Concurrent Learning Adaptive Control for Linear Switched Systems

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Abstract— **A concurrent learning adaptive control architecture for uncertain linear switched dynamical systems is presented. Like other concurrent learning adaptive control architectures, the adaptive weight update law uses both recorded and current data concurrently for adaptation. In addition, a verifiable condition on the linear independence of the recorded data is shown to be sufficient to guarantee global exponential stability and adaptive weight convergence. Furthermore, it is shown that the recorded data eventually meets this condition after a system switch without any additional excitation from the exogenous reference input or knowledge of the switching signal if there is sufficient time in between switches. That is, after a switch the system will be automatically excited and sufficiently rich data will be recorded. As a result, data that is irrelevant to the current subsystem will be overwritten. Thus, reference model tracking error and adaptive weight error will eventually become globally exponential stable for all switched subsystems. Numerical examples are presented to illustrate the effectiveness of the proposed architecture.**

I. Introduction

Adaptive control of linear uncertain dynamical systems has been widely studied for systems with significant uncertainties. The widely studied Model Reference Adaptive Control (MRAC) framework has in particular many theoretical results, proving asymptotic tracking error convergence or ultimate boundedness of tracking error in the presence of significant modeling uncertainty, and several experimental results [1]–[4]. However, typically authors have considered that the parameters of the uncertain dynamical system are fixed, and have assumed the existence of an ideal controller that recovers the reference model dynamics under matching assumptions. In real world scenarios however, assuming that the parameters of the unknown linear system are fixed, may be an unrealistic assumption. In fact, it is known that environmental and configuration changes often cause the parameters to switch between different values in possibly unknown ways. Specific examples of such variations include hybrid dynamical systems, networked systems with agents entering and leaving the network, and systems where adaptation to external disturbances is required. In the presence of such time-variations in the uncertainty, it can be shown (see e.g. [5]) that the standard MRAC adaptive law does not guarantee asymptotic convergence.

Several authors have discussed the adaptive control of linear parameter varying systems (see e.g. [6]–[9]). However, these methods have focused on instantaneous suppression of the uncertainty through a reactive adaptive signal. No extended term learning has been demonstrated (in fact, it may not be possible to do so for a purely time varying system). However, if the system switches between (possibly unknown) modes of operation, it should be possible to simultaneously stabilize the system and learn the ideal parameters for each of the modes. Classical and recent adaptive control laws rely on the system states being persistently excited (PE) in order to guarantee convergence of adaptive weights [1], [2], [10]–[12]. Boyd and Sastry have shown that the condition on PE states can be directly related to a condition on the spectral properties of the exogenous reference input [13] and [14]. However, in the case where the nominal reference input does not meet the spectral condition, an additional exciting signal must be added in order to guarantee convergence of adaptive weights. In general, it may be difficult to select an appropriate exciting signal even if the switching signal was known. Furthermore, this added exciting signal may lead to poor reference model tracking since the system may track the exciting signal [15].

Concurrent learning adaptive control has been proven, under certain conditions, to lead to guaranteed global exponential stability and adaptive weight convergence for linear systems without requiring the system state to be PE [16]. Unlike most adaptive control laws that only use instantaneous system data, in concurrent learning a *history stack* is populated with online recorded system data and is used concurrently with instantaneous data for adaptation. It has been shown that a sufficient condition for global exponential stability of both the tracking error and weight error dynamics is that the history stack be full ranked [17]. It has also been shown in [18] that a singular value maximizing online data recording algorithm can guarantee that the history stack will be full ranked if the system states are exciting over a *finite* period (i.e. persistence of excitation is not needed). However, in the case of a switched system with an unknown switching signal the recorded data may be erroneous. This is, if data is recorded previous to a switch the history stack will be irrelevant to the current system and exponential stability and adaptive weight convergence is not guaranteed.

In this paper, a concurrent learning adaptive control architecture and an associated data recording algorithm for uncertain linear switched dynamical systems are presented. Like other concurrent learning adaptive control architectures, the adaptive weight update law makes use of recorded data and linear independence of the recorded data is shown to be a sufficient condition to guarantee global exponential stability and adaptive weight convergence. Furthermore, it is shown that the recorded data eventually meets the rank condition

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after a switch without any additional excitation from the exogenous reference input or knowledge of the switching signal if there is sufficient time in between switches. That is, after a switch the system will be automatically excited and sufficiently rich data will be recorded. As a result, data that is irrelevant to the current subsystem will be overwritten. Therefore the proposed architecture ensures that tracking error and adaptive weight error will eventually become globally exponential stable without any knowledge of the switching signal or additional excitation from the reference input, under mild assumptions.

In Section II the proposed architecture is presented and it is shown that, under certain conditions, concurrent learning adaptive control leads to guarantee global exponential stability and adaptive weight convergence for linear switched systems. In Section III, we relax assumptions made in Section II and show that concurrent learning can lead to global exponential stability and adaptive weight convergence even if the switching signal is not known. In Section IV presents the results of an exemplary simulation study. The paper is concluded in Section V.

In this paper $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{R}_+$ (resp., $\overline{\mathbb{R}}_+$) denotes the set of positive (resp., nonnegative) real numbers, $\mathbb{R}_+^{n\times m}$ (resp., $\overline{\mathbb{R}}_+^{n\times m}$) denotes the set of positive-definite (resp., nonnegative-definite) $n \times m$ real matrices, $\mathbb{S}^{n\times n}$ denotes the set of $n \times n$ symmetric real matrices, $(\cdot)^T$ denotes the transpose, $(\cdot)^{-1}$ denotes the inverse, "$\triangleq$" denotes equality by definition, and $\equiv$ denotes equivalency. In addition, we write $\|\cdot\|$ for the euclidean norm, $\max(\cdot)$ for the maximum, $\min(\cdot)$ for the minimum, $\mathrm{tr}(\cdot)$ for the trace operator, $\det(\cdot)$ for the determinate of a square matrix, $\mathrm{vec}(\cdot)$ denotes the vectorization of a matrix, $\lambda_{min}(\cdot)$ denotes the smallest eigenvalue of a matrix, $\lambda_{max}(\cdot)$ denotes the largest eigenvalue of a matrix, $\sum$ denotes summation, and Π denotes product.

