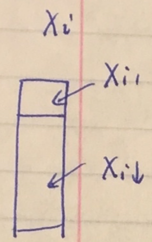
Settings.

① Each row of X_i follow i.i.d. Gaussian distribution. $N(0, \sigma^2)$

② M is the sum function.

③ DDP. $X_i \sim X_i'$. Assign the ith row r_i of X_i to be 0 in X_i'.



$$M\Big|_{X_{i\downarrow}=z}(X_i) = \underbrace{\sum_{r_j \in X_{i\uparrow}} r_j}_{M(X_{i\uparrow})} + M(z), \quad \text{where } \sum_{r_j \in X_{i\uparrow}} r_j \ (M(X_{i\uparrow})) \text{ follow } \underline{N(0, cn\sigma^2)}$$

$$M\Big|_{X_{i\downarrow}=z}(X_i') = \underbrace{\sum_{r_j \in X_{i\uparrow}'} r_j}_{M(X_{i\uparrow}')} + M(z), \quad \text{where } M(X_{i\uparrow}') \text{ follows } \underline{N(0, c(n-1)\sigma^2)}.$$

① Base Case.

(a).

Consider.
$$\frac{P_r(M(X_i)=y \mid z)}{P_r(M(X_i')=y \mid z)} = \exp\left(\frac{(y-M(z))^2}{2cR} + \frac{1}{2}\log^{(1-\frac{1}{n})}\right)$$

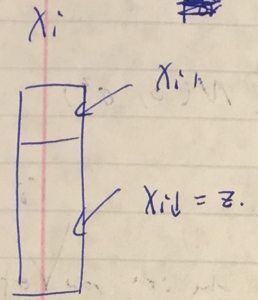
where $R = n(n-1)\sigma^2$.

Pick any $t>0$, for y such that $|y-M(z)| \le t$, we have.

$$\frac{(y-M(z))^2}{2cR} \le \frac{t^2}{2cR}.$$

Let $\varepsilon(t) = \frac{t^2}{2cR} + \frac{1}{2}\log^{(1-\frac{1}{n})}$. $\nearrow t$

(b) Consider. $\Pr[\, |y - M(z)| > t \mid z \,]$



$$= \Pr[\, | M(X_i) - M(z) | > t \mid z \,]$$

$$= \Pr[\, | M(X_{i1}) | > t \,].$$

$$= 2\int_{t}^{+\infty} f_{M(X_{i1})} \cdot d_{M(X_{i1})}$$

For any $X \sim N(0, \sigma^2)$, we have.

$$\Pr[X > t] = \frac{1}{\sqrt{2\pi}\,\sigma} \int_t^{+\infty} e^{-\frac{x^2}{2\sigma^2}} dx$$

$$< \frac{1}{\sqrt{2\pi}\,\sigma} \int_t^{+\infty} \frac{x}{t} \cdot e^{-\frac{x^2}{2\sigma^2}} dx = \frac{\sigma}{\sqrt{2\pi}\, t} e^{-\frac{t^2}{2\sigma^2}}$$

Since $M(X_{i1}) \sim N(0, cn\sigma^2)$, we have.

$$\Pr[\, |y - M(z)| > t \mid z \,] < \sqrt{\frac{2cn}{\pi}} \frac{\sigma}{t} \cdot e^{-\frac{t^2}{2cn\sigma^2}} \quad \downarrow t$$

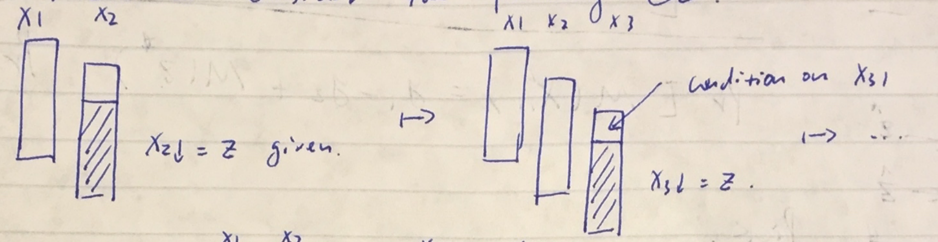
Let $\delta(t) = \sqrt{\frac{2cn}{\pi}} \frac{\sigma}{t} \cdot e^{-\frac{t^2}{2cn\sigma^2}}$

