Concurrent Learning Adaptive Control of Linear Systems with Exponentially Convergent Bounds

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***Abstract*— We develop concurrent learning adaptive controllers, which uses recorded and current data concurrently for adaptation, for model reference adaptive control of uncertain linear dynamical systems. We show that a verifiable condition on the linear independence of the recorded data is sufficient to guarantee global exponential stability. We use this fact to develop exponentially decaying bounds on the tracking error and weight error, and estimate upper bounds on the control signal. These results allow the development of adaptive controllers that ensure good tracking without relying on high adaptation gains, and can be designed to avoid actuator saturation. Simulations validate the presented theory.**

I. INTRODUCTION

In Model Reference Adaptive Control (MRAC) of uncertain multivariable linear dynamical systems, the design objective is to make the linear system behave like a chosen reference model. The underlying assumption is that a set of ideal linear weights exists that guarantees this design objective can be met. This assumption is often known as the *matching condition* (see for example [1], [2]). If the adaptive weights do converge to their ideal values as prescribed by the matching conditions, then the closed loop system behaves like the reference model, and the desired transient response and stability properties of the chosen reference model are dynamically recovered. However, many classical and recent adaptive controllers that use only instantaneous data for adaptation (see [3], [4], [5], [1], [6], [7] and references therein), require that the system states be Persistently Exciting (PE) to guarantee that the adaptive weights converge to these ideal values. Boyd and Sastry have shown that the condition on PE states can be directly related to a condition on the spectral properties of the exogenous reference input [8]. However, enforcing PE through exogenous excitation is not always feasible, particularly in applications which require high precision or smooth operation. Furthermore, in event based systems, it is often infeasible to monitor online whether a signal will remain PE. Hence, weight convergence cannot be guaranteed in many adaptive control applications.

In this paper we describe an approach to guarantee exponential stability of MRAC of uncertain linear multivariable dynamical systems by utilizing the idea of concurrent learning [9], [10]. Particularly, we show that a concurrent learning adaptive controller, that uses both current and past data concurrently for adaptation, can guarantee global exponential stability of the zero solution of the tracking error and the weight error dynamics subject to a verifiable condition on linear independence of the recorded data; *without requiring PE states*. Furthermore, we show that the guaranteed exponential stability results in guaranteed bound on how large the transient error can be, and that this bound reduces exponentially fast in time. These results show that the inclusion of memory can significantly improve the performance and stability guarantees of adaptive controllers. Furthermore, these results also compliment our previous work in concurrent learning (see for example [9], [11]), and extend concurrent learning to adaptive control of linear systems. The performance of a concurrent learning adaptive controller is compared with a classical adaptive controller in an exemplary simulation study. Finally, These results have significant importance in verification and validation of adaptive control systems.

The organization of this paper is as follows, in Section II we pose the classical problem of MRAC of linear systems. In Section III we present the novel concurrent learning adaptive controller for linear systems, and establish its properties using Lyapunov analysis. In Section IV we present the results of an exemplary simulation study. The paper is concluded in Section V.

In this paper, $f(t)$ represents a function of time t. Wherever appropriate, we will drop the argument t consistently over an entire for ease of exposition. The operator $|.|$ denotes the absolute value of a scalar and the operator $\|.\|$ denotes the Euclidian norm of a vector. Furthermore, for a vector $x(t) \in \mathbb{R}^n$, the infinity norm is defined as $\|x(t)\|_\infty = \max_{i=1,..,n} |x_i(t)|$, and the L_∞ norm of the vector signal x can be defined as $\|x\|_{L_\infty} = \max_{i=1,..,n}(\sup_{t\geq 0} |x_i(t)|)$. The operator $\mathrm{vec}(.)$ stacks the columns of a matrix into a vector, and operators $\lambda_{\min}(.)$ and $\lambda_{\max}(.)$ return the smallest and the largest eigenvalue of a matrix.

II. MODEL REFERENCE ADAPTIVE CONTROL OF LINEAR SYSTEMS

In this section we review results on classical MRAC of uncertain multivariable (inclusive of multi-input, multi-output) linear dynamical systems (see for example [3], [4], [5], [1], [2]).

A. The Classical Linear MRAC Problem

Let $x(t) \in \mathbb{R}^n$ be the state vector, let $u(t) \in \mathbb{R}^m$ denote the control input, and consider the following linear time-invariant system:

$$\dot{x} = Ax(t) + Bu(t), \tag{1}$$

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where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. We assume that the pair (A, B) is controllable and that B has full column rank. For the case when $m \le n$, this condition is normally satisfied by physical systems. The assumption can be restrictive when $m > n$, in this case, it can be relaxed by using matrix inverse and pseudoinverse approaches, constrained control allocation, pseudocontrols, or daisy chaining [12], [13], [14], to reduce the dimension of the control input vector. We assume that $u(t)$ is restricted to the class of admissible control inputs consisting of measurable functions, and $x(t)$ is available for full state feedback.

