Concurrent Learning Adaptive Control of Linear Systems with Noisy Measurements

Maximilian Mühlegg*, Girish Chowdhary†and Eric N. Johnson‡

Direct model reference adaptive control of linear uncertain dynamic systems with concurrent learning update laws in the presence of noisy measurements is investigated. Previous results have shown that in the absence of noise, concurrent learning adaptive controllers, which use specifically selected and online recorded data concurrently with instantaneous data for adaptation, can guarantee exponential tracking and weight error convergence without requiring persistency of excitation. Here the robustness of a concurrent learning adaptive controller to noisy measurements is established. A Lyapunov framework is employed to show boundedness of tracking and weight error dynamics in a compact ball around the origin in the presence of bounded measurement noise. Furthermore, it is shown that when concurrent learning is used, additional modification terms such as σ- or e-modification are not required for guaranteeing robustness in the presence of noise. Numerical simulations and experimental results demonstrate improved performance.

Nomenclature

A, A_{rm}	Plant and reference model system matrix
B, B_{rm}	Plant and reference model input matrix
$e, \dot{e}$	Tracking error and its time derivative
e_N	Tracking error corrupted by measurement noise
K_x, K_r, Θ	Adaptive weights
K_x^*, K_r^*, Θ^*	Ideal adaptive weights
$\tilde{K}_x, \tilde{K}_r, \tilde{\Theta}$	Parameter error
P	Solution of the Lyapunov equation
p	Number of stored points in the history stack
Q	Positive definite matrix
r	Reference signal
t	Time
u	Control input
u_{rm}, u_{pd}	Feedforward and feedback part of the control input
v	Band-limited unbiased white measurement noise
$w_{(\cdot)}$	Upper bound on the estimation error and the band-limited unbiased white measurement noise
$x, \dot{x}$	State vector and its time derivative
$x_{rm}, \dot{x}_{rm}$	Reference model state vector and its time derivative
$\hat{x}, \dot{\hat{x}}$	Estimate of the state vector and its derivative
y	State vector corrupted by measurement noise
Γ_Θ	Learning rates
$\Delta_{(\cdot)}$	Estimation error
ϵ	Training signal based on stored data
$\theta, \dot{\theta}$	Pitch angle and pitch rate
λ_{min}	Operator; returns minimum eigenvalue
φ	Vector containing the states and reference signal
Ψ, χ	Estimate of the uncertainty term; error between the uncertainty term and its estimate
φ_H, Ω	History stack

*Student, Technical University of Munich, Germany.
†Postdoctoral associate, Massachusetts Institute of Technology, MA.
‡Assoc. Professor, Georgia Institute of Technology, GA.

I. Introduction

The design objective in Model Reference Adaptive Control (MRAC) of linear plants is to adjust the controller gains in a stable and robust way, such that the dynamics of the closed loop plant mimic the behavior of a reference model with chosen characteristics. The underlying assumption is that there exists an ideal set of linear gains which ensures that the design objective can be met. This assumption is often referred to as matching condition. The aim is to guarantee closed loop stability and to adapt the gains online such that they approach these ideal values. If the adaptive gains do converge to their ideal values, then the closed loop plant behaves like the reference model and the chosen characteristics are obtained. In most classic and some recent MRAC approaches, such as in Narendra[1] , Ioannou[2] , Aström[3] , Lavretsky[4] , Tao[5] and Cao[6] , only instantaneous data is used for adaptation. According to Boyd and Sastry in Ref. 7 , the plant states are required to be persistently exciting in order for the adaptive gains to converge to their ideal values. Not only does this inflict physical stress on the system, but it is also often problematic to monitor whether a signal is persistently exciting.

Chowdhary[8,9] introduced an approach which uses instantaneous data concurrently with recorded data to achieve exponential stability of the zero solution of tracking and parameter error dynamics. It was shown that the plant states do not have to be persistently excited for this purpose. In fact, an online verifiable condition on the linear independence of the recorded data is sufficient to ensure exponential stability for a wide class of modeling uncertainties.

Inherent factors such as measurement noise have not been extensively studied in stability analysis of model reference adaptive controllers. These factors can present serious difficulties for the practical application of adaptive control schemes. Systems which are subject to unknown disturbances have been shown to be stable under certain conditions, such as parametrizable disturbances by Ge[10] or requirements on the persistency of excitation of the system signals (see Ref 11). Another possibility to achieve boundedness of all system signals is to modify the adaptive learning laws by additional damping terms, such as the classic σ-modification of Ioannou[2] or the e-modification of Narendra[12] . However, these modifications only ensure boundedness of the weights around a preselected value and therefore prevent the parameters to converge to their ideal values. In contrast, with concurrent learning the weights are bounded around their true values, thus leading to increased performance of the closed loop system.

The aim of this paper is to study MRAC of linear uncertain dynamical systems subject to measurement noise. We establish robustness of a concurrent learning adaptive controller against bounded measurement noise. Particularly, we show that the adaptive gains converge to a compact domain around the ideal gains and the tracking error stays ultimately bounded in a compact neighborhood of the origin using Lyapunov analysis. We accomplish this by including an Optimal Fixed Point Smoother[13] to accurately estimate the first derivative of the state for recorded data points. With numerical simulation we show improved performance compared to only instantaneous update laws with damping terms. Finally the robustness of the proposed concurrent learning adaptive controller is validated on a real world system.

The organization of the paper is as follows: In section II we outline the classical problem of MRAC of linear plants when subject to measurement noise. In section III we present the concurrent learning adaptive controller for linear plants and show the robustness against measurement noise using Lyapunov analysis. In section IV we present the results of a numerical simulation. In section V we implement the presented controller in a Quanser 3 DOF helicopter and present the results. The paper is concluded in section VI.

II. Linear Direct Model Reference Adaptive Control

A basic explanation of the classical linear direct MRAC is given in this section. The reader is referred to Aström[3] , Narendra[1] and Tao[5] for detailed explanations. Consider the following time-invariant linear system of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $u(t) \in \mathbb{R}^m$ is the control input. We assume that the pair (A, B) in Eq. 1 is controllable and that B has full column rank. Furthermore, we assume that $u(t)$ is restricted to the class of admissible control inputs consisting of measurable functions. Suppose the full state vector $x(t)$ corrupted by additive band-limited unbiased white measurement noise $v(t) \in \mathbb{R}^n$ is available as sensor measurement:

$$y(t) = x(t) + v(t) \tag{2}$$

Consider the following reference model dynamics, which characterizes the desired response of the system

$$\dot{x}_{rm}(t) = A_{rm}x_{rm}(t) + B_{rm}r(t) \tag{3}$$

where $x_{rm}(t), \dot{x}_{rm}(t) \in \mathbb{R}^n$, $A_{rm} \in \mathbb{R}^{n\times n}$, $B_{rm} \in \mathbb{R}^{n\times m}$ and $r(t) \in \mathbb{R}^m$ is a bounded, piecewise continuous reference signal. The state matrix of the reference model A_{rm} is chosen to be Hurwitz with desired transient and stability properties of the closed loop system. The control input of Eq. 1 consists of a feedforward part $u_{rm}(t) = K_r^T(t)r(t)$ with time varying weights $K_r \in \mathbb{R}^{m\times m}$ and a feedback part $u_{pd}(t) = K_x^T(t)y(t)$ with time varying weights $K_x(t) \in \mathbb{R}^{n\times m}$ such that

$$u(t) = u_{rm}(t) + u_{pd}(t). \tag{4}$$

Substituting Eq. 4 and 2 in Eq. 1 yields

$$\dot{x}(t) = Ax(t) + BK_r^T(t)r(t) + BK_x^T(t)x(t) + BK_x^T(t)v(t). \tag{5}$$

The design objective is to match the closed loop plant dynamics of Eq. 5 and the reference model dynamics in Eq. 3. Therefore, assume the following matching conditions:

