Done

CS1231(S) Tutorial 5: Mathematical Induction

National University of Singapore

2019/20 Semester 1

1. Prove by induction that $n^3 + 11n$ is divisible by 3 for all $n \in \mathbb{Z}^+$.

2. Prove by induction that every positive integer can be written as a sum of *distinct* non-negative integer powers of 2, i.e.

$$\forall n \in \mathbb{Z}^+ \ \big(\exists k \in \mathbb{Z}^+ \ \exists i_1, i_2, \ldots, i_k \in \mathbb{Z}_{\geqslant 0} \ (i_1 < i_2 < \cdots < i_k) \wedge (n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_k})\big)$$

3. Let $a_0, a_1, a_2, \ldots$ be the sequence such that $a_0 = 0$, $a_1 = 2$ and $a_2 = 7$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all $n \in \mathbb{Z}_{\geqslant 3}$. Prove by induction that $a_n < 3^n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

4. Prove by induction on n the following statement:

$$\forall n \in \mathbb{Z}_{\geqslant 8} \ \big(\exists x, y \in \mathbb{Z}_{\geqslant 0} \ (n = 3x + 5y)\big).$$

5. Let a be an odd integer. Prove that $a^{2^n} - 1$ is divisible by 2^{n+2} for all $n \in \mathbb{Z}^+$.

6. Prove by induction on n the following statement:

$$\forall n \in \mathbb{Z}_{\geqslant 0} \ \big(\exists x, y \in \mathbb{Z}_{\geqslant 0} \ (n = \tfrac{1}{2}(x + y)(x + y + 1) + y)\big).$$

7. I need to climb a flight of stairs of n steps. I can go up 1 or 2 steps with every stride. Let s_n be the number of ways that I have to climb n steps (so $s_2 = 2$ since I can climb 2 steps in 1 stride going up 2 steps, or in 2 strides each going up 1 step).

   (a) Find a recurrence relation for s_n.

   (b) What is the name of the sequence $s_1, s_2, \ldots$?

8. (D.R. Hofstadter) The following rules govern which strings of letters you can write down. x, y can also be empty strings

   (I) You can write down MI.

   (II) After writing down xI for some string x, you can write down xIU.

   (III) After writing down Mx for some string x, you can write down Mxx.

   (IV) After writing down xIIIy for some strings x, y, you can write down xUy.

   (V) After writing down xUUy for some strings x, y, you can write down xy.

   (VI) You cannot write down a string unless it is allowed by one of the rules above.

   (a) According to these rules, can you write down MUIIU? Prove that your answer is correct.

   (b) According to these rules, can you write down MU? Prove that your answer is correct. (Hint: count the number of I's in the strings you can write down.)