Lecture 4
Part I

SETS

Definition

A **set** is an unordered collection of objects.
The objects in a set are called **elements** or **members** of the set.

Notation

$x \in A$: Object x (is) in the set A.
$x \notin A$: Object x (is) not in the set A.

Two ways of writing down the elements of a set:

- List all its elements (in any order, possibly repetitively) within a pair of braces. For example:
  - $\{1, -2, 5\} = \{-2, 1, 5\} = \{-2, 1, 5, -2, 5\}$
  - $\{1, 2, 3, \dots\}$
- Describe its elements using a predicate $P(x)$. For example, $\{x \in A \mid P(x)\}$[1] is the set containing those elements x of A such that $P(x)$ is true.

[1] Some authors write $\{x \in A : P(x)\}$.

Some common sets and their standard notations

- $\mathbb{C}, \mathbb{R}, \mathbb{Q}$: the sets of complex, real and rational numbers respectively.
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$: the set of integers.
- $\mathbb{Z}^+ = \{x \in \mathbb{Z} \mid x > 0\} = \{1, 2, \dots\}$: the set of positive integers.
- $\mathbb{Z}_{\geq 0} = \{x \in \mathbb{Z} \mid x \geq 0\} = \{0, 1, 2, \dots\}$: the set of positive integers.
- We also have analogous notations $\mathbb{R}^+$ and $\mathbb{R}_{\geq 0}$ for real numbers, as well as those for $\mathbb{Q}$.
- $\varnothing$: the empty set, containing no element.
- $\mathbb{N}$: the set of natural numbers. In this module, $\mathbb{N} = \mathbb{Z}_{\geq 0}$.
- $\mathscr{U}$: the universal set, containing all elements under discussion.

There are still controveries about whether N contains 0 or not
=> Try to avoid using N by using Z+ or Z>=0 instead.

Example

It is possible that an element of a set may be a set itself.

Eg: $\{\varnothing\}$ is a set whose only element is $\varnothing$.

Example

Let $A = \{\{1, 2\}, 3\}$. Determine which of the following statements is true:

- $\{1, 2\} \in A$; True
- $1 \in A$. False

Primary Method For Proving an Element of a Set

Let A be a set. If the elements of A can be easily listed down, then proving that $a \in A$ is merely an observation that a is among of the list of elements of A.

If $A = \{x \mid p(x)\}$, then to prove that $a \in A$, we need to show that $p(a)$ is true.

Example

Let $A = \{x \in \mathbb{R} \mid x^2 - 5x + 6 > 0\}$.
Is $5 \in A$?
Check: Let $p(x) = x^2 - 5x + 6 > 0$. Then $P(5) = 5^2 - 5(5) + 6 > 0$ is true. Thus $5 \in A$.
Is $2.5 \in A$?
Check: $P(2.5) = (2.5)^2 - 5(2.5) + 6 > 0$ is false. Thus $\boxed{2.5} \notin A$.

Subsets

Definition

Let A and B be two sets. We say that A is a **subset** of B, denoted $A \subseteq B$, if every element of A is also an element of B. Symbolically,

$$A \subseteq B \quad \equiv \quad \forall x \ (x \in A \rightarrow x \in B).$$

We write $A \not\subseteq B$ for 'A is not a subset of B'. Thus

$$A \not\subseteq B \quad \equiv \quad \exists x \ (x \in A \wedge x \notin B) \quad \equiv \quad \exists x \in A \ (x \notin B).$$

Note

- For any set A, $\varnothing \subseteq A$ and $A \subseteq A$ are always true.
- If $a \in A$ and $A \subseteq B$, we may summarise this as

  $$a \in A \subseteq B,$$

  from which we infer also that $a \in B$.

Example

The following are all the subsets of $\{1, 2, 3\}$:

$$\varnothing,\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{1,2,3\}.$$

Note

Let $p(x)$ and $q(x)$ be predicates. Let $A = \{x \mid p(x)\}$ and $B = \{x \mid q(x)\}$. Then

$$A \subseteq B \quad \equiv \quad \forall x,\ p(x) \to q(x).$$

Lemma

Let A, B and C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Direct proof.

We need to show that if $x \in A$ then $x \in C$.

1. Let $x \in A$.
2. Since $A \subseteq B$ ($\equiv x \in A \to x \in B$), we have $x \in B$ (modus ponens).
3. since $B \subseteq C$ ($\equiv x \in B \to x \in C$), we have $x \in C$ (modus ponens).