A chosen reference model that characterizes the desired closed-loop response of the system is given by:

$$\dot{x}_{rm} = A_{rm}x_{rm}(t) + B_{rm}r(t), \tag{2}$$

where $A_{rm} \in \mathbb{R}^{n\times n}$ is Hurwitz. Furthermore, $r(t) \in \mathbb{R}$ denotes a bounded, piecewise continuous, reference signal. An adaptive control law consisting of a linear feedback part $u_{pd}(t) = K^T(t)x(t)$, and a linear feedforward part $u_{rm}(t) = K_r(t)r(t)$ with time varying weights $K(t) \in \mathbb{R}^{n\times m}$ and $K_r(t) \in \mathbb{R}^m$ is proposed to have the following form

$$u(t) = u_{rm}(t) + u_{pd}(t). \tag{3}$$

Substituting (3) in (1) we have

$$\dot{x} = (A + BK^T(t))x(t) + BK_r(t)r(t). \tag{4}$$

The design objective is to have (4) behave as the chosen reference model of (2). To that effect, we introduce the following matching conditions:

Assumption 1 There exists $K^* \in \mathbb{R}^{n\times m}$ and $K_r^* \in \mathbb{R}^m$ such that

$$A + BK^{*T} = A_{rm} \tag{5}$$

$$BK_r^* = B_{rm}. \tag{6}$$

Adding and subtracting $BK^{*T}(t)x(t)$ and $BK_r^*r(t)$ in (4) and letting $\tilde{K} = K - K^*$ and $\tilde{K}_r = K_r - K_r^*$ we have

$$\dot{x} = A_{rm}x(t) + B_{rm}r(t) + B\tilde{K}^T(t))x(t) + B\tilde{K}_r(t)r(t). \tag{7}$$

Note that if $\tilde{K} = 0$ and $\tilde{K}_r = 0$, then $\dot{x} = A_{rm}x(t) + B_{rm}r(t)$ and the design objective of having the system behave as the reference model is be satisfied. Furthermore, defining the tracking error as $e(t) \triangleq x(t) - x_{rm}(t)$ yields

$$\dot{e}(t) = A_{rm}e(t) + B\tilde{K}^T(t)x(t) + B\tilde{K}_r(t)r(t). \tag{8}$$

As noted, the tracking error converges to zero exponentially fast if $\tilde{K} = 0$ and $\tilde{K}_r = 0$, since A_{rm} is Hurwitz. It follows from converse Lyapunov theory that there exists a unique positive definite $P \in \mathbb{R}^{n\times n}$ satisfying the Lyapunov eqauation

$$A_{rm}^T P + PA_{rm} + Q = 0 \tag{9}$$

for any positive definite matrix $Q \in \mathbb{R}^{n\times n}$.

Let $\Gamma_x > 0$ and $\Gamma_r > 0$ denote positive definite learning rates, and consider a well known adaptive law for MRAC of linear systems which uses only instantaneous data for adaptation:

$$\dot{K}(t) = -\Gamma_x x(t)e^T(t)PB, \tag{10}$$

$$\dot{K}_r(t) = -\Gamma_r r(t)e^T(t)PB. \tag{11}$$

It is a classical result that the adaptive law of (10) will guarantee the weight error $\tilde{K}(t) \to 0$ and $\tilde{K}_r(t) \to 0$ exponentially, and the tracking error $e(t) \to 0$ exponentially if and only if the system states are persistently exciting [1], [5], [15], [16], [4]. Furthermore, it is also well known that the classical adaptive law of (10) does not guarantee that the weights remain bounded in a predefined domain unless further modifications are made. Commonly used modifications to the adaptive law include σ modification of Ionnou [15], e modification of Narendra [3], or artificial restriction of the weights using parameter projection [1], [17]. Various equivalent definitions of excitation and the persistence of excitation of a bounded vector signal exist in the literature [5], [4], we will use the definitions proposed by Tao in [1]:

Definition 1 A bounded vector signal $x(t)$ is exciting over an interval $[t, t+T]$, $T > 0$ and $t \ge t_0$ if there exists $\gamma > 0$ such that

$$\int_t^{t+T} x(\tau)x^T(\tau)d\tau \ge \gamma I. \tag{12}$$

Definition 2 A bounded vector signal $x(t)$ is persistently exciting if for all $t > t_0$ there exists $T > 0$ and $\gamma > 0$ such that

$$\int_t^{t+T} x(\tau)x^T(\tau)d\tau \ge \gamma I. \tag{13}$$

As an example, consider that in the two dimensional case, vector signals containing a step in every component are exciting, but not persistently exciting; whereas the vector signal $\Phi(t) = [\sin(t), \cos(t)]$ is persistently exciting.

III. Concurrent Learning for Exponential Convergence in MRAC of Linear Systems

The key idea in concurrent learning is to use recorded data concurrently with current data for adaptation. Intuitively, one can argue that if the recorded data is sufficiently rich, then convergence should occur without requiring persistency of excitation. Here, we formalize this argument, and shows that a verifiable condition on the linear independence of the recorded data is sufficient to guarantee exponential stability of the tracking error and weight error dynamics.

