UNIVERSITÀ DEGLI STUDI DI PADOVA
DIPARTIMENTO DI FISICA E ASTRONOMIA "GALILEO GALILEI"
MASTER DEGREE IN PHYSICS OF DATA
COURSE: STOCHASTIC METHODS FOR FINANCE

A study of the Monte Carlo option model

GITHUB REPO: HTTPS://GITHUB.COM/AIDINATTAR/FINANCIAL-MATHEMATICS

Attar Aidin[1]

[1]Dipartimento di Fisica e Astronomia "Galileo Galilei", Università degli Studi di Padova

May 14, 2022

Abstract

The Monte Carlo option model uses Monte Carlo methods to calculate the value of an option with multiple sources of uncertainty or with complicated features. The Monte Carlo model is widely used, in particular, to evaluate the price of path-dependent options. In this report, we will use Monte Carlo methods to generate a number of trajectories for the Geometric Brownian Motion in a Black-Scholes market, and we will use those trajectories to price several options, such as Vanilla options, Asian Options, Lookback Options, and Barrier Options. The results of this work will show the powerfulness of the Monte Carlo model for this kind of tasks.

Keywords
Monte Carlo, European Option, Asian Option, Lookback option, Barrier option, Euler-Scheme, Geometric-Brownian-Motion

1 Introduction

Monte Carlo Simulation can be used to price various financial instruments such as derivatives.

In this report, we will calculate the price of different types of options using the Monte Carlo Simulation.

Even though the option value can be easily calculated using the Black-Scholes Option pricing formula, we can make use of the Monte Carlo Simulation technique to achieve comparable results.

According to [1], when used to value an option, the Monte Carlo simulation uses the risk-neutral valuation result. We sample paths to obtain the expected payoff in a risk-neutral world and then discount this payoff at the risk-free rate. Consider a derivative dependent on a single market variable S that provides a payoff at time T. Assuming that interest rates are constant, we can value the derivative as follows:

1. Sample a random path for S in a risk-neutral world.
2. Calculate the payoff from the derivative.
3. Repeat steps 1 and 2 to get many sample values of the payoff from the derivative in a risk-neutral world.
4. Calculate the mean of the sample payoffs to get an estimate of the expected payoff in a risk-neutral world.
5. Discount this expected payoff at the risk-free rate to get an estimate of the value of the derivative.

Suppose that the process followed by the underlying market variable in a risk-neutral world is

$$dS = \hat{\mu} S dt + \sigma S dz \tag{1}$$

where dz is a Wiener process, $\hat{\mu}$ is the expected return in a risk-neutral world, and σ is the volatility. To simulate the path followed by S, we can divide the life of the derivative into N short intervals of length t and approximate equation 1 as

$$S(t+\Delta t) - S(t) = \hat{\mu} S(t)\Delta t + \sigma S(t)\epsilon\sqrt{\Delta t} \tag{2}$$

where $S(t)$ denotes the value of S at time t, ϵ is a random sample from a normal distribution with mean zero and standard deviation of 1.0. This enables the value of S at time t to be calculated from the initial value of S, the value at time $2\Delta t$ to be calculated from the value at time t, and so on.

One simulation trial involves constructing a complete path for S using N random samples from a normal distribution. In practice, it is usually more accurate to simulate $\log S$ rather than S. From Ito's lemma the process followed by $\log S$ is

$$d\log S = \left(\hat{\mu} - \frac{\sigma^2}{2}\right) dt + \sigma dz \tag{3}$$

so that

$$\log S(t+\Delta t) - \log S(t) = \left(\hat{\mu} - \frac{\sigma^2}{2}\right)\Delta t + \sigma\epsilon\sqrt{\Delta t} \tag{4}$$

or equivalently

$$S(t+\Delta t) = S(t)\exp\left\{\left[\left(\hat{\mu} - \frac{\sigma^2}{2}\right)\Delta t + \sigma\epsilon\sqrt{\Delta t}\right]\right\} \tag{5}$$

The key advantage of Monte Carlo simulation is that it can be used when the payoff depends on the path followed by the underlying variable S as well as when it depends only on the final value of S. (For example, it can be used when payoffs depend on the average value of S between time 0 and time T.) Payoffs can occur at several times during the life of the derivative rather than all at the end. Any stochastic process for S can be accommodated. The procedure can also be extended to accommodate situations where the payoff from the derivative depends on several underlying market variables. The drawbacks of Monte Carlo simulation are that it is computationally very time consuming and cannot easily handle situations where there are early exercise opportunities.

