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RESEARCH ARTICLE

Exponential Parameter and Tracking Error Convergence Guarantees for Adaptive Controllers without Persistency of Excitation

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In Model Reference Adaptive Control (MRAC) the modeling uncertainty is often assumed to be parameterized with time-invariant unknown ideal parameters. The convergence of parameters of the adaptive element to these ideal parameters is beneficial, as it guarantees exponential stability, and makes an online learned model of the system available. Most MRAC methods however, require Persistent Excitation (PE) of the states to guarantee that the adaptive parameters converge to the ideal values. Enforcing PE may be resource intensive and often infeasible in practice. This paper presents theoretical analysis and illustrative examples of an adaptive control method that leverages the increasing ability to record and process data online by using specifically selected and online recorded data concurrently with instantaneous data for adaptation. It is shown that when the system uncertainty can be modeled as a combination of known non-linear bases, simultaneous exponential tracking and parameter error convergence can be guaranteed if the system states are exciting over finite intervals such that rich data can be recorded online; PE is not required. Furthermore, the rate of convergence is directly proportional to the minimum singular value of the matrix containing online recorded data. Consequently, an online algorithm to record and forget data is presented and its effects on the resulting switched closed loop dynamics are analyzed. It is also shown that when radial basis function Neural Networks (NNs) are used as adaptive elements, the method guarantees exponential convergence of the NN parameters to a compact neighborhood of their ideal values without requiring PE. Flight-test results on a fixed wing unmanned aerial vehicle demonstrate the effectiveness of the method.

Keywords: Model Reference Adaptive Control; persistence of excitation; long-term learning; neuroadaptive control; robust adaptive control; unmanned aerial vehicles

1 INTRODUCTION

Adaptive controllers in the widely studied class of Model Reference Adaptive Control (MRAC) algorithms are aimed at making a nonlinear uncertain dynamical system behave like a chosen reference model, see e.g. (Narendra and Annaswamy 1989, Aström and Wittenmark 1995, Ioannou and Sun 1996, Tao 2003, Johnson and Kannan 2005, Cao and Hovakimyan 2008, Volyanskyy et al. 2009, Yucelen and Calise 2010, Chowdhary and Johnson 2011). In MRAC, the system uncertainty is approximated using a weighted combination of basis functions, with the numerical values of the parameters adapted online to guarantee closed loop stability. The underlying assumption is that an ideal set of parameters exists which best captures the uncertainty. Although boundedness of MRAC based methods has been repeatedly established, boundedness does not always guarantee a convergence rate and that the adaptive parameters will approach the ideal parameters over the long term during a normal course of operation. It is useful to drive the parameters closer towards their ideal values, as the resulting representation forms a good approximation of the uncertainty, which results in improved performance, and can be used for planning and health-monitoring purposes. In particular, for MRAC formulations where the

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basis of the uncertainty is known, and the uncertainty function is time-invariant, classical and modern MRAC methods (including (Cao and Hovakimyan 2008, Volyanskyy et al. 2009, Yucelen and Calise 2010)) require a condition on Persistency of Excitation (PE) in the system states to guarantee exponential stability of the tracking error and parameter error dynamics (Boyd and Sastry 1986).

Conceptually, the requirement on PE originates from the fact that most adaptive control methods use only instantaneous data for adaptation and control. Specifically, they minimize a quadratic cost on the instantaneous tracking error (e.g. $J(t) = e^T(t)e(t)$). The resulting gradient based adaptive laws are at most rank-1, and therefore, the system states have to continually span the spectrum of the parameter space to guarantee that the parameter error converges to zero (Boyd and Sastry 1986). The requirement on PE is also commonplace in other control and estimation problems, including in adaptive backstepping (Krstić et al. 1995), and adaptive model predictive control, where PE is required to guarantee correct estimation of the system model used for prediction (Adetola et al. 2009). However, this condition is restrictive and often difficult to implement. For example, in adaptive flight control applications PE reference inputs would waste energy, and potentially cause undue stress on the aircraft.

It is well known that without PE inputs, parameters may diverge in the presence of noise (see e.g. (Aström and Wittenmark 1995, Ioannou and Sun 1996, Narendra and Annaswamy 1986)). To counter this, most MRAC methods rely on various modifications to guarantee ultimate boundedness of tracking error and parameters without PE, including the classic σ-modification of (Ioannou and Sun 1996) and the e-modification of (Narendra and Annaswamy 1987). Other approaches include the projection and dead-zone based approaches (see e.g. (Tao 2003, Cao and Hovakimyan 2008)) which guarantee that the adaptive parameters stay bounded in a compact set even when the system states are not PE.

The claim in general has been that if the parameters stay bounded then an application of Barbalat's lemma results in asymptotic tracking error convergence. In fact, it has been recently argued that instantaneous domination of the uncertainty, sometimes using high adaptation gain, is sufficient to guarantee asymptotic tracking, and guaranteed parameter convergence is not required in MRAC schemes (Cao and Hovakimyan 2008, Volyanskyy et al. 2009, Yucelen and Calise 2010). However, this purely direct view of MRAC does not exploit the potential of the adaptive element to learn the uncertainty over longer term. Alternative views that bring direct and indirect approaches together in a composite-MRAC framework have long claimed that learning the uncertainty improves performance (see e.g. (Duarte and Narendra 1989, Lavretsky 2009)). However, these approaches also require a condition on PE of the system states to guarantee convergence.

The concurrent learning adaptive control method presented in this paper is motivated by the increasing computational power available on modern onboard computers. It uses online recorded data concurrently with instantaneous data to guarantee exponential convergence of both parameter and tracking error for nonlinear uncertain dynamical systems with matched, nonlinear, time-invariant uncertainty that is linearly parameterizable. The main benefit of concurrent learning MRAC is that PE or high adaptation gains are not required. Particularly, it is shown that if the reference input is exciting over a finite interval such that sufficiently rich data can be recorded *online* in a dynamic *history stack*, then exponential closed loop stability is guaranteed. The rate of convergence is shown to be related to the *quality* of online recorded data, and online verifiable metrics are presented to record and forget data under memory constraints. In this sense, the question of which data is useful to record (and safe to forget) online under memory constraints is answered, allowing the method to go beyond that of simply using data buffer based least squares to estimate the uncertainty. Robustness to switched time-variation in the parameters is established. The method requires knowledge of the first derivative of the state for a recorded data point a finite time interval after it is recorded, which can be obtained through fixed point smoothing. Note that knowledge of the first derivative of the state at the *current* time instant is not required. It is also shown that if the basis functions are not known *a-priori*, and a Radial Basis Function (RBF) Neural Network (NN) is used as the adaptive element, then the method guarantees that the tracking error and parameter error are exponentially 2^{th} ultimately bounded. This indicates that the NN parameters approach exponentially fast and remain within a compact set around the ideal parameters. This is a stronger result than that of previously studied neuroadaptive control methods, which only guarantee that the parameters are bounded around an *a-priori* selected value (often chosen as 0) (see e.g. (Kim and Lewis 1998, Nardi 2000, Sundararajan et al. 2002)).

With the approach presented in this paper, a wide class of adaptive systems would not have to rely on PE or high adaptation gains to guarantee exponential stability. The exponential tracking error and parameter error convergence afforded by this method can lead directly into achieving online verifiable stability and performance guarantees for adaptive controllers (in terms of the quality of the data in the history stack). (Sastry and Bodson 1989), and (Cao et al. 2007) have stated that exponential parameter error convergence results in exponential tracking and in implicit transient response bounds. (Anderson 1977), (Bodson and Sastry 1984), (Boyd and Sastry 1986), (Nicolao et al. 2000), (Khasminskii 2012) have reported that exponential stability guarantees are useful for robustness of controllers with adaptation. (Cao et al. 2007) have also stated that parameter convergence is needed for a class of adaptive control problems where the reference input is dependent on the unknown parameters. It is expected therefore that this method will be beneficial to a number of previously studied MRAC applications, including control of robotics arms (Kim and Lewis 1998), flight vehicle control (Johnson and Kannan 2005), and control of medical processes (Haddad et al. 2011).

The convergence of parameters to their ideal values afforded by this method is beneficial particularly for Unmanned Aerial Vehicle (UAV) applications. Along with better trajectory tracking, the online learned uncertainty makes available a model of the aircraft that can be used for path planning and health-monitoring purposes. Therefore, the method is implemented and validated through flight-tests on a fixed-wing UAV, and its performance is compared with existing MRAC implementation (see e.g. (Johnson and Kannan 2005)) with e-modification. An early version of this work appeared in conference papers (Chowdhary and Johnson 2010) and (Chowdhary and Johnson 2011) new contributions here include the history stack management algorithm and its effect on stability and transient performance, relating the quality of recorded data to convergence rate, extension to neuroadaptive control, analysis of robustness to parameter variations and flight-test results on a fixed wing UAV. Finally, this paper meaningfully extends our previous work in this direction. Particularly, in (Chowdhary et al. 2012) concurrent learning adaptive control of linear systems was studied, as opposed to nonlinear systems considered here. In (Chowdhary and Johnson 2011) ultimate boundedness results of concurrent learning controllers for general class of nonlinear uncertainties with bounded domain were presented (as opposed to stronger results for linearly parameterized uncertainty here) along with flight test results on a rotorcraft UAV.

The outline of the paper is as follows. Section 2 states some preliminaries. Section 3 formulates the problems. In Section 4 it is shown that a concurrent learning adaptive controller guarantees exponential stability with only finite excitation for linearly parameterized uncertainties with known bases. The exponential boundedness of concurrent learning neuroadaptive controllers for uncertainties with unknown bases is presented in Section 5. Secton 6 compares concurrent learning to a least squares based approach. Robustness to ideal parameter variation due to faults etc. is established in Section 7. Section 8 connects concurrent learning to σ-modification. Results from numerical simulation are presented in Section 9. Flight test results on a fixed-wing UAV are presented in Section 10. The paper is concluded in Section 11.

2 Preliminaries

In this section some definitions, lemmas, and other mathematical preliminaries required for latter development are introduced. In this paper, $f(t)$ represents a function of time t, we often drop the argument t consistently over an entire equation for ease of exposition. The operator $\|.\|$ denotes the Euclidian norm of vectors and matrices unless otherwise stated, tr(.) denotes the trace operator, $(\vec{.})$ denotes the operator that stacks the columns of a matrix into a vector, SVD denotes the operator that returns the singular values of a matrix, and $\lambda_{\min}(.)$ and $\lambda_{\max}(.)$ denote operators that return the smallest and the largest eigenvalue of a matrix, while $\sigma_{\min}$ and $\sigma_{\max}$ denote the operators that return the smallest and the largest singular value of a matrix.

The following Lemma relates the minimum singular value of a matrix to the sum of its columns. This lemma is required in the proof of Theorems 4.1 and 5.1.

Lemma 2.1: *Let $Z_k \in \mathbb{R}^{m \times p_H}$ be given by $Z_k = [\Phi_1, ..., \Phi_{p_H}]$ where $\Phi_j \in \mathbb{R}^m$ are a set of vectors. Then $\sigma_{\min}(Z_k) = (\sigma_{\min}(\sum_{j=1}^{p_H} \Phi_j \Phi_j^T))^{1/2}$.*

Proof We have that $\sigma_{\min}(Z_k) = (\lambda_{\min}(Z_k Z_k^T))^{1/2}$. The proof now follows by noting that $\sum_{j=1}^{p_H} \Phi_j \Phi_j^T = Z_k Z_k^T$ and $\lambda_{\min}(Z_k Z_k^T) = \sigma_{\min}(Z_k Z_k^T)$ for non complex eigenvalues. □

The following definitions are used in the results presented later. Let $\dot{x}(t) = f(x(t))$ be a nonlinear dynamical system which is globally Lipschitz continuous with an equilibrium at the origin. Exponential stability of the origin of this dynamical system is defined as follows (see (Haddad and Chellaboina 2008)).

Definition 2.2: The origin is globally exponentially stable if for some positive scalars α, ϵ, there exists a time T such that $\|x(t)\| \leq \alpha \|x(0)\| e^{-\epsilon t}$, for $t \geq T$ and for all $x(0)$. If the origin is exponentially stable for all $T \in [0, \infty]$ the origin is said to be globally uniformly exponentially stable.

Exponential ultimate boundedness of a solution to the dynamical system is now defined.

Definition 2.3: Let $x(t)$ be a solution to the nonlinear system $\dot{x}(t) = f(x(t))$ with $x(0) = x_0$, then $x(t)$ is said to be exponentially p^{th} ultimately bounded if $\|x(t)\|^p \leq \alpha \|x(0)\|^p e^{-\epsilon t} + k$ for some positive constants α, ϵ, k.