A. A Concurrent Learning Adaptive Controller

Let (x_j, r_j) denote the j^{th} recorded data pair of the state and reference signal. Recall that since we have assumed B is full column ranked, $(B^TB)^{-1}$ exists, and define for each recorded data point the error variables $\epsilon_{K_j}(t) \in \mathbb{R}^m$, and $\epsilon_{K_{r_j}}(t) \in \mathbb{R}^m$ as follows:

$$\epsilon_{K_j}(t) = (B^TB)^{-1}B^T(\dot{x}_j - A_{rm}x_j - B_{rm}r_j - B\epsilon_{K_{r_j}}(t)), \tag{14}$$

and

$$\epsilon_{K_{r_j}}(t) = K_r^T r_j - (B^T B)^{-1} B^T B_{rm} r_j. \quad (15)$$

The concurrent learning weight update law is then given by:

$$\dot{K}(t) = -\Gamma_x(x(t)e^T(t)PB + \sum_{j=1}^{p} x_j \epsilon_{K_j}^T(t)), \quad (16)$$

$$\dot{K}_r(t) = -\Gamma_r(r(t)e^T(t)PB + \sum_{j=1}^{p} r_j \epsilon_{K_{r_j}}^T(t)). \quad (17)$$

The above concurrent learning law can be evaluated if the first derivative of the state ($\dot{x}_j$) for the j^{th} recorded data point is known. Note that one does not need to know the state derivative $\dot{x}(t)$ at the current time t, rather, only the state derivative of a data point recorded in the past. If measurements are not available, $\dot{x}_j$ can be estimated using a fixed point optimal smoother which uses a forward and a backward Kalman filter to accurately estimate $\dot{x}_j$ in presence of noise. This point stresses the benefit of using memory, as recorded states can undergo further processing to extract relevant information (projected weight error in this case) in the background which can be used for adaptation. Furthermore, since ϵ_j does not directly affect the tracking error at time t, this delay does not adversely affect the instantaneous tracking performance of the controller. Details of this process can be found in [10].

Furthermore, note that by rearranging (7), we see that $\epsilon_{K_j}(t) = \tilde{K}^T(t)x_j$, and by rearranging (2) we see that $\epsilon_{K_{r_j}}(t) = \tilde{K}_r^T(t)r_j$. Hence, in both cases, the m dimensional terms in equations (14) and (15) contain the information of the m by n dimensional weight error $\tilde{K}$ and the m dimensional weight error $\tilde{K}_r$ respectively. The key point to note here is that although the concurrent learning law in (16) will be affected by the weight errors, explicit knowledge of the weight errors (and hence the ideal weights K^* and K_r^*) is not required for evaluating (16). Noting this fact, and recalling $\tilde{K}(t) = K(t) - K^*$, and $\tilde{K}_r = K_r - K_r^*$, the weight error dynamics can be written as a function of the tracking error and the weight error as follows

$$\dot{\tilde{K}}(t) = -\Gamma_x(x(t)e^T(t)PB + \sum_{j=1}^{p} x_j x_j^T \tilde{K}(t)), \quad (18)$$

$$\dot{\tilde{K}}_r(t) = -\Gamma_r(r(t)e^T(t)PB + \sum_{j=1}^{p} r_j r_j \tilde{K}_r(t)). \quad (19)$$

B. Data Recording

Let (x_j, r_j) denote the j^{th} recorded data pair and assume that the data are contained in history stack matrices $X_k = [x_1, x_2, ..., x_p]$ and $R_k = [r_1, r_2, ..., r_p]$, where the subscript $k \in \mathbb{N}$ is incremented every time a history stack is updated. Due to memory considerations, it is assumed that a maximum of m data points can be stored. The idea is to update the history stack by adding data points to empty slots or by replacing an existing point if no empty slot is available to maximize the minimum singular value of X, $\sigma_{\min}(X)$. This is motivated by the fact that the convergence rate of the adaptive controller is directly proportional to $\sigma_{\min}(X)$, this relationship is further explored in proof of Theorem 1. This can be achieved using the singular value maximizing data recording algorithm described below.

Algorithm 1 Singular Value Maximizing Algorithm for Recording Data Points

Require: $p \geq 1$
 if $\frac{\|x(t)-x_p\|^2}{\|x(t)\|} \geq \epsilon$ **then**
  $p = p + 1$
  $X_k(:, p) = x(t)$; {store r_j, $\epsilon_{K_j}(t)$, $\epsilon_{K_{r_j}}(t)$}
 end if
 if $p \geq \bar{p}$ **then**
  $T = X_k$
  $S_{old} = \min SVD(X_k^T)$
  for $j = 1$ to p **do**
   $X_k(:, j) = x(t)$
   $S(j) = \min SVD(X_k^T)$
   $X_k = T$
  end for
  find $\max S$ and let k denote the corresponding column index
  if $\max S > S_{old}$ **then**
   $X_k(:, k) = x(t)$
   $p = p - 1$
  else
   $p = p - 1$
   $X_k = T$
  end if
 end if

C. Stability Analysis

The following theorem contains the main contribution of this paper. We assume that the controller starts with a pre-populated history stack that contains n linearly independent data points, where the recorded data are populated through open loop testing.