Asian Options Asian options are options where the payoff depends on the arithmetic average of the price of the underlying asset during the life of the option. The payoff from an average price call is $\max(0, S_{avg} - K)$ and that from an average price put is $\max(0, K - S_{avg})$, where S_{avg} is the average price of the underlying asset.

There exist another type of Asian option: the floating strike version, whose payoff depends on the average price of the underlying asset during the life of the option. In other words, floating-strike Asian options do not have a fixed strike, but instead, the strike is determined by the value of the underlying at maturity. The option's payoff is the positive difference between the final underlying price and the average underlying price.

Lookback Options The payoffs from lookback options depend on the maximum or minimum asset price reached during the life of the option. The payoff from a floating lookback call is the amount that the final asset price exceeds the minimum asset price achieved during the life of the option. The payoff from a floating lookback put is the amount by which the maximum asset price achieved during the life of the option exceeds the final asset price. Valuation formulas have been produced for floating lookbacks (the one used in this report is reported in Table 2).

Barrier Options Barrier options are options where the payoff depends on whether the underlying asset's price reaches a certain level during a certain period of time.
A number of different types of barrier options regularly trade in the over-the-counter market. They are attractive to some market participants because they are less expensive than the corresponding regular options. These barrier options can be classified as either knock-out options or knock-in options. A knock-out option ceases to exist when the underlying asset price reaches a certain barrier; a knock-in option comes into existence only when the underlying asset price reaches a barrier.

Milstein Scheme Out of the Euler scheme described above to have a numerical solution of the stochastic differential equation, we use another method which has a stronger convergence, the Milstein scheme. We do not report here the derivation but just the result:

$$S(t+\Delta t) = S(t) + rS(t)dt + rS(t)dt + \\ +\sqrt{\sigma\Delta t}S(t)\epsilon + \frac{1}{4}S(t)^2 dt(\epsilon^2 - 1)$$

which, following the Ito's Lemma gives the same as (5).

2 Materials & Methods

To price an option we need to simulate the trajectories of the price of the underlying. They follow a Geometric Brownian Motion process. We fix some parameters as reported in Table 1. The generation of the paths is obtained by generating random numbers, according to a normal distribution, and knowing that:

$$S_t = S_{t-1}e^{(r-\frac{\sigma^2}{2})dt+\sigma\sqrt{dt}\epsilon} \tag{6}$$

Table 1: Asian call and put option prices. Prices in US Dollars.

Parameter	Value
S0	100 USD
σ	20%
T	1 year
r	1%
K	99 USD
dt	1 day
U[1]	130 USD
D[2]	90 USD

To compute the price of a vanilla option we generate a number of paths and for each path, we evaluate the payoff:

$$\text{payoff}_{T,call} = (S_T - K)^+ \tag{7}$$

$$\text{payoff}_{T,put} = (K - S_T)^+ \tag{8}$$

Once we computed the payoffs for each trajectory, we can evaluate the price of a vanilla option as the mean of all the payoffs:

$$\text{option price} = \sum_{i=0}^{N} \frac{\text{payoff}_i}{N} \tag{9}$$

with N the number of paths generated.

The Monte Carlo pricing is done with both a multi-step simulation and a one-step simulation. For the former we use the paths generated with the method described above, for the latter, we generate numbers according to a normal distribution and we evaluate the final price as follows:

$$S_T = S_0 \exp\left\{(r - \frac{\sigma^2}{2})T + \sigma\epsilon\sqrt{T}\right\} \tag{10}$$

The same methods are used to evaluate several option types, such as Asian Options with a fixed strike and with a floating strike, lookback options and barrier options both knock-out and knock-in, and double barrier options. In Table 2 we report the payoffs of each option.

Table 2: Payoff of the exotic options.

Type	Call Payoff	Put Payoff
Asian-fixed strike	$(\frac{1}{T}\int_0^T S_t dt - K)^+$	$(K - \frac{1}{T}\int_0^T S_t dt)^+$
Asian-floating strike[3]	$(S_T - \frac{k}{T}\int_0^T S_t dt)^+$	$(\frac{k}{T}\int_0^T S_t dt - S_T)^+$
Lookback	$S_T - S_{min}$	$S_{max} - S_T$
Barrier	Vanilla but conditioned to the crossing of the barrier	

3 Results & Discussion

Here we report the main results of the report.

