CS1231(S) Tutorial 8: Relations

National University of Singapore

2019/20 Semester 1

The questions marked with an asterisk * are more challenging ones intended for discussion purposes during the tutorial class. The other questions are regular ones. Please attempt all questions before the tutorial class.

1. Let $A = \{1, 2, \ldots, 10\}$ and $B = \{2, 4, 6, 8, 10, 12, 14\}$. We define a relation R from A to B by: for all $a \in A$ and $b \in B$, $a \, R \, b$ if and only if a is a prime and divides b.
   (a) Find the subset R of $A \times B$. Your answer should list down all the elements of R.
   (b) Find R^{-1}. You answer should list down all the elements of R^{-1}.
2. Let A be a non-empty set.
   (a) Explain briefly why $\varnothing$ is a relation on A.
   (b) Determine if $\varnothing$ is (as a relation on A) reflexive, symmetric, or transitive.
3. Let R be a relation on a set A. Show that R is symmetric if and only if $R = R^{-1}$.
4. **Claim:** If R is a symmetric and transitive relation on a set A, then R is reflexive.

   Proof: If $x \, R \, y$, then $y \, R \, x$ since R is symmetric, and thus $x \, R \, x$ since R is transitive. Hence R is reflexive.

   Do you agree with the claim and its proof? Justify your answer.
5. For each of the following relations on $\mathbb{R}$, determine if it is (i) reflexive, (ii) symmetric, (iii) transitive, (iv) an equivalence relation.
   (a) R defined by: for all $a, b \in \mathbb{R}$, $a \, R \, b$ if and only if $ab \geqslant 0$.
   (b) S defined by: for all $a, b \in \mathbb{R}$, $a \, S \, b$ if and only if $|a - b| \leqslant 2$.
   (c) T defined by: for all $a, b \in \mathbb{R}$, $a \, T \, b$ if and only if $ab > 0$.
6. Define a relation R on $\mathbb{Q}$ as follows: $x \, R \, y \Leftrightarrow (x - y \in \mathbb{Z})$.
   (a) Show that R is an equivalence relation.
   (b) Find an element a in the equivalence class $[\frac{37}{7}]$ which satisfies $0 \leqslant a < 1$. Are you able to do this in general for an arbitrary equivalence class $[x]$ ($x \in \mathbb{Q}$)? Justify your answer. (Hint: Use Division Algorithm/Theorem.)

7. Let R be a relation on a non-empty set A. Define a relation S on A by $x\,S\,y$ if and only if $x = y$ or $x\,R\,y$. Show that:

   (a) S is reflexive;

   (b) $R \subseteq S$ (recall that both R and S are subsets of $A \times A$);

   (c) if S' is another reflexive relation on A and $R \subseteq S'$, then $S \subseteq S'$.

   The relation S is called the *reflexive closure* of R. As the above exercise shows, this is the smallest relation on A that is reflexive and contains R as a subset.

8.* Let R be a relation on a non-empty set A. Use recursion to define the *transitive closure* of R, i.e. the smallest relation on A that is transitive and contains R as a subset.

9. Let R be an antisymmetric and transitive relation on a non-empty set A. Let $\widetilde{R}$ denote the reflexive closure of R (see Qn 7). Show that $\widetilde{R}$ is a partial order on A.

10. **Definition.** Let $\preccurlyeq$ be a partial order on a set P, and $a, b \in P$.

    - We say a, b are *comparable* if $a \preccurlyeq b$ or $b \preccurlyeq a$.
    - We say a, b are *compatible* if there exists $c \in P$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

    (a) Is it true that, in all partially ordered sets, any two comparable elements are compatible? Justify your answer.

    (b) Is it true that, in all partially ordered sets, any two compatible elements are comparable? Justify your answer.

11. Consider the 'divides' relation on each of the following subsets A of $\mathbb{Z}^+$. For each subset, draw the Hasse diagram, and find all largest, smallest, maximal and minimal elements.

    (a) $A = \{1, 2, 4, 5, 10, 15, 20\}$;

    (b) $A = \{2, 3, 4, 6, 8, 9, 12, 18\}$.