Assumption 1. *There exist optimal constant weight matrices $K_r^* \in \mathbb{R}^{m\times m}$ and $K_x^* \in \mathbb{R}^{n\times m}$ such that the following well known matching conditions hold*

$$\begin{aligned} A_{rm} &= A + BK_x^{*T} \\ B_{rm} &= BK_r^{*T}. \end{aligned} \tag{6}$$

The weight errors $\tilde{K}_x(t) \in \mathbb{R}^{n\times m}$ and $\tilde{K}_r(t) \in \mathbb{R}^{m\times m}$ between the current and optimal weights can be expressed as

$$\begin{aligned} \tilde{K}_x(t) &= K_x(t) - K_x^* \\ \tilde{K}_r(t) &= K_r(t) - K_r^* \end{aligned} \tag{7}$$

Adding and subtracting $BK_x^{*^T}y(t)$ and $BK_r^{*^T}r(t)$ in Eq. 5 and using Eq. 7 as well as Assumption 1 we obtain

$$\dot{x}(t) = A_{rm}x(t) + B_{rm}r(t) + B\tilde{K}_x^T(t)x(t) + B\tilde{K}_r^T(t)r(t) + B{K_x}^T(t)v(t). \tag{8}$$

Note, that if $\tilde{K}(t) = 0$, $\tilde{K}_r(t) = 0$ and $v(t) = 0$, Eq. 8 reduces to the reference model dynamics of Eq. 3 and the design objective would be met. Define $\Theta(t) \in \mathbb{R}^{(m+n)\times m}$, $\hat{\varphi}(t) \in \mathbb{R}^{(m+n)}$ and $\varphi(t) \in \mathbb{R}^{(m+n)}$ such that

$$\Theta(t) = \begin{bmatrix} K_r(t) \\ K_x(t) \end{bmatrix}; \varphi(t) = \begin{bmatrix} r(t) \\ x(t) \end{bmatrix}; \hat{\varphi}(t) = \begin{bmatrix} r(t) \\ y(t) \end{bmatrix}. \tag{9}$$

Along with $\Theta^* = [K_r^{*T} K_x^{*T}]^T$ and Eq. 9, Eq. 7 can be expressed as $\tilde{\Theta}(t) = \Theta(t) - \Theta^*$. Hence, we can rewrite Eq. 8 and obtain

$$\dot{x}(t) = A_{rm}x(t) + B_{rm}r(t) + B\tilde{\Theta}^T(t)\varphi(t) + B{K_x}^T(t)v(t). \tag{10}$$

Define the tracking error $e(t) \in \mathbb{R}^n$ as $e(t) = x(t) - x_{rm}(t)$. Differentiating $e(t)$ with respect to time yields the error dynamics

$$\dot{e}(t) = A_{rm}e(t) + B\tilde{\Theta}^T(t)\varphi(t) + B{K_x}^T(t)v(t). \tag{11}$$

It follows from converse Lyapunov theory[14] that there exists a unique positive definite matrix $P \in \mathbb{R}^{n\times n}$ satisfying the Lyapunov equation

$$0 = A_{rm}^T P + PA_{rm} + Q \tag{12}$$

for any positive definite matrix $Q \in \mathbb{R}^{n\times n}$. Let $\Gamma_\Theta \in \mathbb{R}^{(n+m)\times(n+m)}$ be a matrix containing positive definite learning rates on the main diagonal. Consider the well known adaptive law for MRAC of linear systems $\dot{\Theta}(t) = -\Gamma_\Theta\varphi(t)e^T(t)PB$, which uses only instantaneous data for adaptation (see e.g. Aström[3] , Narendra[11] , Ioannou[2]). The vector $\varphi(t)$ and the tracking error $e(t)$ are not available due to the presence of noise. Instead, with $e_N(t) = y(t) - x_{rm}(t) = e(t) + v(t)$ only the following update law can be used:

$$\dot{\Theta}(t) = -\Gamma_\Theta\hat{\varphi}(t)e_N^T PB \tag{13}$$

It is well known that in the presence of bounded disturbances, classic gradient based learning laws for direct model reference adaptive control alone do not guarantee boundedness of the system signals. In fact, additional terms, such as the classic σ-modification of Ioannou[2] or the e-modification of Narendra,[12] have to be added in order to ensure boundedness of the weight and tracking error. The following section shows that if concurrent learning adaptive control is used, these damping terms become unnecessary.

III. Concurrent Learning Adaptive Control in the presence of measurement noise

The key idea behind concurrent learning is that online recorded data is used concurrently with current data. The state vector and the reference input are stored in a history stack $\varphi_H \in \mathbb{R}^{(m+n)\times p}$ such that $\varphi_H = [\varphi_1\varphi_2...\varphi_p]$, where φ_j denotes the vector $\varphi(t_j)$ at the time instant t_i and p is the number of stored points. Once the history stack is full, an algorithm is applied in order to store only points which increase the minimum singular value of the history stack (see Chowdhary et al.[9,15]). By estimating the scalar terms $\Theta^{*T}\varphi_j$ for each stored point j and comparing them with the current terms $\Theta^T(t)\varphi_j$, a gradient decent based adaptation can be established for every stored point to reduce the weight error. It is important to note that knowledge of Θ^* is not required at any time. Once the history stack contains at least as many linearly independent points as the dimension of the regressor vector $\varphi(t)$ (referred to as the rank condition), in the absence of noise, concurrent adaptation on instantaneous and recorded data guarantees exponential tracking and convergence of the adaptive parameters (see e.g. Chowdhary et al.[8,9]). In the following, we show that in the presence of measurement noise, concurrent learning adaptive control ensures convergence of the tracking and weight error dynamics to a compact set and hence boundedness of all system signals.

In order to estimate $\Theta^{*T}\varphi_j$, Eq. 10 needs to be solved. For this, a measurement of $\dot{x}_j$ for each stored data point φ_j is required. In most cases the state derivative cannot be measured directly (an example where it is possible to measure the state derivative directly is aircraft velocity). Several methods, such as the Nonlinear Adaptive Variable Structure Derivative Estimator of Xu,[16] exist to estimate the state derivative. However, online methods that rely only on instantaneous data $x(t)$ tend to be affected by measurement noise. Furthermore, estimating $\dot{x}(t)$ with only a Forward Kalman Filter (or a low pass filter) invariably adds a delay to the estimates. On the other hand, the optimal fixed point smoothing is a method is a method that arrives at an estimate of the state derivative without delay (see Gelb[13]).

For concurrent learning, the state derivative does not need to be available at the current time instant t. Rather, the state derivative only needs to be estimated for specifically selected points from the past. Therefore, it is possible to use methods such as optimal fixed point smoothing to obtain good estimates of $\dot{x}_j$ for a data point after a finite time has elapsed after it has been stored.

Optimal fixed point smoothing is a method for arriving at a state estimate at some time t, where $0 \leq t \leq T$, by using all available data up to time T. In particular, the smoother combines a forward Kalman Filter, which operates on all data before time t, and a backward Kalman Filter, which operates on all data after time t up to T, to arrive at an estimate of the state derivative by using all available information (see Gelb[13] ; see Appendix for smoother equations). Note that with full state feedback assumed, the smoother is not intended to be a state observer, but as a noise filter and a reliable estimator of $\dot{x}$ in the presence of band-limited unbiased white noise.