Various equivalent definitions of excitation and the persistence of excitation of a bounded vector signal exist (see e.g. (Aström and Wittenmark 1995, Narendra and Annaswamy 1989)), here we use Tao's definition (Tao 2003):

Definition 2.4: **Exciting Signal**: A bounded vector signal $\Phi(t)$ is *exciting* over a time sequence set $\{t_0, t_0 + 1, ..., t_0 + \delta\}$, $\delta > 0$ if there exists $\gamma > 0$ such that

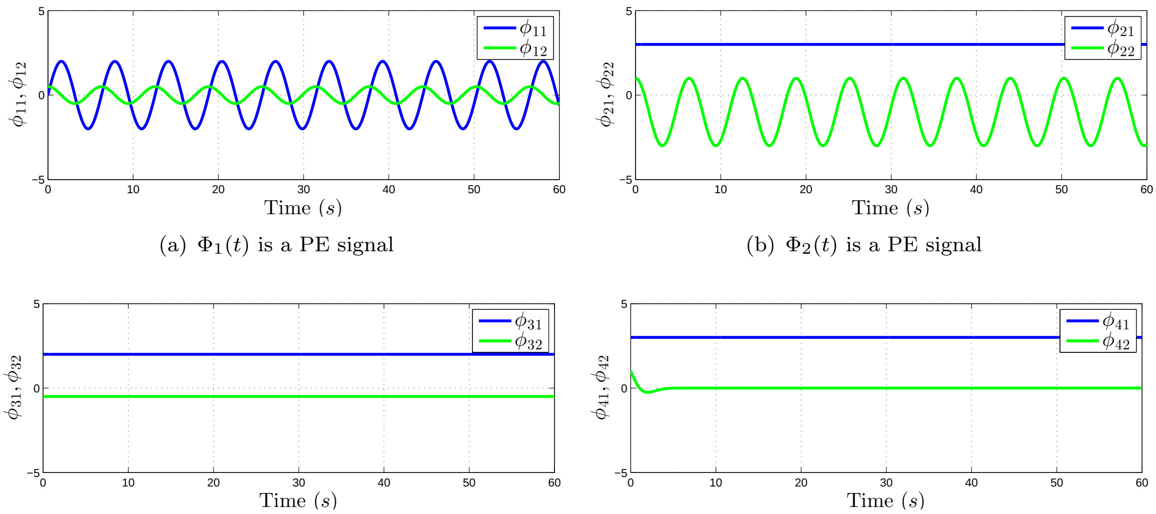
$$\sum_{\tau=t_0}^{t_0+\delta} \Phi(\tau)\Phi^T(\tau) \geq \gamma I. \tag{1}$$

Definition 2.5: **Persistently Exciting Signal**: A bounded vector signal $\Phi(t)$ is *persistently exciting* if there exists a $\delta > 0$ and $\gamma > 0$ such that

$$\sum_{\tau=t}^{t+\delta} \Phi(\tau)\Phi^T(\tau) \geq \gamma I, \quad \forall\, t \geq t_0. \tag{2}$$

The above definitions are for discretely sampled signals. Their continuous analogue is formed by replacing the summations by integrals. Particularly, a continuous signal $\Phi(t)$ is said to be PE if for for all $t \geq t_0$ there exists $T > 0$ and $\gamma > 0$ such that $\int_t^{t+T} \Phi(\tau)\Phi^T(\tau)d\tau \geq \gamma I$. Note that the above definition requires that the matrix $\int_t^{t+T} \Phi(\tau)\Phi^T(\tau)d\tau \in \mathbb{R}^{m \times m}$ is positive definite over any finite interval.

For example, in the two dimensional case where $\Phi_i(t) = [\phi_{i1},\ \phi_{i2}]$, the vector signals $\Phi_1(t) = [2\sin(t),\ 0.5\cos(t)]$ (Figure 1(a)) and $\Phi_2(t) = [3,\ 2(-0.5+\cos(t))]$ (Figure 1(b)) are PE. The vector signal $\Phi_3(t) = [2,\ -0.5]$ (Figure 1(c)) is not exciting over any finite interval, whereas the vector signal $\Phi_4(t) = [3,\ 2e^{-t}(-0.5+\cos(t))]$ (Figure 1(d)) is exciting over a finite interval, but not PE.



(a) $\Phi_1(t)$ is a PE signal

(b) $\Phi_2(t)$ is a PE signal

(c) $\Phi_3(t)$ is not PE

(d) $\Phi_4(t)$ is exciting over an interval but not PE

Figure 1. Two dimensional persistently exciting, not exciting, and exciting over finite interval signals

3 Approximate Model Inversion based Model Reference Adaptive Control

In this section we formulate Approximate Model Inversion based Model Reference Adaptive Control (AMI-MRAC) with full state feedback and multiple inputs. AMI-MRAC is an approximate feedback-linearization based MRAC method that allows the design of adaptive controllers for a general class of nonlinear plants (see e.g. (Johnson and Kannan 2005, Calise et al. 2001)). The concurrent learning MRAC approach is introduced in the framework of AMI-MRAC, although it should be noted that it is applicable to other MRAC architectures e.g. (Narendra and Annaswamy 1989, Aström and Wittenmark 1995, Ioannou and Sun 1996, Tao 2003, Yucelen and Calise 2010).

Let $x(t) = [x_1^T(t), x_2^T(t)] \in D_x \subset \mathbb{R}^n$ be the state vector, with $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$, and let D_x be a compact subset of $\mathbb{R}^n$. Let $\delta(t) \in \mathbb{R}^l$, $l = n_2$, denote the control input, and consider the following multiple-input control-affine nonlinear dynamical system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= f(x(t)) + G(x(t))\delta(t),\end{aligned} \tag{3}$$

where the fully or partially unknown functions f, $f(0) = 0$, and G are assumed to be Lipschitz over D_x and $G(x)$ is assumed to be nonsingular for all $x \in D_x$. The control input δ is assumed to be bounded and piecewise continuous. Note, that the afore defined class of systems can be extended to include non-affine systems of the form, where the input is described by an invertible function (Slotine and Li 1991). For the purpose of this paper it is assumed that $n_2 = l$. This assumption can be restrictive for over-actuated systems. In this case, it may be relaxed by using matrix inverse and pseudo-inverse approaches, constrained control allocation, pseudo-controls, or daisy chaining (see e.g. (Durham 1993, Wise 1995, Yucelen et al. 2011)), to reduce the dimension of the control input vector. It is further assumed that $\delta(t)$ is limited to the class of admissible control inputs and $x(t)$ is available for full state feedback.

In AMI-MRAC a pseudo-control input (desired acceleration) is designed $\nu(t) \in \mathbb{R}^{n_2}$ that can be used to find the control input δ such that the system states track the output of a reference model. If the exact system model $(f(x), G(x))$ in (3) was available for a given $\nu(t)$, $\delta(t)$ could have been found by inverting the system dynamics. However, since the exact system model is often not available, we let $\delta(t)$ be the output of an approximate inversion model

$$\delta(t) = \hat{G}^{-1}(x)(\nu(t) - \hat{f}(x)), \tag{4}$$

where $\hat{G}(x)$ is chosen to be nonsingular for all $x \in D_x$. Let $z = [x, \delta]^T$. For the general system in (3), the use of an approximate inversion model results in

$$\dot{x}_2 = \nu + \Delta(z), \tag{5}$$

where $\Delta(z)$ is the modeling error. The modeling error captures the difference between the system dynamics and the approximate inversion model:

$$\Delta(z) = f(x) - \hat{f}(x) + (G(x) - \hat{G}(x))\delta(t). \tag{6}$$

Note that if the control assignment function $G(x)$ were known and invertible with respect to δ, then an inversion model can be chosen such that the modeling error in (6) is only a function of the state x. A reference model can be designed that characterizes the desired response of the system

$$\begin{aligned} \dot{x}_{1_{rm}} &= x_{2_{rm}}, \\ \dot{x}_{2_{rm}} &= f_{rm}(x_{rm}, r), \end{aligned} \tag{7}$$

where $f_{rm}(x_{rm}(t), r(t))$ denote the reference model dynamics which are assumed to be continuously differentiable in x_{rm} for all $x_{rm} \in D_x \subset \mathbb{R}^n$. The exogenous reference command $r(t)$ is assumed to be bounded and piecewise continuous, furthermore, f_{rm} is assumed to be such that x_{rm} is bounded for a bounded reference input.

Let the tracking error be given by $e(t) = x_{rm}(t) - x(t)$. The pseudo-control input ν consisting of a linear feedback part $\nu_{pd} = [K_1, K_2]e$ with $K_1 \in \mathbb{R}^{n_2 \times n_2}$ and $K_2 \in \mathbb{R}^{n_2 \times n_2}$, a linear feedforward part $\nu_{rm} = \dot{x}_{2_{rm}}$, and an adaptive part $\nu_{ad}(z)$ is chosen to have the form

$$\nu = \nu_{rm} + \nu_{pd} - \nu_{ad}. \tag{8}$$

Since $\Delta(z)$ is a function of ν_{ad} as per (6), and ν_{ad} needs to be designed to cancel Δ, the following assumption needs to be satisfied:

Assumption 3.1 The existence and uniqueness of a fixed-point solution ν_{ad} is assumed for $\Delta(\cdot, \nu_{ad})$ such that $\nu_{ad} = \Delta(\cdot, \nu_{ad})$.

Sufficient conditions for satisfying this assumption are available in (Kim 2003). It should be noted that assumption 3.1 implicitly requires the sign of the control effectiveness to be known. Furthermore, (Kim 2003) presents a way to reformulate the uncertainty such that $\Delta(\cdot, \nu_{ad})$ not longer is an explicit function of ν_{ad}.

Using (5) and (3) the tracking error dynamics can be written as

$$\dot{e} = \dot{x}_{rm} - \begin{bmatrix} x_2 \\ \nu + \Delta \end{bmatrix}. \tag{9}$$

Let $A = \begin{bmatrix} 0 & I_1 \\ -K_1 & -K_2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ I_2 \end{bmatrix}$ where $0 \in \mathbb{R}^{n_2 \times n_2}$, $I_1 \in \mathbb{R}^{n_2 \times n_2}$, and $I_2 \in \mathbb{R}^{n_2 \times n_2}$ are the zero and identity matrices. Substituting $\dot{x}_{rm}$ on the right hand side of equation (7) and using (8) we obtain the following tracking error dynamics that are linear in e (Johnson and Kannan 2005)

$$\dot{e} = Ae + B[\nu_{ad}(z) - \Delta(z)]. \tag{10}$$

The baseline full state feedback controller ν_{pd} is chosen such that A is a Hurwitz matrix. Hence for any positive definite matrix $Q \in \mathbb{R}^{n \times n}$, a positive definite solution $P \in \mathbb{R}^{n \times n}$ exists to the Lyapunov equation

$$0 = A^T P + PA + Q. \tag{11}$$

The uncertainty $\Delta(z)$ can be characterized using the following two cases.

3.1 *Case I: Structured Uncertainty*

Consider the case where it is known that the uncertainty is linearly parameterized and its structure is known. This case is captured through the following assumption.

Assumption 3.2 The uncertainty $\Delta(z)$ can be linearly parameterized, that is there exists a unique matrix of constants $W^* \in \mathbb{R}^{m \times n_2}$ and an m dimensional vector of continuously differentiable regressor functions $\Phi(z) = [\phi_1(z), \phi_2(z), ..., \phi_m(z)]$ such that there exists a positive interval $[t, t+\Delta t]$, $\Delta t \in \mathbb{R}$ over which the integral $\int_t^{t+\Delta t} \Phi(x(t))\Phi^T(x(t))dt$ can be made positive definite for bounded $\Phi(x(t))$, and $\Delta(z)$ can be uniquely represented as

$$\Delta(z) = W^{*T}\Phi(z). \tag{12}$$

A large class of nonlinear uncertainties can be expressed in the above form (see e.g. the nonlinear wing-rock dynamics model (Singh et al. 1995, Monahemi and Krstic 1996)). Note that the requirement on unique W^* for a given basis of the uncertainty $\Phi(x(t))$ ensures that the representation of (12) is minimal, that is functions such as $\Delta(x) = w_1^* \sin(x(t)) + w_2^* \cos(x) + w_3^* \sin(x)$ are represented as $\Delta(x) = [w_1^* + w_3^*, w_2^*]^T[\sin(x), \cos(x)]$. In this case, by letting $W \in \mathbb{R}^{m \times n_2}$ denote the estimate of W^*, the output of the adaptive element can be written as

$$\nu_{ad}(x) = W^T\Phi(z). \tag{13}$$

For this case it is well known that the baseline adaptive law (Narendra and Annaswamy 1989, Tao 2003, Aström and Wittenmark 1995)

$$\dot{W}(t) = -\Gamma_W \Phi(z(t))e^T(t)PB, \tag{14}$$

where Γ_W is a positive definite learning rate matrix, results in $e(t) \to 0$; however (14) does not guarantee the convergence (or even the boundedness in presence of noise) of $W(t)$ (Narendra and Annaswamy 1989, Aström and Wittenmark 1995, Ioannou and Sun 1996, Tao 2003), neither does it guarantee a rate of convergence for $e(t)$ under general operating conditions. Particularly, $e \to 0$ and $W \to W^*$ as $t \to \infty$ exponentially fast if and only if $\Phi(z(t))$ is Persistently Exciting (Boyd and Sastry 1986, Narendra and Annaswamy 1989, Ioannou and Sun 1996, Tao 2003).

3.2 *Case II: Unstructured Uncertainty*

In the more general case where it is only known that the uncertainty $\Delta(z)$ is continuous and defined over a compact domain $D \subset \mathbb{R}^{n+l}$, the adaptive part of the control law (8) is represented using a Radial Basis Function (RBF) Neural Network (NN), the output of which is given by

$$\nu_{ad}(z) = W^T \sigma(z), \tag{15}$$

where $W \in \mathbb{R}^{q \times n_2}$ and $\sigma(z) = [1, \sigma_2(z), \sigma_3(z), ..., \sigma_q(z)]^T$ is a q dimensional vector of chosen radial basis functions. For $i = 2, 3..., q$ let c_i denote the RBF centroid and μ_i denote the RBF widths then for each i the radial basis functions are given as

$$\sigma_i(z) = e^{-\|z-c_i\|^2/\mu_i}. \tag{16}$$

According to the universal approximation property of Radial Basis Function Neural Networks (Park and Sandberg 1991) we have that given a fixed number of radial basis functions q there exists ideal parameters $W^* \in \mathbb{R}^{q \times n_2}$ and a vector of representation error $\tilde{\eta} \in \mathbb{R}^{n_2}$ such that the following approximation holds for all $z \in D \subset \mathbb{R}^{n+l}$ where D is compact

$$\Delta(z) = W^{*T} \sigma(z) + \tilde{\eta}(z). \tag{17}$$

Furthermore $\bar{\eta} \triangleq \sup_{z \in D} \|\tilde{\eta}(z)\|$ can be made arbitrarily small given sufficient number of radial basis functions. We say that this model of the uncertainty is unstructured because the number of RBFs needed and the location of centers is not always clear. Similar to the case of structured uncertainty, the baseline adaptive law of (14) (with $\Phi(x(t))$ replaced by $\sigma(x(t))$) guarantees uniform ultimate boundedness of the tracking error, and guarantees that the adaptive parameters stay bounded within a neighborhood of the ideal parameters only if the system states are PE (see e.g. (Sanner and Slotine 1992, Nardi 2000, Kim and Lewis 1998)). Note that in this case $e(t)$ cannot converge to 0 since $\bar{\eta} \neq 0$. If the system states are not persistently exciting, additional modifications such as e-mod (Narendra and Annaswamy 1987), σ-mod (Ioannou and Sun 1996), or projection operator based modifications (see (Tao 2003)) are required to guarantee that the adaptive parameters stay bounded around a neighborhood of an *a-priori* selected guess of the ideal parameters (usually set to 0). In the next section, we describe how the requirement on persistency of excitation can be relaxed through the use of concurrent learning adaptive control.

