March 20, 2022

Abstract

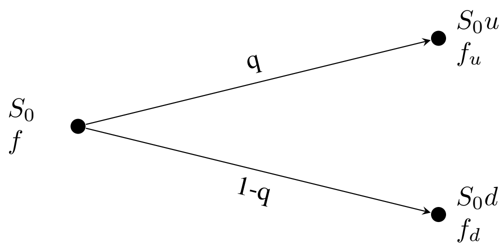
The binomial tree model is a simple option-pricing toy-model based on some assumptions. In this paper, we will derive the closed-form expression and use it to price a call option with 3 and 6 months of maturity of an underlying asset represented by the BioNTech stocks, using first a one-step and then a n-steps binomial recombining tree. The obtained results will not be satisfying compared to the real prices. To investigate that a study of the behaviour of the price of each strike will be done.

Keywords
Binomial-Model, BNTX, Binomial-Tree, Option-Pricing

1 Introduction

The binomial option pricing model, introduced by Cox, Ross and Rubenstein in 1979, is a widely used method to value options. Unlike other models which involve the solution of differential equations, the binomial model is based on a relatively simple argument, based only on the assumption needed that arbitrage opportunities do not exist. We set up a portfolio of the stock and the option in such a way that there is no uncertainty about the value of the portfolio at the end of the maturity. We then argue that, since the portfolio has no risk, the return it earns must equal the risk-free interest. This enables us to work out the cost of setting up the portfolio and therefore the option's price. Because there are two securities (the stock and the stock option) and only two possible outcomes, it is always possible to set up a riskless portfolio.

Following [2], we can consider a stock whose price is S_0 and an option on the stock whose current price is f. We suppose that the option lasts for time T and that during the life of the option the stock price can either move up from S_0 to a new level, S_0u, where $u > 1$, or down to S_0d, where $d < 1$. If the stock price moves up to S_0u, we suppose that the payoff from the option is f_u; if the stock price moves up to S_0d, we suppose that the payoff from the option is f_d, as illustrated in the Figure below.



We imagine a portfolio consisting of a long position in Δ shares and a short position in one option. We calculate the value of Δ that makes the portfolio riskless. Respectively, if there is an up or down movement in the stock price, the value of the portfolio at the end of the life of the option is

$$S_0u\Delta - f_u \qquad S_0d\Delta - f_d$$

The two are equal when

$$S_0u\Delta - f_u = S_0d\Delta - f_d$$

or

$$\Delta = \frac{f_u - f_d}{S_0u - S_0d} \tag{1}$$

In this case, the portfolio is riskless and, for there to be no arbitrage opportunities, it must earn the risk-free interest rate. Following the notation of [1], if we denote the risk-free interest rate by r, the present value of the portfolio is

$$\frac{(S_0u\Delta - f_u)}{1 + rT}$$

The cost of setting up the portfolio is

$$S_0u\Delta - f$$

it follow that

$$S_0\Delta - f = \frac{(S_0u\Delta - f_u)}{1 + rT}$$

and after some maths

$$f = \frac{qf_u + (1-q)f_d}{1 + rT} \tag{2}$$

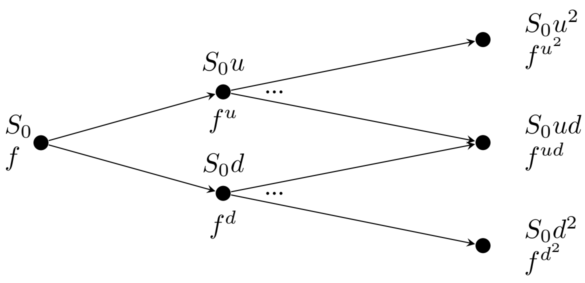
where

$$q = \frac{(1 + rT) - d}{u - d} \tag{3}$$

Where $u = e^{+\sigma\sqrt{T}}$ and $d = 1/u$. Equations 2 and 3 enable an option to be priced when stock price movements are given by a one-step binomial tree.