Let $\hat{x}_j \in \mathbb{R}^n$ be the estimate of x_j and let $\dot{\hat{x}}_j \in \mathbb{R}^n$ be the estimate of $\dot{x}_j$ of the j^{th} stored data point obtained using an Optimal Fixed Point Smoother. Note here, that these are not functions of time, but constant vectors stored in the history stack. The Kalman Filter estimation error and covariance are bounded (with an appropriate choice of process and noise covariances, see Gelb[13] , Jazwinski[17]). Then, we have $\hat{x}_j = x_j + \Delta_{SM,j}$ where $\|\Delta_{SM,j}\| \leq w_2$, and $\dot{\hat{x}}_j = \dot{x}_j + \Delta_{SMD,j}$ where $\|\Delta_{SMD,j}\| \leq w_3$. Since we have assumed B to have full column rank, $(B^TB)^{-1}$ exists. Then, solving Eq. 10 to $\Theta^{*T}\varphi_j$ for the j^{th} data point yields

$$\Theta^{*T}\varphi_j = (B^TB)^{-1}B^T(A_{rm}x_j - \dot{x}_j + B\Theta_j^T\varphi_j + BK_{x,j}^Tv_j + B_{rm}r_j). \tag{14}$$

In the absence of measurement noise, the term $\Theta^{*T}\varphi_j$ could have been recreated for the j^{th} data point by calculating the right hand side of Eq. 14. However, since all terms aside from $B_{rm}r(t)$ are unknown due to the presence of noise, define instead $\Psi_j \in \mathbb{R}^m$ as an estimate of $\Theta^{*T}\varphi_j$:

$$\Psi_j = (B^TB)^{-1}B^T(A_{rm}\hat{x}_j - \dot{\hat{x}}_j + BK_r^Tr_j + BK_x^Ty_j + B_{rm}r_j) \tag{15}$$

For the j^{th} data point, define the error between $\Theta^{*T}\varphi_j$ and its estimate as $\chi_j = \Psi_j - \Theta^{*T}\varphi_j$. Then with $B\Theta^T\varphi_j + BK_{x,j}^Tv_j = BK_{r,j}^Tr_j + BK_{x,j}^Ty_j$ we have

$$\chi_j = (B^TB)^{-1}B^T(A_{rm}\Delta_{SM,j} - \Delta_{SMD,j}). \tag{16}$$

Note, that once the j^{th} point is stored, χ_j is constant. The background training signal for the j^{th} data point in the nominal case is created by $\epsilon_j(t) = \Theta(t)\varphi_j - \Theta^{*T}\varphi_j$ (see e.g. Chowdhary[9]). Again, note that explicit knowledge on Θ^* is not required, only on the term $\Theta^{*T}\varphi_j$. However, due to the presence of noise, both terms $\Theta(t)\varphi_j$ and $\Theta^{*T}\varphi_j$ are unknown. Define $\hat{\hat{\varphi}}_j \in \mathbb{R}^{(m+n)}$ such that

$$\hat{\hat{\varphi}}_j = \begin{bmatrix} r_j \\ \hat{x}_j \end{bmatrix}. \tag{17}$$

Define the modified background signal $\hat{\epsilon} \in \mathbb{R}^m$ for the j^{th} data point:

$$\hat{\epsilon}_j = \Theta(t)\hat{\hat{\varphi}}_j - \Psi_j \tag{18}$$

Noting that $\Psi_j = \chi_j + \Theta^{*T}\varphi_j$ we have

$$\hat{\epsilon}_j(t) = \tilde{\Theta}^T(t)\hat{\hat{\varphi}}_j + K_x^{*T}\Delta_{SM,j} - \chi_j. \tag{19}$$

The concurrent learning weight update law in the presence of measurement noise is then given by:

$$\dot{\Theta}(t) = -\Gamma_\Theta\hat{\varphi}(t)e_N^T(t)PB - \Gamma_\Theta\sum_{j=1}^{p}\hat{\hat{\varphi}}_j\hat{\epsilon}_j^T(t) \tag{20}$$

The following theorem contains the main results and shows that the concurrent learning adaptive controller with the update law in Eq. 20 ensures that all system signals are uniformly ultimately bounded. The proof is as follows: First boundedness of the states in a compact ball B_α is assumed such that $B_\alpha = \{x(t)|\|x(t)\| \leq \alpha\}$. A compact positive invariant set is constructed that is valid for all $x(t) \in B_\alpha$. Finally, it is proven that a requirement on the exogenous input and the reference model exists, such that if $x(0)$ then $x(t) \in B_\alpha \forall t \geq t_0$. Since $x(t)$ does not leave B_α, uniform ultimate boundedness of the closed loop system is established. For simplicity the following theorem assumes that the rank condition in the history stack is met at t_0. Assume, that in order to populate the empty history stack up to t_0 the update law of Eq. 13 was used with additional damping term, such as σ-modification:

$$\dot{\Theta}(t) = -\Gamma_\Theta\hat{\varphi}(t)e_N^T(t)PB - \Gamma_\Theta\kappa(\Theta(t) - \Theta_0) \tag{21}$$

Here $\kappa > 0$ and Θ_0 is a preselected value around which the weights are bounded.

Theorem 1. *Consider the system in Eq. 1, the reference model in Eq. 3, the control law of Eq. 4 and the output of the system corrupted by band-limited unbiased white measurement noise in Eq. 2. Assume that an Optimal Fixed Point Smoother is used to estimate the state and its first time derivative. Furthermore, assume that the measurement noise as well as the estimation error of the Optimal Fixed Point Smoother are bounded by $\|v(t)\| \leq w_1$, $\|\Delta_{SM,j}\| \leq w_2$ and $\|\Delta_{SMD,j}\| \leq w_3$. Let $\varphi_H = [\hat{\hat{\varphi}}_1, \hat{\hat{\varphi}}_2, ..., \hat{\hat{\varphi}}_p]$ be the history stack matrix containing recorded states and reference signals. Let $p \geq n + m$ at t_0 be the number of recorded linear independent data points, which were recorded during a phase where the update law of Eq. 21 was used. Define $B_\alpha = \{x(t)|\|x(t)\| \leq \alpha\}$, where B_α contains the origin and let $x(t_0) \in B_\alpha$. Let $V(e, \tilde{\Theta})$ be a positive definite and radially unbounded function. Let P be the solution to the Lyapunov equation in Eq. 12 for a positive definite matrix Q. If the exogenous reference input $r(t)$ is such that the state of the reference model is bounded in a compact ball $B_\gamma = \{x_{rm} : \|x_{rm}\| \leq \gamma\} \forall t \geq t_0$ and if for all $x(t) \in B_\alpha$ there exists a compact positively invariant set $\Omega_\beta = \{(e, \tilde{\Theta})|V(e, \tilde{\Theta}) \leq \beta\}$ with $\beta \geq 0$ and a positive scalar $\delta \geq 0$ with $\delta = \sqrt{\frac{\beta}{\lambda_{min}(P)}}$ such that $\gamma \leq \alpha - \delta$, the concurrent learning weight update laws of Eq. 20 guarantee uniform ultimate boundedness of all system signals.*

Proof. Consider the following positive definite and radially unbounded Lyapunov candidate:

$$V(e(t), \tilde{\Theta}(t)) = \frac{1}{2}e^T(t)Pe(t) + \frac{1}{2}tr(\tilde{\Theta}^T(t){\Gamma_W}^{-1}\tilde{\Theta}(t)) \tag{22}$$

Taking the time derivative of Eq. 22 along the trajectory of the tracking error dynamics of Eq. 11 yields

$$\dot{V}(e(t), \tilde{\Theta}(t)) = -\frac{1}{2}e^T(t)Qe(t) + e^T(t)PB\tilde{\Theta}^T(t)\varphi(t) + e^T(t)PB{K_x}^T(t)v(t) + tr(\tilde{\Theta}^T(t){\Gamma_\Theta}^{-1}\dot{\Theta}(t)). \tag{23}$$