4 Concurrent Learning Exponentially Stable Adaptive Control for the Case of Structured Uncertainty

Concurrent learning adaptive controllers leverage the increasing ability of modern onboard computers to record and process large amounts of data online. During operation, each measurement is evaluated for recording based on a metric as it becomes available online, and stored in a dynamic *history stack* if chosen. Modeling error, which is an invariant state-dependent property of the system, is inferred from recorded data for each recorded point. Adaptation then occurs concurrently on instantaneous tracking error and the inferred modeling error. Let p_H denote the number of recorded data points, then from (5) we have $\Delta(z_j) \approx \hat{\dot{x}}_{2_j} - \nu(z_j)$, where $\hat{\dot{x}}_{2_j}$ is the estimate of $\dot{x}_{2_j}$. Let $\epsilon_j(t) = W^T(t)\Phi(z_j) - \Delta(z_j)$. A concurrent learning adaptive law that uses both recorded and instantaneous data concurrently for adaptation is chosen to have the following form

$$\dot{W}(t) = -\Gamma_W \Phi(z(t)) e^T(t) P B - \sum_{j=1}^{p_H} \Gamma_W \Phi(z_j) \epsilon_j^T(t). \tag{18}$$

The history stack can be time varying due to replacement of old data with newly selected data (see Algorithm **1**), hence the concurrent learning update law of (18) exhibits switching in parameter update dynamics.

Remark 1:
For evaluating the adaptive law of (18) the term $\epsilon_j(t) = W^T(t)\Phi(z_j) - \Delta(z_j)$ is required for the j^{th} recorded data point where $j \in [1, 2, ..p_H]$. The model error $\Delta(z_j)$ can be estimated by using (6) and noting that $\Delta(z_j) \approx \dot{\hat{x}}_{2_j} - \nu(z_j)$. Since $\nu(z_j)$ is known, the problem of estimating the system uncertainty can be reduced to that of estimation of $\dot{x}_{2_j}$. Note that the time derivative of $\dot{x}_2(t)$ at the *current* time instant is *not* needed. When measurements are not available, $\dot{x}_{2_j}$ can be estimated using an implementation of a fixed point smoother which provides more accurate estimates of $\dot{x}_{2_j}$ than numeric differentiation (Gelb 1974). Fixed point smoothing uses a forward and a backward Kalman filter for accurately estimating $\dot{x}_{2_j}$ in presence of noise and entails a selectable time delay before $\epsilon_j(t)$ can be calculated for that data point. This point stresses the benefit of using memory, as recorded states can undergo further processing to extract relevant information (modeling error in this case) in the background which can then be used for adaptation. Furthermore, since ϵ_j does not directly affect the tracking error at time t, this delay does not adversely affect the instantaneous tracking performance of the controller.

Remark 2: In the following two sections it is assumed that $\dot{x}_{2_j} = \dot{\hat{x}}_{2_j}$ to facilitate discussion. This assumption is relaxed in (Mühlegg et al. 2012, Chowdhary et al. 2013) where robustness properties of concurrent learning adaptive controller to noise and estimation errors are analyzed. Furthermore, results from numerical simulation presented in this paper are in presence of noise.

Define the parameter error as $\tilde{W} = W - W^*$, then the parameter error dynamics for the case of structured uncertainty can be written as

$$\dot{\tilde{W}}(t) = -\Gamma_W \sum_{j=1}^{p_H} \Phi(z_j)\Phi^T(z_j)\tilde{W}(t) - \Gamma_W \Phi(z(t))e^T(t)PB. \tag{19}$$

Note that the history stack is not a buffer containing past states over a time-window, rather the data in the history stack are specifically selected and retained to maintain a sparse representation of the system's operating space. The key questions therefore are: Which data points are useful to record and safe to forget? What is the minimum number of data points required for convergence? From the analysis of the adaptive law in (18) it will be seen that the minimum singular value of the history stack directly affects rate of convergence of the parameter update law in (18), and that the minimum number of data points required is equal to the dimension of m, which is the dimension of ϕ (see proof of Theorem 4.1). Based on these insights, an online algorithm for selecting data points to record is now described, its effect on convergence is analyzed immediately afterwards in Theorem 4.1. Let $p_H \in \mathbb{N}$ denote the number of stored points. Let $p^* \in \mathbb{N}$ denote the subscript of the last point stored. For ease of exposition, for a stored data point z_j, let $\Phi_j = \Phi(z_j)$, which is the data point to be stored. Let $Z_t = [\Phi_1, ..., \Phi_{p_H}]$ denote the matrix containing the recorded state information in the history stack at time t. The j^{th} column of Z_t will be denoted by $Z_t(:, j)$. It is assumed that the maximum allowable number of recorded data points is limited due to memory or processing power considerations. Therefore, Z_t has a maximum of $\bar{p} \in \mathbb{N}$ columns, and $\bar{p} \geq m$ so that with appropriate selection of recorded data, rank(Z_t) can be made equal to m. For the j^{th} data point, the associated modeling error $\Delta(z_j)$ is assumed to be stored in the j^{th} column of the matrix $\bar{\Delta}$.

The history stack is populated using an algorithm that is designed to maximize the minimum singular value of the symmetric matrix $\Omega = \sum_{j=1}^{p_H} \Phi(z_j)\Phi^T(z_j)$. Note that along with Lemma 2.1 this can be accomplished by maximizing the minimum singular value of Z_t. The algorithm also ensures that any data point linearly independent of the data stored in the history stack is included in the history stack. At the initial time t_0 the algorithm begins by setting $Z_t(:, 1) = \Phi(z(t_0))$. The algorithm then selects sufficiently different points for storage. A data point is considered to be sufficiently different if it is linearly independent of the points in the history stack and in the sense of the Euclidean distance from the last point stored. If the number of stored points equals the maximum allowable number, the algorithm incorporates new data points such that the minimum singular value of Z_t is increased by doing an exhaustive search over all recorded points to find the data point that can be replaced. Algorithm **1** describes the details of the algorithm. The implementation can be sped up in various ways which we do not get into here.

Algorithm 1 Singular Value Maximizing Algorithm for Recording Data Points

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if $\frac{\|\Phi(z(t))-\Phi_{p*}\|^2}{\|\Phi(z(t))\|} \geq \epsilon$ **or** $\mathrm{rank}([Z_t, \Phi(z(t))]) > \mathrm{rank}([Z_t])$ **then**
  if $p_H < \bar{p}$ **then**
    $p_H = p_H + 1$
    $Z_t(:, p_H) = \Phi(z(t))$; {store $\bar{\Delta}(:, p_H) = \epsilon_j$}
  else
    $T = Z_t$
    $S_{old} = \min \mathrm{SVD}(Z_t^T)$
    for $j = 1$ to p_H **do**
      $Z_t(:, j) = \Phi(z(t))$
      $S(j) = \min \mathrm{SVD}(Z_t^T)$
      $Z_t = T$
    end for
    find $\max S$ and let k denote the corresponding column index
    if $\max S > S_{old}$ **then**
      $Z_t(:, k) = \Phi(z(t))$, {store $\bar{\Delta}(:, k) = \epsilon_j$}
    end if
  end if
end if

Theorem 4.1 :
Consider the system in (3), the reference model in (7), the inversion model of (4), the control law of (8), the case of structured uncertainty with the uncertainty given by (12), the switching parameter update law of (18), and Algorithm **1** *for specifically selecting and recording data points $\Phi(z_j)$, then $(e(t), \tilde{W}(t))$ are bounded. If in addition $r(t)$ is such that $\Phi(z(t))$ is exciting over a finite interval $(0, T + \delta t)$, such that, for $T_2 = T + \delta t$, m linearly independent data points $\Phi(z_j) \in \mathrm{R}^m$ are selected by Algorithm* **1**, *then the zero solution $(e, \tilde{W}) \equiv 0$ of the closed loop system given by (10) and (19) is globally uniformly exponentially stable.*

Proof Consider the following positive definite and radially unbounded Lyapunov like candidate

$$V(e, \tilde{W}) = \frac{1}{2} e^T P e + \mathrm{tr}(\frac{1}{2} \tilde{W}^T {\Gamma_W}^{-1} \tilde{W}). \tag{20}$$

Let $\xi = [e^T, \vec{\tilde{W}}^T]^T$, then (20) can be bounded from below and above by

$$\frac{1}{2} \min(\lambda_{\min}(P), \lambda_{\min}({\Gamma_W}^{-1})) \|\xi\|^2 \leq V(e, \tilde{W}) \leq \frac{1}{2} \max(\lambda_{\max}(P), \lambda_{\max}({\Gamma_W}^{-1})) \|\xi\|^2. \tag{21}$$

Differentiating (20) along the trajectories of (10) and (19) and using the Lyapunov equation (11), we have

$$\begin{aligned} \dot{V}(e, \tilde{W}) &= -\frac{1}{2} e^T Q e + e^T P B (\nu_{ad} - \Delta) \\ &+ \mathrm{tr}(\tilde{W}^T (-\sum_{j=1}^{p_H} \Phi(z_j) \Phi^T(z_j) \tilde{W} - \Phi(z) e^T P B)). \end{aligned} \tag{22}$$

Using the representation of $\Delta(z)$ for structured uncertainties (12), the output of the adaptive element (13), canceling like terms and simplifying we have

$$\dot{V}(e, \tilde{W}) = -\frac{1}{2}e^T Q e - \mathrm{tr}(\tilde{W}^T (\sum_{j=1}^{p_H} \Phi(z_j)\Phi^T(z_j))\tilde{W}). \tag{23}$$

Remember, $\Omega(t) = \sum_{j=1}^{p_H} \Phi(z_j)\Phi^T(z_j)$, then it follows that Ω is non-negative definite for all t. Hence, $\dot{V}(e, \tilde{W}) \leq 0$. It follows therefore that $(e(t), \tilde{W}(t))$ are bounded. Assume now that in addition $r(t)$ is such that $\Phi(z(t))$ is exciting over a finite interval $(0, T + \delta t)$. Furthermore, assume that the optimal fixed point smoother induces a time delay δt in order to estimate $\dot{x}_{2_j}$. Then Algorithm **1** ensures that for $t \geq T_2$, where $T_2 = T + \delta t$, the history stack contains as many linearly independent elements as the basis of the uncertainty (m). Hence, for all $t \geq T_2$ the matrix $\Omega(t) = Z_t Z_t^T$ is positive definite. Furthermore, Algorithm **1** guarantees that $\sigma_{\min}(Z_t)$ is monotonically increasing. Let $\lambda_{\min}(\Omega(t))$ denote the minimum eigenvalue of Ω at time $t \geq T_2$ and note that with Lemma 2.1 $\lambda_{\min}(\Omega(t)) \geq \lambda_{\min}(\Omega(T_2))$, then

$$\dot{V}(e, \tilde{W}) \leq -\frac{1}{2}\lambda_{\min}(Q)e^T e - \lambda_{\min}(\Omega)(t)\,\mathrm{tr}(\tilde{W}^T\tilde{W}). \tag{24}$$

Further,

$$\dot{V}(e, \tilde{W}) \leq -\frac{1}{2}\min(\lambda_{\min}(Q), 2\lambda_{\min}(\Omega(t)))\|\xi\|^2. \tag{25}$$

Multiplying and dividing (25) by $\max(\lambda_{\max}(P), \lambda_{\max}(\Gamma_W^{-1}))$ and considering equation (21) we have

$$\dot{V}(e, \tilde{W}) \leq -\frac{\min(\lambda_{\min}(Q), 2\lambda_{\min}(\Omega(t)))}{\max(\lambda_{\max}(P), \lambda_{\max}({\Gamma_W}^{-1}))}V(e, \tilde{W}). \tag{26}$$

Therefore, $V(e, \tilde{W})$ is a common Lyapunov function (Liberzon 2005) and $(e(t), \tilde{W}(t)) \to 0$ uniformly exponentially fast as $t \to \infty$ for all $t \geq T_2$. Since $r(t)$ is such that $\Phi(z(t))$ is exciting over a finite interval $(0, T_2)$, $e(t)$ and $\tilde{W}$ are guaranteed to reduce exponentially in the interval $(0, T_2)$. It follows that the zero solution $(e, \tilde{W}) \equiv 0$ of the closed loop system given by (10) and (19) is exponentially stable. Since $V(e, \tilde{W})$ is radially unbounded, this result is global. □

Remark 3: It can be seen from the proof of Theorem 4.1 that exponential stability is guaranteed as soon as the history stack Z_t is full ranked, that is, $rank(Z_t) = m$. Furthermore, the rate of convergence is dependent directly on the minimum singular value of the history stack. The condition that the history stack be full-ranked will be referred to as the rank-condition. Algorithm **1** guarantees that if the system states are exciting over a finite interval such that a full-ranked history stack can be recorded, then exponential stability is guaranteed without requiring *persistent* excitation. On an intuitive level, the concurrent learning adaptive controller "remembers" excitation by maintaining a sparse representation of the exciting states in the form of the online recorded full-ranked history stack. It then updates the weights to reduce the (estimated) modeling error concurrently for all recorded data in the history stack.

Remark 4: Any time delay δt between when a point is recorded and when it is used (e.g. as induced due to time required for smoothing) does not affect the stability properties. The main requirement that ensures exponential stability is that $\Omega(t)$ is positive definite. This criterion is met as soon as the rank condition is met. However, in order for Theorem 4.1 to hold uniformly, the time delay δt requires $r(t)$ to be excited over an interval $(0, T + \delta t)$ instead of $(0, T)$.