II. Concurrent Learning Adaptive Control for Linear Switched Systems

A concurrent learning architecture can be used to control uncertain linear systems when it was assumed that the ideal adaptive weights are time-invariant or slowly changing [17]. However, in a linear switched system the ideal adaptive weights change instantaneously and in some cases unpredictably. Therefore, the assumption of slowly varying weights may not always be satisfied. In this section, a concurrent learning adaptive law is developed that guarantees exponential stability of the linear switched systems subject to a rank condition on the recorded data.

A. Concurrent Learning Adaptive Controller

Consider a class of uncertain dynamical switched systems given by

$$\dot{x}(t) = A_i x(t) + b_i u(t), \quad i \in \{1,\ldots,v\}, \quad t \in \bar{\mathbb{R}}_+, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector available for feedback, $u(t) \in \mathbb{R}^m$ is the control input restricted to the class of admissible controls consisting of measurable functions, $A_i \in \mathbb{R}^{n\times n}$ is an unknown system matrix and $b_i \in \mathbb{R}^{n\times m}$ is an known control input matrix for every switching index, $i \in \{1,\ldots,v\}$. We assume that the pair (A_i, b_i) is controllable and $\det(b_i^T b_i) \neq 0$ for every switching index, $i \in \{1,\ldots,v\}$. For the purposes of this paper, we assume that there are finite subsystems, furthermore infinitely many switches between subsystems are not permitted in any finite time interval. In addition, it is assumed that the switching time and the switching index of current active subsystem is known, or, in other words, the switching signal is known. This is a strong assumption and will be relaxed in Section III. Next, consider the designer chosen reference system selected to capture the desired response of the closed loop system given by

$$\dot{x}_{\mathrm{r}}(t) = A_{\mathrm{r}_i} x_{\mathrm{r}}(t) + b_{\mathrm{r}_i} r(t), \; i \in \{1,\ldots,v\}, \; t \in \bar{\mathbb{R}}_+, \quad (2)$$

where $r(t) \in \mathbb{R}^m$ denotes a bounded, piecewise continuous reference signal, $A_{\mathrm{r}_i} \in \mathbb{R}^{n\times n}$ and $b_{\mathrm{r}_i} \in \mathbb{R}^{n\times m}$ for every switching index, $i \in \{1,\ldots,v\}$. Furthermore, A_{r_i} is assumed to be Hurwitz and the pair $(A_{\mathrm{r}_i}, b_{\mathrm{r}_i})$ is controllable for every switching index, $i \in \{1,\ldots,v\}$. The following matching conditions are assumed to be met.

Assumption 1 There exists $K = [K_1,\ldots,K_v], K_i \in \mathbb{R}^{n\times m}$ and $K_r = [K_{r_1},\ldots,K_{r_v}], K_{r_i} \in \mathbb{R}^{m\times m}$ such that

$$A_i + b_i {K_i}^T = A_{\mathrm{r}_i} \quad (3)$$

$$b_i K_{r_i} = b_{\mathrm{r}_i}. \quad (4)$$

for each $i \in \{1,\ldots,v\}$.

Define the time varying adaptive weights as $\hat{K}(t) = [\hat{K}_1(t),\ldots,\hat{K}_v(t)], \hat{K}_i(t) \in \mathbb{R}^{m\times n}$. Let the feedback control law when subsystem i is active be given as

$$u_i(t) = u_{rm_i}(t) + u_{pd_i}(t), \quad (5)$$

where $u_{rm_i}(t) = K_{r_i} r(t)$ and $u_{pd_i}(t) = \hat{K}_i^T(t) x(t)$.

From (1) and (5) the uncertain dynamical switched system can be written as

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + b_i (K_{\mathrm{r_i}} r(t) + \hat{K}_i^T(t) x(t)) \\ \dot{x}(t) &= A_{\mathrm{r_i}} x(t) + b_{\mathrm{r}_i} r(t) + b_i \tilde{K}_i^T(t) x(t) \end{aligned} \quad (6)$$

where $\tilde{K}_i^T(t) \triangleq \hat{K}_i^T(t) - K_i^T$. The system error dynamics are given using $e(t) \triangleq x(t) - x_{\mathrm{r}}(t)$, (2) and (6) as

$$\dot{e}(t) = A_{\mathrm{r_i}} e(t) + b_i \tilde{K}_i^T(t) x(t), \quad i \in \{1,\ldots,v\}. \quad (7)$$

Next, define $\epsilon_{K_{i_j}}(t) \in \mathbb{R}^m$ as

$$\epsilon_{K_{i_j}}(t) = (b_i^T b_i)^{-1} b_i^T (\dot{x}_{i_j} - A_{r_i} x_{i_j} - b_{r_i} r_{i_j}), \quad (8)$$

and (x_{i_j}, r_{i_j}) denotes the j^{th} recorded pair when subsystem i is active. A method for data recording is discussed later. From (6), $\tilde{K}_i^T(t)x(t)$ can be defined as $\tilde{K}_i^T(t)x(t) = (b_i^T b_i)^{-1} b_i^T (\dot{x}(t) - A_{\mathrm{r_i}} x(t) + b_{\mathrm{r}_i} r(t))$. Thus,

$$\epsilon_{K_{i_j}}(t) = \hat{K}_i^T(t) x_{i_j} - K_i^T x_{i_j} = \tilde{K}_i^T(t) x_{i_j}. \quad (9)$$