Inserting the update law of Eq. 20, the training signal of Eq. 19 and canceling like terms, we have

$$\begin{aligned}\dot{V}(e(t), \tilde{\Theta}(t)) = &-\frac{1}{2}e^T(t)Qe(t) + e^T(t)PBK_x^{*T}v(t) - tr[\tilde{\Theta}^T(t)\hat{\varphi}v^T(t)PB \\ &\tilde{\Theta}^T(t)\sum_{j=1}^{p}\hat{\hat{\varphi}}_j\hat{\hat{\varphi}}_j^T\tilde{\Theta}(t) + \tilde{\Theta}^T(t)\sum_{j=1}^{p}\hat{\hat{\varphi}}_j\Delta_{SM,j}^T K_x^* - \tilde{\Theta}^T(t)\sum_{j=1}^{p}\hat{\hat{\varphi}}_j\chi_j^T].\end{aligned} \tag{24}$$

Note, that $\sum_{j=1}^{p} \hat{\varphi}_j \hat{\varphi}_j^T = \varphi_H \varphi_H^T$ and let $\Omega = \varphi_H \varphi_H^T$. Assume, that the reference input is bounded by $\|r(t)\| \leq \bar{r}$ and let $\|\varphi(t)\| \leq \bar{\varphi}\ \forall x(t) \in B_\alpha$ and $\|r(t)\| \leq \bar{r}$. Let the band-limited white noise be bounded by $\|v(t)\| \leq w_1$ and assume an upper bound on the ideal parameters with $\|\Theta^*\| \leq \bar{\Theta}$. Furthermore, the error between ${\Theta^*}^T \varphi_j$ and its estimate Ψ_j can be bounded by

$$\|\chi_j\| \leq \|(B^T B)^{-1} B^T\| (\|A_{rm}\| w_2 + w_3) = \bar{\chi}. \tag{25}$$

Then Eq. 24 can be bounded by

$$\begin{aligned} \dot{V}(e(t), \tilde{\Theta}(t)) \leq & -\frac{1}{2}\lambda_{min}(Q)\|e\|^2 + w_1 \bar{\Theta}\|PB\|\|e\| + w_1 \bar{\varphi}\|PB\|\|\tilde{\Theta}\| + w_1^2 \|PB\|\|\tilde{\Theta}\| \\ & - \lambda_{min}(\Omega)\|\tilde{\Theta}\|^2 + p w_2 (\bar{\varphi} + w_2)\bar{\Theta}\|\tilde{\Theta}\| + p\bar{\chi}(\bar{\varphi} + w_2)\|\tilde{\Theta}\|. \end{aligned} \tag{26}$$

For convenience define the following simplification:

$$c = w_1 \bar{\varphi}\|PB\| + w_1^2\|PB\| + p w_2(\bar{\varphi} + w_2)\bar{\Theta} + p\bar{\chi}(\bar{\varphi} + w_2) \tag{27}$$

Then Eq. 26 reduces to

$$\dot{V}(e(t), \tilde{\Theta}(t)) \leq -\frac{1}{2}\lambda_{min}(Q)\|e\|^2 + w_1\bar{\Theta}\|PB\|\|e\| + c\|\tilde{\Theta}\| - \lambda_{min}(\Omega)\|\tilde{\Theta}\|^2. \tag{28}$$

Consider the following set:

$$\begin{aligned} \Omega_V = \{(e(t), \tilde{\Theta}(t)) | & \frac{1}{2}\lambda_{min}(Q)(\|e\| - \frac{w_1\bar{\Theta}\|PB\|}{\lambda_{min}(Q)})^2 + \lambda_{min}(\Omega)(\|\tilde{\Theta}\| - \frac{c}{2\lambda_{min}(\Omega)})^2 \\ & \leq \frac{w_1^2\bar{\Theta}^2\|PB\|^2}{2\lambda_{min}(Q)} + \frac{c^2}{4\lambda_{min}(\Omega)}\} \end{aligned} \tag{29}$$

Where $\dot{V}(e(t), \tilde{\Theta}(t)) < 0$ outside of Ω_V. Consider the contour of Ω_V given by:

$$\frac{1}{2}\lambda_{min}(Q)(\|e\| - \frac{w_1\bar{\Theta}\|PB\|}{\lambda_{min}(Q)})^2 + \lambda_{min}(\Omega)(\|\tilde{\Theta}\| - \frac{c}{2\lambda_{min}(\Omega)})^2 - \frac{w_1^2\bar{\Theta}^2\|PB\|^2}{2\lambda_{min}(Q)} - \frac{c^2}{4\lambda_{min}(\Omega)} = 0 \tag{30}$$

The equation above describes a rotated and translated ellipse. Hence, the largest $\|e\|$ on the contour of Ω_V, denoted with e_{lim}, is obtained by setting the second bracketed term in Eq. 30 to zero and solving to $\|e\|$:

$$e_{lim} = \frac{w_1\bar{\Theta}\|PB\|}{\lambda_{min}(Q)} + \sqrt{\frac{w_1^2\bar{\Theta}^2\|PB\|^2}{\lambda_{min}^2(Q)} + \frac{c^2}{2\lambda_{min}(\Omega)\lambda_{min}(Q)}} \tag{31}$$

Similarly, the largest $\|\tilde{\Theta}\|$ on the contour of Ω_V, denoted with $\tilde{\Theta}_{lim}$, is obtained by setting the first bracketed term in Eq. 30 to zero and solving to $\|\tilde{\Theta}\|$:

$$\tilde{\Theta}_{lim} = \frac{c}{2\lambda_{min}(\Omega)} + \sqrt{\frac{w_1^2\bar{\Theta}^2\|PB\|^2}{2\lambda_{min}(Q)\lambda_{min}(\Omega)} + \frac{c^2}{4\lambda_{min}^2(\Omega)}} \tag{32}$$

Let $\beta \geq max_{\|e\| \leq e_{lim}, \|\tilde{\Theta}\| \leq \tilde{\Theta}_{lim}}(V(e, \tilde{\Theta}))$ and define the following set:

$$\Omega_\beta = \{(e(t), \tilde{\Theta}(t)) | V(e, \tilde{\Theta}) \leq \beta\} \tag{33}$$

Then $\Omega_V \subseteq \Omega_\beta$ where Ω_β is positively invariant. The analysis above holds for all $x(t) \in B_\alpha$. Hence, in order to complete the proof, it has to be shown that if $x(t_0) \in B_\alpha$, $x(t) \in B_\alpha \forall t \geq t_0$. Therefore define the maximum tracking error for which $V(e, \tilde{\Theta}) = \beta$ with δ, where

$$\delta = \sqrt{\frac{\beta}{\lambda_{min}(P)}}. \tag{34}$$

If the exogenous input $r(t)$ is such that the state $x_{rm}(t)$ of the bounded input bounded output reference model of Eq. 3 remains bounded in the compact ball $B_\gamma = \{x_{rm} | \|x_{rm}\| \leq \gamma\} \forall t \geq t_0$ and $\gamma \leq \alpha - \delta$, then $x(t) \in B_\alpha \forall t \geq t_0$. Hence, the bounds on $\varphi(t)$ hold and following LaSalle's invariance principle the closed loop system is uniformly ultimately bounded. $\square$

Remark 1. *With concurrent learning adaptive control the weights are bounded around their true values. Whereas, with damping terms, such as σ-modification or e-modification the weights are bounded around a preselected value (often chosen to be 0). Hence, the performance of the control law is expected to improve.*

Remark 2. *The output of the fixed point smoother is not used in the closed loop system until after estimation has finished. The estimates are used to calculate the (constant) uncertainty term Ψ_j for the j^{th} data point, which is then stored along with the history stack φ_H. Hence, with an appropriate choice of process and noise covariances, the estimation error and covariance of the Kalman Filter are bounded. Furthermore, the states of the smoother were not included in the Lyapunov analysis of the closed loop system in order to prove stability.*

Remark 3. *The limits of the tracking and parameter error in Eq. 31 and 32 for which $V(e, \tilde{\Theta})$ is still negative are primarily dependent on the performance of the Optimal Fixed Point Smoother. Better estimation reduces the size of the positive invariant set Ω_β and therefore leads to increased performance of the closed loop system.*