Theorem 1 Consider the system in (1), the control law of (3), and let $p \geq n$ be the number of recorded data points. Let $X_k = [x_1, x_2, ..., x_p]$ be the matrix containing recorded states, and $R_k = [r_1, r_2, ..., r_p]$ be the matrix containing recorded reference signals. Assume that initially at $t = 0$, $rank(X_0) = n$ and for at least one of the recorded r_j, $r_j \neq 0$, and the history stacks are consequently updated using Algorithm 1 then the following concurrent learning weight update laws guarantee that the zero $(e(t), \tilde{K}(t), \tilde{K}_r(t)) \equiv 0$ is globally exponentially stable:

Proof: Let $\mathrm{tr}(.)$ denote the trace operator, and consider the following positive definite and radially unbounded Lyapunov candidate

$$V(e, \tilde{K}, \tilde{K}_r) = \frac{1}{2}e^T Pe + \frac{1}{2}\mathrm{tr}(\tilde{K}^T \Gamma_x^{-1} \tilde{K}) + \frac{1}{2}\tilde{K}_r^T \Gamma_r^{-1} \tilde{K}_r. \quad (20)$$

Let $\xi = [e^T, \mathrm{vec}(\tilde{K}), \tilde{K}_r]^T$ then we can bound the Lyapunov candidate above and below with positive definite functions as follows

$$\frac{1}{2}\min(\lambda_{\min}(P), \lambda_{\min}({\Gamma_x}^{-1}), \Gamma_r^{-1})\|\xi\|^2 \leq V(e, \tilde{K}, \tilde{K}_r) \leq \frac{1}{2}\max(\lambda_{\max}(P), \lambda_{\max}({\Gamma_x}^{-1}), \Gamma_r^{-1})\|\xi\|^2. \tag{21}$$

Taking the time derivative of the Lyapunov candidate along the trajectories of system (8), (18), and (19), and using the Lyapunov equation (9) and (18) we have

$$\begin{aligned}\dot{V}(e(t), \tilde{K}(t), \tilde{K}_r(t)) = &-\frac{1}{2}e^T(t)Qe(t) + e^T(t)PB(\tilde{K}^T(t)x(t) + \tilde{K}_r^T(t)r(t)) \\ &- \mathrm{tr}(\tilde{K}^T(t)(x(t)e^T(t)PB + \sum_{j=1}^{p} x_j x_j^T \tilde{K}(t)) \\ &- \tilde{K}_r^T(t)(r(t)e^T(t)PB + \sum_{j=1}^{p} r_j r_j \tilde{K}_r(t)).\end{aligned} \tag{22}$$

Simplifying further and canceling like elements the time derivative of the Lyapunov candidate reduces to

$$\begin{aligned}\dot{V}(e(t), \tilde{K}(t), \tilde{K}_r(t)) = &-\frac{1}{2}e^T(t)Qe(t) - \mathrm{tr}(\tilde{K}^T(t)\sum_{j=1}^{p} x_j x_j^T \tilde{K}(t)) \\ &- \tilde{K}_r^T(t)\sum_{j=1}^{p} r_j r_j \tilde{K}_r(t).\end{aligned} \tag{23}$$

Let $\Omega_K = \sum_{j=1}^{p} x_j x_j^T$, then since $X = [x_1, x_2, ..., x_p]$ is linearly independent, we have $\Omega_K > 0$ (that is Ω_K is positive definite), similarly since there exists at least one non-zero r_j we have $\sum_{j=1}^{p} r_j r_j > 0$. Hence we have

$$\dot{V}(e, \tilde{K}, \tilde{K}_r) \leq -\frac{1}{2}\lambda_{\min}(Q)\|e\|^2 - \lambda_{\min}(\Omega_K)\|\tilde{K}\|^2 - \sum_{j=1}^{p} r_j^2\|\tilde{K}_r\|^2. \tag{24}$$

This can be reduced to

$$\dot{V}(e, \tilde{K}, \tilde{K}_r) \leq -\frac{\min\left(\lambda_{\min}(Q), 2\lambda_{\min}(\Omega_K), 2\sum_{j=1}^{p} r_j^2\right)}{\max\left(\lambda_{\max}(P), \lambda_{\max}(\Gamma_x^{-1}), \Gamma_r^{-1}\right)} V(e, \tilde{K}, \tilde{K}_r). \tag{25}$$

Note that Algorithm 1 guarantees that $\lambda_{\min}(\Omega_K)$ is monotonically increasing. Hence, (25) guarantees that the equation (20) is a common Lyapunov function, and establishes uniform exponential stability of the zero solution $e \equiv 0$ and $\tilde{K}(t) \equiv 0$, $\tilde{K}_r(t) \equiv 0$ [18]. Since $V(e, \tilde{K}, \tilde{K}_r)$ is radially unbounded, the result is global and x tracks x_{ref} exponentially and $K(t) \to K^*$, and $K_r(t) \to K_r^*$ exponentially fast as $t \to \infty$. ■

Remark 1 For the concurrent learning adaptive controller of Theorem 1, a sufficient condition for guaranteeing exponential convergence of tracking error $e(t)$ and weight estimation error $\tilde{K}(t)$ and $\tilde{K}_r(t)$ to zero is that the matrix $X = [x_1, x_2, ..., x_p]$ contain n linearly independent data points, and for at least one of the recorded r_j, $r_j \neq 0$. This condition, referred to as the rank condition, requires only that the states be exciting (see Definition 1) over the finite interval when the data was recorded. The condition however, is not equivalent to a condition on PE states (see Definition 2) which requires the states to be exciting over *all* finite intervals. Particularly, the rank condition is only concerned with online recorded data in the past, whereas, PE is concerned with how the states behave over all past and future intervals. Finally, note that while the rank of a matrix can be calculated online, Definition 2 does not yield easily to online verification of whether a signal is PE.