The proposed concurrent learning adaptive law for uncertain dynamical linear switched systems when subsystem i is active is given as

$$\dot{\hat{K}}_k(t) = -\Gamma_{x_k}(xe^T Pb_k + \sum_{j=1}^{p} x_{k_j}\epsilon^T_{K_{k_j}}(t)), k = i \quad (10)$$

$$\dot{\hat{K}}_k(t) = -\Gamma_{x_k}\sum_{j=1}^{p} x_{k_j}\epsilon^T_{K_{k_j}}(t), k \neq i, \quad (11)$$

where $\Gamma_{xi} \in \mathbb{R}_+$ and $\Gamma_{ri} \in \mathbb{R}_+$ denote adaptive learning rates and $P \in \mathbb{R}_+^{n\times n} \cap \mathbb{S}^{n\times n}$ satisfies the Lyapunov equation

$$0 = A_{\rm r}^T P + PA_{\rm r} + Q \quad (12)$$

for any positive definite $Q \in \mathbb{R}_+^{n\times n} \cap \mathbb{S}^{n\times n}$.

Remark 1 Note that while the first derivative of a data point $\dot{x}_{k_j}$ recorded at a past time-instant t_k is required to evaluate the adaptive law in 10, the first derivative of the state at the *current* time instant $\dot{x}(t)$ is *not* required. Estimating the former when $\dot{x}_{k_j}$ cannot be measured, is a far simpler problem than estimating the latter. In particular, it is well known that a fixed point optimal smoother that uses a forward and/or a backward Kalman filter can be used to estimated $\dot{x}_{k_j}$ reliably using data before and after a data point has been recorded [19]. This method has been experimentally validated to yield good estimates of $\dot{x}_{k_j}$ [4]. Furthermore, theoretical guarantees of good closed loop performance when using this method for estimating $\dot{x}_{k_j}$ in presence of noisy measurements have been presented in [20]. Finally, note that the delay between recording a point x_{k_j} and incorporating it in the adaptive law after estimating $\dot{x}_{k_j}$ will not be detrimental to system performance since $\epsilon_{K_{k_j}}$ does not affect the tracking error at time t. Details of this process can be found in [4].

Remark 2 From (9), the adaptive weight error dynamics are given as $\dot{\tilde{K}}_k(t) = -\Gamma_{x_k}(xe^T Pb_k + \sum_{j=1}^{p} x_{k_j}x^T_{k_j}\tilde{K}_i(t)), k = i$ and $\dot{\tilde{K}}_k(t) = -\Gamma_{x_k}\sum_{j=1}^{p} x_{k_j}x^T_{k_j}\tilde{K}_i(t), k \neq i$. Unlike most MRAC architectures, the adaptive weight update law is directly affected by model reference tracking error *and* adaptive weight error. In the following sections, it is shown that both the tracking error and the adaptive weight error can be global uniform exponential stable under certain conditions.

B. Data Recording

Let (x_{i_j}, r_{i_j}) denote the j^{th} recorded pair when the subsystem i is active. Define $\mathbf{X}_i$, as $\mathbf{X}_i = [x_{i_1}, x_{i_2}, ..., x_{i_p}], i \in \{1, \ldots, v\}$. Throughout the paper $\mathbf{X}_i$ is referred to as the i^{th} subsystem's *history stack*. To facilitate real-time implementation, it is assumed that a maximum of $p \geq n$ data points can be stored for each subsystem, therefore, the number of columns in the history stack are limited to p. The history stack is updated by adding data point to empty positions or by replacing existing data points. Data points are replaced such that the matrix's minimum singular value is maximized. As shown in Theorem 1, maximizing the smallest singular value of the history stack is beneficial, since the convergence rate of the adaptive controller is directly proportional to the smallest singular value overall $\mathbf{X}_i$, $i \in \{1, \ldots, v\}$. When the switching signal is known, the recording points can be easily placed into the appropriate history stack and the singular value maximizing algorithm presented in [18] can be trivially extended. However if the switching signal is unknown the algorithm cannot be easily extended. An outline of how data recording can be done in this case is given in Section III.

C. Stability Analysis

To begin our stability analysis we will assume that all of the system's history stacks, $\mathbf{X}_1, \ldots, \mathbf{X}_v$, are each pre-populated with n linearly independent data points where the recorded data points were populated through open loop testing. This is a strong assumption which will be relaxed in Section III.

Theorem 1 Consider the system in (1), the control law of (5) and the adaptive control laws (10) and (11). If $\mathbf{X}_i$ contains n linearly independent data points for $i \in \{1, \ldots, v\}$. Then the zero solution $(e(t), \tilde{K}(t)) \equiv 0$ is globally uniformly exponentially stable.

Proof: We will assume there are two subsystems and an arbitrary switching sequence. Consider the following positive definite and radially unbounded candidate Lyapunov function

$$\begin{aligned} V(e, \tilde{K}) \;\; &= \;\; \frac{1}{2}e^T Pe + \frac{1}{2}\mathrm{tr}(\tilde{K_1}^T \Gamma_{x_1}^{-1}\tilde{K}_1) \\ &\quad + \frac{1}{2}\mathrm{tr}(\tilde{K_2}^T \Gamma_{x_2}^{-1}\tilde{K}_2). \end{aligned} \quad (13)$$

Let $\xi = [e^T, \mathrm{vec}(\tilde{K})]^T$, we can now bound the Lyapunov candidate above and below with positive definite functions as follows

$$\begin{aligned} &\frac{1}{2}\min(\lambda_{\min}(P), \min_{i=1,2}(\lambda_{\min}(\Gamma_{x_i}{}^{-1})))\|\xi_i\|^2 \\ &\qquad\qquad \leq V(e, \tilde{K}) \leq \\ &\frac{1}{2}\max(\lambda_{\max}(P), \max_{i=1,2}(\lambda_{\max}(\Gamma_{x_i}{}^{-1})))\|\xi_i\|^2. \end{aligned} \quad (14)$$