IV. Demonstration through Numerical Simulation

In this section we present simulation results for the control of a simple linear system with concurrent learning subject to noisy measurements. We use the following second order plant:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 4 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{35}$$

Assume that the state matrix is unknown for the purpose of the control design. Note that as the eigenvalues are $\lambda_1 = 1.236$ and $\lambda_2 = -3.236$, the plant is unstable. Furthermore, the output is corrupted such that

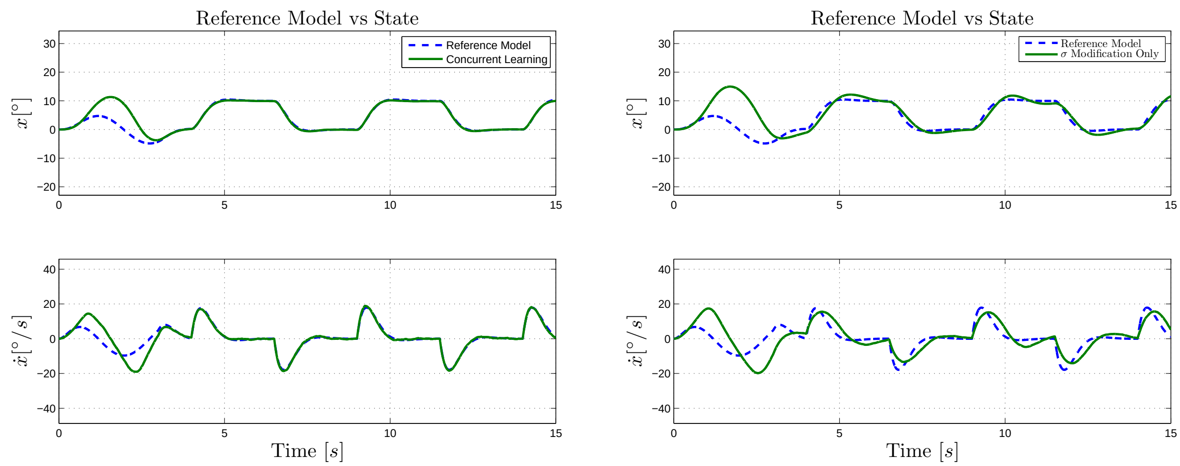
$$y(t) = x(t) + v(t) \tag{36}$$

where $v(t)$ is unbiased band-limited white noise. The design objective is to make the plant behave like the following second order reference model with natural frequency of $3.9 rad/sec$ and damping ratio of 0.7:

$$\dot{x}_{rm}(t) = \begin{bmatrix} 0 & 1 \\ -15.21 & -5.46 \end{bmatrix} x_{rm}(t) + \begin{bmatrix} 0 \\ 15.21 \end{bmatrix} r(t) \tag{37}$$

The simulation runs for 15 seconds with a time-step of 0.001 seconds. The reference signal $r(t)$ is comprised out of 2 parts. The first 3 seconds are occupied by a sine function with frequency $2rad/sec$ and an amplitude of $5°$ in order to provide sufficient excitation in the history stack. The reference command is equal to $0°$ between second 3 and 4. The second part is composed out of three step inputs with amplitude $10°$ at seconds 4, 9 and 14. Each step input lasts for 2.5 seconds. The control input of Eq. 4 was used, along with the concurrent learning weight update law of Eq. 20 with $\Gamma_\Theta = 20$. The history stack is empty for t_0 and contains at most 8 data points. Until the history contain at least as many linearly independent points as the dimension of the regressor vector $\varphi(t)$, a σ-modification term with the gain $\kappa = 0.01$ (see Ioannou[2]) is added to the update law in order to guarantee boundedness of all system signals. Along with the states and the reference input, the uncertainty term Ψ_j is stored by evaluating Eq. 15. Only data points which are sufficiently different from the last stored point were considered for storage. Furthermore, data points were stored using an algorithm that adds or replaces existing data points only if the minimum singular value $\sigma_{min}(\varphi_H)$ increases (see Chowdhary[15]). The background training signal is constructed with Eq. 19. The adaptive weights K_x were initialized at zero and K_r was initialized at 7.5. The simulation results of concurrent learning are compared to the results of the instantaneous update law of Eq. 13 with σ modification term, where $\kappa = 0.01$, and the same learning rate Γ_Θ.

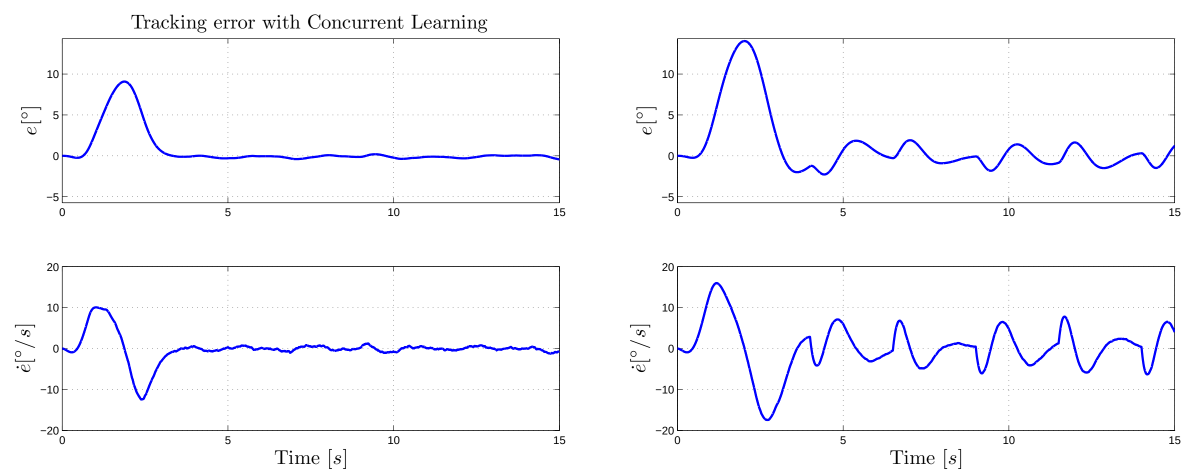
Figure 1(a) shows the tracking performance of the adaptive controller with concurrent learning. It can be seen that the states of the reference model and the plant states are hardly distinguishable after about 3 seconds. In comparison, the plant in Figure 1(b) does not track the reference model as adequately when only the instantaneous update law is used. The performance difference is also apparent in the evolution of the tracking error in Figures 2(a) and 2(b). The reason for that becomes clear if we examine the evolution of the adaptive weights in Figures 3(a) and 3(b). After the rank of the recorded data is equal to the dimension of the regressor vector $\varphi(t)$ (in this case equal to 3), the tracking and weight error dynamics are guaranteed to be exponentially stable if there is no measurement noise (as shown in Chowdhary[9]). In the presence of measurement noise, the tracking and weight error dynamics are bounded. While in the case of concurrent learning the weights converge to a compact set around their ideal values (see Figure 3(a)) in



(a) (b)

Figure 1. Comparison of reference model states and filtered plant states with concurrent learning in Figure 1(a) during the numerical simulation. Comparison of reference model states and filtered plant states with σ-modification only in Figure 1(b) during the numerical simulation.