5 Concurrent Learning for Systems with Unstructured Uncertainty

In many real-world applications, the basis of the uncertainty may not be known a priori. A widely used approach in this case is to use an RBF-NN adaptive element (see section 3.2). The analysis in this section shows that a concurrent learning adaptive law guarantees that the NN weights approach exponentially fast and stay bounded in a compact neighborhood of the ideal weights that approximate the uncertainty best in sense of the universal approximation theorem. Note that in this case, it is not possible to guarantee convergence of the parameter error $\tilde{W}$ to zero due to the approximations inherent to any universal approximator (see equation (17)). In the following let κ denote a positive constant.

Theorem 5.1: *Consider the system in (3), the reference model in (7), the inverting controller of (4) and the control law of (8). Assume that the structure of the system uncertainty is unknown and the uncertainty is approximated over a compact domain D using a Radial Basis Function NN as in (15) with $\bar{\eta} = \sup_{z \in D} \|\tilde{\eta}(z)\|$. Furthermore, assume that Algorithm **1** is used to specifically select and record data points $\sigma(z_j)$, and consider the following switching concurrent learning parameter update law*

$$\dot{W} = \begin{cases} -\Gamma_W \sigma(z) e^T PB - \sum_{j=1}^{p_H} \Gamma_W \sigma(z_j) \epsilon_j^T - \kappa W & \textit{if } \operatorname{rank}(Z_t) < q \\ -\Gamma_W \sigma(z) e^T PB - \sum_{j=1}^{p_H} \Gamma_W \sigma(z_j) \epsilon_j^T & \textit{if } \operatorname{rank}(Z_t) = q. \end{cases} \tag{27}$$

*Let $V(e, \tilde{W})$ be a positive definite and radially unbounded function. Furthermore, let $\mathcal{B}_\alpha$ be the largest compact ball in D. Assume $x(0) \in \mathcal{B}_\alpha$ and let β be a positive scalar. If there exists a positive invariant set $\Theta_\beta = \{(e, \tilde{W}) | V(e, \tilde{W}) \leq \beta\}$, a constant $\delta = \left(\frac{2\beta}{\lambda_{min}(P)}\right)^{1/2}$ and if the exogenous input $r(t)$ is such that the state $x_{rm}(t)$ of the bounded input bounded output reference model (7) remains bounded in the compact ball $\mathcal{B}_m = \{x_{rm} : \|x_{rm}\| \leq m_{rm}\}$ such that $m_{rm} \leq \alpha - \delta$ holds for all $t \geq 0$ then the solution $(e(t), \tilde{W}(t))$ of the closed loop system of (10) and (27) is ultimately bounded. If in addition $r(t)$ is such, that $\sigma(z(t))$ is exciting over a finite interval $(0, T + \delta t)$ so that for $T_2 = T + \delta t$, q linearly independent data points $\sigma(z_j)$ are selected by Algorithm **1**, then the solution $(e(t), \tilde{W}(t))$ of the closed loop system of (10) and (27) is exponentially 2^{th} ultimately bounded for $t \geq T_2$.*

Proof Let $\xi = [e^T, \vec{\tilde{W}}^T]^T$ and consider the following positive definite and radially unbounded Lyapunov like candidate of (20):

$$V(e, \tilde{W}) = \frac{1}{2} e^T P e + \operatorname{tr}(\frac{1}{2} \tilde{W}^T {\Gamma_W}^{-1} \tilde{W}). \tag{28}$$

Consider first the case when $\operatorname{rank}(Z_t) < q$, then noting with equations (15) and (17) that $\nu_{ad}(z_j) - \Delta(z_j) = \tilde{W}^T \sigma(z_j) + \tilde{\eta}(z_j)$, the parameter error dynamics take the following form:

$$\dot{\tilde{W}}(t) = -\Gamma_W (\sum_{j=1}^{p_H} \sigma(z_j) \sigma^T(z_j) + \kappa I) \tilde{W}(t) - \sum_{j=1}^{p_H} \sigma(z_j) \tilde{\eta}^T(z_j) - \Gamma_W \sigma(z(t)) e^T(t) PB - \kappa W^*. \tag{29}$$

Differentiating (28) along the trajectory of tracking and parameter error dynamics in (10) and (29), using the Lyapunov equation (11), noting that $\nu_{ad}(z) - \Delta(z) = \tilde{W}^T \sigma(z) + \tilde{\eta}(z)$ and canceling like terms we have

$$\begin{aligned}\dot{V}(e,\tilde{W}) =& -\frac{1}{2}e^TQe + e^TPB\tilde{\eta}(z)\\ &-\mathrm{tr}(\tilde{W}^T(\sum_{j=1}^{p_H}\sigma(z_j)\sigma^T(z_j)+\kappa I)\tilde{W}+\sum_{j=1}^{p_H}\sigma(z_j)\tilde{\eta}^T(z_j)-\kappa W^*).\end{aligned} \tag{30}$$

Note that $\|\sigma(z(t))\| \leq q^{1/2}$ due to the definition of RBFs (equation (16)). Formulating an upper bound on $\dot{V}(e,\tilde{W})$ and simplifying further yields

$$\dot{V}(e,\tilde{W}) \leq -\frac{1}{2}\lambda_{\min}(Q)\|e\|^2 - \kappa\|\tilde{W}\|^2 + \kappa\|W^*\|\|\tilde{W}\| + p_H\|\tilde{W}\|\bar{\eta}q^{1/2} + \|e\|\|PB\|\bar{\eta}. \tag{31}$$

Let $c_1 = \|PB\|\bar{\eta}$, $c_2 = \kappa\|W^*\|$, $c_3 = p_H\bar{\eta}q^{1/2}$, then (31) becomes

$$\dot{V}(e,\tilde{W}) \leq -\|e\|(\frac{1}{2}\lambda_{\min}(Q)\|e\| - c_1) - \|\tilde{W}\|(\kappa\|\tilde{W}\| - c_2 - c_3). \tag{32}$$

The compact set Θ_γ, outside which $\dot{V}(e,\tilde{W}) < 0$, is constructed by first setting the left hand side of (32) to zero and adding and subtracting $\frac{(c_2+c_3)^2}{4\kappa}$ as well as $\frac{c_1^2}{2\lambda_{min}(Q)}$:

$$\begin{aligned}0 \leq& -\|e\|(\frac{1}{2}\lambda_{\min}(Q)\|e\| - c_1) + \frac{c_1^2}{2\lambda_{min}(Q)} - \frac{c_1^2}{2\lambda_{min}(Q)}\\ &-\|\tilde{W}\|(\kappa\|\tilde{W}\| - c_2 - c_3) + \frac{(c_2+c_3)^2}{4\kappa} - \frac{(c_2+c_3)^2}{4\kappa}.\end{aligned} \tag{33}$$

Hence, we find that for all $t < T_2$, $\dot{V}(e,\tilde{W}) < 0$ outside of the compact set

$$\Theta_\gamma = \{(e,\tilde{W}) : \frac{1}{2}\lambda_{min}(Q)[\|e\| - \frac{c_1}{\lambda_{min}(Q)}]^2 + \kappa[\|\tilde{W}\| - \frac{c_2+c_3}{2\kappa}]^2 \leq \frac{c_1^2}{2\lambda_{min}(Q)} + \frac{(c_2+c_3)^2}{4\kappa}\}. \tag{34}$$

Let $\beta = \max_{(e,\tilde{W})\in\Theta_\gamma} V(e,\tilde{W})$ and define the compact set $\Theta_\beta = \{(e,\tilde{W}) : V(e,\tilde{W}) \leq \beta\}$. Note that, $\Theta_\gamma \subseteq \Theta_\beta$ with $\dot{V}(e,\tilde{W}) < 0$ outside of Θ_β. It follows that the compact set Θ_β is positively invariant with respect to $V(e,\tilde{W})$. Furthermore, note that $\frac{1}{2}\lambda_{min}(P)\|e\|^2 \leq V(e,\tilde{W})$ and define δ to be $\delta = \left(\frac{2\beta}{\lambda_{min}(P)}\right)^{1/2}$. If the exogenous input $r(t)$ is such that the state $x_{rm}(t)$ of the bounded input bounded output reference model of (7) remains bounded in the compact ball $\mathcal{B}_m = \{x_{rm} : \|x_{rm}\| \leq m_{rm}\}$ for all $t \geq 0$ and $m_{rm} \leq \alpha - \delta$, then $x(t) \in D\ \forall t \geq 0$. Hence, the NN approximation of (17) holds and the solution of the closed loop system of (10) and (27) $(e(t), \tilde{W}(t))$ is ultimately bounded.

Consider now that at time T_2 the history stack satisfies $\mathrm{rank}(Z_t) = q$, and note that $(e(T_2), \tilde{W}(T_2))$ are bounded due to the previous result. In this case, the parameter error dynamics become

$$\dot{\tilde{W}}(t) = -\Gamma(\sum_{j=1}^{p_H}\sigma(z_j)\sigma^T(z_j))\tilde{W}(t) - \sum_{j=1}^{p_H}\sigma(z_j)\tilde{\eta}^T(z_j) - \Gamma_W\sigma(z(t))e^T(t)PB. \tag{35}$$

Let $\Omega(t) = Z_tZ_t^T = \sum_{j=1}^{p_H}\sigma(z_j)\sigma^T(z_j)$, and note that $\Omega(t)$ is positive definite for all $t \geq T_2$ since $\mathrm{rank}(Z_t) = q$ due to Algorithm **1**. Algorithm **1** guarantees that $\sigma_{\min}(Z_t)$ is monotonically increasing for all $t \geq T_2$. Let $\lambda_{\min}(\Omega(t))$ denote the minimum eigenvalue of Ω at time $t \geq T_2$, then with Lemma 2.1 $\lambda_{\min}(\Omega(t)) \geq \lambda_{\min}(\Omega(T_2))$. Hence,

$$\dot{V}(e,\tilde{W}) \leq -\|e\|(\frac{1}{2}\lambda_{\min}(Q)\|e\| - c_1) - \|\tilde{W}\|(\lambda_{\min}(\Omega(T_2))\|\tilde{W}\| - c_3). \tag{36}$$

Similar to the previous case, the compact set Ω_γ, outside which $\dot{V}(e,\tilde{W}) < 0$, is constructed by setting the left hand side of (36) to zero and adding and subtracting $\frac{c_1^2}{2\lambda_{min}(Q)}$ as well as $\frac{c_3^2}{4\lambda_{min}(\Omega(T_2))}$:

$$\begin{aligned} 0 \leq &- \|e\|(\frac{1}{2}\lambda_{\min}(Q)\|e\| - c_1) + \frac{c_1^2}{2\lambda_{min}(Q)} - \frac{c_1^2}{2\lambda_{min}(Q)} \\ &- \|\tilde{W}\|(\lambda_{\min}(\Omega(T_2))\|\tilde{W}\| - c_3) + \frac{c_3^2}{4\lambda_{min}(\Omega(T_2))} - \frac{c_3^2}{4\lambda_{min}(\Omega(T_2))}. \end{aligned} \tag{37}$$

Hence, we find that for all $t \geq T_2$, $\dot{V}(e,\tilde{W}) < 0$ outside of the compact set

$$\begin{aligned} \Omega_\gamma = \{(e,\tilde{W}) : &\frac{1}{2}\lambda_{min}(Q)[\|e\| - \frac{c_1}{\lambda_{min}(Q)}]^2 \\ &+ \lambda_{min}(\Omega(T_2))[\|\tilde{W}\| - \frac{c_3}{2\lambda_{min}(\Omega(T_2))}]^2 \leq \frac{c_1^2}{2\lambda_{min}(Q)} + \frac{c_3^2}{4\lambda_{min}(\Omega(T_2))}\}. \end{aligned} \tag{38}$$

Let $\beta = \max_{(e,\tilde{W})\in\Omega_\gamma} V(e,\tilde{W})$ and define the compact set $\Omega_\beta = \{(e,\tilde{W}) : V(e,\tilde{W}) \leq \beta)$. Note that, $\Omega_\gamma \subseteq \Omega_\beta$ with $\dot{V}(e,\tilde{W}) < 0$ outside of Ω_β. It follows that the compact set Ω_β is positively invariant with respect to $V(e,\tilde{W})$. Furthermore, note that $\frac{1}{2}\lambda_{min}(P)\|e\|^2 \leq V(e,\tilde{W})$ and define δ to be $\delta = \left(\frac{2\beta}{\lambda_{min}(P)}\right)^{1/2}$. If the exogenous input $r(t)$ is such that the state $x_{rm}(t)$ of the bounded input bounded output reference model of (7) remains bounded in the compact ball $\mathcal{B}_m = \{x_{rm} : \|x_{rm}\| \leq m_{rm}\}$ for all $t \geq 0$ and $m_{rm} \leq \alpha - \delta$, then $x(t) \in D\ \forall t \geq 0$. Hence, the NN approximation of (17) holds and the solution of the closed loop system of (10) and (27) $(e(t),\tilde{W}(t))$ is ultimately bounded.

Now, let $\bar{c} = \frac{\min(\lambda_{\min}(Q),2\lambda_{\min}(\Omega(T_2)))}{\max(\lambda_{\max}(P),\lambda_{\max}(\Gamma_W{}^{-1}))}$, and note that due to the boundedness guarantee the quantity

$$k = \sup_\tau \int_{T_2}^{\tau} e^{-\bar{c}\ (t-\tau)}(c_1\|e(\tau)\| + c_3\|\tilde{W}(\tau)\|)d\tau \tag{39}$$

exists. Let $\bar{k} = c_1\|e\| + c_3\|\tilde{W}\|$, then from (36) we have

$$\dot{V}(e,\tilde{W}) \leq -\bar{c}V(e(T_2),\tilde{W}(T_2)) + \bar{k}. \tag{40}$$

Since $V(e(T_2),\tilde{W}(T_2)) \leq \frac{1}{2}\max(\lambda_{\max}(P),\lambda_{\max}(\Gamma_W{}^{-1}))\|\xi(T_2)\|^2$ and $\frac{1}{2}\min(\lambda_{\min}(P),\lambda_{\min}(\Gamma_W{}^{-1}))\|\xi\|^2 \leq V(e,\tilde{W})$ we have that

$$\|\xi\|^2 \leq \frac{1}{2}\max(\lambda_{\max}(P),\lambda_{\max}(\Gamma_W{}^{-1}))\|\xi(T_2)\|^2 e^{-\bar{c}t} + k. \tag{41}$$

Therefore, for $t \geq T_2$ the solution of the closed loop system of (10) and (27) $(e(t),\tilde{W}(t))$ is exponentially 2^{th} ultimately bounded in the sense of Definition 2.3.