Assuming that subsystem 1 is active and taking the time derivative of the Lyapunov function candidate along the trajectories of system (7), (10), (11), and using the Lyapunov equation (12) we have

$$\begin{aligned} \dot{V}(e, \tilde{K}) &= -\frac{1}{2}e^T Qe + e^T Pb_1\tilde{K}_1^T x - \mathrm{tr}(\tilde{K_1}^T xe^T Pb_1) \\ &\quad -\mathrm{tr}[\tilde{K_1}^T \sum_{j=1}^{p} x_{1_j}x^T_{1_j}\tilde{K}_1 - \tilde{K_2}^T \sum_{j=1}^{p} x_{2_j}x^T_{2_j}\tilde{K}_2] \\ &= -\frac{1}{2}e^T Qe - \mathrm{tr}[\tilde{K_1}^T \sum_{j=1}^{p} x_{1_j}x^T_{1_j}\tilde{K}_1 \\ &\quad \tilde{K_2}^T \sum_{j=1}^{p} x_{2_j}x^T_{2_j}\tilde{K}_2] \end{aligned}$$

Note the same expression results if subsystem 2 is active. Let $\Omega_{K_i} = \sum_{j=1}^{p_i} x_{i_j} x_{i_j}^T$ for $i \in \{1,2\}$. $\Omega_{K_i} \in \mathbb{R}_+^{n\times n}$ since $\mathbf{X_i}$ contains n linearly independent data points for $i \in \{1,2\}$. Let $\Omega_{K_{min}} = \min_{i=1,2}(\lambda_{\min}(\Omega_{K_i}))$. A bound on the derivative of the Lyapunov function candidate is now defined as $\dot{V}(e,\tilde{K}) \leq -\frac{1}{2}\lambda_{\min}(Q)\|e\|^2 - \Omega_{K_{min}}\|\tilde{K}\|^2$. This bound can also defined as

$$\dot{V}(e,\tilde{K}) \leq -\frac{\min\left(\lambda_{\min}(Q), 2\Omega_{K_{min}}\right)}{\max\left(\lambda_{\max}(P), \Gamma_{x_{max}}^{-1}\right)} V(e,\tilde{K}),$$

where $\Gamma_{x_{max}}^{-1} = \max_{i=1,2}\Gamma_{x_i}^{-1}$. Note that data recording algorithm guarantees that $\Omega_{K_{min}}$ is monotonically increasing. Notice that the bound of the derivative of the Lyapunov function candidate is independent of the switching sequence. Also note that the Lyapunov function and its derivative are always continuous regardless of the switching signal. Thus, the Lyapunov function candidate is a *common Lyapunov function* and establishes global uniform exponential stability of the zero solution $e(t) \equiv 0$ and $\tilde{K}(t) \equiv 0$ [21]. The extension to the general case where $v > 2$ is trivial. ■

Remark 3 Theorem 1 states that $\tilde{K} = [\tilde{K}_1, \ldots, \tilde{K}_v]$ is globally uniformly exponentially stable. Therefore, the proposed adaptive law allows the adaptive weights of each subsystem to update and converge independent of the switching sequence of the system. Thus, assuming the system's state history stacks, $\mathbf{X}_1, \ldots, \mathbf{X}_v$, are populated with n linearly independent data points the adaptive weights $\hat{K}_i$ will converge to K_i for every switching index, $i \in \{1, \ldots, v\}$, regardless of the system's switching signal.

III. Population of History Stacks

We will now relax the strong assumptions made in Section II. In particular, we will no longer assume that the switching signal is known. Furthermore, we will no longer assume that the system's history stacks, $\mathbf{X}_1, \ldots, \mathbf{X}_v$, are each prepopulated with n linearly independent data points. Before presenting our final results we review some results from harmonic analysis.

A. Results in Harmonic Analysis

Consider a signal $f(\cdot) = f_{\text{p}}(\cdot) + f_{\text{a}}(\cdot)$ where $f_{\text{p}}(\cdot)$ represents the periodic portion of the signal with period T and $f_{\text{a}}(\cdot)$ represents the aperiodic portion of the signal. We can defined the signal's frequency spectrum as the frequency spectrum of the periodic signal $f_{\text{p}}(\cdot)$ and the frequency spectrum of the aperiodic signal $f_{\text{a}}(\cdot)$. A periodic signal $f_{\text{p}}(\cdot)$ with period T can be represented as $f_{\text{p}}(t) = \sum_{n=-\infty}^{\infty} F_n e^{\frac{-2\pi j v t}{T}}$ where $F_k = \frac{1}{T}\int_{t_0}^{t_0+T} f_{\text{p}}(t) e^{\frac{-2\pi k j t}{T}} \mathrm{d}t$. The periodic signal is said to have a spectral line at frequency $\frac{-2\pi k}{T}$ if $F_k \neq 0$. An aperiodic signal $f_{\text{p}}(\cdot)$ can be represented as $f_{\text{a}}(t) = \int_{-\infty}^{+\infty} F(v) e^{-2\pi j v t}\mathrm{d}v$ where $F(v) = \int_{-\infty}^{+\infty} f(t) e^{-2\pi j v t}\mathrm{d}t$. Finally, $f(t) = \sum_{n=-\infty}^{\infty} F_n e^{\frac{-2\pi j v t}{T}} + \int_{-\infty}^{+\infty} F(v) e^{-2\pi j v t}\mathrm{d}v$. Thus, the frequency spectrum of the signal $f(t)$ is made up of a discrete spectrum (line spectrum) and a continuous spectrum. If $f_{\text{a}}(\cdot) \neq 0$ then we say $f(\cdot)$ is aperiodic. Furthermore, we let $\bar{f}(v)$ denote the value of the spectrum of $f(t)$ at frequency v.