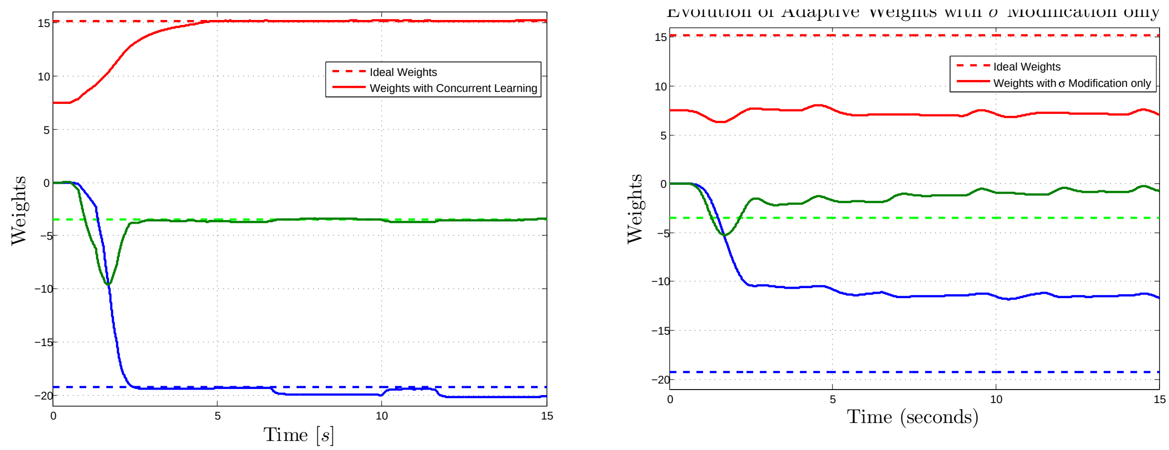
the case of only instantaneous adaptation the weights stay bounded around a preselected value (see Figure 3(b)), here chosen to be zero.



(a) (b)

Figure 2. Tracking Error with concurrent learning in Figure 2(a) during the numerical simulation. Tracking Error with σ Modification only in Figure 2(b) during the numerical simulation.

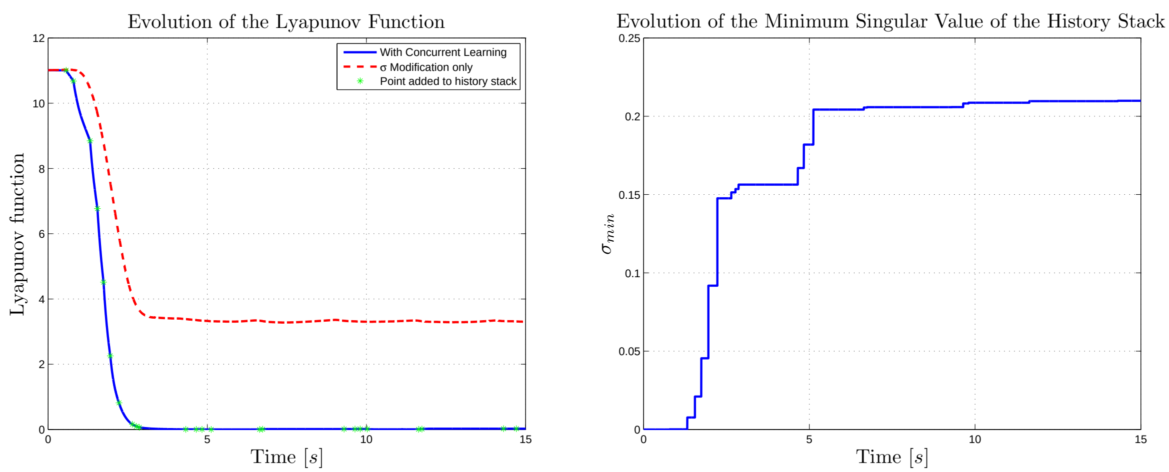
The convergence of the tracking error and the weights to a compact domain can also be seen in the evolution of the Lyapunov function in Figure 4(a). Even though for concurrent learning a deviation from zero is not visible in the figure after 5 seconds, the Lyapunov function never entirely stays equal to zero. This is again due to the measurement noise. Figure 4(b) shows the evolution of the singular value when the singular value maximization approach of Chowdhary[15] is used to update the history stack. As the minimum singular value increases, the size of the compact domain around the true weights, in which the current weights reside, decreases. Figure 6 shows the evolution of the plant input. Even though the input is strongly excited, its absolute value is bounded. Finally, Figure 5(a) shows the unfiltered measured output of the system. As can be seen, the performance of the Optimal Fixed Point Smoother is superior to a Forward Kalman Filter. Notably, the Optimal Fixed Point Smoother estimates the time derivative of the state more precisely, which thereby reduces the training signal error.



(a)

(b)

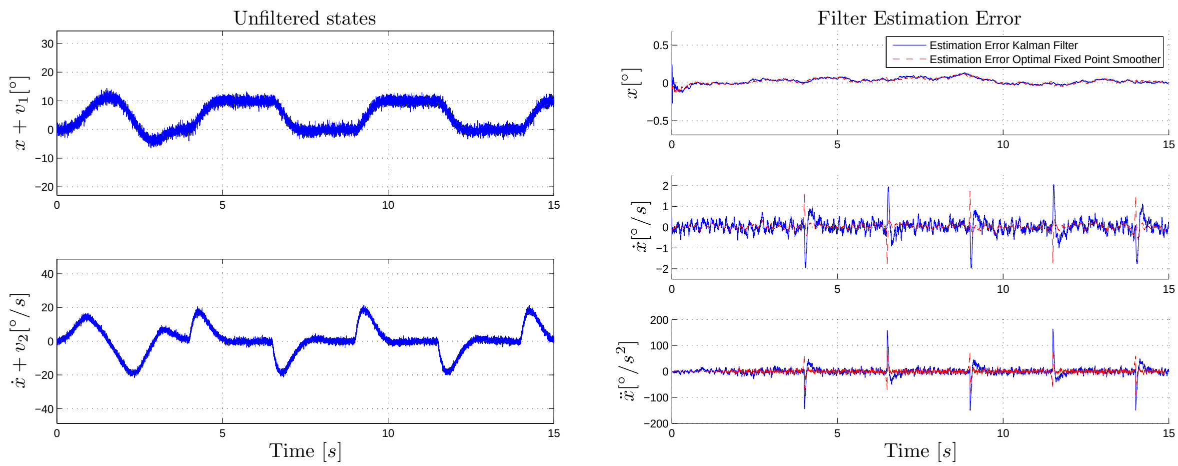
Figure 3. Evolution of the weights with concurrent learning in Figure 3(a) during the numerical simulation. Note that the weights converge to a closed set around their true values and stay there. Evolution of the weights with σ-modification only in Figure 3(b) during the numerical simulation. Note that even though the weights are bounded, they do not converge.



(a)

(b)

Figure 4. Evolution of the Lyapunov Function comparing concurrent learning with an instantaneous update with σ-modification only in Figure 4(a) during the numerical simulation. Evolution of the Minimum Singular Value of the history stack φ_H in Figure 4(b) during the numerical simulation.



(a) (b)

Figure 5. Unfiltered states corrupted by measurement noise in Figure 5(a) during the numerical simulation. Estimation Error of the Forward Kalman Filter and the Optimal Fixed Point Smoother in Figure 5(b) during the numerical simulation.

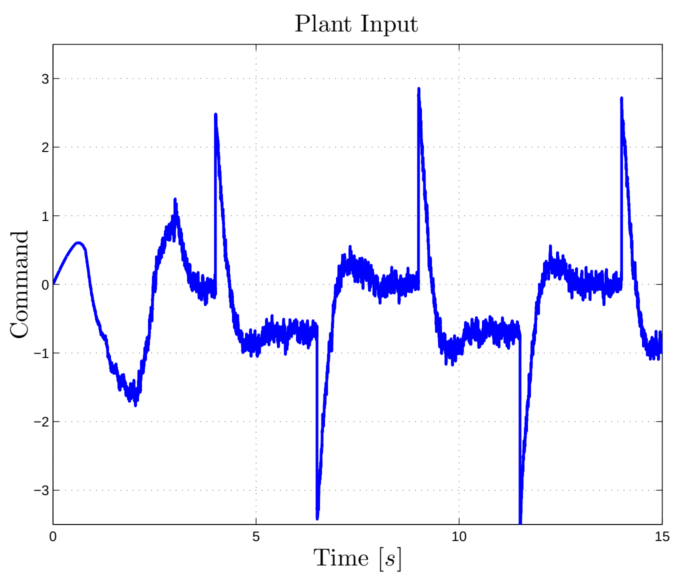


Figure 6. Plant input with concurrent learning during the numerical simulation.