$\square$

Note

If $A \subseteq B$ and $B \subseteq C$, we may summarise this as

$$A \subseteq B \subseteq C,$$

from which (according to the above lemma) we infer also that $A \subseteq C$.

Definition

A subset A of B is **proper** if $A \neq B$. Thus A is a proper subset of B, denoted $A \subsetneq B$, if and only if

$$(\forall x \in A,\ x \in B) \wedge (\exists y \in B,\ y \notin A).$$

Example

$\mathbb{Z} \subsetneq \mathbb{Q}$; $\mathbb{Z}_{\geq 0} \subsetneq \mathbb{Z}^{+}$.

Remarks

We do not use the symbol $\subset$ in this module, because it is a little ambiguous.

Definition

Two sets A and B are **equal** if they contain the same elements. More precisely,

$$A = B \quad \equiv \quad A \subseteq B \ \wedge \ B \subseteq A \quad \equiv \quad \forall x \, (x \in A \leftrightarrow x \in B).$$

Example

Let $A = \{n \in \mathbb{R} \mid n^2 = 1\}$ and $B = \{-1, 1\}$. Then $A = B$.

Exercise

Is $\varnothing = \{\varnothing\}$?

No, because the LHS contains no element, while the RHS contains 1 element

Primary Method For Proving Set (In)Equalities

To show that $A \subseteq B$, we start with $x \in A$ and aim for the conclusion $x \in B$.
In other words, we aim to prove $x \in A \Rightarrow x \in B$.

To show that $A \boxed{=} B$, we show $A \subseteq B$ and $B \subseteq A$.
More precisely, we start with $x \in A$ and aim for the conclusion $x \in B$; then start with $y \in B$ and aim for the conclusion $y \in A$.
Sometimes, when we are lucky, we may be able to cut this short by showing $x \in A \Leftrightarrow x \in B$.

We call this method of proof the **element method**.

Differentiate between => and ->:
When we use ->, the statement we say may be true or false; but when we use =>, we want to state that the statement is true

Set Operations

Definition

Let A and B be sets.

- The **union** of A and B, denoted $A \cup B$, is the set

$$A \cup B = \{x \in \mathcal{U} \mid x \in A \text{ or } x \in B\}.$$

- The **intersection** of A and B, denoted $A \cap B$, is the set

$$A \cap B = \{x \in \mathcal{U} \mid x \in A \text{ and } x \in B\}.$$

- The **complement** of B in A, denoted $A - B$ (or $A \setminus B$), is the set

$$A - B = \{x \in A \mid x \notin B\}.$$

We further write $\overline{B}$ for $\mathcal{U} - B$, simply called the complement of B.

Set identities

Theorem (Theorem 6.2.2)

Let A, B and C be subsets of the universal set $\mathcal{U}$.

- *Commutative Laws:*

$$A \cup B = B \cup A, \qquad A \cap B = B \cap B.$$

- *Associative Laws:*

$$(A \cup B) \cup C = A \cup (B \cup C), \qquad (A \cap B) \cap C = A \cap (B \cap C).$$

- *Distributive Laws:*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C);$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

- *Identity Laws:*

$$A \cup \varnothing = A, \qquad A \cap \varnothing = \varnothing.$$

Set identities (Cont'd)

Theorem (Cont'd)

- *Complement Laws:*

$$A \cup \overline{A} = \mathcal{U}, \qquad A \cap \overline{A} = \varnothing.$$

- *Double Complement Law:*

$$\overline{(\overline{A})} = A.$$

- *Idempotent Laws:*

$$A \cup A = A, \qquad A \cap A = A.$$

- *Universal Bound Laws:*

$$A \cup \mathcal{U} = \mathcal{U}, \qquad A \cap \mathcal{U} = A.$$

Set identities (Cont'd)

Theorem (Cont'd)

- *De Morgan's Laws:*

$$\overline{A \cup B} = \overline{A} \cap \overline{B}, \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}.$$

- *Absorption Laws:*

$$A \cup (A \cap B) = A, \qquad A \cap (A \cup B) = A.$$

- *Set Difference Law:*

$$A - B = A \cap \overline{B}.$$

-

$$\overline{\mathscr{U}} = \varnothing, \qquad \overline{\varnothing} = \mathscr{U}.$$

Proof of Set Identities

All identities can be proved by the element method or using the truth table as illustrated below.