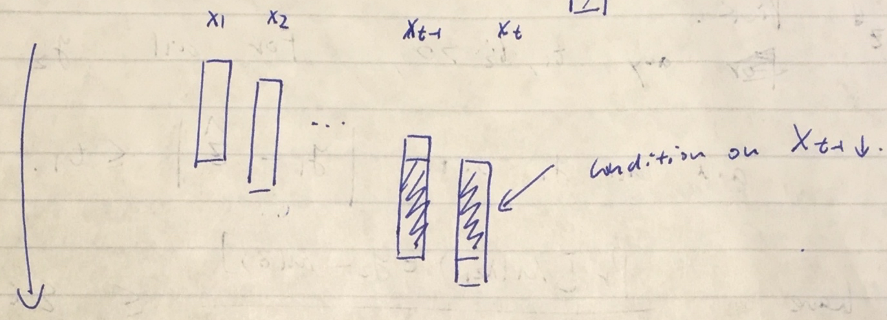
Not hard to see

For any z, $t > 0$., any set S

$$\Pr[\, M(X_i) \in S \mid z \,] \le e^{\varepsilon(t)} \Pr[\, M(X_i') \in S \mid z \,] + \delta(t).$$

(2) Need to consider the following case.

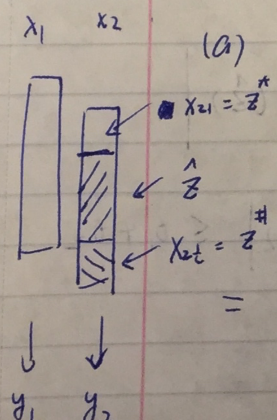




NTS: $\Pr[M(x_1) \in S_1,\ M(x_2) \in S_2 \mid z]$

$\leq e^{\varepsilon(t_1, t_2)} \Pr[M(x_1') \in S_1,\ M(x_2') \in S_2 \mid z]$

$+ \delta(t_1, t_2)$.

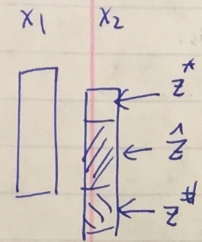


(a) Consider.
$$\frac{\Pr(M(x_1) = y_1,\ M(x_2) = y_2 \mid z)}{\Pr(M(x_1') = y_1,\ M(x_2') = y_2 \mid z)}.$$

$$= \frac{\sum_{M(z^*)} \Pr(M(x_1) = y_1 \mid z, M(z^*)) \cdot \Pr(M(x_2) = y_2 \mid M(z^*), z) \cdot \Pr[M(x_{21}) = M(z^*)]}{\text{''}}$$

$$= \frac{\Pr(M(x_1) = y_1 \mid z, M(x_{21}) = y_2 - M(z)) \cdot \Pr[M(x_{21}) = y_2 - M(z)]}{\Pr(M(x_1') = y_1 \mid z, M(x_{21}') = y_2 - M(z)) \cdot \Pr[M(x_{21}') = y_2 - M(z)]}$$

$$= \frac{\Pr[M(x_{11}) = y_1 - y_2 + M(z^\#)]}{\Pr[M(x'_{11}) = y_1 - y_2 + M(z^\#)]} \cdot \frac{\Pr[M(x_{21}) = y_2 - M(z)]}{\Pr[M(x'_{21}) = y_2 - M(z)]}$$



Pick.

~~For~~ any $t_1, t_2 > 0$, For all y_2 s.t. $|y_2 - M(z)| \le t_2$

and all y_1 s.t. $|y_1 - \hat{z}| \le t_1$.

We have
$$\frac{\Pr[M(x_{21}) = y_2 - M(z)]}{\Pr[M(x'_{21}) = y_2 - M(z)]} \le e^{\varepsilon(t_2)}.$$

$$\frac{\Pr[M(x_{11}) = y_1 - y_2 + M(z^\#)]}{\Pr[M(x'_{11}) = y_1 - y_2 + M(z^\#)]} = \exp\left(\frac{(y_1 - y_2 + M(z^\#))^2}{2cR} + \frac{1}{2}\log\left(1 - \frac{1}{n}\right) \right).$$

Consider. $|y_1 - y_2 + M(z^\#)| = |y_1 - (y_2 - M(z)) - M(\hat{z})|$

$$\le |y_1 - M(\hat{z})| + |y_2 - M(z)| \le t_1 + t_2$$

Hence,

$$\le e^{\varepsilon(t_1 + t_2)}.$$

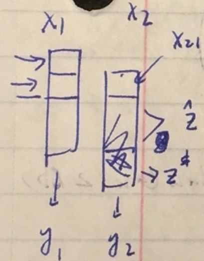
Let $\varepsilon(t_1, t_2) = \underline{\varepsilon(t_2) + \varepsilon(t_1 + t_2)}$.