□

Remark 5 : The term κW in equation (27) is σ-modification (Ioannou and Sun 1996), and it is required to guarantee the boundedness of parameters before the rank-condition ($\text{rank}(Z_t) = q$) is satisfied. Once the rank condition is satisfied, the concurrent learning adaptive law no longer requires the σ-modification, or any other modification term, to guarantee the boundedness of parameters. Furthermore, if the history stack is such that $\lambda_{min}(\Omega(T_2)) > \kappa$ it can be shown that the ultimate bounds on tracking error with concurrent learning will be smaller than for only σ-modification. Furthermore, the bounds on $\tilde{W}$ with concurrent learning can be contrasted with those achieved when using projection based adaptation, or σ and e modifications. The latter only guarantee that the parameters stay bounded around a preselected value (usually set to 0). In contrast, the adaptive law in Theorem 5.1 ensures boundedness of parameters around their ideal values. See Section 8 for further comparison with σ-modification.

6 Discussion on Least Squares Based Approach to Concurrent Learning

It is easy to notice that the rank-condition required for exponential convergence in Theorem 4.1 and that required for finding a least squares based optimal solution to the regression problem in (12) are similar. This similarity could lead one to imagine a control architecture in which once the rank-condition is satisfied, the least-squares regression problem is solved to find the optimal weights and directly use them in the control law (8). In particular, for the case of scalar structured uncertainty with $\Delta(z) \in \mathbb{R}$, let $\theta \in \mathbb{R}^m$ denote an estimate of the ideal weights W^*. Let $Y = [\Delta(1), \Delta(2), ..., \Delta(N)]^T$ and let $Z_t = [\Phi_1, ..., \Phi_p] \in \mathbb{R}^{m \times p}$ denote the matrix containing the recorded state information in the history stack at time t. Then a closed form solution to the least squares problem can be found as $\theta = (Z_t Z_t^T)^{-1} Z_t Y$ if the history stack matrix Z_t satisfies the rank-condition. However, switching the adaptive weights with the least squares estimate once it becomes available can introduce transients due to the sudden change. This can be alleviated by smoothening the transition using a modified version of Ioannou's σ-modification ((Ioannou and Kokotovic 1983)):

$$\dot{W}(t) = -(\Phi(z)e^T(t)PB - \Gamma_\theta(W(t) - \theta))\Gamma_W.$$

In (Chowdhary and Johnson 2010a) we showed that this approach also guarantees exponential stability of the closed loop. However, even with this modification, the least squares based adaptive law suffers from several drawbacks. It requires an inversion, it does not scale well to non-scalar cases, and could lead to undesirable transient performance due to numerical issues encountered while inverting ill-conditioned matrices. The concurrent learning update law on the other hand yields the same solution that least squares would yield (W^*) at a much lower computational burden and without being susceptible to stability issues caused by numerical ill-conditioning. In particular, while the least squares approach is of order $O(n^3)$, the CL approach is of the order $O(n^2)$. Furthermore, in the above mentioned least squares based approach, one would have to wait until the rank-condition is met before inverting. Some of these shortcomings can be alleviated by solving the least squares problem recursively within a MRAC setting ((Chowdhary and Johnson 2010b)). However well known results in least squares regression state that persistency of excitation in the system signal $\Phi(z)$ is required for the recursive least squares approach to converge ((Tao 2003), (Aström and Wittenmark 1995)). Therefore, it can be seen that the concurrent learning adaptive law of Theorems 4.1 and 5.1 guarantees that the adaptive weights approach the least squares estimate at an exponential rate while requiring much less computational cost and guaranteeing closed-loop stability.

7 Robustness to Parameter Variation

Assumption 3.2 (requiring the existence of an ideal weight vector W^*) is widely used in adaptive control. In presence of sudden configuration changes to the system (e.g. structural damage) the dynamcis of the aircraft are altered and hence Assumption 3.2 could be violated. It should be noted that one effective way of handling such variations is to modify Algorithm **1** to prioritize removing older recorded data so that the majority of the data points in the history stack correspond to the current configuration. Furthermore, if the ideal parameters switch in a known or detectable manner, then it is straight forward to extend Theorem 4.1 to show that a concurrent learning controller, with independent history stacks corresponding to each switched system, guarantees closed loop uniform exponential stability of the zero solution. The question that arises is: what happens if irrelevant data points are not removed after a switch in the ideal parameters? In this section robustness of concurrent learning adaptive controllers to sudden ideal parameter variation is established assuming that the uncertainty is structured and irrelevant data points are not removed.

Consider the following switching version of the dynamical system in (3)

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= f_k(x(t)) + G_k(x(t))\delta(t), \qquad k \in \{1,..N\}. \end{aligned} \tag{42}$$

It is assumed that each pair (f_k, G_k) is Lipschitz over D_x, that each $G_k(x)$ is nonsingular for all $x \in D_x$ and that there are finite switches in finite time. Assume further that the reference model does not switch. Then, using the control law of (8) we have that the tracking error dynamics take the form

$$\dot{e} = Ae + B(\nu_{ad}(z) - \Delta_k(z)), \qquad k \in \{1,..N\}, \tag{43}$$

where $\Delta_k(z) = f_k(x) - \hat{f}(x) + (G_k(x) - \hat{G}_k(x))\delta$. Let W_k^* denote an ideal parameter vector for each $k \in \{1,..,N\}$, where N is the number of possible parameter variations. The tracking error dynamics become

$$\dot{e} = Ae + B(W^T\Phi(z) - W_k^{*^T}\Phi(z)), \qquad k \in \{1,..N\}. \tag{44}$$

To simulate a worst-case analysis, assume that data points are added to the history stack when the system $k = 1$ is active, that is, when ideal parameters are given by W_1^*. Let the bounded parameter variation for each system be given by $\Delta W_k^* = W_1^* - W_k^*$, then applying the concurrent learning law of (18) the parameter error dynamics take the following form

$$\dot{\tilde{W}}_k = -\Gamma_W \sum_{j=1}^{p_H} \Phi(z_j)\Phi^T(z_j)(\tilde{W}_k(t) + \Delta W_k^*) - \Gamma_W \Phi(z(t))e^T(t)PB. \tag{45}$$

Theorem 7.1 : *Consider the systems in (42), the reference model in (7), the inversion model of (4), the control law of (8), the case of structured uncertainty with the uncertainty given by $\Delta_k(z) = W_k^*\Phi(z)$ for $k \in \{1,..N\}$. Let for each recorded data point j, $\epsilon_j(t) = W^T(t)\Phi(z_j) - \Delta(z_j)$, with $\Delta(z_j) = \dot{x}_j - \nu(z_j)$. Furthermore, assume that the history stack is populated using Algorithm* **1** *when the system $k = 1$ is active and assume that the recorded data points $\Phi(z_j)$ satisfy the rank-condition, then the zero solutions $(e, \tilde{W}_k)$ $k \in \{1,..N\}$ of the closed loop system of (44) and (45) are ultimately bounded.*

Proof We do the proof only for the case $N = 2$, other cases follow in a similar manner. Consider the following positive definite and radially unbounded Lyapunov like candidates

$$V_k(e, \tilde{W}) = \frac{1}{2}e^T Pe + \mathrm{tr}(\frac{1}{2}\tilde{W}_k^T {\Gamma_W}^{-1}\tilde{W}_k), \quad k \in \{1, 2\}. \tag{46}$$

Let $\xi_k = [e^T, \vec{\tilde{W}}_k^T]^T$, then

$$\frac{1}{2}\min(\lambda_{\min}(P), \lambda_{\min}({\Gamma_W}^{-1}))\|\xi_k\|^2 \leq V_k(e, \tilde{W}) \leq \frac{1}{2}\max(\lambda_{\max}(P), \lambda_{\max}({\Gamma_W}^{-1}))\|\xi_k\|^2. \tag{47}$$

The derivative of the k^{th} Lyapunov-like candidate along the trajectories of the k^{th} system of (44) and (45) is

$$\dot{V}_k(e, \tilde{W}_k) = -\frac{1}{2}e^T Qe - \mathrm{tr}(\tilde{W}_k^T \sum_{j=1}^{p_H} \Phi(z_j)\Phi^T(z_j)(\tilde{W}_k - \Delta W_k^*)). \tag{48}$$

Consider first the case when $k = 1$. In this case $\Delta W_1^* = W_1^* - W_1^* = 0$, let for all t, $\Omega(t) = \sum_{j=1}^{p_H} \Phi(z_j)\Phi^T(z_j)$, where $\Phi(z_j)$ are the elements of the history stack Z_t at time t. Since $\mathrm{rank}(Z_0) = m$, we have $\lambda_{\min}(\Omega(0)) > 0$. Due to Algorithm **1** $\lambda_{\min}(\Omega(t))$ is monotonically increasing, hence

$$\dot{V}_1(e, \tilde{W}_1) \leq -\frac{1}{2}\lambda_{\min}(Q)e^T e - \lambda_{\min}(\Omega(0))\,\mathrm{tr}(\tilde{W}_1^T\tilde{W}_1) < 0. \tag{49}$$

It follows therefore that over any finite time interval $[T, T + \Delta T]$ the solution $(e(t), \tilde{W}_1(t))$ is bounded and approaches the origin for arbitrary $(e(T), \tilde{W}_1(T))$.

Consider now the case when $k = 2$, in this case $\Delta W_2^* = W_1^* - W_2^*$, hence

$$\dot{V}_2(e, \tilde{W}_2) = -\frac{1}{2}e^T Qe - \mathrm{tr}(\tilde{W}_2^T \sum_{j=1}^{p_H} \Phi(z_j)\Phi^T(z_j)(\tilde{W}_2 - \Delta W_2^*)). \tag{50}$$

Therefore, the derivative of the Lyapunov-like candidate and particularly $\dot{V}_2(e, \tilde{W}_2)$ is negative outside of the compact set

$$\begin{aligned}\Theta_\gamma = \{(e, \tilde{W}_2) : &\frac{1}{2}\lambda_{min}(Q)\|e\|^2 + \lambda_{min}(\Omega(t))(\|\tilde{W}_2\| - \frac{1}{2}\frac{\lambda_{max}(\Omega(t))}{\lambda_{min}(\Omega(t))}\|\Delta W_2^*\|)^2 \\ &\leq \frac{(\lambda_{max}(\Omega(t)))^2}{4\lambda_{min}(\Omega(t))}\|\Delta W_2^*\|^2\}.\end{aligned} \tag{51}$$

Let $\beta = \max_{(e,\tilde{W})\in\Theta_\gamma} V(e, \tilde{W})$ and define the compact set Θ_β to be $\Theta_\beta = \{(e, \tilde{W}) : V(e, \tilde{W}) \leq \beta)$. Note that, since $\Theta_\gamma \subseteq \Theta_\beta$ with $\dot{V}(e, \tilde{W}) < 0$ outside of Θ_β and with the defintion of Θ_γ, the compact set Θ_β is positively invariant for arbitrary $(e(T), \tilde{W}_2(T))$. It follows therefore that over any finite time interval $[T, T + \Delta T]$ the solution $(e(t), \tilde{W}_2(t))$ is bounded and approaches the set Θ_β for arbitrary $(e(T), \tilde{W}_1(T))$.

Let $S = \{(t_1, k_1), (t_2, k_2), ..)\}$ be an arbitrary switching sequence (with finite switches in finite time), the sequence denotes that the system k_i was active between time t_i and t_{i+1}. Note that we assume that the sequence S is minimal, that is $k_i \neq k_{i+1}$ (Branicky 1998). Suppose at an arbitrary instant t_i the system switches from $k = 1$ to $k = 2$, then $\tilde{W}_2(t_i) = \tilde{W}_1(t_i) + \Delta W_2^*$. Since $\dot{V}_1(e, \tilde{W}_1) < 0$ for all $t \in [t_{i-1}, t_i]$ it follows that $V_1(e(t_i), \tilde{W}_1(t_i)) < V_1(e(t_{i-1}), \tilde{W}_1(t_{i-1}))$, hence $e(t_i)$, and $\tilde{W}_1(t_i)$ approach zero and consequently $\tilde{W}_2(t_i)$ is bounded.

Now suppose the system switches from $k = 2$ to $k = 1$ at the instant t_{i+1}, then $\tilde{W}_1(t_{i+1}) = \tilde{W}_2(t_{i+1}) - \Delta W_2^*$. Since $\dot{V}_2(e, \tilde{W}_2)$ is negative definite outside of a compact set, it follows from (50) that within the interval $[t_i, t_{i+1}]$, $(e(t), \tilde{W}_2(t))$ either approaches and enters the set Θ_β in finite time, or stays bounded within Θ_β if already inside. Hence both $e(t_{i+1})$ and $\tilde{W}_2(t_{i+1})$ are guaranteed to be bounded, consequently $\tilde{W}_1(t_{i+1})$ is bounded. Since for $k = 1$ concurrent learning adaptive control guarantees exponential stability of the closed loop system $(e, \tilde{W}_1)$, it follows that the set Θ_β is positively invariant in spite of arbitrary switching. Therefore, it follows that the solutions $(e, \tilde{W}_k)$ of the closed loop system of (44) and (45) are ultimately bounded. □

Remark 6: For $N > 2$ a positively invariant set $\bar{\Theta}_\beta$ which guarantees ultimate bounds of the closed loop system (44) and (45) can be constructed by taking the union of each positively invariant set $\Theta_{\beta,k}$ of each system k such that $\bar{\Theta}_\beta = \cup_{k \in N} \Theta_{\beta,k}$.