Proof of Distributive Law by the element method.

$$
\begin{aligned}
x \in A \cap (B \cup C) &\equiv (x \in A) \wedge (x \in B \cup C) \\
&\equiv (x \in A) \wedge (x \in B \vee x \in C) \\
&\Leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \\
&\equiv (x \in A \cap B) \vee (x \in A \cap C) \\
&\equiv x \in (A \cap B) \cup (A \cap C).
\end{aligned}
$$

$\square$

Proof of Set Identities (Cont'd)

Proof of Absorption Law using truth table.

$x \in A$	$x \in B$	$x \in A \cup B$	$x \in A \cap (A \cup B)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

From the table, we see that the first and last columns are the same. Thus $A = A \cap (A \cup B)$. $\square$

The other set identities can be proved similarly, which we leave for you as an exercise.

Proving Set Equalities using Set Identities

We now have a new method of proving equalities of sets, using set identities.

Example

Prove that $(A \cup B) \cap (A \cup \overline{B}) = A$.

Solution:

$$\begin{aligned} (A \cup B) \cap (A \cup \overline{B}) &= A \cup (B \cap \overline{B}) && \text{(Distributive Law)} \\ &= A \cup \varnothing && \text{(Complement Law)} \\ &= A && \text{(Identity Law).} \end{aligned}$$

Conclusion: Proving set equality (3 methods)
- Element method
- Truth table
- Set identities
-Venn Diagram (not recommended)

Venn Diagrams

Sometimes, to understand complicated situations involving 3 sets or less, it helps to draw a Venn diagram.

In such a diagram, a set is usually represented as the region enclosed by a circle (or an ellipse).

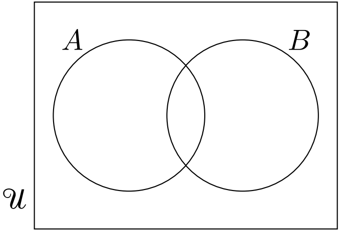

Figure: A Venn diagram involving two sets A and B

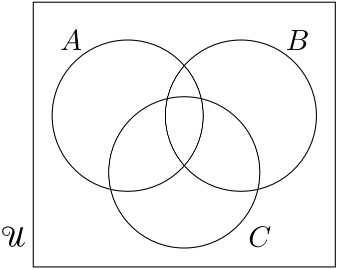

Figure: A Venn diagram involving three sets A, B and C

Warning!

Venn diagrams can only be used as a tool to help us understand, and does not constitute a formal proof of set (in)equalities.

Are Venn diagrams useful for a situation involving four sets?

Disjoint Sets

Definition

Two sets A and B are **disjoint** if $A \cap B = \varnothing$.

A Venn diagram of two disjoint sets.

Definition

Let $\mathscr{C}$ be a collection of sets. The sets in $\mathscr{C}$ are **pairwise disjoint** if and only if for all $A, B \in \mathscr{C}$ with $A \neq B$, we have $A \cap B = \varnothing$.

Union and Intersection of 3 or more sets

By the Associative Laws, we may write $A \cup B \cup C$ and $A \cap B \cap C$ without any ambiguity.

This generalises to any number of sets, and we have the following shorthand:

$$\bigcup_{i=1}^{n} A_i := A_1 \cup A_2 \cup \cdots \cup A_n;$$

$$\bigcap_{i=1}^{n} A_i := A_1 \cap A_2 \cap \cdots \cap A_n.$$

$\bigcup_{i \in I} A_i$

Cartesian Products

Let A and B be sets. The Cartesian product of A and B, denoted $A \times B$, is the set

$$A \times B := \{(a,b) \mid a \in A,\ b \in B\}.$$

The elements of $A \times B$, which are of the form (a,b), are called **ordered pairs**. Note that $(a,b) \neq (b,a)$ unless $a = b$.

Generalising, when $A_1, A_2, \ldots, A_n$ are sets, the Cartesian product of $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \cdots \times A_n$, is the set

$$A_1 \times A_2 \times \cdots \times A_n := \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1,\ a_2 \in A_2,\ \ldots,\ a_n \in A_n\}.$$

The elements of $A_1 \times A_2 \times \cdots \times A_n$, which are of the form $(a_1, a_2, \ldots, a_n)$, are called **n-tuples**.