This procedure can be generalized to a multi-steps recombining binomial tree: if we consider n steps till maturity we obtain

$$f = (1 + r\Delta t)^{-1} \sum_{j=0}^{n} f^{u^j d^{n-j}} \binom{n}{j} q^j (1-q)^{n-j} \tag{4}$$



2 Materials & Methods

In this paper, we try to compute the price of a call option with a maturity of 3 and 6 months using the stock prices of an underlying. The underlying chosen is the BioNTech SE (BNTX in the Nasdaq Stock Market), a biotechnology company, which develops and commercializes immunotherapies for cancer and other infectious diseases. It develops pharmaceutical candidates based on messenger ribonucleic acid (mRNA) for use as individualized cancer immunotherapies, as vaccines against infectious diseases and as protein replacement therapies for rare diseases, and also engineered cell therapy, novel antibodies and small molecule immunomodulators as treatment options for cancer [3].
The company achieved success in 2021 after developing, in partnership with Pfizer, the mRNA vaccine for preventing COVID-19 infections, which since have been approved by the institutional medicines agencies all over the world [3].

To price the option we need to use the historical prices, to compute the volatility σ. The choice of the appropriate number of days to use is not easy: more data generally lead to more accuracy, but the volatility does change over time and data that are too old may not be relevant for predicting future volatility. A compromise that seems to work reasonably well is to use closing prices from daily data over the most recent 90 to 180 days, or, as a rule of thumb, it can be set equal to the number of days to which the volatility is to be applied. We do the latter, taking the historical data of respectively 3 and 6 months.

The daily volatility is computed as the standard deviation of the daily returns, given by

$$\log \frac{S_{t+1}}{S_t} \qquad \text{or} \qquad \frac{S_{t+1} - S_t}{S_t}$$

Then, to annualize the value obtained $\sigma_{annual} = \sigma_{daily} \cdot \sqrt{252}$.

We choose also a strike price to compare the results given by the binomial model and the real prices. We choose a strike price *at the money*, meaning that $K \approx S_0$.

On the other hand, the interest rate was taken from the `global-rates.com` site: since we deal with the American market we use the *USD LIBOR* interest rate. The values used are reported in Table 1

Table 1: Interest rates used

	Consultation date	Interest rate
3-Months	12-03-2022	0.80286%
6-Months	20-03-2022	1.28757%

The data used are taken from the Yahoo Finance site and the analysis involved the use of `Excel` and `Julia 1.7.0` using a `Jupyter` framework and can be found in the `GitHub` repository.

3 Results and Discussion

Here we describe the results of the task. We divide the section in a first part in which we describe the outcome a simple one-step binomial model and the latter is an analysis of the effect of using a multi-step recursive binomial tree.

Both the analysis are made using the historical prices of the stock BNTX described earlier, whose values can be found in Figure 1(a)

3.1 One-step binomial model

In Figure 2 are reported the returns over the 6 months. As we said earlier we use the rule of thumb to select the data we consider to compute the volatility, therefore the 3 months call option is evaluated considering only 3 months of data.

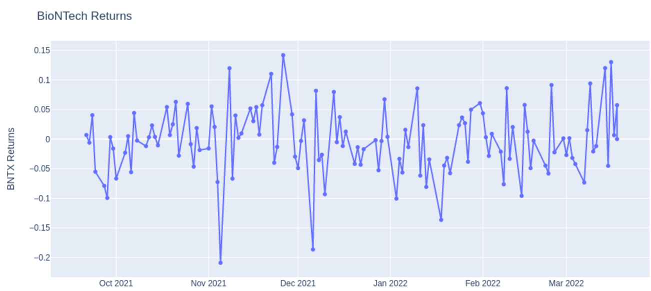


Figure 2: Daily return of the underlying stock considered in a 6 months range.

The values for the daily and annual volatility are reported in Table 2

Table 2: Daily and Annualized volatility for a 3 and 6 months range.

	Daily Vol.	Annualized Vol.
3-Months	.049	.779
6-Months	.057	.912

Using that information we can infer the price of a call option with maturity of 3 and 6 months with strike price $\sim S_0$ as explained earlier. The results are reported in Table 3.

3.2 Multi-steps binomial model

The generalization to *n-steps* is straightforward, given the theoretical result we obtained in Eq. 4.

Figure 1: Top (a): Stock price trend for the last 6 months. The boxes represent the spread between the open and close values and the lines represent the spread between the low and high values. Increasing candles are drawn in green whereas decreasing are drawn in red; Call Option price for 3 (center (b)) and 6 (bottom (c)) months as a function of the number of steps used till maturity and the comparison with the real price.

Table 3: Daily and Annualized volatility for a 3 and a 6 months range.

	Binomial model [$]	Real Call price [$]
3-Months	25.36	21.70
6-Months	30.02	14.00

The plot in Figures 1(b) and (c) shows the converging behaviour of the binomial recursive tree model (it should converge to the *Black-Scholes* call option price, this will be investigated in a future paper). From the graphical visualization one can observe the poor results obtained. This is probably due to a divergence between the historical volatility, based on the past data of the underlying prices, and the implied volatility, *i.e.* the market expectation of the future volatility.

