Lecture 9
Part I

RELATIONS

Definition

Let A and B be two sets.

- A **relation R from A to B** is a subset of $A \times B$.
- Let R be a relation from A to B. For each $(a, b) \in A \times B$, we write $a\,R\,b$ if $(a, b) \in R$, and $a \not{R}\, b$ if $(a, b) \notin R$.

Example

- Let $R = \{(1,3), (2,2), (2,3), (3,1)\}$, a relation from $\mathbb{Z}$ to $\mathbb{Z}$. We have $2\,R\,3$ but $3 \not{R}\, 2$.
- Let $f : A \to B$ be a function. Recall the graph $\Gamma(f)$ of f as introduced in Tut 4, Qn 8:

$$\Gamma(f) = \{(a, f(a)) \mid a \in A\}.$$

Then $\Gamma(f)$ is a relation from A to B. This example shows that functions can be thought of as special types of relations.

Definition

Let R be a relation from A to B. The **domain of R** is the set

$$\{a \in A \mid \exists b \in B \;\; a\,R\,b\}.$$

The **range of R** is the set

$$\{b \in B \mid \exists a \in A \;\; a\,R\,b\}.$$

Note

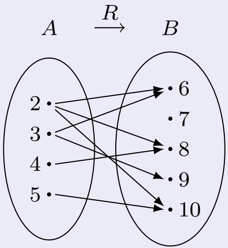
When R is the graph $\Gamma(f)$ of a function $f : A \to B$, then the domain of R is exactly the domain of the function f, and the range of R is exactly the range of the function f.

Arrow Diagrams

Generalising the idea of arrow diagrams for functions, we also have arrow diagrams for a relation R from A to B, where we get an arrow from $a \in A$ to $b \in B$ if and only if $a \, R \, b$.

Example

Let $A = \{2, 3, 4, 5\}$ and $B = \{6, 7, 8, 9, 10\}$. Define the relation R from A to B by $a \, R \, b$ if and only if $a \mid b$, where $a \in A$ and $b \in B$. Then R may be depicted by the following arrow diagram:



Inverse of a Relation

Definition

Let R be a relation from A to B. Then the **inverse of R**, denoted R^{-1}, is the relation from B to A defined by

$$R^{-1} = \{(b, a) \in B \times A \mid \underline{a \, R \, b}\}.$$

Note

- $\forall a \in A \; \forall b \in B \; (a \, R \, b \Leftrightarrow b \, R^{-1} \, a)$.
- The arrow diagram of R^{-1} can be obtained by reversing the arrows in the arrow diagram of R.
- Let $f : A \to B$ be a function. Then $(\Gamma(f))^{-1}$ is a relation from B to A. Furthermore, $(\Gamma(f))^{-1}$ is the graph of a function $g : B \to A$ if and only if f is bijective, if and only if $g = f^{-1}$.

(Binary) Relations on a Set

Definition

Let A be a set. A **(binary) relation on A** is a relation from A to A, i.e. a subset of $A \times A$.

Example

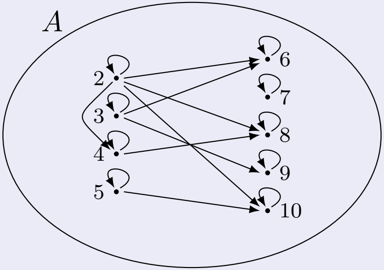
- On $\mathbb{R}$, we have relations $\leq$, $\geq$ (the inverse of $\leq$), $<$ and $>$ (the inverse of $<$).
- Let A be a set. Then on $\mathscr{P}(A)$, we have the relation $\subseteq$.
- Let $n \in \mathbb{Z}^+$. On $\mathbb{Z}$, we have the relation $\equiv \pmod{n}$.

Arrow Diagram

The arrow diagram of a relation R on a set A displays the set A only once, with an arrow going from a_1 to a_2 if and only if $a_1 \, R \, a_2$.

