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CS1231(S) Tutorial 6: Number theory 1

National University of Singapore

2019/20 Semester 1

The questions marked with an asterisk * are more challenging ones intended for discussion purposes during the tutorial class. The other questions are regular ones. Please attempt all questions before the tutorial class.

1. Let $a, b \in \mathbb{Z}$. Show that if $a \mid b$ and $b \mid a$, then $a = b$ or $a = -b$.

2. Find the quotient and the remainder when
   (a) 44 is divided by 8;
   (b) 777 is divided by 21;
   (c) -123 is divided by 19;
   (d) 0 is divided by 17;
   (e) -100 is divided by 101.

3. Show that for all odd integers $n \in \mathbb{Z}$,
$$n^2 \, \mathbf{div} \, 4 = \frac{n^2 - 1}{4}.$$

4. Is 107 prime? Is 113 prime?

5. Find an integer $n \geqslant 1231$ that share no prime divisor with 15811090783488000. Prove that your answer is correct. Did you implicitly or explicitly use the Fundamental Theorem of Arithmetic (i.e., the fact that every positive integer greater than 1 has a unique factorization into a product of primes) in your proof? If yes, then can you avoid it (possibly by choosing a different n)?

6.* An integer n is said to be a *perfect square* if $n = k^2$ for some $k \in \mathbb{Z}$. Prove that a positive integer n is a perfect square if and only if it has an odd number of positive divisors. (Hint: pair up the divisors strictly bigger than $\sqrt{n}$ and the divisors strictly smaller than $\sqrt{n}$.)

7. Find the binary, octal and hexadecimal expansions of 1231.

8. Find the decimal expansions of
   (a) $(1101001)_2$;
   (b) $(156)_8$;
   (c) $(74)_{16}$.

9.* Let $n \in \mathbb{Z}_{\geqslant 0}$ with decimal expansion $(a_k a_{k-1} \dots a_0)_{10}$. Prove that $9 \mid n$ if and only if $9 \mid (a_0 + a_1 + \cdots + a_k)$. (Hint: for example,

$$\begin{aligned}
7524 &= 7 \cdot 1000 + 5 \cdot 100 + 2 \cdot 10 + 4 \\
&= 7 \cdot (999 + 1) + 5 \cdot (99 + 1) + 2 \cdot (9 + 1) + 4 \\
&= (7 \cdot 999 + 7) + (5 \cdot 99 + 5) + (2 \cdot 9 + 2) + 4 \\
&= (7 \cdot 999 + 5 \cdot 99 + 2 \cdot 9) + 7 + 5 + 2 + 4 \\
&= 9 \cdot (7 \cdot 111 + 5 \cdot 11 + 2 \cdot 1) + (7 + 5 + 2 + 4).
\end{aligned}$$

You may use without proof the fact that

$$10^i = 9 \cdot 10^{i-1} + 9 \cdot 10^{i-2} + \cdots + 9 \cdot 10^0 + 1$$

for all $i \in \mathbb{Z}^+$.)