CS1231/CS1231S: Discrete Structures

Tutorial #10: Counting and Probability II

Week 12: 4 – 8 November 2019

I. Discussion Questions

These are meant for you to discuss on the LumiNUS Forum. No answers will be provided.

D1. How many one-to-one functions are there from a set of m elements to a set with n elements, where $m \leq n$?

D2. Suppose a random sample of 2 lightbulbs is selected from a group of 8 bulbs in which 3 are defective, what is the expected value of the number of defective bulbs in the sample? Let X represent of the number of defective bulbs that occur on a given trial, where $X = 0,1,2$. Find $E[X]$.

II. Tutorial Questions

1. (CS1231 Past Year's Exam Question)
   You wish to select five persons from seven men and six women to form a committee that includes at least three men.
   (a) In how many ways can you form the committee?
   (b) If you randomly choose five persons to form the committee, what is the probability that you will get a committee with at least three men? Give your answer correct to 4 significant figures.

2. Think of a set with $m + n$ elements as composed of two parts, one with m elements and the other with n elements. Give a combinatorial argument to show that

$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0}$$

   where $m, n \in \mathbb{Z}^+$, $r \leq m$ and $r \leq n$.

   Call the above equation (A). Using equation (A), prove that for all integers $n \geq 0$,

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

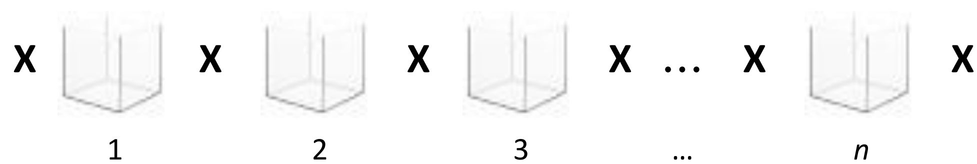
3. Let's revisit Question 5 of Tutorial #8:

   Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

   Last week, the answer given was

   For k $(1 \le k \le n)$ consecutively numbered boxes that contain white balls, there are $n - k + 1$ ways. Therefore, total number of ways is $\sum_{k=1}^{n}(n-k+1) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

   Now, let's use another approach to solve this problem. Draw crosses on the side of the boxes as shown below. How do you use these crosses?



4. How many solutions are there to the following equation?

$$x_1 + x_2 + x_3 + x_4 = 30, \text{ where } x_i \text{ are integers and } x_i \ge 2.$$

5. One urn contains 10 red balls and 25 green balls, and a second urn contains 22 red balls and 15 green balls. A ball is chosen as follows: First an urn is selected by tossing a loaded coin with probability 0.4 of landing heads up and probability of 0.6 of landing tails up. If the coin lands heads up, the first urn is chosen; otherwise, the second urn is chosen. Then a ball is picked at random from the chosen urn.

   Write your answers correct to three significant figures.

   (a) What is the probability that the chosen ball is green?

   (b) If the chosen ball is green, what is the probability that it was picked from the first urn?

6. Expected value.

   (a) A lottery game offers \$1 million to the grand prize winner, \$1000 to each of the 100 second prize winners, \$500 to each of the 300 third prize winners and \$10 to each of the 1,000 consolation prize winners. The cost of the lottery is \$3 per ticket and 500,000 tickets were sold. What is the expected gain or loss of a ticket?

   (b) An urn contains five balls numbered 1, 2, 2, 8 and 8. If a person selects a set of two balls at random, what is the expected value of the sum of the numbers on the balls? Write your answer correct to three significant figures.

7. (AY2015/16 Semester 1 exam question)
   Let $A = \{1, 2, 3, 4\}$. Since each element of $\wp(A \times A)$ is a subset of $A \times A$, it is a binary relation on A.

   Assuming each relation in $\wp(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is (a) reflexive? (b) symmetric?

   Can you generalize your answer to any set A with n elements?

8. (This is an optional question. This question may be skipped if there is not enough time.)
   Let's revisit Question 1 of Tutorial #8:

   > In a certain tournament, the first team to win four games wins the tournament. Suppose there are two teams A and B, and team A wins the first two games. How many ways can the tournament be completed?

   The solution given last week uses a possibility tree to depict the 15 ways. Now, let's approach this problem using combination.

   Let us define a function $W(a, b)$ to be the number of ways the tournament can be completed if team A has to win a more games to win, while team B has to win b more games to win. Hence,

$$W(a, b) = \begin{cases} 1, & \text{if } a = 0 \text{ or } b = 0. \\ W(a, b-1) + W(a-1, b), & \text{otherwise.} \end{cases}$$

   We observe that this is similar to the computation of the Pascal's triangle. In fact,

$$W(a, b) = \binom{a+b}{a}.$$

   (verify the above yourself.)

   Now, we denote the function $T(n, k)$ to be the number of ways the tournament can be completed, given that the first team to win n games wins the tournament, and team A wins the first k ($k \le n$) games.

   Derive a simple combination formula for $T(n, k)$ (hint: relate function T to function W), and hence solve $T(4, 2)$ which is the problem in question 1 of tutorial #8.