Remark 7: It can be shown that the adaptive law of Theorem 5.1 for unstructured uncertainties also guarantees the ultimate boundedness $(e, \tilde{W}(t))$ in presence of parameter variations. We omit the proof here due to space limitation, it can be formed using arguments similar to Theorem 7.1.

8 A Connection to σ-Modification

A key shortcoming of the traditional MRAC adaptive law of (14) is that it does not guarantee boundedness of the adaptive parameters. It is well known that if the parameters are not guaranteed to be bounded, they can drift, and for sufficiently large learning rates, the bursting phenomena may be encountered (Aström and Wittenmark 1995, Narendra and Annaswamy 1989). Bursting is a result of unbounded weight error growth and is characterized by high frequency oscillations. Various modifications have been proposed to counter weight drift, these include the classical σ-modification of (Ioannou and Sun 1996), e modification of (Narendra and Annaswamy 1986), and projection based algorithms (see e.g. (Tao 2003)). These modification guarantee that the weights stay bounded in a compact neighborhood of a preselected value. In this section we make a connection between the concurrent learning adaptive law presented before and the classical σ-modification, which can be viewed as a way to add damping to the parameter dynamics. Consider the traditional adaptive law with σ-modification:

$$\dot{W}(t) = -\Gamma_W(\Phi(z(t))e^T(t)PB + \kappa(W(t) - \bar{W})), \tag{52}$$

where $\kappa > 0$ denotes the σ-modification gain and $\bar{W}$ is an a-*priori* selected desired value (often set to zero) around which the parameters are desired to be bounded. Let $\mathring{W} = \bar{W} - W^*$, then the weight error dynamics for the σ-modification adaptive law in (52) can be written as

$$\dot{\tilde{W}}(t) = -\Gamma_W \sigma \tilde{W}(t) - \Gamma_W(\Phi(z(t))e^T(t)PB - \kappa \mathring{W}). \tag{53}$$

Since Γ_W and κ are assumed to be positive definite, and since the equation contains a term linear in $\tilde{W}$, one can intuitively argue that the weight error dynamics will be bounded-input-bounded-state stable if the tracking error e stays bounded. In fact, through Lyapunov arguments it can be shown that the update law in (52) will guarantee that the tracking error and the weight error stay bounded. This result is classical, and we do not intend to get into its details here (they can be found for example in (Ioannou and Sun 1996)). The point that we intend to make here follows from noting that if $\bar{W} = W^*$, that is, if the ideal weights are known a-*priori*, then σ-modification based adaptive law of (52) will also guarantee exponential parameter error and tracking error convergence. Now if we compare the weight error dynamics in (53) with those obtained in (19) with concurrent learning it can be seen that once Z_t is full ranked, the weight error dynamics with concurrent learning also contain a linear term in $\tilde{W}$ which is multiplied by a Hurwitz matrix. Furthermore, the weight error dynamics with concurrent learning do not contain the $\mathring{W}$ term. This suggests that the concurrent learning update law in (18) is comparable in its behavior to a σ-modification based adaptive law when one knows a-*priori* the ideal weights W^*, and hence contains inherently the desirable robustness properties of σ-modification. This analysis goes to show how exponential weight convergence results in robustenss of MRAC schemes and is in agreement with the classical finding of (Boyd and Sastry 1986, Anderson 1977, Sastry and Bodson 1989).

It should be noted however that the concurrent learning approach provides several benefits over that of σ-modification. In particular, the main drawback of σ-modification is that in the presence of excitation the parameter estimates would approach their real values, but cannot achieve them due to the added damping. Furthermore, if the excitation is removed, the parameter estimates return to zero. In contrast, the second term in the concurrent learning law of Theorems 4.1 and 5.1 attracts the parameters towards their true value even when there is no excitation in the system after the rank-condition has already been satisfied. This helps avoid steady state errors, which can be common to σ-modification.

9 Numerical Simulation of Trajectory Tracking in the Presence of Wing Rock Dynamics

In this section we present results of numerical simulations that support the developed theory. In particular, we focus on concurrent learning adaptive control in the presence of structured uncertainties as defined in section 3.1. Modern highly swept-back or delta wing fighter aircraft are susceptible to lightly damped oscillations in roll angle known as "Wing Rock". Wing rock often occurs at conditions commonly encountered at landing (Saad 2000); making precision control in presence of wing rock critical for safe landing. In this section we use concurrent learning control to track a sequence of roll commands in the presence of wing rock dynamics. Let ϕ denote the roll attitude of an aircraft, p denote the roll rate, δ_a denote the aileron control input, then a model for wing rock dynamics is (Monahemi and Krstic 1996)

$$\dot{\phi} = p \tag{54}$$

$$\dot{p} = L_{\delta_a}\delta_a + \Delta(x). \tag{55}$$

Here $\Delta(x) = W_0^* + W_1^*\phi + W_2^*p + W_3^*|\phi|p + W_4^*|p|p + W_5^*\phi^3$ describes wingrock motion, and $L_{\delta_a} = 3$. The parameters for $\Delta(x)$ are adapted from (Singh et al. 1995), they are $W_1^* = 0.2314, W_2^* = 0.6918, W_3^* = -0.6245, W_4^* = 0.0095, W_5^* = 0.0214$. In addition to these parameters, a trim error is introduced by setting $W_0^* = 0.8$. The ideal parameter vector W^* is assumed to be unknown. The chosen inversion model has the form $\delta_a = \frac{1}{L_{\delta_a}}\nu$. This choice results in the modeling uncertainty of (6) to be given by $\Delta(x)$. The adaptive controller uses the control law of (8). The linear gain K of the control law is given by $[0.5, 0.4]$, a second order reference model with natural frequency of 1 rad/s and damping ratio of 0.5 is chosen, and the learning rate is set to $\Gamma_W = 2$. The simulation uses a time-step of 0.05 seconds. The concurrent learning controller uses the parameter update law of (18), while the adaptive controller without concurrent learning uses the parameter update law of (14). Note, that the control design parameters are chosen equally for both approaches in order to ensure a fair comparison. In the following sections numerical simulation is performed for concurrent learning without and with measurement noise.

9.1 *Trajectory Tracking without Measurement Noise*

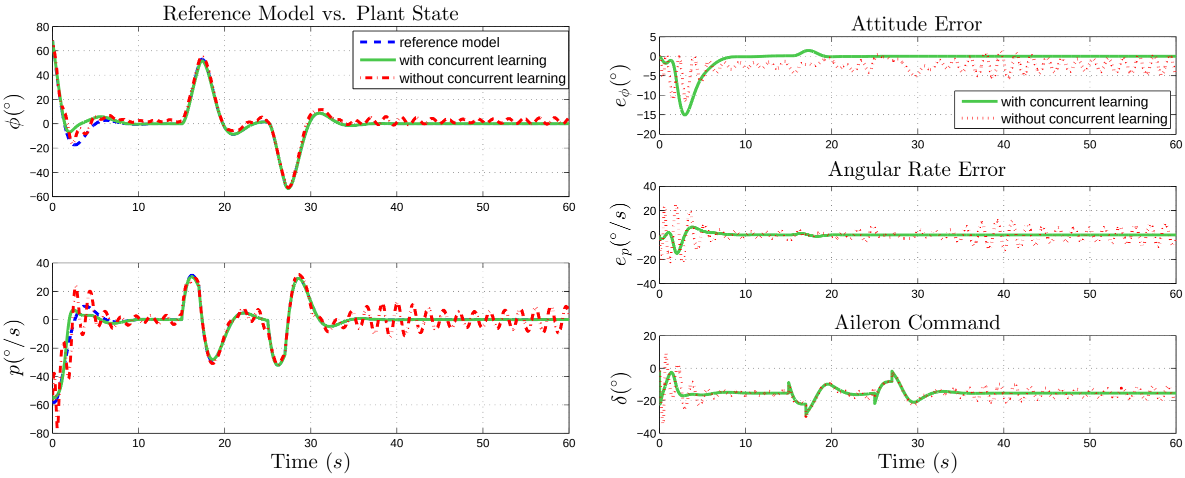
The aim here is to validate the claim of Theorem 4.1. In this section we assume that the measurements are not noisy and that no previously recorded data points are available. Algorithm **1** is used to populate and update the history stack online, the size of which is restricted to 30 recorded data points. An attitude tracking command of 57 ° and −57 ° is commanded between seconds 15 to 17 and seconds 25 to 27 s respectively. The initial conditions are set to $\phi(0) = 68$ ° and $p(0) = -57$ °/s.

Figure 2(a) compares the trajectory tracking capabilities of the adaptive controller of (8) with and without concurrent learning. It can be seen that the plant still exhibits oscillatory motion in the roll angle if the MRAC update law of (14) is used. This is attributed to the fact that there are no guarantees for parameter convergence if only an instantaneous update law is used and the regressor vector is not PE. It should be noted, that natural oscillations of the system alone do not guarantee that the regressor vector is PE. In fact, if the approximation of the uncertainty becomes better, the system oscillations would decrease, and so would the excitation. Thus, the PE condition in Definition 2.5 does not hold $\forall t \geq t_0$. Since this would further slow down parameter convergence, the uncertainty is not canceled uniformly and the rate of oscillation reduction would continue to decrease. In contrast, it can be seen that the plant states hardly differ from the reference model states if concurrent learning is used. In particular, the exponential parameter convergence of concurrent learning result in significant improvement in tracking as the weights converge to their true values, particularly when tracking the two commanded steps in attitude. Due to parameter convergence the uncertainty is uniformly cancelled, yielding linear exponentially stable error dynamics. Figure 2(b) explicitly compares the tracking errors and the control commands. It can be seen that with concurrent learning, the tracking error rapidly approaches zero. This is in agreement with Theorem 4.1, where exponential stability of the tracking and weight error dynamics was established. Due to the cancellation of the uncertainty, the commanded control inputs are also seen to be less oscillatory when concurrent learning is used.

Figure 3(a) compares the evolution of the adaptive parameters with and without concurrent learning. With concurrent learning the parameters rapidly converge to their true values. This again complies with the claim of exponentially stable weight error dynamics in Theorem 4.1 and is the foundation for reducing the error dynamics to a linear form by cancellation of the uncertainty. In contrast, the instantaneous update law does not lead to parameter convergence since the states are not persistently excited. Hence, the uncertainty is not completely learned and this affects the tracking error performance. The claim of Theorem 4.1 can also be validated by observing the evolution of the Lyapunov candidate in Figure 3(b). The figure shows that the Lyapunov function with concurrent learning decays monotonically to zero with an exponential bound, which is not the case without concurrent learning. With σ-modification in particular, the tracking and parameter error dynamics are guaranteed to be bounded in a positively invariant set, this is why the Lyapunov function for σ-modification does not converge to zero but stays abounded after about 10 seconds. Finally, Figure 4 shows that Algorithm **1** adds and replaces data in the history stack such that its minimum singular value increases monotonically.

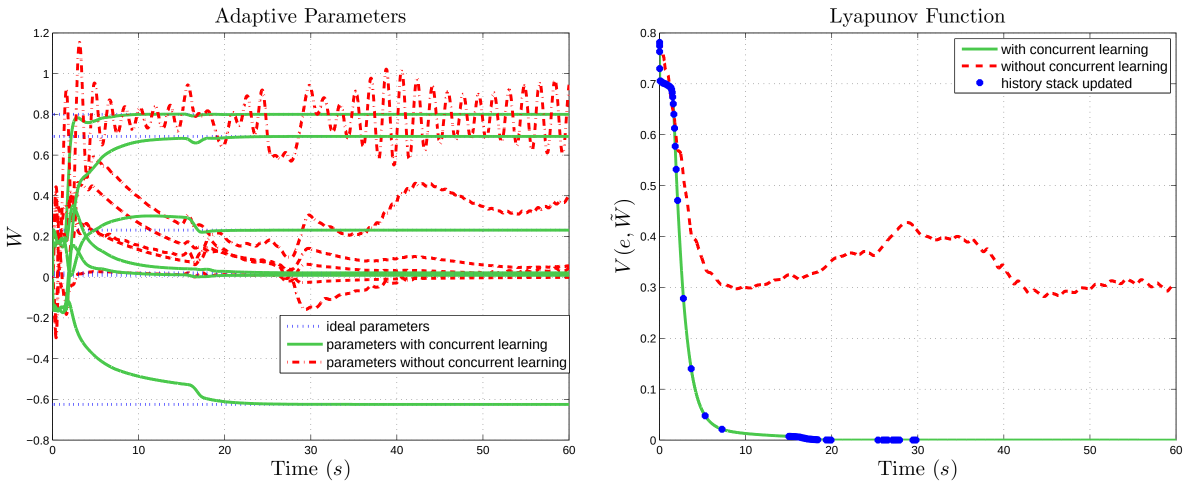
Note, that increasing the learning rates for the instantaneous update can potentially decrease the oscillations by increasing the damping. Indeed, other authors have argued through simulation and theoretical analysis that it is possible to get good tracking response by dominating the uncertainty with large adaptation gains. When large adaptation gains are used, small oscillations may still occur, which are masked by large or rapidly changing control inputs. Such inputs could be undesirable in flight conditions especially in the presence of actuator rate and position constraints. In theory, input filtering can alleviate this problem, however, we focus on the alternative: learning the uncertainty, so that high gain control is not required. In particular, concurrent learning achieves uncertainty cancellation through learning and hence does not need to employ high gain control.

Also note, that the convergence estimates are conservative, as they are concerned with the exponentially decaying upper bounds on the tracking and parameter error (see Definition 2.2, which defines exponential stability). As is common with exponentially stable systems, this means that as long as $\dot{V} < 0$ and the exponentially decaying upper bounds are not violated the weight error of a single adapative parameter might as well increase. This actually happens to some weights in Figure 3(a).



(a) (b)

Figure 2. Comparison of tracking capability of adaptive controller with and without concurrent learning in Figure 2(a). Note that the concurrent learning adaptive controller has significantly improved tracking over time, indicating long term learning. The tracking errors for the two controllers are compared in Figure 2(b).



(a) (b)

Figure 3. Comparison of the evolution of the adaptive parameters with and without concurrent learning in Figure 3(a). Note that all the parameters of the concurrent learning adaptive controller converge to the true parameters, while the parameters without concurrent learning do not. The Lyapunov functions for the two controllers are compared in Figure 3(b).