Example

Let $A = \{n \in \mathbb{Z} \mid 2 \leq n \leq 10\}$. Define the relation R on A by $x \, R \, y$ if and only if $x \mid y$. Below is the arrow diagram of R:



Definition

Let R be a relation on a set A. We say that R is:

- **reflexive** if, and only if,

$$\forall x \in A \;\; x\,R\,x;$$

- **symmetric** if, and only if,

$$\forall x, y \in A \; (x\,R\,y \rightarrow y\,R\,x);$$

- **transitive** if, and only if,

$$\forall x, y, z \in A \; (x\,R\,y \wedge y\,R\,z \rightarrow x\,R\,z).$$

Exercise

How do you check if R is reflexive and/or symmetric from its arrow diagram?

Can you also tell if R is transitive from its arrow diagram?

Example

Let R, S, T, U be relations on $\mathbb{Z}$ defined as follows:

$$x\,R\,y \Leftrightarrow x \mid y; \qquad x\,S\,y \Leftrightarrow x \equiv y \pmod 7;$$
$$x\,T\,y \Leftrightarrow x < y; \qquad x\,U\,y \Leftrightarrow \gcd(x,y) = 1.$$

Then:

- R is reflexive and transitive, but not symmetric.
- S is reflexive, symmetric and transitive.
- T is transitive, but neither reflexive nor symmetric.
- U is symmetric, but neither reflexive nor transitive.

Caution!

Terms like irreflexive, asymmetric and intransitive mean more than not reflexive, not symmetric and not transitive when used to describe a relation. As such do not use them unless you really know what they mean.

Equivalence Relations

Definition

A relation on a set A is an **equivalence relation** if and only if it is reflexive, symmetric and transitive.

Example

- The relation $\equiv \pmod{n}$ on $\mathbb{Z}$ is an equivalence relation.
- Define R on $\mathbb{R}$ by: $a\,R\,b$ if and only if $\lfloor a \rfloor = \lfloor b \rfloor$. Then R is an equivalence relation.

Equivalence Classes

Definition

Let R be an equivalence relation on a set A (assumed non-empty). For each $a \in A$, the **equivalence class of a** (with respect to R), denoted $[a]$ (or $[a]_R$ to be more precise), is the set

$$[a] = \{x \in A \mid a\, R\, x\}.$$

Note

An equivalence relation on a set A is necessarily reflexive. Thus if $a \in A$ then $a \in [a]$, so that $[a] \neq \varnothing$.

Lemma

Let R be an equivalence relation on a set A, and let $x, y \in A$.

1. *If $x \, R \, y$, then $[x] = [y]$.*
2. *If $x \not R \, y$, then $[x] \cap [y] = \varnothing$.*

Proof.

1. If $x \, R \, y$, then:
    1. If $a \in [x]$, then:
        1. $x \, R \, a$ (definition of $[x]$).
        2. $y \, R \, x$ (by (1) and symmetricity of R).
        3. $y \, R \, a$ (by (1.1.2) and (1.1.1), and transitivity of R.)
        4. $a \in [y]$ (definition of $[y]$).
    2. Thus $[x] \subseteq [y]$.
    3. If $a' \in [y]$, then:
        1. $y \, R \, a'$ (definition of $[y]$).
        2. $x \, R \, a'$ (by (1) and (1.3.1), and transitivity of R.)
        3. $a' \in [x]$ (definition of $[y]$).
    4. Thus $[y] \subseteq [x]$.
    5. By (1.2) and (1.4), $[x] = [y]$.

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Proof.

2. If $x \not R y$ then:
   1. If $[x] \cap [y] \neq \varnothing$, then:
      1. there exists $a \in A$ such that $a \in [x] \cap [y]$.
      2. $a \in [x] \wedge a \in [y]$ (definition of $\cap$).
      3. $x R a \wedge y R a$ (definition of $[x]$ and $[y]$).
      4. $x R a \wedge a R y$ (by symmetricity of R).
      5. $x R y$ (by transitivity of R), a contradiction.
   2. Hence $[x] \cap [y] = \varnothing$.

$\square$

Definition

Let R be an equivalence relation on a non-empty set A. A subset S of A is an **equivalence class of R** if, and only if, $S = [a]$ for some $a \in A$. The set of all equivalence classes of R shall be denoted as A/R.

Note

A/R is a collection of subsets of A. Thus, $A/R \subseteq \mathscr{P}(A)$, the power set of A.

