Lecture 9
Part II

PARTIAL ORDERS

Anti-symmetricity

Definition

Let R be a relation on a set A. We say that R is **anti-symmetric** if, and only if,

$$\forall x, y \in A\ (x\,R\,y \wedge y\,R\,x \Rightarrow x = y).$$

Note

R is not anti-symmetric if and only if there exist distinct $x, y \in A$ such that $x\,R\,y$ and $y\,R\,x$.

Example

- $<$, $\leq$, $>$, $\geq$ are all anti-symmetric relations on $\mathbb{R}$.
- The 'divides' relation $|$ is not anti-symmetric on $\mathbb{Z}$, but is anti-symmetric on $\mathbb{Z}^+$.
- If a relation R on A is both symmetric and anti-symmetric, then $x \not{R}\, y$ for all distinct $x, y \in A$.

Partial Orders

Definition

A **partial order on A** is a relation on A that is reflexive, anti-symmetric and transitive.

Example

- On $\mathbb{R}$, both $\geq$ and $\leq$ are partial orders.
- On $\mathbb{Z}^+$, $|$ is a partial order.
- On $\mathscr{P}(A)$, $\subseteq$ is a partial order.

Lemma

If R is a partial order on A, then so is R^{-1}.

Proof.

Easy exercise. $\square$

Note

Partial orders generalises the idea of $\leq$ on $\mathbb{R}$. As such, we shall denote a general partial order as $\preceq$ (or sometimes $\trianglelefteq$), and think of $x \preceq y$ as x is 'smaller' (or 'less') than or equal to y.

We further write $x \prec y$ for ($x \preceq y$ and $x \neq y$).

Definition

Let $\preceq$ be a partial order on a non-empty set A, and let $x, y \in A$. We say that x and y are **comparable** if $x \preceq y$ or $y \preceq x$.

If $x \npreceq y$ and $y \npreceq x$ we say that x and y are **incomparable**.

Definition

Let $\preceq$ be a partial order on a non-empty set A. Let $a \in A$. We say that a is:

consider the divide relation in Z+:1|x for all x => 1 is the smallest element
consider the divide relation in Z>1:
2 is not divides by every x, but there is no x that divides 2 => 2 is a minimal element, but not the smallest. (3,5, 7,.. is also a minimal element)

- **minimal** if nothing is smaller than a, i.e.

$$\sim(\exists x \in A\ (x \prec a)) \equiv \forall x \in A\ (x \preceq a \rightarrow x = a).$$

- **smallest** (or **least** or **minimum**) if a is smaller than everything else, i.e.

$$\forall x \in A\ a \preceq x.$$

- **maximal** if nothing is larger than a, i.e.

$$\sim(\exists x \in A\ (a \prec x)) \equiv \forall x \in A\ (a \preceq x \rightarrow x = a).$$

- **largest** (or **greatest** or **maximum**) if a is larger than everything else, i.e.

$$\forall x \in A\ x \preceq a.$$

Lemma

Let $\preceq$ be a partial order on a non-empty set A. Then A has at most one largest element and one smallest element.

Proof.

If both x and y are smallest, then $x \preceq y$ (since x is smallest), and $y \preceq x$ (since y is smallest), so that $x = y$ by anti-symmetricity of $\preceq$.
Similar proof if both x and y are largest. $\square$

Note

It is possible to have no largest or smallest element. It is also possible to have more than one maximal element, and/or more than one minimal element.

Exercise

Prove that if there are two or more maximal (resp. minimal) elements then there cannot be any largest (resp. smallest) element.

Hasse diagram

The Hasse diagram is used to depict a partial order $\preceq$ on a non-empty set A. It is similar in nature to the arrow diagrams of general relations on A except that the arrows are replaced by lines and there are no loops. The Hasse diagram is much less cluttered than the arrow diagram while retaining all the pertinent information about $\preceq$.

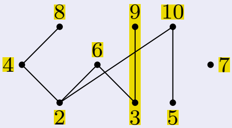
More precisely, for the Hasse diagram, we draw a line from x to y if and only if $x \prec y$ and there does not exist z such that $x \prec z$ and $z \prec y$, in which case we also place x lower than y in the diagram, so that the line from x to y goes upwards.