For the following analysis we say a signal's spectrum is *not concentrated* on q points if there exist a set $\mathbf{v} \in \{v_1, \ldots, v_{q+1}\}, v_m \in \mathbb{R}$ such that $\bar{f}(v_m) \neq 0$ for each $m \in \{1, \ldots, q+1\}$. Furthermore, we say a signal's spectrum is *concentrated* on *at most* q points if there does not exist a set $\mathbf{v} \in \{v_1, \ldots, v_{q+1}\}, v_m \in \mathbb{R}$ such that $\bar{f}(v_m) \neq 0$ for each $m \in \{1, \ldots, q+1\}$.

Remark 4 Let $u(t)$, $y(t)$ be the input and output, respectively, of a stable linear time-invariant system with transfer function $\hat{L}(s)$. The value of the spectrum of $y(t)$ at v is

$$\bar{y}(v) = \bar{L}(v)\bar{u}(v). \tag{15}$$

Note that the affect of the initial condition is neglected here since it exponentially decays [13].

Definition 1 [2] A bounded vector signal $x(t)$ is persistently exciting (PE) if for all $t > t_0$ there exists $T > 0$ and $\alpha > 0$ such that $\int_t^{t+T} x(\tau)x^T(\tau)d\tau \geq \alpha I$.

Equivalently if $t^\star = t + \delta$ and $\delta = \frac{T}{2}$ a bounded vector signal $x(t)$ is PE if all $t^\star - \delta > t_0$ there exists $\delta > 0$ and $\alpha > 0$ such that

$$\int_{t^\star-\delta}^{t^\star+\delta} x(\tau)x^T(\tau)d\tau \geq \alpha I. \tag{16}$$

Remark 5 Note that, assuming sampling occurs at a sufficiently fast rate, if a signal is PE while subsystem i is active, the history stack for subsystem i will become populated with n linearly independent data points.

Lemma 1 Let the spectrum of $x(t) \in \mathbb{R}^N$ be defined at $v_1, v_2, \ldots v_N$. Also, let $\bar{x}(v_1), \ldots, \bar{x}(v_N)$ be linear independent in $\mathbb{C}^N$. Then, $x(t)$ is PE [13].

Proof: We will take $t_0 = -\infty$. Define $X(t,\delta) \in \mathbb{R}_{n\times n}$ as

$$X(t,\delta) = \int_{t-\delta}^{t+\delta} \begin{bmatrix} e^{-jv_1\tau} \\ \cdot \\ \cdot \\ \cdot \\ e^{-jv_n\tau} \end{bmatrix} x^T(\tau)\mathrm{d}\tau. \tag{17}$$

The limit as $\delta \to \infty$, by definition, is

$$X_0 = \begin{bmatrix} \bar{x}^T(v_1) \\ \cdot \\ \cdot \\ \cdot \\ \bar{x}^T(v_n) \end{bmatrix}. \tag{18}$$

By assumption X_0 is non-singular. Thus, for a large enough $\delta^\star$, $X(t,\delta^\star)$ is invertible and $\exists W > 0 \in \mathbb{R}$ such that

$||X(t,\delta^\star)^{-1}|| \leq W||X_0^{-1}||$ for $\delta \geq \delta^\star$ and all t. Now for $z \in \mathbb{R}^n$ with $||z|| = 1$ and any $v \in \mathbb{R}$ we have

$$\begin{aligned}\int_{t-\delta}^{t+\delta} (x^T z)^2 \mathrm{d}\tau = & \int_{t-\delta}^{t+\delta} |x^T z e^{-jv\tau}|^2 \mathrm{d}\tau \\ \geq & \ |\int_{t-\delta}^{t+\delta} x^T z e^{-jv\tau} \mathrm{d}\tau|^2. \end{aligned} \tag{19}$$

Using (19) for $v = v_1, \ldots, v_n$ we have

$$\begin{aligned}\int_{t-\delta}^{t+\delta} (x^T z)^2 \mathrm{d}\tau &\geq \frac{1}{n}\sum_{k=1}^{n} |\int_{t-\delta}^{t+\delta} x^T z e^{-jv_n\tau}\mathrm{d}\tau|^2 \\ &= \frac{1}{n}||X(s,\delta)||^2 \\ &\geq \frac{1}{n}||X(s,\delta)^{-1}||^{-2}, \delta \geq \delta^\star \\ &\geq \frac{1}{W^2 n}||X_0^{-1}||^{-2}.\end{aligned}$$

Equation (16) will hold for $\delta = \delta^\star$ and $\alpha = \frac{1}{W^2 n}||X_0^{-1}||^{-2} > 0$. ■

Remark 6 Rather than limiting the result to signals whose *discrete spectrum* satisfy certain conditions as done in [13], we generalized the result to include (aperiodic) signals whose entire spectrum satisfies similar conditions.

The following result is similar to one found in [13]. Again the conditions are relaxed to include the aperiodic portion of a signal.

Lemma 2 Let $A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n$ be a controllable pair and assume that the spectrum of the input $u(t)$ to the system $\dot{x}(t) = Ax(t) + bu(t)$ is not concentrated on less than n points. Then, if A is exponentially stable, $x(t)$ is persistently exciting.

Proof: Since the pair (A, b) are controllable the system can be written in the controllable canonical form and the system's transfer function from $u(s)$ to $x(s)$ is

$$\frac{1}{p(s)}\begin{bmatrix} 1 \\ s \\ . \\ . \\ s^{N-1}\end{bmatrix}, \quad p(s) = det(sI - A). \tag{20}$$

The transfer function is exponentially stable. If spectrum of $u(t)$ is defined at $v_1, \ldots, v_N$, so is spectrum of $x(t)$ as

$$\hat{x}(v_i) = \frac{\bar{u}(v_i)}{p(v_i)}\begin{bmatrix} 1 \\ . \\ . \\ . \\ (v_i)^{N-1}\end{bmatrix}, i = 1, \ldots, N. \tag{21}$$

Due to Lemma 1, $x(t)$ is persistently exciting since

$$\det\begin{bmatrix} 1 & \ldots & 1 \\ v_1 & & v_N \\ . & & . \\ . & & . \\ v_1^{N-1} & \ldots & v_N^{N-1}\end{bmatrix} = \prod_{i<j}(v_i - v_j) \neq 0. \tag{22}$$

■

Remark 7 Lemma 2 suggests that as long as the input to a system is aperiodic the system states will be PE.

