Done

CS1231(S) Tutorial 4: Functions

National University of Singapore

2019/20 Semester 1

1. Which of the following formulas define a function $f\colon \mathbb{Q} \to \mathbb{Q}$?

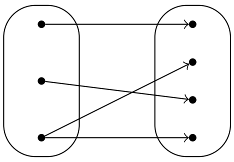
   (a) $f(n) = \pm n$. NO

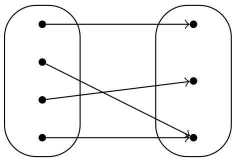
   (b) $f(n) = 2^n$. no, i.e. sqrt(2) = 2^(1/2) is not a rational number

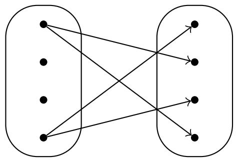
   (c) $f(n) = \frac{1}{n^2+1}$. Yes

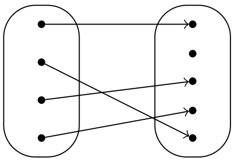
   (d) $f(n) = \lfloor \sin n \rfloor$. Yes

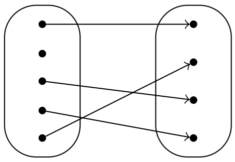
2. Each of the diagram below represents a way of assigning the LHS points to the RHS points: a point x on the left is assigned a point y on the right if and only if there is an arrow from x to y.

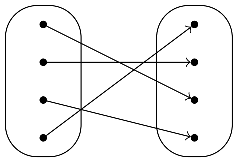
   (a)  NO

   (b)  Yes, surjection

   (c)  No

   (d)  Yes, injection

   (e)  no

   (f)  yes, bijection

   Which of the above represent a function from the LHS set to the RHS set? Amongst those that represent a function, which ones represent injections, which ones represent surjections, and which ones represent bijections?

3. Let U be a set and $A \subseteq U$ such that $\varnothing \neq A \neq U$. Define the function $\chi\colon U \to \mathbb{Z}$ by setting, for all $x \in U$,

$$\chi(x) = \begin{cases} 0, & \text{if } x \notin A; \\ 1, & \text{if } x \in A. \end{cases}$$

   domain: U
   codomain: Z
   image: {0, 1}

   Find the domain, the codomain, and the image of χ.

4. Which of the functions defined in the following are injective? Which are surjective? Prove that your answers are correct. If a function defined below is both injective and surjective, then find a formula for the inverse of the function. Here denote by Bool the set $\{\textbf{true}, \textbf{false}\}$.

$$f\colon \mathbb{Q} \to \mathbb{Q}; \quad x \mapsto 12x + 31, \qquad g\colon \mathrm{Bool}^2 \to \mathrm{Bool}; \quad (p,q) \mapsto p \wedge {\sim}q, \qquad h\colon \mathrm{Bool}^2 \to \mathrm{Bool}^2; \quad (p,q) \mapsto (p \wedge q, p \vee q),$$

$$k\colon \mathbb{Z} \to \mathbb{Z}; \quad x \mapsto \begin{cases} x, & \text{if } x \text{ is even}; \\ 2x - 1, & \text{if } x \text{ is odd}. \end{cases}$$

5. Find the values of the following:

   (a) $\lfloor 12.31 \rfloor$; 12

   (b) $\lceil 12.31 \rceil$; 13

   (c) $\lfloor -12.31 \rfloor$; -13

   (d) $\lceil -12.31 \rceil$. -12

6. Let $A = \{1, 2, 3\}$. The *order* of a bijection $f\colon A \to A$ is defined to be the least $n \in \mathbb{Z}^+$ such that

$$\underbrace{f \circ f \circ \ldots \circ f}_{n\text{-many } f\text{'s}} = \mathrm{id}_A.$$

   Define functions $g, h\colon A \to A$ by setting, for all $x \in A$,

$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

   Find the orders of g, h, $g \circ h$, and $h \circ g$.

7. Let A, B, C be sets. Show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ for all bijections $f\colon A \to B$ and all bijections $g\colon B \to C$.

8. Fix sets A, B. Define the *graph* of a function $f\colon A \to B$ to be

$$\{(x, y) \in A \times B : x \in A \text{ and } y = f(x)\}.$$

   (a) Assuming $A \neq \varnothing$, find a subset $S \subseteq A \times B$ that cannot be the graph of any function $f\colon A \to B$.

   (b) Find a necessary and sufficient condition for a subset $S \subseteq A \times B$ to be the graph of some function $A \to B$. Show that your condition is indeed necessary and sufficient.

9. Let $f\colon A \to B$ be a function. Let $X \subseteq A$ and $Y \subseteq B$.

   (a) Compare the sets X and $f^{-1}(f(X))$.

      (i) Is one always a subset of the other? Justify your answer.

      (ii) Propose, with justification, a necessary and sufficient condition on X for the two sets to be equal.

   (b) Compare the sets Y and $f(f^{-1}(Y))$.

      (i) Is one always a subset of the other? Justify your answer.

      (ii) Propose, with justification, a necessary and sufficient condition on Y for the two sets to be equal.