V. Demonstration through Experimentation on a Quanser 3 DOF Helicopter

The purpose of this section is to validate the presented concurrent learning adaptive controller on a real world system, namely the Quanser 3-DOF helicopter. The system is depicted in Figure 7. The system consists of an arm mounted on a base with two rotors and a counterweight at each end. The system can pitch and yaw about the mount of the arm by varying the voltage on the two rotors attached to the arm. The system can also roll about the axis of the arm. Yawing motion is achieved by applying a rolling moment to the arm by applying different voltages to each of the rotors. The focus here is on the control of the pitch axis of the helicopter. The relevant state vector is defined as $x = [\theta, \dot{\theta}]$, where θ is the pitch angle, and $\dot{\theta}$ is the pitch rate. Actuation is achieved by varying the voltage to the motors and is denoted by u.

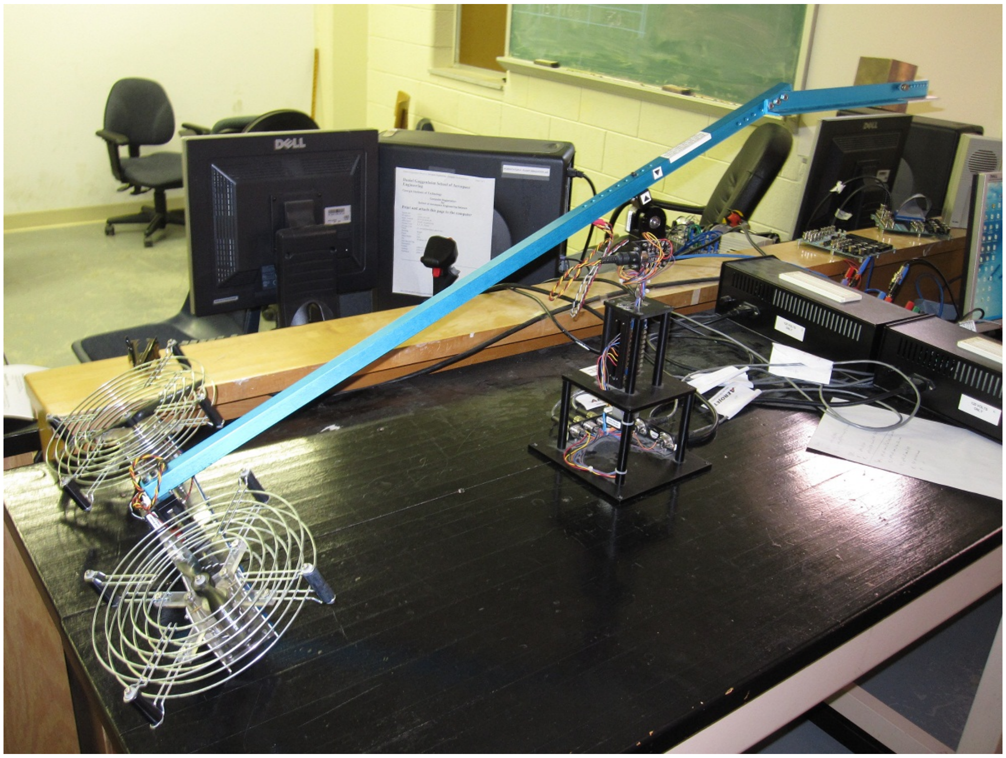


Figure 7. The Quanser 3-DOF helicopter. The system can roll about the axis of the arm as well as pitch and yaw about the mount of the arm by varying the voltages to the two rotors attached to the arm.

A linear model for the system can be assumed to have the following form:

$$\dot{x} = Ax + \begin{bmatrix} 0 \\ M_\delta \end{bmatrix} u. \tag{38}$$

The state matrix A of the system is unknown, the control effectiveness derivative relating the pitch actuation to pitch rate M_δ has been estimated as $M_\delta = 0.5$. The pitch angle θ and the pitch rate $\dot{\theta}$ are available for measurement through an encoder. For concurrent learning the time derivative of the state vector is estimated using a Kalman filter based fixed point smoother using a simplified process model for the system given below:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x. \tag{39}$$

Further details on how such a smoother can be implemented can be found in Chowdhary.[18] Control objective is to make the system behave like the following second order reference model with natural frequency of $2.8 rad/s$ and damping ratio of 0.7,

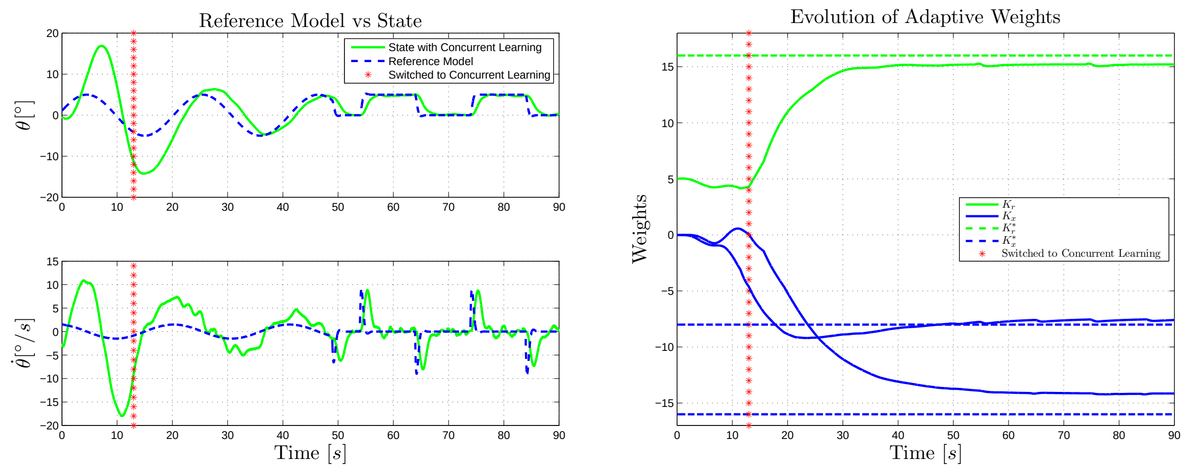
$$\dot{x}_{rm} = \begin{bmatrix} 0 & 1 \\ -8 & -4 \end{bmatrix} x_{rm}(t) + \begin{bmatrix} 0 \\ 8 \end{bmatrix} r(t). \tag{40}$$

For these values, an estimate of the ideal gains was found to be $K_x^* = [-16 \ -8]$ and $K_r^* = 16$. Note that the estimates of K_x^*,K_r^* are not required for the implementation of the controller, they were estimated for the purpose of comparison. The experiment runs for 90 seconds with a time step of 0.0025 seconds. The reference signal is a cosine function with frequency $0.3 rad/s$ and amplitude $5°$ for 47 seconds, followed by two step inputs of magnitude $5°$ after 54 and 74 seconds, each lasting 10 seconds. The control law of Eq. 4 is used along with the adaption law of Eq. 20. The learning rates are held constant throughout the experiment and chosen to be $\Gamma_x = 6$ and $\Gamma_r = 10$. A pre-recorded history stack is not available. Hence, until the history stack is verified to meet the rank condition, a σ-modification term with $\kappa = 0.1$ is added to the adaptive update law in order to guarantee boundedness of all system signals. The

σ-modification term is removed after the history stack reaches full rank. The adaptive weights K_x were initialized at zero and K_r was initialized at 5.