B. *Outdated History Stack*

We now relax assumptions made in Section II. Therefore, we will no longer assume that the switching signal is known and that the history stacks are prepopulated. As a result, the adaptive weight update law (10-11) is no longer valid. For the remainder of the paper we will replace (10-11) with

$$\dot{\hat{K}}(t) = -\Gamma(xe^T P b_k + \sum_{j=1}^{p} x_k \epsilon_{K_k}^T(t)) \tag{23}$$

where

$$\epsilon_{K_i}(t) = (b_i^T b_i)^{-1} b_i^T(\dot{x}_i - A_r x_i - b_r r_i). \tag{24}$$

Note that a unique reference model cannot be selected for each subsystem and a single reference model must be selected for all the subsystems. Since the switching signal is unknown, only one adaptive weight estimate can update and there is a single history stack. This formulation presents issues since the history stack used may be *outdated* (i.e. contain erroneous data). In other words since the switching signal is unknown the history stack being used at a particular time may have been recorded when another subsystem was active. In the following results we show that the system remains bounded if an erroneous history stack is used and we also show, under a mild assumption and a change in the recording algorithm, that the history stack eventually becomes *current* and full rank.

Theorem 2 Consider the system in (1), the control law of (5) and the adaptive control law (23). Assume that $\sum_{k=1}^{p} x_k x_k^T$ is nonsingular, but arbitrary. Then the system tracking error and adaptive weight estimate error are bounded.

Proof: First note that the arbitrary history stack can be viewed as random errors in the recorded data points (8)

$$\epsilon_{K_{i_j}}(t) = (b_i^T b_i)^{-1} b_i^T(\dot{x}_i - A_{rm} x_i - b_{rm} r_i + \delta_i), \tag{25}$$

where δ_i represents the *error* is data recording. Next, consider the candidate Lyapunov function $V(e, \tilde{K}) = \frac{1}{2}e^T P e + \frac{1}{2}\mathrm{tr}(\tilde{K}^T \Gamma_{x_1}^{-1}\tilde{K})$ and taking the time derivative of the Lyapunov function candidate along the trajectories of system (7), (23), and using the Lyapunov equation (12) we have

$$\dot{V}(e, \tilde{K}) = -\frac{1}{2}e^T Q e - \mathrm{tr}[\tilde{K}^T \sum_{j=1}^{p} x_i x_i^T \tilde{K} + \|\tilde{K}\|\bar{c} \tag{26}$$

where $\bar{c} = \|\sum_{j=1}^{p} x_i (b_i^T b_i)^{-1} b_i^T \delta_i\|$. It follows that the tracking error is ultimately bounded and the adaptive weights $\hat{K}(t)$ approach and stay bounded in a compact ball around the ideal weights $K(t)$. ■

Remark 8 Notice that Theorem 2 also implies that error in the recorded data points will not lead to instability [17].

Before presenting the final result, we will first state a needed assumption and argue that this assumption is not restrictive and will be satisfied almost always in application.

Assumption 2 The effective control signal input (6) $b_{r_i}r(t) + b_i\tilde{K}_i^T(t)x(t)$ is aperiodic in a time interval $t \in [t_0, t_0+\tau]$ for some τ after a switch.

At first this assumption seems relatively harsh and restrictive. However, consider that the adaptive control law governing the evolving of $\tilde{K}_i^T(t)$ is nonlinear. Also, $x(t)$, in practice, is never perfectly periodic (even with a periodic input) due to nonlinear unmodeled dynamics, initial conditions and digitalization. Thus, it is not unreasonable to assume that $b_i\tilde{K}_i^T(t)x(t)$ will in fact be aperiodic. Finally, note unless $r(t)$ is selected such that it cancels the aperiodic portion of $b_i\tilde{K}_i^T(t)x(t)$ perfectly that effective control input $b_{r_i}r(t) + b_i\tilde{K}_i^T(t)x(t)$ will almost always be aperiodic.

Instead of replacing individual data points, we replace the entire history stack throughout the evolution of the system according to the following algorithm.

Algorithm 1 Data Recording with Unknown Switching

$X_{rec} = \text{singmax}(X_{rec}, x(t), p)$
if $\text{minsvd}(\Omega(X_{rec})) > \text{minsvd}(\Omega(X_{use}))c_1e^{-c_2(t-\bar{t})} + c_3$ **then**
$X_{use} = X_{rec}$
$X_{rec} = 0_{n\times n}$
$\bar{t} = t$
end if

Where p is the number of maximum data points allowed to be stored in X_{rec}, $c_1, c_2, c_3 \in \mathbb{R}_+$ describe the switching surface and X_{use} is utilized for adaptive weight update. The function $\text{minsvd}(\cdot)$ gives the minimum singular value of a matrix and $\Omega(X) = \sum_{j=1}^{p} x_i x_i^T, X = [x_1, \ldots, x_p]$. The function $\text{singmax}(\cdot)$ is the singular value maximizing algorithm presented in [18]. The algorithm simply replaces the entire history stack according to switching surface based on the minimum singular values of X_{rec} and X_{use}. Note if $c_3 > 0$ then X_{use} is guaranteed to be full rank.

Theorem 3 Consider the system in (1), the control law of (5), the adaptive control law (23) and suppose data points are recorded as described in Algorithm 1. Assume that the history stack $\sum_{k=1}^{m} x_k x_k^T$ after a switch is nonsingular, but arbitrary. If the current subsystem remains active for a sufficiently large time then there exist time $t_0+\tau$ such that the history stack will be current and full rank.