9.2 *Numerical Simulation in the Presence of Errors in the Recorded Data*

The aim of this section is to show robustness of the concurrent learning adaptive controller even when the stored data is corrupted and the measurements are noisy. A theoretical treatment of this topic is given in (Chowdhary et al. 2013, Mühlegg et al. 2012). In order to introduce errors to the history stack, it is assumed that the output of the wing rock dynamics model is corrupted by band limited white noise with a standard deviation of 2.5° for the roll angle and $10^\circ/s$ for the roll rate. The tracking capability of the concurrent learning adaptive controller is compared with a baseline adaptive controller augmented with σ-modification ($\kappa = 0.05$) to prevent bursting.

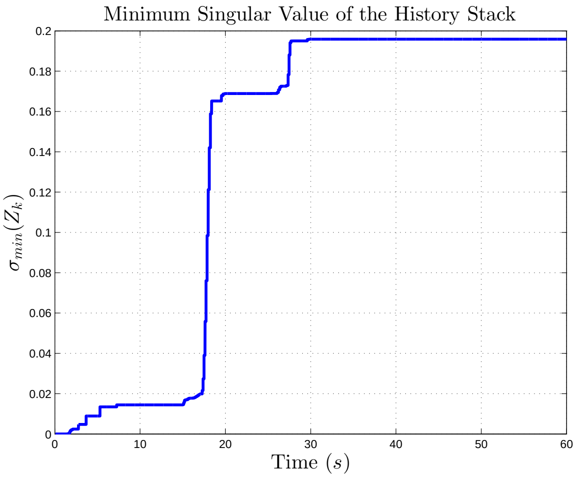


Figure 4. Evolution of minimum singular value of the history stack of adaptive controller with concurrent learning.

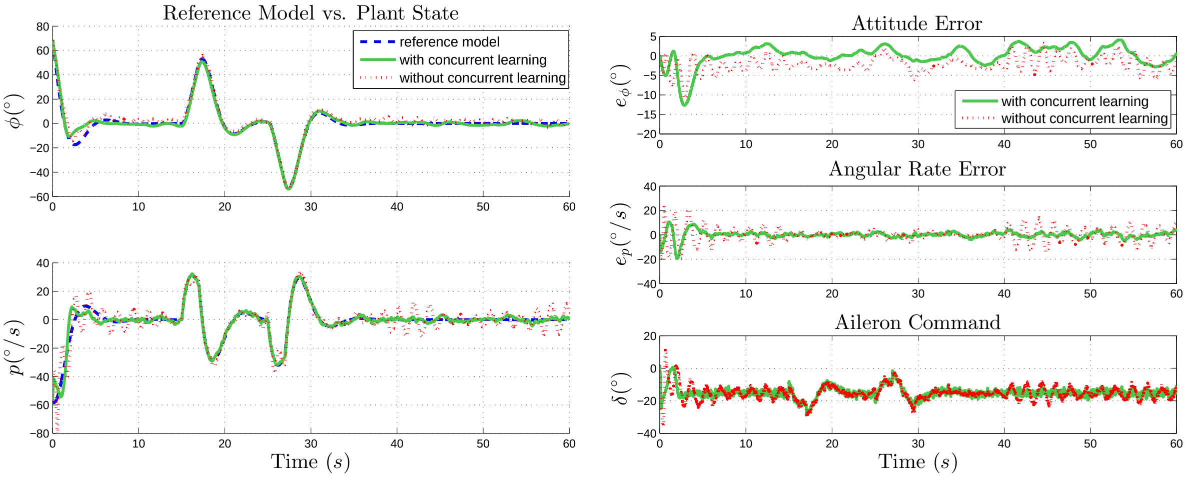
The concurrent learning adaptive controller uses a forward-backward Kalman filter (smoother) in order to estimate $\dot{p}_j$.

Figure 5(a) compares the trajectory tracking capability of the adaptive controller with and without concurrent learning. It can be observed that the states are less oscillatory with concurrent learning than with instantaneous update laws only. However, even with concurrent learning the plant does not track the reference model perfectly but with minor deviations proportional to the noise intensity. This also becomes apparent if the tracking error in Figure 5(b) is taken into account. It can be seen that the tracking error with concurrent learning is bounded around 0, while the tracking error for the instantaneous update law is slightly biased in ϕ and highly oscillatory in p. Furthermore, the instantaneous update law requires more control action than concurrent learning. The difference in performance is due to how both adaptive laws react to noise in the state feedback and the convergence of the adaptive parameters.

Figure 6(a) shows the evolution of the parameters for both cases. With concurrent learning, the parameters are observed to converge to a bounded set around their true values, while without concurrent learning, the parameters stay bounded around the origin. Hence, in general the concurrent learning adaptive controller approximates the uncertainty better than the instantaneous update law. Due to noise, and hence a corrupted history stack, the uncertainty is not uniformly canceled with concurrent learning. As a result the uncertainty still effects the error dynamics, which, together with the noisy feedback, explains the small oscillations in the states. However, it can be seen that the concurrent learning adaptive controller is better at handling the noisy measurements than instantaneous update laws only. Figure 6(b) compares the evolution of the Lyapunov like energy functions of the adaptive controller. With concurrent learning the energy function decreases to a much smaller ball around the origin. The times at which the history stack was updated is also depicted in Figure 6(b). The estimation error of $\dot{p}_j$ and p_j can directly be related to the magnitude of the bounds on the tracking and parameter error. With a better estimate the quality of the uncertainty estimation $\Delta_j(z_j) \approx \dot{\hat{p}}_j - \nu_j(z_j)$ and thus the quality of the stored data increases. Figure 7(a) compares the measured and real state, whereas Figure 7 7(b) compares the estimation error in the case that either a forward Kalman filter or a smoother is used to estimate $\dot{p}_j$. On average, the estimation error is reduced by about 20% over the Forward Kalman Filter if a smoother is applied.

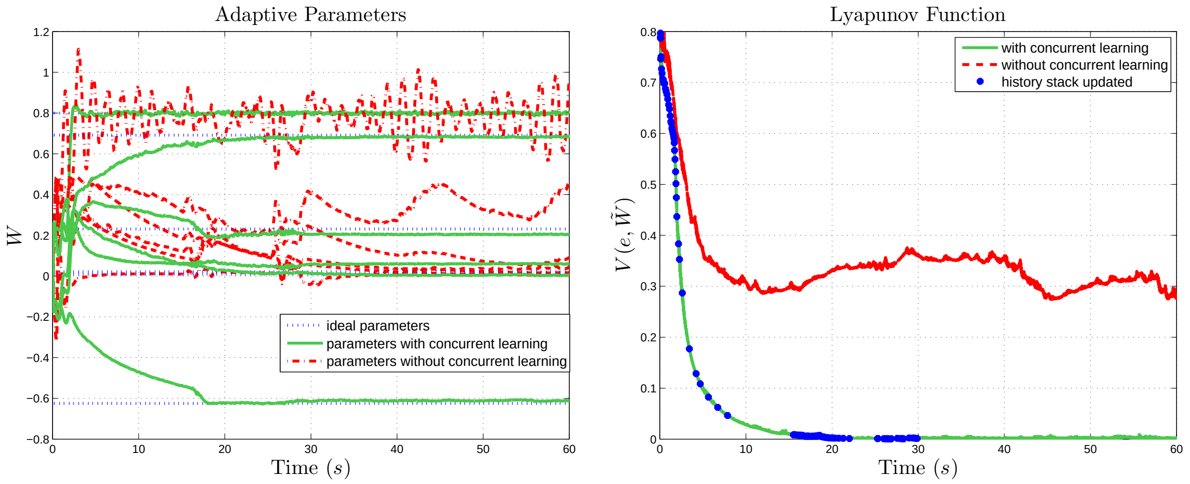
9.3 *Evaluation of the Simulation Results*

The aim of this section is to give a summary of the previous simulation runs and evaluate the performance of the concurrent learning adaptive controller compared to MRAC with instantaneous update law only. Unfortunately, common and reliable metrics, such as phase and gain margins



(a) (b)

Figure 5. Comparison of tracking capability of adaptive controller in presence of noisy measurements with and without concurrent learning in Figure 5(a). Note that tracking is significantly improved with concurrent learning. The tracking errors are compared in Figure 5(b).



(a) (b)

Figure 6. Comparison of evolution of adaptive parameters with and without concurrent learning in presence of noisy measurement in Figure 6(a). Note that all the parameters of the concurrent learning adaptive controller converge to a compact set around their true values, while the parameters without concurrent learning do not. The Lyapunov like functions for the two adaptive controllers are compared in Figure 6(b).

for linear systems, do not yet exist for nonlinear controllers. However, in order to compare the tracking capabilities of the controllers define E to be the integral of the squared tracking error 2-norm from $t_0 = 0$ until the end of the simulation at $t = 60s$:

$$E = \int_{t_0=0}^{t=60} \|e\|_2^2 dt. \tag{56}$$

Table 1 summarizes the result for the 4 simulations performed in the paper. This includes instantaneous update only with and without noise as well as concurrent learning with and without noise. It can be seen that E is significantly smaller for both cases when concurrent learning is used than for instantaneous update laws only. This is mainly attributed to the fact that with concurrent learning the weights do converge to their true values (no noise) or stay bounded around their optimal values (noisy case), which does not happen if only instantaneous

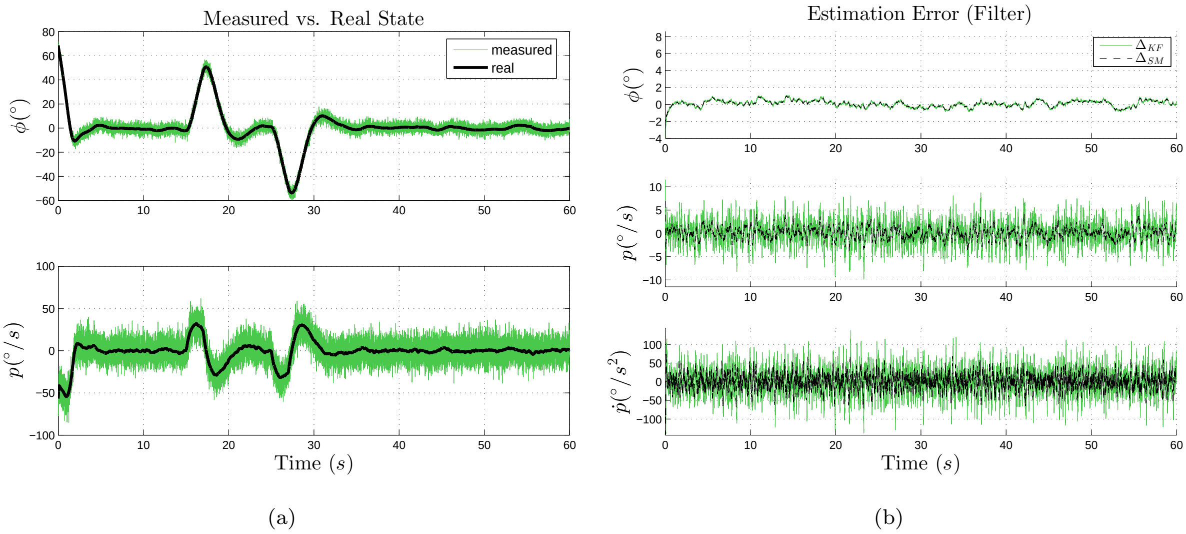


Figure 7. Comparison of measured and real states in Figure 7(a). Estimation error of $\dot{p}$ if a forward Kalman filter or a smoother is used in Figure 7(b). Note the improvement obtained over a forward Kalman filter by a combined forward-backward Kalman filter (smoother) in estimating $\dot{p}$; this results in a better estimate of the uncertainty for the recorded data.

update laws are used.

Also note, that for the instantaneous update E is slightly smaller for the noisy case than for the simulation without noise. This leads to the assumption that noise has only a minor influence on the instantaneous update laws. This can be explained by the fact that the weight update laws act like a first order filter for the weight update. In the end, if E is used to compare the tracking capabilities, concurrent learning still yields better performance than the instantaneous update law.

	Instantaneous Update	Concurrent Learning
Without Noise	336.05	79.02
With Noise	327.26	111.09

Table 1. Integral of the tracking error norm for different numerical simulations

10 Flight Test Validation of a Concurrent Learning Adaptive Autopilot for a Fixed-Wing UAV

In this section flight test results of concurrent learning adaptive controllers implemented for attitude control of a fixed wing aircraft are presented. The flight test vehicle is the GT Twinstar (Figure 8) foam built, twin engine UAV (Chowdhary et al. 2010), equipped with an autopilot. The available state information includes velocity and position in global and body reference frames, accelerations along the body x, y, z axes, roll, pitch, yaw rates and attitude, barometric altitude, and air speed information. These measurements can be further used to determine the aircraft's velocity with respect to the air mass, and the flight path angle. For the implementation of concurrent learning adaptive controllers derivatives of the angular rates and accelerations are estimated using a fixed point smoother (Chowdhary and Johnson 2011).