This shows us that the binomial tree model is a useful toy-model but it has some limitations that should be corrected relaxing some assumptions such as the constantness of the volatility.

3.3 Binomial-tree model for several strikes

We do a further study to show that the choice of the strike for the previous estimation was in some ways unlucky. To show that we see Figure 3, which reports a comparison between the prediction of a 60-steps binomial-tree and the real call option prices for a 6 months maturity option. We observe that for the first part of the curve ($K \lesssim 400$) the behaviour is more or less respected, while for the second part ($K \gtrsim 400$) it quasi-totally diverges from the predicted behaviour. For the latter, we can guess that the divergence is either due to *fake prices* or to speculation phenomena which we can't explain.

For the first curve, one can see also a weird behaviour around

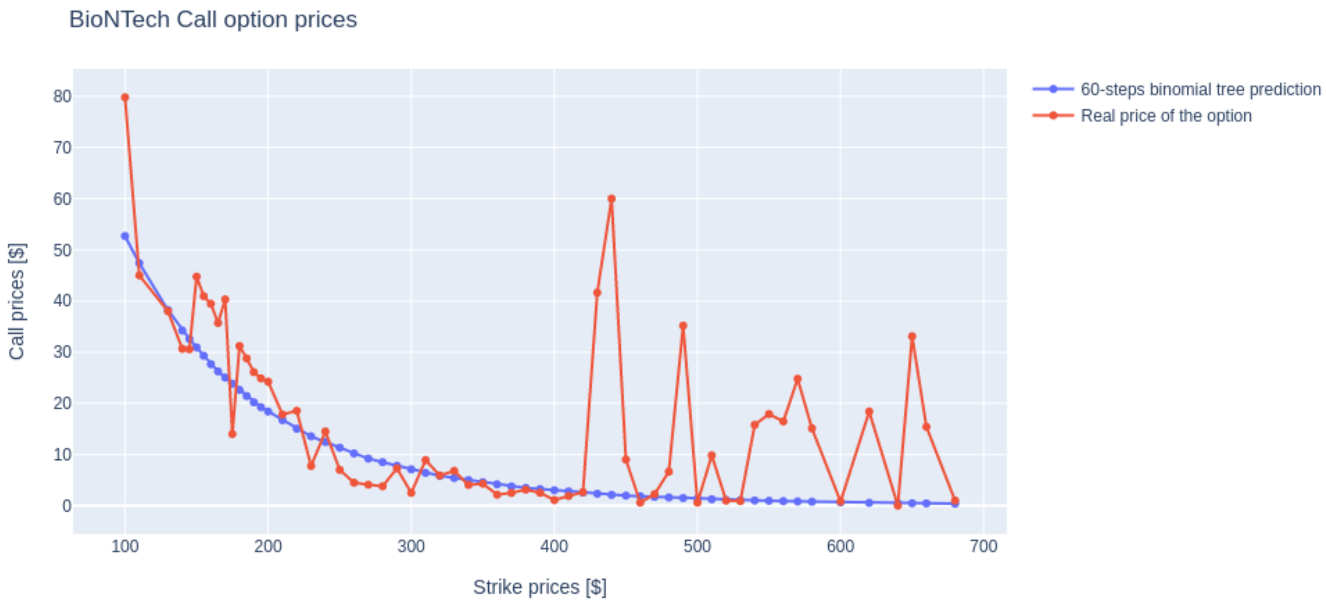


Figure 3: Comparison between the theoretical curve obtained through a 60-steps binomial tree model and the real call option prices with maturity 6 months as a function of the strike prices.

the strike *at the money* in which we have a fall, while both before and after the prices are higher than the expected one. This could be due to the divergence of the implied volatility from the historical one proposed earlier or even due to other market behaviours.

Conclusions

We implemented a binomial recombining tree model to price a call option of an underlying represented by the *BioNTech* stock asset. The results obtained are not satisfying in particular for the 6 months maturity prediction. As we explained this can be due to a divergence between the historical volatility and the implied volatility that a simple toy-model such as the one we consider cannot predict and should be managed using more complex models, such as the *Heston Model*.

The future development of the work done should be the implementation of the Black-Scholes model prediction and the demonstration that the binomial model should converge to it for the number of steps $n \longrightarrow \infty$.
Another future work should be the implementation of the binomial model, considering a dividend-paying asset.

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