Remark 2 The minimum singular value of the history stack $\sigma_{\min}(X)$ is directly proportional to the rate of convergence, and can be used as an online performance metric that relates the richness of the recorded data to guaranteed exponential convergence.

Remark 3 If the rank condition is met, $\tilde{K}(t)$ and $\tilde{K}_r(t)$ converge to zero, and hence no additional modifications to the adaptive law, such as σ-mod or e-mod, or a projection operator than bounds the weights in a compact set are required to guarantee the boundedness of weights for concurrent learning adaptive controllers. We have assumed that the history-stack can be pre-populated with selected system states from a set of recorded data from either open-loop or closed-loop experiments, or even from simulation testing. This allows for a nice way of incorporating test data into the control design process. If however, a pre-recorded data is not available, the previously mentioned modifications can be used to guarantee the boundedness of weights until Algorithm 1 records data online such that $rank(X) = n$. The stability proof for this case follows by using arguments similar to that of Theorem 1 and is omitted due to space limitations.

Remark 4 A connection can be established between the concurrent learning adaptive control law of (16-17) and the classical σ-modification based adaptive laws (see for example [15]). To see this, note that since $\epsilon_{K_j}(t)$ can be shown to be equal to $\tilde{K}^T(t)x_j$, we can rewrite (16) as $\dot{K}(t) = -\Gamma_x(x(t)e^T(t)PB + \sum_{j=1}^{p} x_j x_j^T(K(t) - K^*)$. Note that the ideal weights K^* need not be known for implementation of (16-17). A σ-modification enabled adaptive law on the other hand is: $\dot{K}(t) = -\Gamma_x(x(t)e^T(t)PB + \sigma I(K(t) - \bar{K}))$, where $\sigma > 0$ denotes the σ modification gain, I denotes the identity matrix, and $\bar{K}$ is an a-*priori* selected desired value around which the law tries to bound the weights (usually selected as 0). On comparing the two adaptive laws, it can be seen that once X_k is full ranked, then replacing ϵI term in the

σ-modification adaptive law with the positive definite term $\sum_{j=1}^{p} x_j x_j^T$, and replacing $\bar{K}$ with *a-priori* estimates of the ideal weights, results in the concurrent learning law of (16). This point suggests a connection between the two approaches and suggests that the concurrent learning adaptive law contains inherently the robustness properties of σ-modification.

Remark 5 In the adaptive control law of Theorem 1 we assumed that B was known. Alternatively, the adaptive law can be reformulated to only require the knowledge of $sign(B)$ (see [2]) or $sign(K^*)$ (see [1]).

Theorem 1 guarantees that the tracking error and weight error dynamics are globally exponentially stable subject to a rank-condition. This property of concurrent learning adaptive controllers allows us to further characterize their response through transient performance bounds. The quantities $\|x - x_{rm}\|_{L_\infty}$ and $\|\tilde{K}\|_{L_\infty}$ have previously been used to characterize transient performance bounds of adaptive controllers [7]. We will follow a similar approach. For the ease of exposition, we assume without loss of generality that $e(0) = 0$, $K(0) = 0$, $K_r(0) = 0$, $\Gamma_x = \gamma_x I$, and $\Gamma_r = \gamma_r I$, where γ_x and γ_r are positive scalars, while I is the identity matrix of appropriate dimensions.

Corollary 2 If Theorem 1 holds, then the quantities $\|e(t)\|_{L_\infty}$, $\|\tilde{K}(t)\|_{L_\infty}$, and $\|\tilde{K}_r(t)\|_{L_\infty}$ are bounded from above by exponentially decreasing functions.

Proof: Assume for sake of exposition, $K(0) = 0$ and $K_r(0) = 0$, it follows from (20) that

$$
\begin{aligned}
&V(e(0), \tilde{K}(0), \tilde{K}_r(0)) = \frac{1}{2} e^T(0) P e(0) + \frac{1}{2} \mathrm{tr}({K^*}^T \Gamma_x^{-1} K^*) \\
&+ \frac{1}{2} {K_r^*}^T \Gamma_r^{-1} K_r^* \\
&\leq \frac{1}{2} \lambda_{\max}(P) \|e(0)\|_2^2 + \frac{1}{2} \lambda_{\max}(\Gamma_x^{-1}) \|K^*\|_2^2 \\
&+ \frac{1}{2} \lambda_{\max}(\Gamma_r^{-1}) \|K_r^*\|_2^2
\end{aligned}
\tag{26}
$$