From the Hasse diagram, one can determine if $x \prec y$ by checking if we can get from x to y by using lines going upwards.

The minimal (resp. maximal) elements can be easily determined from the Hasse diagram, as they correspond to those elements having no lines going downwards (resp. upwards).

Example

Let $A = \{n \in \mathbb{Z} \mid 2 \leq n \leq 10\}$. Define the relation $\preceq$ on A by $x \preceq y$ if and only if $x \mid y$. Below is the Hasse diagram of $\preceq$:



From the diagram, we see that 2, 3, 5, 7 are minimal elements while 6, 7, 8, 9, 10 are maximal elements.

Definition

Let $\preceq$ be a partial order on a non-empty set A.

- We say that $\preceq$ is a **total order** (or A is **totally ordered by** $\preceq$) if and only if every two elements of A are comparable, i.e.

$$\forall x, y \in A\ (x \preceq y \vee y \preceq x) \equiv \forall x, y \in A\ (x \prec y \vee x = y \vee y \prec x).$$

- We say that a subset B of A is a **chain** if and only if every two elements of B are comparable, i.e.

$$\forall x, y \in B\ (x \preceq y \vee y \preceq x) \equiv \forall x, y \in B\ (x \prec y \vee x = y \vee y \prec x).$$

Example

- $\leq$ is a total order on $\mathbb{R}$.
- $|$ is a partial order, but not a total order, on $\mathbb{Z}^+$. $\{1, 2, 4, 12, 60\}$ is a chain (with respect to $|$).
- On the Hasse diagram, a chain is a subset consisting of elements that lie on a single ascending (or descending) line.

Orders on Cartesian Products

Let $\preceq$ be a partial order on a non-empty set A, and let $n \in \mathbb{Z}^+$. The set A^n has two natural partial orders induced by $\preceq$:

- **Lexicographic order:** $(x_1, x_2, \ldots, x_n) \preceq_{\text{lex}} (y_1, y_2, \ldots, y_n)$ if and only if $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$, or $x_k \preceq y_k$ where $k = \min\{i \mid x_i \neq y_i\}$.
- **Product order:** $(x_1, x_2, \ldots, x_n) \preceq_{\text{prod}} (y_1, y_2, \ldots, y_n)$ if and only if $x_i \preceq y_i$ for all (admissible) i.

Note

- $\preceq_{\text{prod}} \subseteq \preceq_{\text{lex}}$.
- It is not difficult to see that if $\preceq$ is a total order on A, then $\preceq_{\text{lex}}$ is a total order on A^n. However, $\preceq_{\text{prod}}$ is never a total order when A has at least 2 elements.
- The lexicographic order is sometimes called the **dictionary** order.

Example

Consider the natural order $\leq$ on $\mathbb{R}$. This is a total order on $\mathbb{R}$. On $\mathbb{R}^2$, we have $(1,1) \preceq_{\text{lex}} (3,0)$ and $(1,1) \preceq_{\text{lex}} (2,3)$, while $(1,1) \not\preceq_{\text{prod}} (3,0)$ and $(1,1) \preceq_{\text{prod}} (2,3)$.

Zorn's Lemma (Non-examinable)

Zorn's Lemma: Let $\preceq$ be a partial order on a non-empty set A. If every chain in A has an upper bound[1], then A has a maximal element.

Despite its name, Zorn's Lemma is NOT a result that can be proved. It is equivalent to the Axiom of Choice, whose name describes them more appropriately.

Zorn's Lemma has many applications. Among them is the existence of an injective function between any two sets, and the existence of bases and the uniqueness of their cardinality for infinite-dimensional vector spaces.

[1] $a \in A$ is an **upper bound** for a subset B of A if $b \preceq a$ for all $b \in B$.

Summary

We have covered:

- Anti-symmetricity
- Partial Orders
- Minimal and maximal elements, smallest and largest elements
- Hasse diagram
- Total order and chains
- Lexicographic and product orders