The control algorithm has a cascaded inner and outer loop design. The outerloop, which is integrated with the guidance loop, commands the desired roll angle (ϕ), angle of attack (α), and angle of sideslip (β) to achieve desired waypoints (Chowdhary and Johnson 2011). The innerloop attitude controller ensures that the states of the aircraft track these desired quantities using the adaptive control architecture described in Section 3. A RBF NN with 10 radial basis functions whose centers are spaced with a uniform distribution in the region of expected operation is employed as the adaptive element. The baseline adaptive controller uses

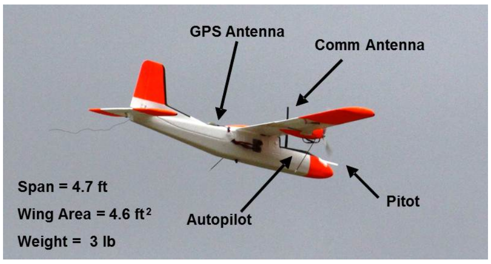


Figure 8. The Georgia Tech Twinstar UAS.

the e-modification adaptive control law with the e-modification gain set to 0.01 (Narendra and Annaswamy 1987) and has been extensively flight test validated (Johnson and Kannan 2005). The aircraft is commanded to track an elliptical pattern while holding altitude at 61 m (200 feet) in low gust conditions. The concurrent learning adaptive controller modifies the baseline using the learning law of Theorem 5.1. The flight test results presented are from the same flight, the only difference between the two results is that in one concurrent learning is on, and in the other it is not on. The RBF parameters, the learning rate of the baseline adaptive law etc. remain the same. The ground tracks of both controllers are compared in Figure 9. In that Figure, the circles denote the commanded way points, the dotted line connecting the circles denotes the path the aircraft is expected to take between waypoints. While turning at the waypoints, the onboard guidance law smooths the trajectory (Chowdhary and Johnson 2011) by commanding circles of 24 m (80 feet) radius. From that Figure, it can be seen that the concurrent learning adaptive controller has better tracking performance, especially improved cross-tracking performance, that is, it has less tracking error in the direction perpendicular to the flight path. Furthermore, it was found that the concurrent learning controller is better at eliminating steady-state errors in attitude than the baseline adaptive controller, as can be inferred from Table 2. This is one reason why the concurrent learning controller has better cross-tracking performance than the baseline adaptive controller. The plots showing inner loop performance and actuator inputs are presented in (Chowdhary et al. 2010), they are omitted here for brevity.

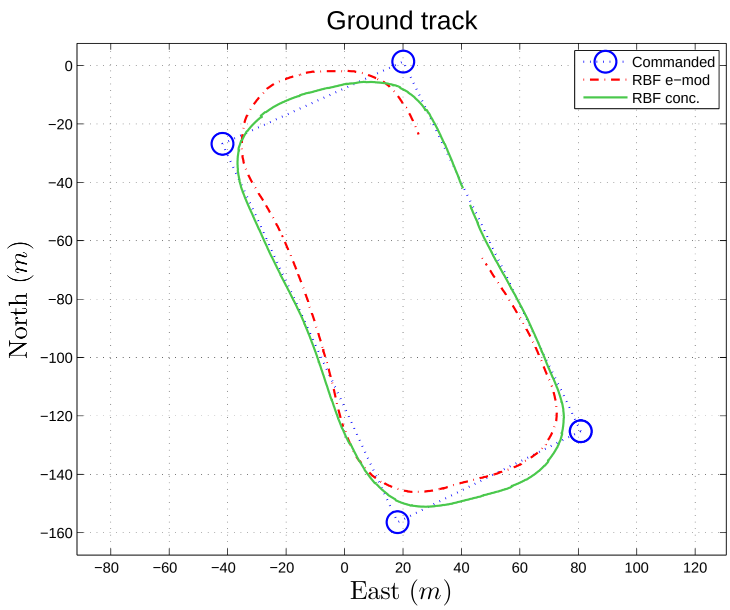


Figure 9. Comparison of ground track for baseline adaptive controller with concurrent learning adaptive controller. Note that the concurrent learning controller has better cross-tracking performance than the baseline adaptive controller

Variable	Mean error with baseline (deg)	Mean error with concurrent learning (deg)
roll angle ϕ	5.54	0.5
angle of attack α	-0.5	-0.2
sideslip angle β	-3.3	-0.3

Table 2. Comparison of mean tracking error for baseline and concurrent learning adaptive controllers

11 CONCLUSION

A concurrent learning adaptive controller that uses online selected, recorded and processed data concurrently with instantaneous data for guaranteeing parameter and tracking error convergence in Model Reference Adaptive Control without requiring persistent excitation is presented. The key question of which data is useful to record and safe to forget is answered through stability analysis of the effects of a data recording algorithm on the switched closed loop system. It is shown that if the system uncertainty can be represented as an unknown combination of known basis functions with fixed parameters, the controller guarantees exponential stability of the zero solution of the closed loop dynamics if the states are exciting over a finite interval; persistency of excitation is not needed. It is also shown that when a radial basis function neural network is used as an adaptive element, the method guarantees that the tracking error and weight error are exponentially 2^{th} ultimately bounded around the origin. This is a stronger result than previously studied neuroadaptive control methods, since it indicates that the NN parameter approach exponentially fast and remain within a compact set around the ideal parameters as opposed to staying bounded around an *a-priori* selected value (often chosen as 0). Numerical simulations and flight test results indicate that the concurrent learning adaptive controllers can outperform Model Reference Adaptive Controllers that rely only on current data. Therefore the key conclusion of this paper is that concurrent use of recorded and current data can guarantee learning of the modeling uncertainty in the model reference adaptive control framework. The learned model is useful for planning and health monitoring purposes. Furthermore, ensuring learning of the uncertainty in this manner provides exponentially decaying bounds on the tracking error and modeling error. This ensures good transient response and excellent tracking in steady-state as the weights converge. Hence, the presented method, and its possible variants, provide a pathway to create provably stable learning-focused adaptive control architecture.

References

Adetola, V., DeHaan, D., and Guay, M. (2009), "Adaptive model predictive control for constrained nonlinear systems," *Systems and Control Letters*, 58, 320–326.

Anderson, B. (1977), "Exponential stability of linear equations arising in adaptive identification," *IEEE Transactions on Automatic Control*, 22, 83–88.

Aström, K.J., and Wittenmark, B., *Adaptive Control*, 2 ed., Readings: Addison-Weseley (1995).

Bodson, M., and Sastry, S. (1984), "Exponential convergence and robustness margins in adaptive control," in *Decision and Control, 1984. The 23rd IEEE Conference on*, dec., Vol. 23, pp. 1282 –1285.

Boyd, S., and Sastry, S. (1986), "Necessary and Sufficient Conditions for Parameter Convergence in Adaptive Control," *Automatica*, 22, 629–639.

Branicky, M.S. (1998), "Multiple Lyapunov Functions and Other Analysis Tools for Switched and Hybrid Systems," *IEEE Transactions on Automatic Control*, 43, 475–482.

Calise, A., Hovakimyan, N., and Idan, M. (2001), "Adaptive Output Feedback Control of Nonlinear Systems Using Neural Networks," *Automatica*, 37, 1201–1211, Special Issue on Neural Networks for Feedback Control.

Cao, C., and Hovakimyan, N. (2008), "Design and Analysis of a Novel Adaptive Control Architecture With Guaranteed Transient Performance," *Automatic Control, IEEE Transactions on*, 53, 586 –591.

Cao, C., Hovakimyan, N., and Jiang, W. (2007), "Intelligent Excitation for Adaptive Control With Unknown Parameters in Reference Input," *IEEE Transactions on Automatic Control*, 52, 1525 –1532.

Chowdhary, G., Mühlegg, M., and Johnson, E. (2013), "On the robustness of Concurrent Learning Adaptive Controllers to measurement noise," *International Journal of Control*, Unpublished manuscript.

Chowdhary, G., Yucelen, T., Mühlegg, M., and Johnson, E. (2012), "Concurrent Learning Adaptive Control of Linear Systems with Exponentially Convergent Bounds," *International Journal of Adaptive Control and Signal Processing*, accepted.

Chowdhary, G., and Johnson, E.N. (2010), "Concurrent Learning for Convergence in Adaptive Control Without Persistency of Excitation," in *49th IEEE Conference on Decision and Control*.

Chowdhary, G., and Johnson, E.N. (2010a), "Least Squares Based Modification Term for Adaptive Control," in *IEEE Conference on Descision and Control*, Atlanta.

Chowdhary, G., and Johnson, E.N. (2010b), "Recursively Updated Least Squares Based Modification Term for Adaptive Control," in *American Control Conference 2010*.

Chowdhary, G., and Johnson, E.N. (2011), "A Singular Value Maximizing Data Recording Algorithm for Concurrent Learning," in *American Control Conference*, submitted.

Chowdhary, G., and Johnson, E.N. (2011), "Theory and Flight Test Validation of a Concurrent Learning Adaptive Controller," *Journal of Guidance Control and Dynamics*, 34, 592–607.

Chowdhary, G., Johnson, E.N., Kimbrell, S.M., Chandramohan, R., and Calise, A.J. (2010), "Flight Test Results of Adaptive Controllers in the Presence of Significant Aircraft Faults," in *AIAA Guidance Navigation and Control Conference*, Toronto, Canada, *Invited*.

Duarte, M.A., and Narendra, K.S. (1989), "Combined Direct and Indirect Approach to Adaptive Control," *IEEE Transactions on Automatic Control*, 34, 1071–1075.

Durham, W. (1993), "Constrained Control Allocation," *AIAA Journal of Guidance, Control, and Dynamics*, 16, 717–772.

Gelb, A., *Applied Optimal Estimation*, Cambridge: MIT Press (1974).

Haddad, W.M., and Chellaboina, V., *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*, Princeton: Princeton University Press (2008).

Haddad, W., Volyanskyy, K., Bailey, J., and Im, J.J. (2011), "Neuroadaptive Output Feedback Control for Automated Anesthesia With Noisy EEG Measurements," *Control Systems Technology, IEEE Transactions on*, 19, 311 –326.

Ioannou, P.A., and Kokotovic, P.V., *Adaptive Systems with Reduced Models*, Secaucus, NJ: Springer Verlag (1983).

Ioannou, P.A., and Sun, J., *Robust Adaptive Control*, Upper Saddle River: Prentice-Hall (1996).

Johnson, E., and Kannan, S. (2005), "Adaptive Trajectory Control for Autonomous Helicopters," *Journal of Guidance Control and Dynamics*, 28, 524–538.

Khasminskii, R., *Stochastic Stability of Differential Equations*, Berlin: Springer (2012).

Kim, N. (2003), "Improved Methods in Neural Network Based Adaptive Output Feedback Control, with Applications to Flight Control," Georgia Institute of Technology, Atlanta Ga.

Kim, Y.H., and Lewis, F., *High-Level Feedback Control with Neural Networks*, Vol. 21 of *Robotics and Intelligent Systems*, Singapore: World Scientific (1998).

Krstić, M., Kanellakopoulos, I., and Kokotović, P., *Nonlinear and Adaptive Control Design*, New York: John Wiley and Sons (1995).

Lavretsky, E. (2009), "Combined/Composite Model Reference Adaptive Control," *IEEE Transactions on Automatic Control*, 54, 2692 –2697.

Liberzon, D. (2005), "Switched Systems," *Handbook of Networked and Embedded Control Systems*, Boston: Birkhauser, pp. 559–574.

Monahemi, M.M., and Krstic, M. (1996), "Control of Wingrock Motion Using Adaptive Feedback Linearization," *Journal of Guidance Control and Dynamics*, 19, 905–912.

Mühlegg, M., Chowdhary, G., and Johnson, E. (2012), "Concurrent Learning Adaptive Control of Linear Systems with Noisy Measurement," in *Proceedings of the AIAA Guidance, Navigation and Control Conference*, August, MN.

Nardi, F. (2000), "Neural Network based Adaptive Algorithms for Nonlinear Control," Georgia Institute of Technology, School of Aerospace Engineering, Atlanta, GA 30332.

Narendra, K.S., and Annaswamy, A.M. (1986), "Robust Adaptive Control in the Presence of Bounded Disturbances," *IEEE Transactions on Automatic Control*, AC-31, 306–315.

Narendra, K., and Annaswamy, A. (1987), "A New Adaptive Law for Robust Adaptation without Persistent Excitation," *IEEE Transactions on Automatic Control*, 32, 134–145.

Narendra, K.S., and Annaswamy, A.M., *Stable Adaptive Systems*, Englewood Cliffs: Prentice-Hall (1989).

Nicolao, G.D., Magi, L., and Scattolini, R. (2000), Vol. 26 of *Progress in Systems and Control Theory*, "Stability and Robustness of Nonlinear Receding Horizon Control," *Nonlinear Model Predictive Control*, Basel-Boston-Berlin: Birkhäuser, pp. 3–22.

Park, J., and Sandberg, I. (1991), "Universal approximation using radial-basis-function networks," *Neural Computatations*, 3, 246–257.

Saad, A.A. (2000), "Simulation and analysis of wing rock physics for a generic fighter model with three degrees of freedom," Air Force Institute of Technology, Air University, Wright-Patterson Air Force Base, Dayton, Ohio.

Sanner, R., and Slotine, J.J. (1992), "Gaussian networks for direct adaptive control," *Neural Networks, IEEE Transactions on*, 3, 837 –863.

Sastry, S., and Bodson, M., *Adaptive Control: Stability, Convergence, and Robustness*, Upper Saddle River: Prentice-Hall (1989).

Singh, S.N., Yim, W., and Wells, W.R. (1995), "Direct Adaptive Control of Wing Rock Motion of Slender Delta Wings," *Journal of Guidance Control and Dynamics*, 18, 25–30.

Slotine, J.J.E., and Li, W., *Applied Nonlinear Control*, Prentice Hall (1991).

Sundararajan, N., Saratchandran, P., and Yan, L., *Fully Tuned Radial Basis Function Neural Networks for Flight Control*, Springer (2002).

Tao, G., *Adaptive Control Design and Analysis*, New York: Wiley (2003).

Volyanskyy, K.Y., Haddad, W.M., and Calise, A.J. (2009), "A New Neuroadaptive Control Architecture for Nonlinear Uncertain Dynamical Systems: Beyond σ and e-Modifications," *IEEE Transactions on Neural Networks*, 20, 1707–1723.

Wise, K. (1995), "Applied controls research topics in the aerospace industry," in *Decision and Control, 1995., Proceedings of the 34th IEEE Conference on*, dec, Vol. 1, pp. 751 –756 vol.1.

Yucelen, T., Kim, K., Calise, A.J., and Nguyen, N. (2011), "Derivative-free output feedback adaptive control of an aeroelastic generic transport model," in *Guidance Navigation and Control Conference*, August, Portland, OR: AIAA.

Yucelen, T., and Calise, A.J. (2010), "Kalman Filter Modification in Adaptive Control," *Journal of Guidance, Control, and Dynamics*, 33, 426–439.