Proof: This is a direct result from Assumption 2, Lemma 2 and Remark 7. ■

Remark 9 In practice, the time τ will depend directly on the physical system and the inputs to the system. It has been our observation that only a few more than n samples are needed to recorded a full rank history stack if the system is displaying transient behavior. In order words, in our observations, τ is unnoticeably small.

Remark 10 Theorem 3 implies if a linear switched system subjected to concurrent learning is switched sufficiently slow tracking error and adaptive weight estimate error will eventually tend to zero after each switch.

IV. Simulation Results

In this section we present numerical examples to illustrate the effectiveness of our proposed architecture. In particular, improved model reference tracking with a less oscillatory input is seen in our architecture when compared to a MRAC gradient based weight update law with the projection operator based modification [22]. Consider the uncertain linear switched system given by $\dot{x}(t) = A_r x(t) + b_r r(t) + [\hat{W}^T(t) - W(t)]x(t)$, $B^T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-K_1}{M1} & \frac{-B_1-B_2}{M1} & 0 & \frac{B_2}{M_1} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{B_2}{M_2} & \frac{-K_2}{M_2} & \frac{-B_2}{M_2} \end{bmatrix} \tag{27}$$

where $M_1 = 0.1, M_2 = 0.3, K_1 = 2.5, K_2 = 3, B_1 = 0.05, B_2 = 0.01$. Furthermore, the nominal reference input is given as $r(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ \sin(\pi t/2) \end{bmatrix}$ and the system's ideal adaptive weights switch at an unknown time and are given as

$$W(t)^T = \begin{cases} \begin{bmatrix} 0 & 0.3 & 0.9 & 0.8 \\ -0.8 & -0.5 & -0.8 & 0.3 \end{bmatrix}, \text{ if } 0 \le t < 50 \\ \begin{bmatrix} 1 & 0.3 & 0.3 & -0.8 \\ 0.2 & -0.25 & -0 & 0.5 \end{bmatrix}, \text{ if } 50 \le t \le 100 \end{cases}.$$

Therefore, the system switches at $t = 50$.

It is assumed that no data from open-loop testing is available. The history stack was initialized such that $\mathbf{X} = [x_1, \ldots, x_4] = \mathrm{I}_4$ and $\epsilon_{K_j}(t) = \hat{W}(t)^T x_i - 1$, $j \in [1, \ldots, 4]$. The parameters of the recording algorithm were initialized as $\bar{t} = -\infty$, $c_1 = 1$, $c_2 = 0.01$ and $c_3 = 1 \times 10^{-6}$.

For all cases the concurrent learning architecture is compare with to a MRAC gradient based weight update law with the projection operator based modification with $\theta_{max} = 2$ [22]. This comparison highlights the benefit of using recorded data even in the case of uncertain linear switched system with an unknown switching signal. In the following three cases it is shown that reference model tracking is significantly better when concurrent learning is used. The cases also demonstrate that the system can *self-excite*. That is, the history stack will become full rank without any additional excitation from the exogenous reference input.

This feature allows concurrent learning to be used without any prior knowledge of the system or, in the case of linear switched systems, knowledge of the switching signal. Finally, the following are defined $[e_1, \ldots, e_4]^T = x - x_r$, $u(t) = [u_1, u_2]^T = r(t) + \tilde{K}(t)x(t)$, $\Omega^2_{K_{min}} = \lambda_{\min}(\sum_{j=1}^{p_i} x_i x_i^T)^2$.

A. Case 1

For the first case, the adaptive learning rate $\Gamma = 5$ for both the concurrent learning and MRAC architecture. As shown in Figure 1, concurrent learning show significantly better tracking ability in all states. Figure 2 shows that, unlike the MRAC architecture, concurrent learning is able to decrease the adaptive weight error. It can also be seen that tracking error decreases over time suggesting that decreasing the adaptive weight error improves the tracking ability of the system. In addition, the inputs to the system are less oscillatory when concurrent learning is used. Notice that since the input to the system is aperiodic for some time Assumption 2 is satisfied. Finally notice that the system is able to acquire a new full rank history stack in a relatively small time at initialization, $t = 0$, and when the system switches, $t = 50$.

B. Case 2

For the second case, the adaptive learning rate of the MRAC architecture was increased to $\Gamma = 15$. However, the concurrent learning still displayed significantly better tracking ability in all states even with a smaller adaptive learning rate as shown in Figure 3. Also, since the adaptive learning rate was increased MRAC can be expected to produce inputs that are even more oscillatory then before.

C. Case 3

For the final case $r_2(t) \equiv 0$ to demonstrate that even with limited excitation from the reference input the concurrent learning architecture will still be able to acquire a full rank history stack and show improved performance over MRAC. As seen in Figure 4 and 5 concurrent learning displays better tracking ability in all states with less oscillatory inputs. Also, as seen in the first case, the system is able to acquire a new full rank history stack in a relatively small time at initialization, $t = 0$, and when the system switches, $t = 50$. As expected, since the nominal reference input has limited excitation the minimum singular value of the recorded history stack decreases and the history stack is replaced less often. As a result the convergence rate is slower than the first case. However, the adaptive weight error is still decreasing and will eventually converge (if no further switching occurs). Finally, notice that even with a constant input the system was able to self-excite after an unforeseen switch.

V. Conclusion

In this paper, we presented a concurrent learning adaptive control architecture and an associated data recording algorithm for uncertain linear switched dynamical systems. Under strong assumptions we showed that our architecture ensures that the model reference tracking error and adaptive weight error are globally uniformly exponentially stable regardless of the switching signal. We then relaxed these assumptions and showed that this architecture ensures that tracking error and adaptive weight error will eventually become globally exponential stable without any knowledge of the switching signal or additional excitation from the reference input, under mild assumptions. Numerical examples illustrate the effectiveness of the proposed architecture and its advantages over the model reference control architecture. Future work will include calculation of the time (or upper bound of the time) required to populate a full rank current history stack after a switch and modification of the adaptive process to detect recurring subsystems.