Let $\bar{\epsilon} = \frac{\min\left(\lambda_{\min}(Q), 2\lambda_{\min}(\Omega_K), 2\sum_{j=1}^{p} r_j^2\right)}{\max\left(\lambda_{\max}(P), \lambda_{\max}(\Gamma_x^{-1}), \Gamma_r^{-1}\right)}$, then from (25) we have $\dot{V}(e, \tilde{K}, \tilde{K}_r) \leq -\bar{\epsilon} V(e, \tilde{K}, \tilde{K}_r)$. Therefore from Theorem 1 it follows that

$$
V(e(t), \tilde{K}(t), \tilde{K}_r(t)) \leq V(e(0), \tilde{K}(0), \tilde{K}_r(0)) e^{-\bar{\epsilon} t}. \tag{27}
$$

Let $\Psi(t) = V(e(0), \tilde{K}(0), \tilde{K}_r(0)) e^{-\bar{\epsilon} t}$, since $\frac{1}{2}\lambda_{\min}(P)\|e\|_\infty^2 \leq \frac{1}{2}\lambda_{\min}(P)\|e\|_2^2 \leq \frac{1}{2} e^T P e \leq \Psi$ it follows that

$$
\begin{aligned}
\|e\|_{L_\infty} \leq &\Big(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \|e(0)\|_2^2 + \frac{\lambda_{\max}(\Gamma_x^{-1})}{\lambda_{\min}(P)} \|K^*\|_2^2 + \\
&\frac{\lambda_{\max}(\Gamma_r^{-1})}{\lambda_{\min}(P)} \|K_r^*\|_2^2\Big)^{\frac{1}{2}} e^{\frac{\bar{\epsilon} t}{2}}
\end{aligned}
\tag{28}
$$

If $e(0) = 0$, $\Gamma_x = \gamma_x I$, and $\Gamma_r = \gamma_r I$, the above equation reduces to

$$
\|e\|_{L_\infty} \leq \sqrt{\frac{\|K^*\|_2^2}{\gamma_x \lambda_{\min}(P)} + \frac{\|K_r^*\|_2^2}{\gamma_r \lambda_{\min}(P)}}\; e^{\frac{\bar{\epsilon} t}{2}} \tag{29}
$$

Similarly, $\frac{1}{2}\lambda_{\min}(\Gamma_x^{-1})\|\tilde{K}\|_\infty^2 \leq \Psi$ and $\frac{1}{2}\lambda_{\min}(\Gamma_r^{-1})\|\tilde{K}_r\|_\infty^2 \leq \Psi$. Therefore, it follows that

$$
\|\tilde{K}\|_{L_\infty} \leq \sqrt{\|K^*\|_2^2 + \frac{\gamma_x}{\gamma_r}\|K_r^*\|_2^2}\; e^{\frac{\bar{\epsilon} t}{2}}, \tag{30}
$$

and

$$
\|\tilde{K}\|_{L_\infty} \leq \sqrt{\frac{\gamma_r}{\gamma_x}\|K^*\|_2^2 + \|K_r^*\|_2^2}\; e^{\frac{\bar{\epsilon} t}{2}}. \tag{31}
$$

■

Remark 6 Equation (29) brings an interesting property of the presented controller to light. It shows that the tracking error is bounded above by an exponentially decaying function, hence, even without using high adaptation gains (γ_x and γ_r), concurrent learning can guarantee good tracking performance, unlike other recently proposed controllers (see [7], [19]), that rely on high adaptation weights to guarantee good tracking.

The analysis can be further extended to guarantee the boundedness of the control signal, particularly, let $\bar{x}_{rm}$ be a known upper bound for the bounded input bounded output reference model, then it follows that

$$
\|x\|_{L_\infty} \leq \|e\|_{L_\infty} + \bar{x}_{rm}, \tag{32}
$$

similarly,

$$
\|K\|_{L_\infty} \leq \|\tilde{K}\|_{L_\infty} + \|K^*\|_\infty, \tag{33}
$$

$$
\|K_r\|_{L_\infty} \leq \|\tilde{K}_r\|_{L_\infty} + \|K_r^*\|_\infty. \tag{34}
$$

Since $u = K^T x + K_r^T r$, we have

$$
\|u\|_{L_\infty} \leq \|K\|_{L_\infty}\|x\|_{L_\infty} + \|K_r\|_\infty \|r\|_{L_\infty}. \tag{35}
$$

Due to (29), (30), and (31), as $t \to \infty$ we have,

$$
\|u\|_{L_\infty} \leq \|K^*\|_{L_\infty} \bar{x}_{rm} + \|K_r^*\|_\infty \|r\|_{L_\infty}. \tag{36}
$$

Letting $\bar{x}_{rm} = \alpha\|r\|_{L_\infty}$, where α is a positive constant, we obtain an upper bound on the control input.

$$
\|u\|_{L_\infty} \alpha \leq (\|K^*\|_{L_\infty} + \|K_r^*\|_\infty)\|r\|_{L_\infty}. \tag{37}
$$

This result is of particular significance, since, by estimating conservative bounds on K^* and K_r^*, it allows the design of a reference model to ensure that the control signal does not exceed a predefined value. This fact can be useful, since most physical actuators do tend to saturate.