We also write A^n for

$$\underbrace{A \times A \times \cdots \times A}_{n}.$$

Power Sets

Definition

Let A be a set. The **power set** of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A, i.e.

$$\mathscr{P}(A) = \{X \mid X \subseteq A\}.$$

Example

- $\mathscr{P}(\{1,2,3\}) = \{\varnothing,\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{1,2,3\}\}.$
- $\mathscr{P}(\varnothing) = \{\varnothing\}.$

Partitions

Definition

Let A be a set. A **partition** of A is an Unodered collection of pairwise disjoint, non-empty subsets of A whose union is A.
In other words, P is a partition of A if and only if:

- $P \subseteq \mathscr{P}(A)$;
- $\varnothing \notin P$;
- for all $X, Y \in P$ with $X \neq Y$, we have $X \cap Y = \varnothing$ (distinct elements of P are pairwise disjoint);
- $\boxed{\bigcup}_{X \in P} X = A$.

Example

- Both $\{\{1\}, \{2\}, \{3\}\}$ and $\{\{1\}, \{2, 3\}\}$ are partitions of $\{1, 2, 3\}$.
- Both $\{\{1, 2\}, \{2, 3\}$ and $\{\{1\}, \{3\}\}$ are not partitions of $\{1, 2, 3\}$.
- If A is non-empty, then $\{A\}$ is a partition of A.

Lemma

Let A and B be sets. Then $\{A \cap B, A - B, B - A\}$ is a partition of $A \cup B$.

Proof using truth tables.

Let $S = (A \cap B) \cup (A - B) \cup (B - A)$.

$x \in A$	$x \in B$	$x \in A \cap B$	$x \in A - B$	$x \in B - A$	$x \in S$	$x \in A \cup B$
T	T	T	F	F	T	T
T	F	F	T	F	T	T
F	T	F	F	T	T	T
F	F	F	F	F	F	F

From the table, we find that $A \cap B$, $A - B$ and $B - A$ are pairwise disjoint (at most one T in each row for the columns in the middle), and that $A \cup B = S$. Thus $\{A \cap B, A - B, B - A\}$ is a partition of $A \cup B$. $\square$

Finite and Infinite Sets

Definition

A set is **finite** if it contains only finitely many distinct elements.

A set is **infinite** if it contains infinitely many (distinct) elements.

If S is a finite set, we write $|S|$ for the number of distinct elements in S.

Example

- $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are all infinite sets.
- $\varnothing$, $\{-1, 0.5, \sqrt{2}, -1, \sqrt{2}\}$, $\{x \in \mathbb{Z}^+ \mid x \leq 10^{100}\}$ are all finite sets, and

$\boxed{\text{-1 and sqrt(2) is repeated => only counted once each}}$

$$|\varnothing| = 0,$$
$$|\{-1, 0.5, \sqrt{2}, -1, \sqrt{2}\}| = 3,$$
$$|\{x \in \mathbb{Z}^+ \mid x \leq 10^{100}\}| = 10^{100}.$$

Note

- The union of finitely many finite sets is a finite set.
- If A and B are disjoint finite sets, then $|A \cup B| = |A| \cup |B|$.
- If $A_1, A_2, \ldots, A_n$ are pairwise disjoint finite sets, then $|\bigcup_{i=1}^n A_i| = \sum_{i=1}^n |A_i|$.

Lemma

Let A and B be finite sets, not necessarily disjoint. Then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Proof.

1. Recall that $\{A \cap B, A - B, B - A\}$ is a partition of $A \cup B$. Thus,

$$|A \cup B| = |A \cap B| + |A - B| + |B - A|.$$

2.
   1. $(A \cap B) \cup (A - B) = (A \cap B) \cup (A \cap \overline{B}) = A \cap (B \cup \overline{B}) = A \cap \mathcal{U} = A$.
   2. $A \cap B$ and $A - B$ are disjoint (since they are elements in a partition), so $|A| = |(A \cap B) \cup (A - B)| = |A \cap B| + |A - B|$, or

$$|A - B| = |A| - |A \cap B|.$$

3. Interchanging the roles of A and B in (2), we also get

$$|B - A| = |B| - |A \cap B|.$$

4. Substituting (2.2) and (3) into (1), we get

$$|A \cup B| = |A \cap B| + (|A| - |A \cap B|) + (|B| - |A \cap B|) = |A| + |B| - |A \cap B|.$$

$\square$

Summary

We have covered:

- How to define a set
- How to prove that an element lies in a given set
- Subsets
- How to prove set (in)equalities
- Union and intersection and complement of sets
- Set identities
- Venn diagrams
- Cartesian products of sets
- Power sets and partitions
- (In)finite sets, and number of elements in a finite set