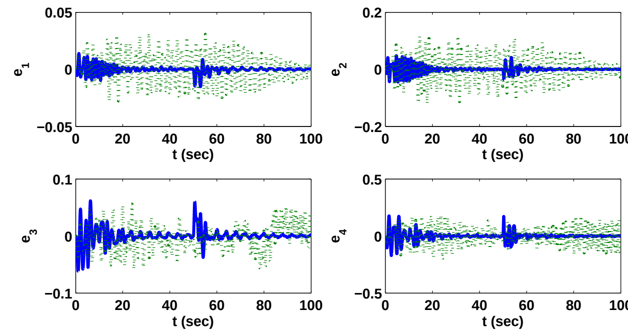

Fig. 1. Tracking error for $\Gamma = 5$, Case 1 (solid and dotted lines show the response of the system subject to concurrent learning and standard MRAC architecture, respectively).

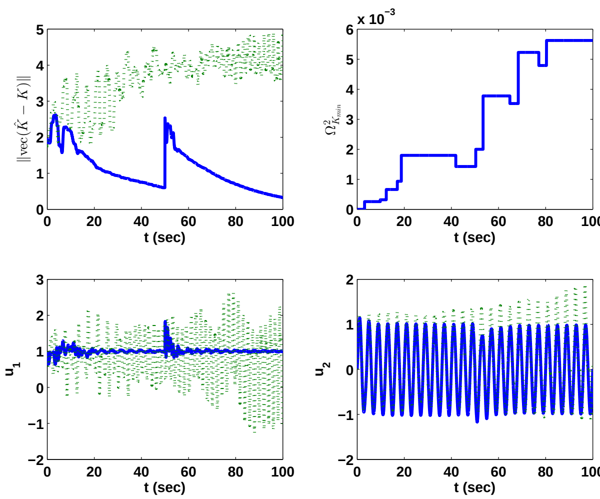

Fig. 2. Norm of the adaptive weight error, minimum singular value of the *current* history stack, and control inputs for $\Gamma = 5$, Case 1 (solid and dotted lines show the response of the system subject to concurrent learning and standard MRAC architecture, respectively).

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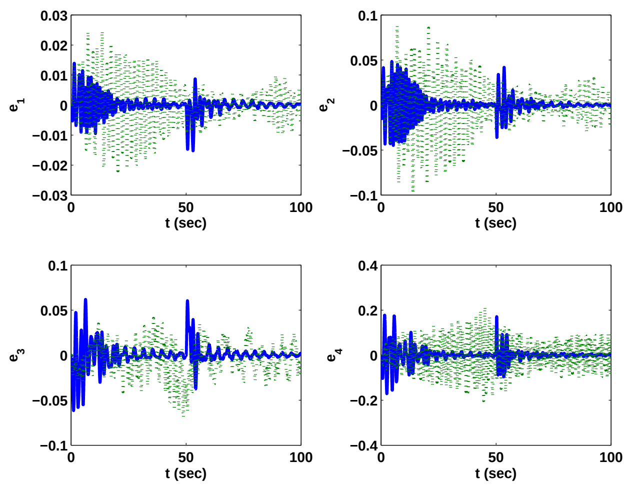

Fig. 3. Tracking error when $\Gamma = 5$ is used in the concurrent learning architecture and $\Gamma = 15$ is used in the MRAC architecture, Case 2 (solid and dotted lines show the response of the system subject to concurrent learning and standard MRAC architecture, respectively).

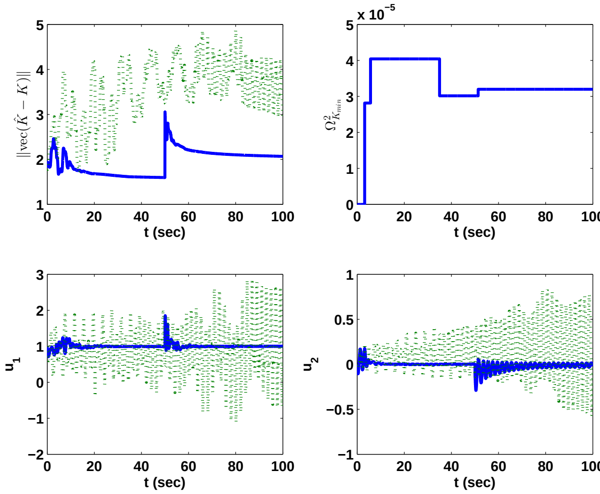

Fig. 5. Norm of the adaptive weight error, minimum singular value of the *current* history stack, and control inputs for $\Gamma = 5$ and $r_2(t) \equiv 0$, Case 3 (solid and dotted lines show the response of the system subject to concurrent learning and standard MRAC architecture, respectively).

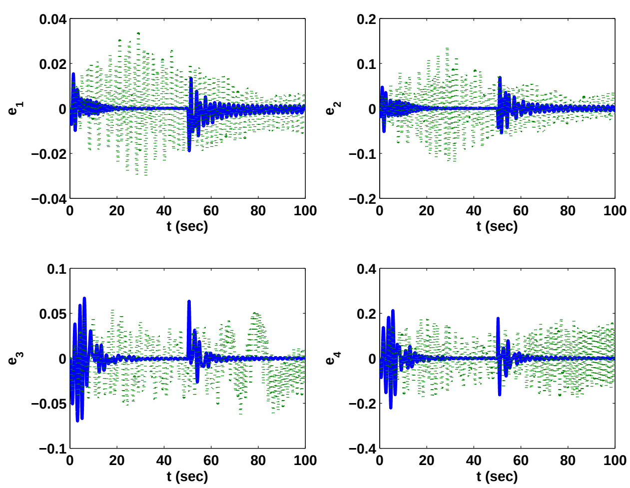

Fig. 4. Tracking error for $\Gamma = 5$ and $r2(t) \equiv 0$, Case 3 (solid and dotted lines show the response of the system subject to concurrent learning and standard MRAC architecture, respectively).

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