Example

- $\mathbb{Z}/\big(\equiv \pmod{n}\big) = \{[0], [1], \ldots, [n-1]\}$, where $[i] = \{k \in \mathbb{Z} \mid k \bmod n = i\}$.
- Let R be the relation on $\mathbb{R}$ defined by $a\,R\,b$ if and only if $\lfloor a \rfloor = \lfloor b \rfloor$. Then $\mathbb{R}/R = \{[m, m+1) \mid m \in \mathbb{Z}\}$, where $[m, m+1) = \{x \in \mathbb{R} \mid m \leq x < m+1\}$.

Corollary

Any two distinct equivalence classes of an equivalence relation are disjoint.

Note

'Distinct' means 'not equal'.

Proof.

Let the two equivalence classes be $[x]$ and $[y]$. If $x\,R\,y$, then $[x] = [y]$ by the last Lemma, contradicting $[x]$ and $[y]$ are distinct. Thus $x \not R\, y$, and $[x] \cap [y] = \varnothing$ by the last Lemma. $\square$

Recall that: A partition P of a non-empty set A is a collection of pairwise disjoint non-empty subsets of A whose union is A.

Theorem

Let R be an equivalence relation on a non-empty set A. Then A/R is a partition of A.

Proof.

1. Each element of A/R is an equivalence class, say $[a]$, which is a non-empty (since $a \in [a]$) subset of A.
2. If $X, Y \in A/R$ with $X \neq Y$, then $X \cap Y = \varnothing$ by the last Corollary.
3.
    1. Clear that the union of any number of subsets of A is a subset of A; thus the union of the equivalence classes of R is a subset of A.
    2. If $a \in A$, then $a \in [a]$, so that a is an element of an equivalence class of R, and hence an element of the union of the equivalence classes of R. Thus A is a subset of the union of the equivalence classes of R.
4. By (3.1) and (3.2), the union of the equivalence classes of R is A.
5. By (1), (2) and (4), A/R is a partition of A.

$\square$

Every Partition is a Set of Equivalence Classes

Let A be a non-empty set, and let $P \subseteq \mathscr{P}(A)$ be a partition of A. We now define a relation R on A as follows:

$$x\,R\,y \Leftrightarrow \exists S \in P\ (x \in S \wedge y \in S)$$

- R is reflexive: For each $a \in A = \bigcup_{S \in P} S$, there exists $S \in P$ such that $a \in S$. Thus

$$\forall a \in A\ \exists S \in P\ (a \in S) \equiv \forall a \in A\ \exists S \in P\ (a \in S \wedge a \in S) \equiv \forall a \in A\ a\,R\,a.$$

- R is symmetric: This is easy.
- R is transitive: If $x\,R\,y$ and $y\,R\,z$, then $x, y \in S$ for some $S \in P$ and $y, z \in T$ for some $T \in P$. Thus $y \in S \cap T$, so that $S \cap T \neq \varnothing$, and hence $S = T$ (since elements of P are pairwise disjoint). Consequently, $x, z \in S$, so that $x\,R\,z$.

Therefore, R is an equivalence relation.

Claim: $A/R = P$.

Proof of Claim.

1. If $x \in S_0$ for some $S_0 \in P$, then $x \in A$, and
$$[x] = \{a \in A \mid x\,R\,a\} = \{a \in A \mid \exists S \in P\ (x \in S \wedge a \in S)\}$$
$$= \{a \in A \mid a \in S_0\} = S_0 \in P.$$
2. If $C \in A/R$, then $C = [a]$ for some $a \in A$. Let S_a be the unique element of P such that $a \in S_a$, so that $C = [a] = S_a \in P$ by (1). Thus $A/R \subseteq P$.
3. If $S \in P$, then $S \neq \varnothing$, so that there exists $s \in S$, and hence $S = [s] \in A/R$ by (1). Thus, $P \subseteq A/R$.
4. By (2) and (3), $A/R = P$.

$\square$

Summary

We have covered:

- Relations from a set to another set
- (Binary) relations on a set
- Reflexive, symmetric and transitive
- Equivalence relations and equivalence classes
- Equivalence classes partition a set, and every partition is a set of equivalence classes