Figure 8(a) shows the tracking performance of the adaptive controller with concurrent learning. It can be seen that the system performance improves significantly once the rank condition on the history stack is met at about 13 seconds. After that the system tracks the magnitude of the cosine function and the steps accurately. However, the system lags slightly behind the reference model. This is possibly due to unknown prevalent hardware time delays in relaying the command to the system and unmodeled actuator dynamics. The reason for the improved performance becomes clear if the evolution of the weights in Figure 8(b) is examined. Once the rank condition is met at about 13 seconds, the weights begin to converge to a compact set around their estimated ideal values rapidly. The feedforward gain K_r needs only 10 seconds to converge to a compact set while the feedback gain K_x takes about 30 seconds longer. The feedforward gain converges to a neighborhood of the estimated ideal value of $K_r^* = 8$ with an error of about 5%. Additionally, the steady-state value of the feedback gain k_1 differs by about 5% and k_2 differs by about 11% from the estimated ideal values of $K_x^* = [-16 - 8]$. This can be attributed to errors in calculating the ideal values and the control effectiveness as well as to the presence of noise, disturbances, and unmodeled dynamics in the system.

It is interesting to note that almost zero tracking error is observed during steady state conditions, which indicates that the adaptive controller can trim the system effectively in presence of disturbances and time delays. In combination, these results validate that the concurrent learning adaptive controller can deliver good tracking performance despite noisy estimates of the pitch angle from the encoder, unmodeled time delays, and the presence of external disturbances.



(a)

(b)

Figure 8. Tracking performance of the adaptive controller with concurrent learning during the experiment in Figure 8(a). Note, that the performance increases drastically after 30 seconds. This corresponds to the time when the weights have converged. Uncertainties and time delays prevent the system to track the reference model perfectly. Evolution of the adaptive weights with concurrent learning during the experiment in Figure 8(b). Note that after the history stack meets the rank condition at about 13 seconds, the weights converge to a set around their theoretically obtained values. Unmodeled uncertainties and disturbances prevent the weights to converge totally.

VI. Conclusion

We showed robustness of a concurrent learning adaptive controller against noisy measurements without requiring additional damping terms, such as σ- or e-modification, in the update law. A concurrent learning controller uses instantaneous data concurrently with recorded data. The result is subject to a condition on the boundedness of the measurement noise and the estimation error of the Optimal Fixed Point Smoother. We showed that this condition results in the convergence of the tracking error and the weight error dynamics to a compact set around the zero solution. We demonstrated the effectiveness of the presented concurrent learning adaptive control through an exemplary simulation of a second-order system. The results indicate that if the recorded data meets a verifiable condition on linear independence and the estimation error of the Optimal Fixed Point Smoother are bounded, then the tracking error is ultimately bounded in a domain around the equilibrium point. Furthermore the adaptive weights converge to a compact domain around their true values. Additionally, we demonstrated the effectiveness of the presented concurrent learning adaptive controller through experimentation on a Quanser 3-DOF Helicopter.

Appendix

Let Z_k denote the measurements, $(-)$ denote predicted values, and $(+)$ denote corrected values, Q and R denote the process and measurement noise covariance matrices. Let P denote the error covariance matrix, let Φ denote the transition matrix for the linear system in Eq.40, let $y = [x, \dot{x}, \ddot{x}]^T$, and $Z_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} y$. Then the forward Kalman filter propagates the state $\hat{y}_k(-) = \Phi_k \hat{y}_{k-1}$ and the error covariance $P_k(-) = \Phi_k P_{k-1} {\Phi_k}^T + Q_k$. The Kalman gain is given by $K_k = P_k(-){H_k}^T[H_k P_k(-){H_k}^T + R_k]^{-1}$, and the corrected state estimate by $\hat{y}_k(+) = \hat{y}_k(-) + K_k[Z_k - H_k\hat{y}_k(-)]$. The error covariance is updated using $P_k(+) = [I - K_k H_k]P_k(-)$. The smoothed state estimate can be given with $\hat{y}_{k|N} = \hat{y}_{k|N-1} + B_N[\hat{y}_N(+) - \hat{y}_N(-)]$, where $\hat{y}_{k|k} = \hat{y}_k$, $N = k+1, k+2, \cdots$, $B_N = \prod_{i=k}^{N-1} P_i(+)\Phi_i^T P_{i+1}^{-1}(-)$. The smoother error covariance matrix is propagated backwards with $P_{k|N} = P_{k|N-1} + B_N[P_k(+) - P_k(-)]B_N^T$, where $P_{k|k} = P_k(+)$.

References

[1] Narendra, K. S. and Annaswamy, A. M., *Stable Adaptive Systems*, Prentice-Hall, Englewood Cliffs, 1989.

[2] Ioannou, P. A. and Sun, J., *Robust Adaptive Control*, Prentice-Hall, Upper Saddle River, 1996.

[3] Aström, K. J. and Wittenmark, B., *Adaptive Control*, Addison-Weseley, Readings, 2nd ed., 1995.

[4] Lavretsky, E., "Combined/Composite Model Reference Adaptive Control," *Automatic Control, IEEE Transactions on*, Vol. 54, No. 11, nov. 2009, pp. 2692 –2697.

[5] Tao, G., *Adaptive Control Design and Analysis*, Wiley, New York, 2003.

[6] Cao, C. and Hovakimyan, N., "Design and Analysis of a Novel Adaptive Control Architecture With Guaranteed Transient Performance," *Automatic Control, IEEE Transactions on*, Vol. 53, No. 2, march 2008, pp. 586 –591.

[7] Boyd, S. and Sastry, S., "Necessary and Sufficient Conditions for Parameter Convergence in Adaptive Control," *Automatica*, Vol. 22, No. 6, 1986, pp. 629–639.

[8] Chowdhary, G., *Concurrent Learning for Convergence in Adaptive Control Without Persistentcy of Excitation*, Ph.D. thesis, Georgia Institute of Technology, Atlanta, GA, 2010.

[9] Chowdhary, G., Yucelen, T., Mühlegg, M., and Johnson, E., "Concurrent Learning Adaptive Control of Linear Systems with Exponentially Convergent Bounds," *International Journal of Adaptive Control and Signal Processing*, 2011, in early access.

[10] Ge, S. and Wang, J., "Robust adaptive neural control for a class of perturbed strict feedback nonlinear systems," *Neural Networks, IEEE Transactions on*, Vol. 13, No. 6, nov 2002, pp. 1409 – 1419.

[11] Narendra, K. S. and Annaswamy, A. M., "Robust Adaptive Control in the Presence of Bounded Disturbances," *IEEE Transactions on Automatic Control*, Vol. AC-31, No. 4, 1986, pp. 306–315.

[12] Narendra, K. and Annaswamy, A., "A New Adaptive Law for Robust Adaptation without Persistent Excitation," *IEEE Transactions on Automatic Control*, Vol. 32, No. 2, February 1987, pp. 134–145.

[13] Gelb, A., *Applied Optimal Estimation*, MIT Press, Cambridge, 1974.

[14] Haddad, W. M. and Chellaboina, V., *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*, Princeton University Press, Princeton, 2008.

[15] Chowdhary, G. and Johnson, E., "A singular value maximizing data recording algorithm for concurrent learning," *American Control Conference (ACC), 2011*, 29 2011-july 1 2011, pp. 3547 –3552.

[16] Xu, J.-X., Jia, Q.-W., and Lee, T. H., "On the design of a nonlinear adaptive variable structure derivative estimator," *Automatic Control, IEEE Transactions on*, Vol. 45, No. 5, may 2000, pp. 1028 –1033.

[17] Jazwinski, A. H., *Stochastic processes and filtering theory*, Acedemic Press, New York, 1974.

[18] Chowdhary, G. and Johnson, E. N., "Theory and Flight Test Validation of a Concurrent Learning Adaptive Controller," *Journal of Guidance Control and Dynamics*, Vol. 34, No. 2, March 2011, pp. 592–607.