[1]Upper barrier
[2]Downer barrier
[3]k is a weighting, here considered as 1

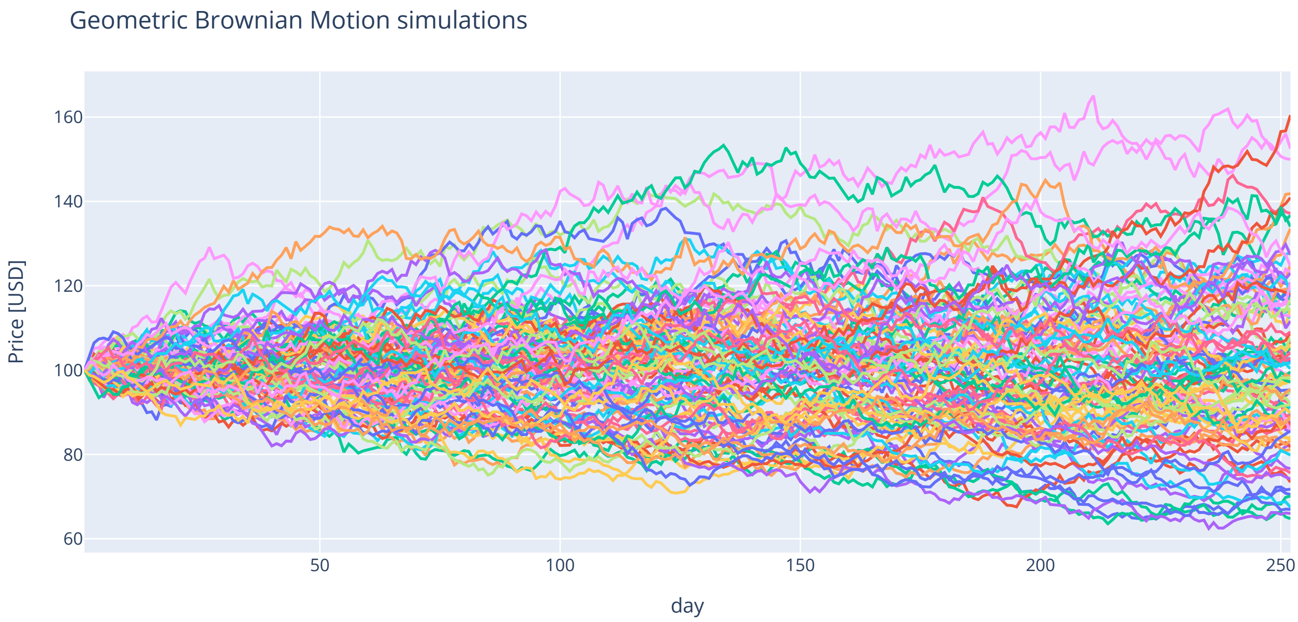

Figure 1: Monte Carlo's generated paths for an underlying asset. We represent in the figure just 100 paths for visualization reasons, but the computations are done using $100k$

