Lecture 4
Part II

FUNCTIONS

Definition

A **function** (or **map**) f from a set A to a set B is a rule that assigns to each element $a \in A$ a unique element $f(a) \in B$. We denote such a function by

$$\begin{aligned} f : A &\to B; \\ a &\mapsto f(a). \end{aligned}$$

The set A is the **domain** of f and the set B is the **codomain** of f.

Note

- Note the difference between the arrows $\to$ and $\mapsto$ in the definition of f.
- When $f(a)$ can be written down as a closed formula in terms of a, we can replace the line $a \mapsto f(a)$ by the explicit formula for $f(a)$. For example,

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}; \\ a &\mapsto a^2 + a + 1. \end{aligned} \quad = \quad \begin{aligned} f : \mathbb{R} &\to \mathbb{R}; \\ f(a) &= a^2 + a + 1. \end{aligned}$$

Exercise

What is wrong with the following function?

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}; \\ f(x) &\mapsto x^2 + 4. \end{aligned}$$

Definition

A function $f : A \to B$ is **well-defined** if and only if

- f defines a unique $f(a)$ for each $a \in A$;
- $f(a) \in B$ for each $a \in A$.

Exercise

The following 'functions' are not well-defined. Why?

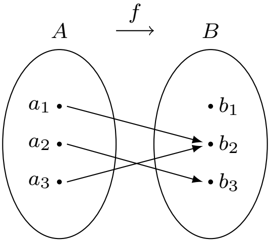
- $f : \mathbb{Z} \to \mathbb{R}$; $f(x) = \sqrt{x}$. sqrt(x) may not be a real number if x<0 => f(a) doesn't belong to B for each a of A
- $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$; $f(x) = a$ where $a^2 = x$. f doesn't defines a unique f(x). f(x) could be sqrt(x) and -sqrt(x)

Note

Every function, by definition, is well-defined. A function that is not well-defined is not a function (a contradiction of terms). Thus checking if a function f is well-defined, is the same as checking if the rule defined by f is a function.

Arrow Diagrams

A function may be depicted using an **arrow diagram**.



In the arrow diagram of a well-defined function:

- all arrows originate from the domain and terminate at the co-domain;
- every element in the domain has one and only one arrow originating from it.

Some Common Functions

- Let A be a set. The function $I_A : A \to A;\ a \mapsto a$ is called the **identity function** on A.
- Let B be a subset of A. Then function $\iota_B^A : B \to A;\ b \mapsto b$ is called the **inclusion map** of B in A.
- For $x \in \mathbb{R}$, define $|x|$, $\lfloor x \rfloor$ and $\lceil x \rceil$ by:

$$|x| := \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0; \end{cases}$$

$$\lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid n \leq x\};$$

$$\lceil x \rceil := \min\{n \in \mathbb{Z} \mid x \leq n\}.$$

These are the **absolute value**, **floor** and **ceiling** of x respectively. We thus have the absolute value function $\mathbb{R} \to \mathbb{R};\ x \mapsto |x|$. Similarly, we also have the floor function and ceiling function.

Sequences

A **sequence** (or more accurately, an infinite sequence) is a function whose domain is $\mathbb{Z}^+$.

For example, a function $f : \mathbb{Z}^+ \to \mathbb{R}$ is a real sequence, and a function $g : \mathbb{Z}^+ \to \mathbb{Z}$ is a sequence of integers.

Sometimes we write a sequence $h : \mathbb{Z}^+ \to B$ as an infinite tuple $(h(1), h(2), h(3), \dots) = (h(n))_{n \in \mathbb{Z}^+}$.

We will see more of sequences later in the course.

Images and Preimages

Let $f : A \to B$ be a function. For $a \in A$, if $f(a) = b$, we say that b is the **image** of a under f, and that a is a **preimage** (or an **inverse image**) of b under f.

Let $X \subseteq A$ and $Y \subseteq B$. Then:

$$f(X) := \{f(x) \mid x \in X\} = \{b \in B \mid \exists x \in X, f(x) = b\};$$
$$f^{-1}(Y) := \{a \in A \mid f(a) \in Y\}.$$

We call $f(X)$ the set of images of X under f, and $f^{-1}(Y)$ the set of preimages of Y under f.