The probability that y_1, y_2 does not ... good region.

(b). Consider $1 - \Pr\left[\, |y_1 - \hat{z}| \le t_1,\ |y_2 - M(z)| \le t_2 \,\middle|\, z \right]$

$$\Pr\left[\, |y_1 - \hat{z}| \le t_1,\ |y_2 - M(z)| \le t_2 \,\middle|\, z \right].$$

$$= \sum_{M(z^*)} \Pr\left[\, |y_1 - \hat{z}| \le t_1 \,\middle|\, M(\hat{z}), z \right] \cdot \Pr\left[\, |y_2 - M(z)| \le t_2 \,\middle|\, M(z^*), z \right] \cdot \Pr\left[M(\hat{z}^*) \right].$$



Since $\Pr\left[\, |y_2 - M(z)| \le t_2 \,\middle|\, M(z^*), z \right]$

$$= \begin{cases} 1 & |M(z^*)| \le t_2. \\ 0 & \text{otherwise.} \end{cases}$$

$$= \int_{-t_2}^{t_2} \underbrace{\Pr\left[\, |y_1 - \hat{z}| \le t_1 \,\middle|\, M(x_{21}), z \right]} \cdot f_{M(x_{21})}\, d_{M(x_{21})}$$

Now, $= \Pr\left[\, |M(x_{11}) + M(x_{21})| \le t_1 \,\middle|\, M(x_{21}), z \right].$

Given $|M(x_{11}) + M(x_{21})| \le t_1$ and $|M(x_{21})| \le t_2$,

we have $|M(x_{11})| \le t_1 + t_2$

Hence, $\Pr\left[\, |M(x_{11}) + M(x_{21})| \le t_1 \,\middle|\, M(x_{21}), z \right] \ge \Pr\left[\, |M(x_{11})| \le t_1 + t_2 \right].$

By the base case (b) part,

$$\Pr[M(X_{11}) \le t_1+t_2] = 1 - \Pr[M(X_{11}) > t_1+t_2]$$
$$\geq 1 - \delta(t_1+t_2).$$

Hence, $\int_{-t_2}^{t_2} \Pr[\,|y_1 - \hat{z}| \le t_1 \mid M(X_{21}), z\,] \cdot f_{M(X_{21})} \cdot d_{M(X_{21})}.$

$$\geq (1-\delta(t_1+t_2)) \underline{\int_{-t_2}^{t_2} f_{M(X_{21})}\, d_{M(X_{21})}}$$

$$= (1-\delta(t_1+t_2)) \left(1 - 2\int_{t_2}^{+\infty} f_{M(X_{21})}\, d_{M(X_{21})}\right).$$

$$\geq \underline{(1-\delta(t_1+t_2)).(1-2\delta(t_2))}.$$ — By Base Case 2 (b)

Hence, $1 - \Pr[\,|y_1 - \hat{z}| \le t_1,\ |y_2 - M(z)| \le t_2 \mid z\,]$

$\leq 1-$ $\quad \leq \underline{\delta(t_1+t_2) + 2\delta(t_2)}$

Let $\delta(t_1, t_2) = \delta(t_1+t_2) + 2\delta(t_2)$.

Then, It is easy to see. for any z, $t_1, t_2 > 0$, and any sets S_1, S_2.

$$\Pr[M(X_1) \in S_1,\ M(X_2) \in S_2 \mid z] \le e^{\varepsilon(t_1, t_2)} \cdot \Pr[M(X_1') \in S_1,\ M(X_2') \in S_2 \mid z] + \delta(t_1, t_2).,$$ where.

$\varepsilon(t_1, t_2) = \varepsilon(t_1+t_2) + \varepsilon(t_2)$, and $\delta(t_1, t_2) = \delta(t_1+t_2) + 2\delta(t_2)$.