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CS1231/CS1231S: Discrete Structures

Tutorial #9: Counting and Probability I

Week 11: 28 October – 1 November 2019

I. Discussion Questions

These are meant for you to discuss on the LumiNUS Forum. No answers will be provided.

D1. A box contains three blue balls and seven white balls. One ball is drawn, its colour recorded, and it is returned to the box. Then another ball is drawn and its colour is recorded as well.

(a) What is the probability that the first ball is blue and the second is white?
(b) What is the probability that both balls drawn are white?
(c) What is the probability that the second ball drawn is blue?

D2. Calculate
(a) the probability that a randomly chosen positive three-digit integer is a multiple of 6.
(b) the probability that a randomly chosen positive four-digit integer is a multiple of 7.

D3. Assuming that all years have 365 days and all birthdays occur with equal probability. What is the smallest value for n so that in any randomly chosen group of n people, the probability that two or more persons having the same birthday is at least 50%?

Write out the equation to solve for n and write a program to compute n.

(This is the well-known *birthday problem*, whose solution is counter-intuitive but true.)

II. Tutorial Questions

1. In a certain tournament, the first team to win four games wins the tournament. Suppose there are two teams A and B, and team A wins the first two games. How many ways can the tournament be completed?

   (We will use possibility tree to solve this problem for now. In the next tutorial, we will approach this problem using combination.)

2. (Past year's exam question.)
   The figure on the right shows a combination lock with 40 positions.

   To open the lock, you rotate to a number in a clockwise direction, then to a second number in the counterclockwise direction, and finally to a third number in the clockwise direction. If consecutive numbers in the combination cannot be the same, how many combinations of three-number codes are there?

3. BayBiz is an office building at HarbourFront with 50 floors. Each floor has 20 offices on the south wing facing Sentosa, and 20 offices on the north wing facing Mount Faber.

   There are three security guards who each check half the offices every night and make sure that all is secure and that the lights are turned out.

   Avery checks all the offices on the north wing on even floors and those on the south wing on odd floors. Bruce checks all the offices on odd floors. Carlson checks all the offices on the top 25 floors.

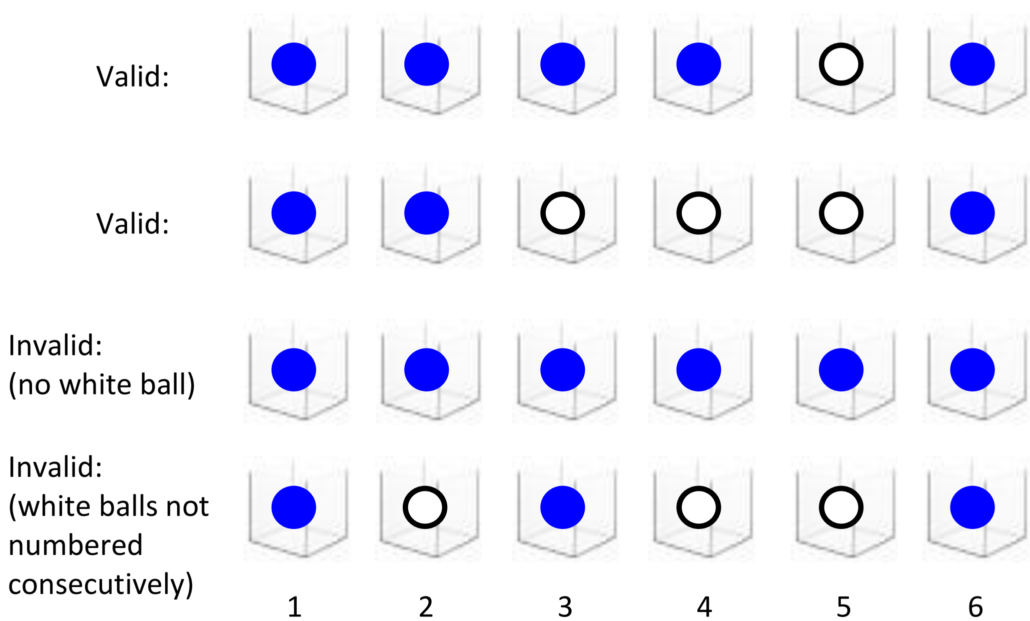
   In the worst case, how many offices have their lights left on?