Note

For $a \in A$ and $Y \subseteq B$, we have

$$a \in f^{-1}(Y) \quad \Leftrightarrow \quad f(a) \in Y.$$

Exercise

Let $f : A \to B$ be a function, $X \subseteq A$ and $Y \subseteq B$.

1. If $X \neq \varnothing$, can $f(X) = \varnothing$? never
2. If $Y \neq \varnothing$, can $f^{-1}(Y) = \varnothing$? yes
3. Show that if $X' \subseteq X$, then $f(X') \subseteq f(X)$.
4. Show that if $Y' \subseteq Y$, then $f^{-1}(Y') \subseteq f^{-1}(Y)$.

Range

Definition

Let $f : A \to B$ be a function. The **range** of f, denoted by $\mathscr{R}(f)$, is the set of all images of f, i.e.

$$\mathscr{R}(f) := \{f(a) \mid a \in A\} = f(A).$$

Note

Clearly $\mathscr{R}(f) \subseteq B$, and $f(X) \subseteq \mathscr{R}(f)$ for all $X \subseteq A$.

Example

1. Let A be a set. Then $\mathscr{R}(I_A) = A$.
2. Let B be a subset of A. Then $\mathscr{R}(\iota_B^A) = B$.
3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ for all $x \in \mathbb{R}$. Then $\mathscr{R}(f) = \mathbb{R}_{\geq 0}$.

How to find $\mathscr{R}(f)$?

When asked to determine the range of a function f:

- Make an intelligent guess what it should be. Let's say it should be the set C.
- Prove that $C = \mathscr{R}(f)$ (i.e. that $C \subseteq \mathscr{R}(f)$ and that $\mathscr{R}(f) \subseteq C$).

Exercise

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + x + 1$ for all $x \in \mathbb{R}$. Determine the range of f.

Solution.

[Note that $f(x) = x^2 + x + 1 = (x - \frac{1}{2})^2 + \frac{3}{4}$. From this we guess that $\mathscr{R}(f) = \{x \in \mathbb{R} \mid x \geq \frac{3}{4}.]$

1. 1. Let $a \in \mathbb{R}$.
   2. $f(a) = (a - \frac{1}{2})^2 + \frac{3}{4} \geq \frac{3}{4}$, so that $f(a) \in \{x \in \mathbb{R} \mid x \geq \frac{3}{4}\}$.
2. Thus, $\mathscr{R}(f) = \{f(a) \mid a \in \mathbb{R}\} \subseteq \{x \in \mathbb{R} \mid x \geq \frac{3}{4}\}$.
3. 1. Let $b \in \{x \in \mathbb{R} \mid x \geq \frac{3}{4}\}$.
      [We need to find $a' \in \mathbb{R}$ such that $f(a') = b$, i.e. $(a' - \frac{1}{2})^2 + \frac{3}{4} = b$, which upon solving yields $a' = \frac{1}{2} \pm \sqrt{b - \frac{3}{4}}$.]
   2. Then $b - \frac{3}{4} \geq 0$ so that $\sqrt{b - \frac{3}{4}} \in \mathbb{R}$.
   3. Let $a' = \frac{1}{2} + \sqrt{b - \frac{3}{4}}$. Then $a' \in \mathbb{R}$, and
      $$f(a') = (a' - \tfrac{1}{2})^2 + \tfrac{3}{4} = b.$$
4. Thus $\{x \in \mathbb{R} \mid x \geq \frac{3}{4}\} \subseteq \mathscr{R}(f)$.
5. From (2) and (4), we get $\mathscr{R}(f) = \{x \in \mathbb{R} \mid x \geq \frac{3}{4}\}$.

□

Equality of functions

Definition

Two functions f and g are **equal**, denoted $f = g$, if and only if:

- the domains of f and g are equal;
- the codomains of f and g are equal;
- $f(x) = g(x)$ for all x in the domain of f ($=$ domain of g).

Exercise

Let B be a subset of A.