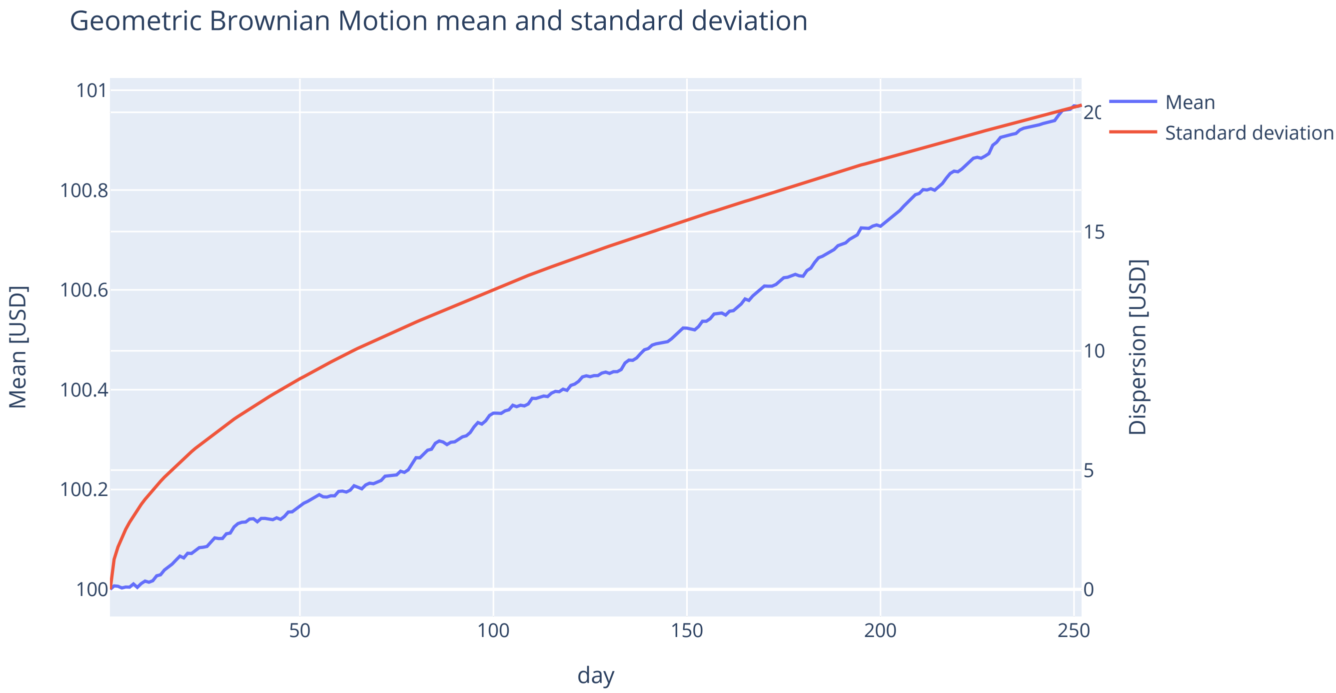

Figure 2: Mean and standard deviation of the price of the underlying as a function of the days elapsed.

In Figure 1 we report the paths generated with the method described above. We generated in the figure just 100 paths for visualization reasons: for the computations of the options prices we used $100k$ paths. As we expect the mean of all the paths follows the drift given by the interest rate. On the other hand, we see, as we expect, that as days elapse the dispersion in the paths. In Figure 2 we see the trend of the mean and the standard deviation of the prices of the underlying based on the total number of simulations. As we see the mean follows quite well a linear trend, in particular, it is in accord with the interest rate as drift (we choose 1% and we get after ~ 1 year a mean of 101USD), while the standard deviation follows almost perfectly a square root trend.

For the pricing of a vanilla option, we followed the methods described above: in Table 3 we report the price obtained for each method we know. We see that there is good compatibility between the multi-step Monte Carlo based both on the Euler scheme and Milstein scheme and the non-Monte Carlo methods (Cox-Ross-Rubenstein binomial tree with 1001 steps, Leisen-Reimer binomial tree with 1001 steps and the Black-Scholes). The Monte Carlo with one-step computation gives results com-

Table 3: Comparison between the different methods. Prices in US Dollars.

Option type	Monte Carlo multi-step	Monte Carlo one-step	Milstein Scheme	CRR	LR	BS
Call	8.96	9.00	8.93	8.92	8.92	8.92
Put	6.99	6.98	6.91	7.00	6.93	6.93