IV. SIMULATION RESULTS

In this section we present simulation results of control of a simple linear system with concurrent learning adaptive law. The focus is on understanding the concurrent learning approach, hence we use an exemplary second order system:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 4 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{38}$$

It is assumed that the state matrix (matrix A in (1)) is unknown for the purpose of control design. Note that the system is unstable as the eigenvalues of the A matrix are 1.9025, and -2.1025. The control objective is to make the system behave like the following second order reference model with natural frequency of $3.9\ rad/sec$ and damping ratio of 0.7,

$$\dot{x}_{rm} = \begin{bmatrix} 0 & 1 \\ -15.21 & -5.46 \end{bmatrix} x_{rm}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t). \tag{39}$$

The simulation runs for 9 seconds with a time-step of 0.001 seconds. The reference signal $r(t)$ is: $r(t) = 10$ between 0 to 2 seconds, $r(t) = -10$ between 2 to 3 seconds, and $r(t) = 10$ thereafter. Control law of (3) is used along with the adaptation law of Theorem 1, with $\Gamma_x = 10$ and $\Gamma_r = 1$. A pre-recorded history-stack of 10 data points was recorded during a previous simulation run with only the baseline adaptive law of (10) and with $r(t) = 5$ for the entire duration of that simulation run. The history-stack is then updated online using Algorithm 1. For the history-stack used, $\sigma_{\min}(\Omega_0) \approx 1.1$, at $t = 0$. The initial values of K and K_r were set to zero.

Figure 1 compares the tracking performance of the adaptive controllers with and without concurrent learning. It can be seen that system states are almost indistinguishable from the reference model when using concurrent learning adaptive controller, particularly when tracking the reference command between 2 and 3 seconds. Whereas, for the classical adaptive law which uses only instantaneous data, the tracking is not as precise. The reason for this becomes clear when we examine the evolution of weights in Figure 2. With concurrent learning, the weights rapidly approach their ideal values, and have converged within 1 second, particularly note that K_r converges within a single time step. Whereas for the classical adaptive law, the weights do not converge to their ideal values. Figure 3 compares the tracking error with and without concurrent learning, we observe that the tracking error rapidly approaches and remains at zero with concurrent learning, in contrast with the tracking error for the classical adaptive law. Particularly, in contrast with the classical adaptive law, for the step at 2 seconds, no visible transient can be seen with concurrent learning, this is in agreement with Corollary 2, which guarantees exponentially decaying bounds on the tracking error. For the presented results, $\bar{\epsilon} = 0.0455$ and $\sqrt{\bar{\gamma}} = 20.166$. Hence, $\|\xi\|_{L_\infty} \leq 20.166$, and $\|\xi(t)\|^2 \leq 20.166^2 e^{-0.046t}$ which turns out to be a conservative bound. Figure 4 compares the control effort required by the two adaptive controllers. We observe that while the peak magnitudes of the required control effort remain comparable, the control effort with concurrent learning does not oscillate as much as that without concurrent learning. This is a result of the weights arriving their ideal values, and indicates that the control effort with concurrent learning requires less energy and is in that sense optimal. Note that increasing the learning rates Γ_x, and Γ_r only make the control effort without concurrent learning more oscillatory and could lead to bursting [5]. Furthermore, note that the reference signal is not persistently exciting. Hence, the simulation results show that if the rank-condition is satisfied, concurrent learning adaptive controllers guarantee exponential stability of the zero solution of tracking error and weight error dynamics without requiring persistency of excitation. Finally, Figure 5 shows that Algorithm 1 ensures that $\sigma_{\min}(X_k)$ increases monotonically.

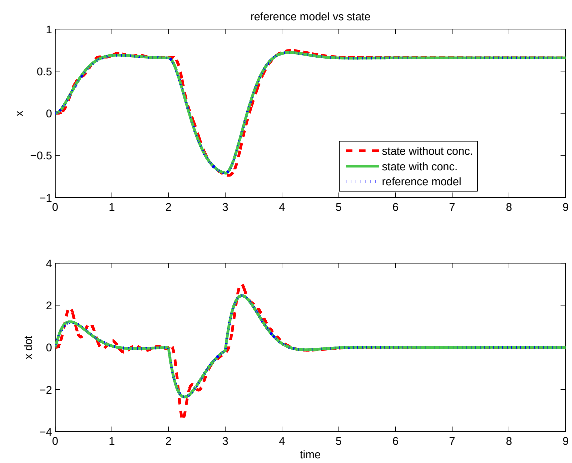

Fig. 1. Comparison of tracking performance of adaptive controllers with and without concurrent learning. Note that the system states and the reference model states are almost indistinguishable with concurrent learning, whereas without concurrent learning this is not the case.