- Is $\iota_B^A = I_A$? true iff B = A, otherwise false since f(x) != g(x)
- Is $\iota_B^A = I_B$? true iff B=A, otherwise false since codomains are not equal.

Composition of Functions

Definition

(codomain of F must be equal to domain of G)

Let $f : A \to B$ and $g : B \to C$ be functions. The **composition of f with g**, denoted $g \circ f$, is the function with domain A and co-domain C that sends $a \in A$ to $g(f(a))$ for all $a \in A$. In other words,

$$g \circ f : A \to C; \quad a \mapsto g(f(a)).$$

Exercise

Check that $g \circ f$ is indeed well-defined.

Note

- In order for $g \circ f$ to be defined, we need the codomain of f to be equal to the domain of g.
- We write f^2 for $f \circ f$ (defined when the domain and the codomain of f are equal).

Example

Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions defined by $f(x) = 2x + 3$ and $g(x) = x^2$ for all $x \in \mathbb{R}$. Then $g \circ f, f \circ g : \mathbb{R} \to \mathbb{R}$, and

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = (2x + 3)^2 = 4x^2 + 12x + 9;$$
$$(f \circ g)(x) = f(g(x)) = f(x^2) = 2x^2 + 3$$

for all $x \in \mathbb{R}$.

Example

Let $f : \mathbb{R} \to \mathbb{Z}$, $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \lfloor x \rfloor + 1, \qquad g(x) = x + 2, \qquad h(x) = \lfloor x \rfloor + 3$$

for all $x \in \mathbb{R}$. Then

$$h(x) = \lfloor x \rfloor + 3 = (\lfloor x \rfloor + 1) + 2 = f(x) + 2 = g(f(x))$$

for all $x \in \mathbb{R}$.

Is $h = g \circ f$? no, since the codomain of f is different from the domain of g

Exercise

Let $f : A \to B$ be a function. Recall the identity functions I_A and I_B. Prove that:

1. $f \circ I_A = f$;
2. $I_B \circ f = f$.

1. Prove:
(i) same domain
(ii) same codomain
(iii) f . Ia (a) = f(Ia(a)) = f(a)

Theorem (Associativity of Composition of Functions)

Let $f : A \to B$, $g : B \to C$ and $h : C \to D$ be functions. Then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Proof.

1. Both $h \circ (g \circ f)$ and $(h \circ g) \circ f$ have domain A.
2. Both $h \circ (g \circ f)$ and $(h \circ g) \circ f$ have codomain D.
3. For all $a \in A$,

$$\begin{aligned}(h \circ (g \circ f))(a) &= h((g \circ f)(a)) = h(g(f(a)));\\ ((h \circ g) \circ f)(a) &= (h \circ g)(f(a)) = h(g(f(a))).\end{aligned}$$

Thus, $(h \circ (g \circ f))(a) = ((h \circ g) \circ f)(a)$.

□

Remarks

- We may thus write $h \circ g \circ f$ without ambiguity.
- For $f : A \to A$ and $n \in \mathbb{Z}^+$, we write f^n for

$$\underbrace{f \circ f \circ \cdots \circ f}_{n}.$$

We further define f^0 to be I_A (so that $f^0(a) = a$ for all $a \in A$) by convention.

Commutativity of Composition of Functions?

Is composition of functions commutative in general?

This question is asking if $g \circ f$ and $f \circ g$ are equal, when f and g are functions.

Of course if $g \circ f$ is defined, then we necessarily must have that the codomain of f equals the domain of g, say $f : A \to B$ and $g : B \to C$. Now, for $f \circ g$ to be defined, we need $C = A$.

Even if $C = A$, we have $g \circ f : A \to A$ and $f \circ g : B \to B$, so that $g \circ f \neq f \circ g$ unless $A = B$.

Exercise

If $f : A \to A$ and $g : A \to A$ are functions, then is $f \circ g = g \circ f$?

Summary

We have covered:

- How to define a function - the correct notation to be used
- Arrow diagram of a function - its characteristics
- Domain, codomain and range of a function
- Images and preimages
- When are two functions equal
- Composition of functions - always associative, seldom commutative