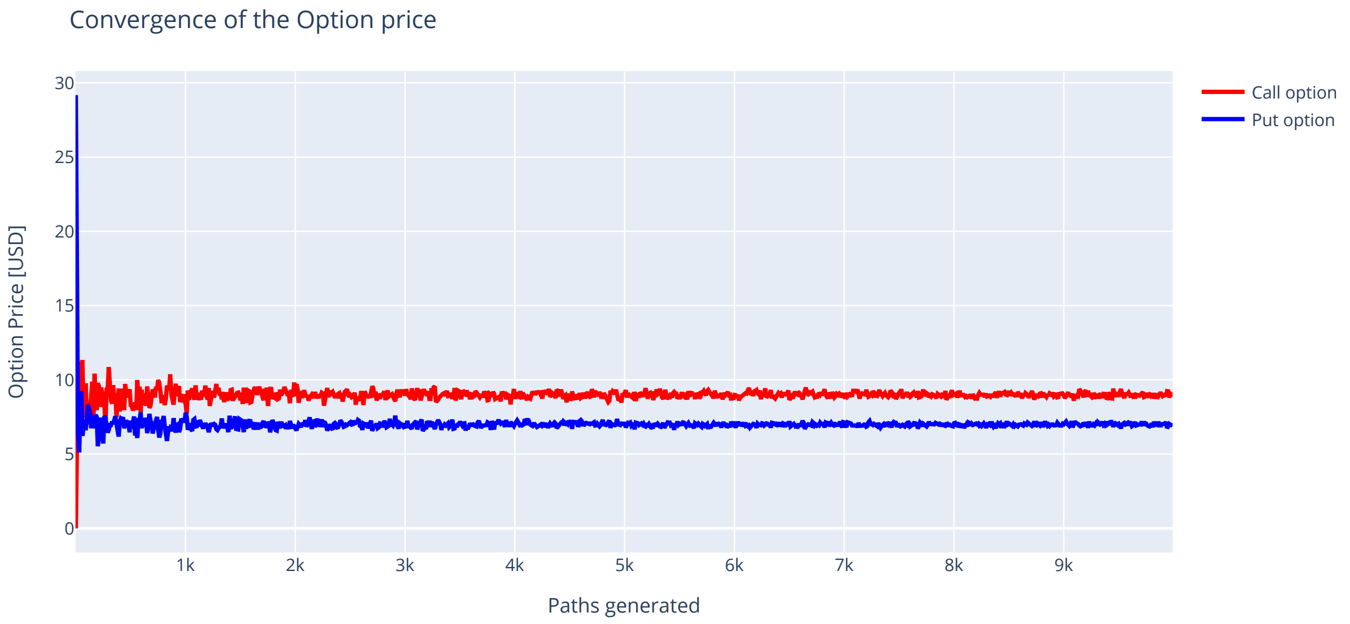

Figure 3: Multi-Step Monte Carlo method Option Price convergence.

Figure 4: One-Step Monte Carlo method Option Price convergence.

parable in terms of the option price, meaning that, if we do not need to know the path followed by the underlying, as happens in the vanilla option case, there is no need to compute all the path, which gives a more computationally expensive method.

In Figure 3 and Figure 4 we report the charts about the convergence of the option price for both a call option and a put option as a function of the number of paths we generate: as we see both the charts show convergence and for $N_{paths} > 5k$ there are no significant changes. Since we used $N_{paths} = 100k$ there is no reason to doubt the convergence of the option prices.
Those charts show also that the multi-step version, in the beginning, has a great variance, while for the one-step version it is less: if we have significant limits in the computation resources it should be better to use the latter method, at the cost of not knowing the paths and so cannot use them to compute the price

of some exotic options.

In Table 4 we report the price for the call and put of some exotic options. Those options cannot be evaluated with a single step Monte Carlo, since their price depends on the path they follow, as described in the theoretical introduction and reported in Table 2.

Table 4: Floating strike call and put option prices. Prices in US Dollars.

	Call [USD]	Put [USD]
Asian - Fixed strike	5.35	3.98
Asian - Floating strike	4.85	4.38
Lookback	14.89	15.71
Barrier - Down Out	7.42	.17
Barrier - Down In	1.40	6.84
Barrier - Up Out	3.47	6.99
Barrier - Up In	5.35	.02
Double Barrier	2.44	3.89

To evaluate the prices we generate $100k$ paths and evaluate the payoffs to compute the option price according to the parameters described in Figure 1.
Since each option is based on a different mechanism it does not make sense to compare the different prices: only the Asian option with a fixed strike is based on the strike proposed and in general we can see that both the call and put cost less than Europan options.
On the other hand, we see the lookback options being expensive to establish and we expect the potential profits often nullified by the costs.

Conclusions

We implemented the Monte Carlo methods to simulate the trajectory of an underlying asset as a function of the time and we used it to evaluate the price of a vanilla option. We saw that for vanilla options, and in general for options which are not path-dependent the one-step Monte Carlo method is a computationally less expensive method which gives comparable results.
We then implemented the methods to evaluate the price of some exotic options such as Asian options, with fixed and floating strike, Lookback options with floating strike, Barrier options knock-in and knock-out and Double Barrier options.

We observe that the results are compatible with the analytical methods such as Leisen-Reimer, Cox-Ross-Rubenstein and Black-Scholes, and this tells us the powerfulness of the method represented by the Monte Carlo method.
We observe that these Monte Carlo methods are quite similar to the binomial tree methods: they solve the computational problem of the binomial trees, which force us to use recombining trees, using stochasticity.

Future improvements of the work can be the implementation of other options, and furthermore the implementation of Monte Carlo methods in different markets, such as the Heston market.

References

[1] HULL, J. *Options, futures, and other derivatives*, 6. ed., pearson internat. ed ed. Pearson Prentice Hall, Upper Saddle River, NJ [u.a.], 2006.