CS1231/CS1231S: Discrete Structures

Tutorial #11: Graphs and Trees

Week 13: 11 – 15 November 2019

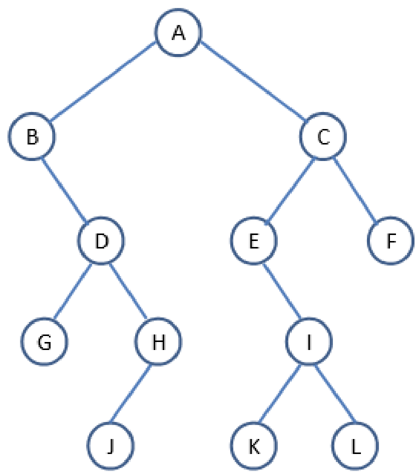
I. Discussion Questions

These are meant for you to discuss on the LumiNUS Forum. No answers will be provided.

D1. For any simple connected graph with n $(n > 0)$ vertices, what is the minimum and maximum number of edges the graph may have?

D2. (AY2016/17 Semester 1 Exam Question)
How many simple graphs on 3 vertices are there? In general, how many simple graphs on n $(n > 1)$ vertices are there?

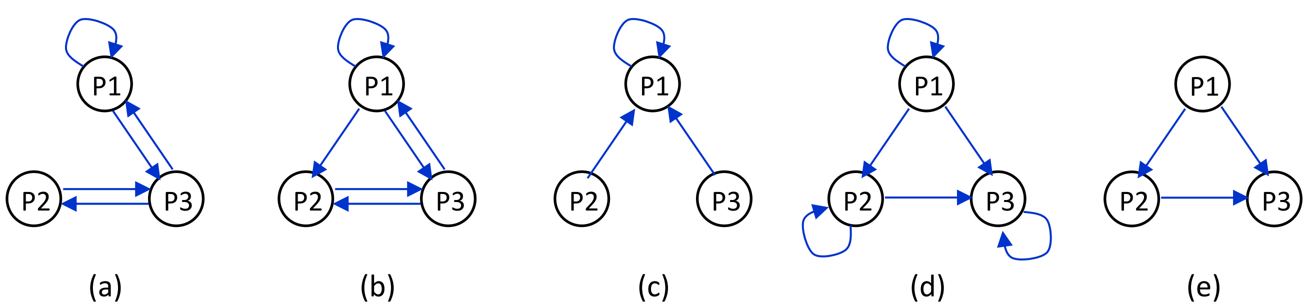
D3. Given the following binary tree, write the pre-order, in-order, and post-order traversals of its vertices.



II. Exploration

Read the document "IdolRank" posted on LumiNUS "Tutorials" Files or the CS1231S website "Tutorials" page.

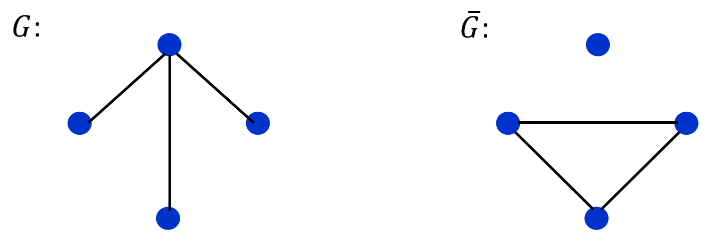
By hand or with a computer program, find out the winner of each of the five graphs below. *P1*, *P2* and *P3* represent three contestants, and an arrow from vertex x to vertex y indicates that x is the referee of y. The second graph is already solved in the above "IdolRank" document.



III. Definitions

Definition 1. If G is a simple graph, the *complement* of G, denoted $\bar{G}$, is obtained as follows: the vertex set of $\bar{G}$ is identical to the vertex set of G. However, two distinct vertices v and w of $\bar{G}$ are connected by an edge if and only if v and w are not connected by an edge in G.

The figure below shows a graph G and its complement $\bar{G}$.



A graph G and its complement $\bar{G}$.

Definition 2. A *self-complementary* graph is isomorphic with its complement.

Definition 3. A simple circuit (cycle) of length three is called a *triangle*.

IV. Tutorial Questions

1. Draw all self-complementary graphs with (a) four vertices; (b) five vertices.

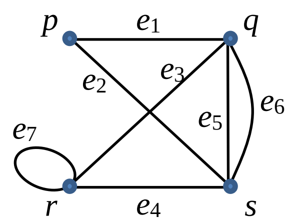
2. (AY2016/17 Semester 1 Exam Question)
   Let G be a simple graph with n vertices where every vertex has degree at least $\left\lfloor \frac{n}{2} \right\rfloor$. Prove that G is connected.

3. Show that every simple graph with at least two vertices has two vertices of the same degree.

   (This is similar to the popular puzzle: "Prove that at a party with at least two persons, there are two people who know the same number of people".)

4. Prove that for any simple graph G with six vertices, G or its complementary graph $\bar{G}$ contains a triangle.

5. Given the graph shown below:



(a) Write the adjacency matrix A for the graph. Let the rows and columns be p, q, r and s.

(b) Find A^2 and A^3.

(c) How many walks of length 2 are there from p to q? From s to itself? List out all the walks.

(d) How many walks of length 3 are there from r to s? From s to p? List out all the walks.

6. (AY2017/18 Semester 1 Exam Question)
Suppose you are given a pile of stones. At each step, you can separate a pile of k stones into two piles of k_1 and k_2 stones. (Obviously, $k_1 + k_2 = k$.) On doing this, you earn $\$(k_1 \times k_2)$.

What is the maximum amount of money you can earn at the end if you start with a pile of n stones? Explain your answer.

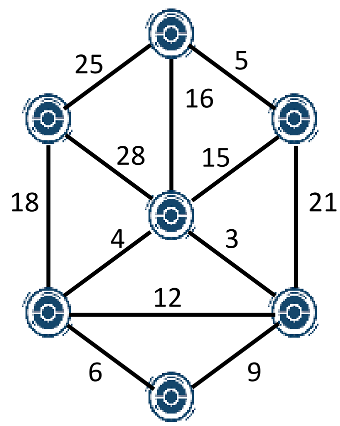
The diagram below illustrates the (incomplete) process of separating a pile of 8 stones.

7. How many edges are there in a forest with v vertices and k components?

8. How many possible binary trees with 4 vertices *A*, *B*, *C* and *D* have this in-order traversal: *A B C D*? Draw them.

9. (AY2016/17 Semester 1 Exam Question)
   The figure below shows a graph where the vertices are Pokestops. Using either Kruskal's algorithm or Prim's algorithm, find its minimum spanning tree (MST). If you use Prim's algorithm, you must start with the top-most vertex.

   Indicate the order of the edges inserted into the MST in your answer.



10. Construct the binary tree given the following in-order and pre-order traversals of the tree:

    In-order: I A D J N H B E K O F L G C M

    Pre-order: H N A I J D O B K E C L F G M

    Draw diagrams to trace the steps of your construction.