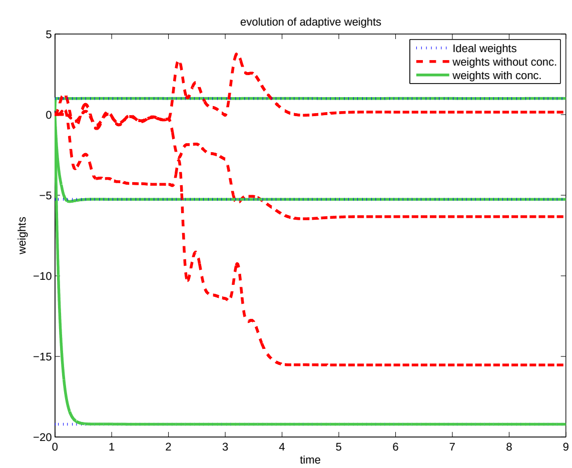

Fig. 2. Comparison of evolution of adaptive weights with and without concurrent learning. Note that the weights converge rapidly to their true values when using concurrent learning, whereas, for the classical adaptive law the weights do not converge to their ideal values.

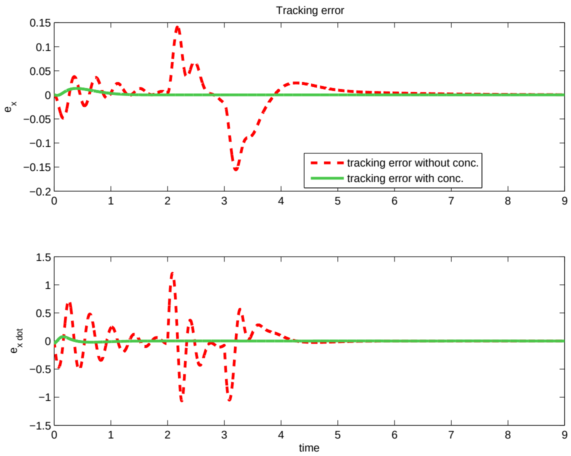

Fig. 3. Comparison of tracking error with and without concurrent learning. Note that no visible transient is observed for the step at 2 seconds with concurrent learning, in contrast with the transient observed without concurrent learning.

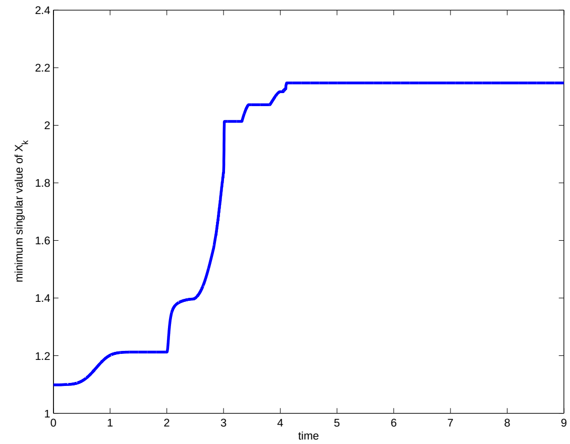

Fig. 5. Evolution of the minimum singular value of the history stack matrix X_k, note that Algorithm 1 guarantees that $\sigma_{\min}(X_k)$ increases monotonically

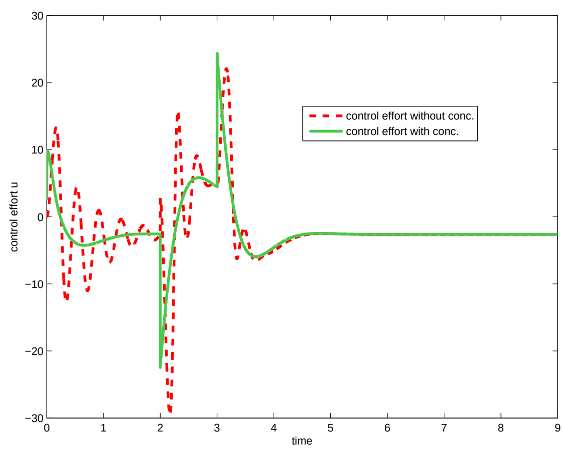

Fig. 4. Comparison of control effort with and without concurrent learning. Note that while the peak magnitude of the control effort remains comparable, the control effort with concurrent learning does not oscillate as much as the control effort without concurrent learning. This indicates that the control effort with concurrent learning requires less energy, an effect arising due to the convergence of adaptive weights to their ideal values.

V. CONCLUSION

We developed concurrent learning adaptive controller for uncertain linear multivariable dynamical systems. The developed concurrent learning adaptive controller uses instantaneous data concurrently with recorded data for adaptation and can guarantee global exponentially stability of the zero solution of the tracking error and weight error dynamics subject to a sufficient verifiable condition on linear independence of the recorded data. This indicates that the tracking error and weight errors exponentially approach zero and their infinity norms are exponentially bounded. This fact was used to formulate exponentially decaying upper bounds on tracking and weight error without relying on high adaptation gain. Furthermore, the exponential convergence also indicates that system dynamics will exponentially approach the reference model dynamics, thereby recovering the desired transient response and stability characteristics of the reference model. We demonstrated the effectiveness of the presented concurrent learning approach through an exemplary simulation study. These results have significant implications in verification and validation of adaptive controllers and in designing adaptive controllers that avoid saturation.

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