4. Among all permutations of n positive integers from 1 through n, where $n \geq 3$, how many of them have integers 1, 2 or 3 in the correct position? An integer k is in the correct position of it is at the k^{th} position in the permutation. For example, the permutation 3, 2, 4, 1, 5 has integers 2 and 5 in their correct positions, and the permutation 12, 1, 3, 9, 10, 8, 7, 6, 2, 4, 11, 5 has integers 3, 7, and 11 in their correct positions.

5. Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

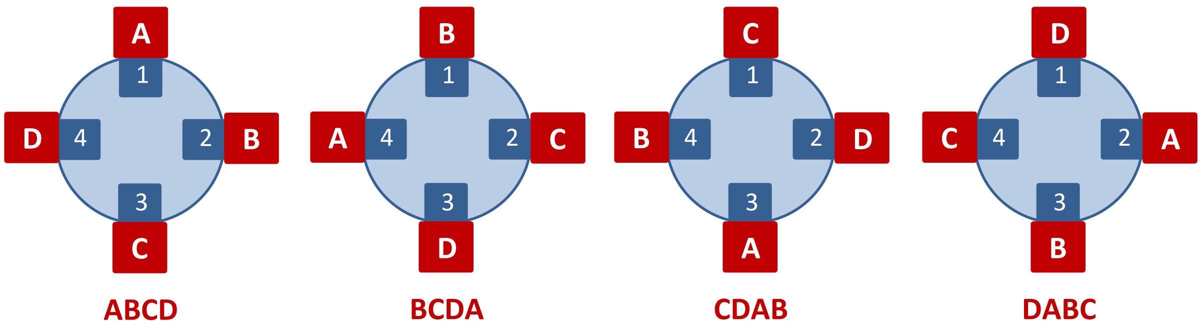
   (Use sum of a sequence to solve this problem. Next tutorial, we will revisit this problem using a different approach.)

   Some examples for $n = 6$ are shown below for your reference.



6. We have learned that the number of permutation of n distinct objects is $n!$, but that is on a straight line. If we seat four guests Anna, Barbie, Chris and Dorcas on chairs on a straight line they can be seated in 4! or 24 ways.

   What if we seat them around a circular table? Examine the figure below.



   The four seating arrangements (clockwise from top) *ABCD*, *BCDA*, *CDAB* and *DABC* are just a single permutation, as in each arrangement the persons on the left and on the right of each guest are still the same persons. Hence, these four arrangements are considered as one permutation.

   This is known as *circular permutation*. The number of linear permutations of 4 persons is four times its number of circular permutation. Hence, there are $\frac{4!}{4}$ or 3! ways of circular permutations for 4 persons. In general, the number of circular permutations of n objects is $(n-1)!$

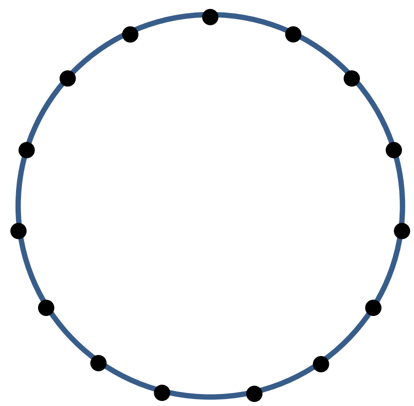
   Answer the following questions:

   a. In how many ways can 8 boys and 4 girls sit around a circular table, so that no two girls sit together?

   b. In how many ways can 6 people sit around a circular table, but Eric would not sit next to Freddy?

   c. In how many ways can $n-1$ people sit around a circular table with n chairs?

7. On the circumference of a circle are 15 dots, evenly spaced out. How many ways can we pick 3 dots from these 15 dots so that they do not form an equilateral triangle?



8. (Past year's exam question.)
   Prove that if you randomly put 51 points inside a unit square, there are always three points that can be covered by a circle of radius 1/7.

9. (Past year's exam question.)
   In a city, houses are randomly assigned distinct numbers between 1 and 50 inclusive. What is the minimum number of houses to ensure that there are 5 houses numbered consecutively?

   To receive full credit, you must define the pigeons and pigeonholes.