CS1231(S) Tutorial 3: Sets

National University of Singapore

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When asked to 'find' a set in the following, the answer should involve a list of all of the elements in the set.

1. Which of the following are true? Which of them are false?

   (a) $\varnothing \in \varnothing$.

   (b) $\varnothing \subseteq \varnothing$.

   (c) $\varnothing \in \{\varnothing\}$.

   (d) $\varnothing \subseteq \{\varnothing\}$.

   (e) $1 \in \{\{1,2\},\{2,3\},4\}$.

   (f) $\{1,2\} \subseteq \{3,2,1\}$.

   (g) $\{3,3,2\} \subsetneq \{3,2,1\}$.

2. Let $A = \{1, \{1,2\}, 2, \{1,2\}\}$. Find $|A|$.

3. Let $A = \{0,1,4,5,6,9\}$ and $B = \{0,2,4,6,8\}$. Find $|A|$, $|B|$, $|A \cap B|$, and $|A \cup B|$.

4. Let $A = \{2n+1 \mid n \in \mathbb{Z}\}$ and $B = \{2n-1 \mid n \in \mathbb{Z}\}$. Is $A = B$? Prove that your answer is correct.

5. Let $A = \{x \in \mathbb{Z} \mid 2 \leqslant x \leqslant 5\}$ and $B = \{x \in \mathbb{R} \mid 2 \leqslant x \leqslant 5\}$. Is $A = B$? Prove that your answer is correct.

6. Let $U = \{5,6,7,\dots,12\}$ and $M_k = \{n \in \mathbb{Z} \mid n = km \text{ for some } m \in \mathbb{Z}\}$ for each $k \in \mathbb{Z}$. Find:

   (a) $\{n \in U \mid n \text{ is even}\}$;

   (b) $\{n \in U \mid n = m^2 \text{ for some } m \in \mathbb{Z}\}$;

   (c) $\{-5,-4,-3,\dots,5\} - \{1,2,3,\dots,10\}$;

   (d) $\overline{\{5,7,9\} \cup \{9,11\}}$, where U is considered the universal set;

   (e) $\big\{(x,y) \in \{1,3,5\} \times \{2,4\} \mid x+y \geqslant 6\big\}$;

   (f) $\mathcal{P}(\{2,4\})$;

   (g) $U - \bigcup_{k=5}^{12} M_k$.

7. Show that for all sets A, B, C,

$$A \cap (B - C) = (A \cap B) - C.$$

8. (2009/10 Semester 2 exam question B) Prove that for all sets A and B,

$$(A \cup \overline{B}) \cap (\overline{A} \cup B) = (A \cap B) \cup (\overline{A} \cap \overline{B}).$$

9. Let A, B be sets. Show that $A \subseteq B$ if and only if $A \cup B = B$.

10. For sets A and B, define $A \oplus B = (A - B) \cup (B - A)$.

    (a) Let $A = \{1, 4, 9, 16\}$ and $B = \{2, 4, 6, 8, 10, 12, 14, 16\}$. Find $A \oplus B$.

    (b) Show that for all sets A, B,

    $$A \oplus B = (A \cup B) - (A \cap B).$$

11. (2015/16 Semester 1 exam question 16(a)) Denote by $|x|$ the absolute value of the integer x, i.e.,

    $$|x| = \begin{cases} x, & \text{if } x \geqslant 0; \\ -x, & \text{if } x < 0. \end{cases}$$

    Given the set $S = \{-9, -6, -1, 3, 5, 8\}$, for each of the following statements, state whether it is true or false, with explanation.

    (a) $\exists z \in S \ \forall x, y \in S \ \ z > |x - y|$.

    (b) $\exists z \in S \ \forall x, y \in S \ \ z < |x - y|$.

12. **Definition.** Let A, B be sets and $R \subseteq A \times B$. For $X \subseteq A$ and $Y \subseteq B$, define

    $$R[X] = \{y \in B \mid (x, y) \in R \text{ for some } x \in X\}, \quad \text{and}$$
    $$R^{-1}[Y] = \{x \in A \mid (x, y) \in R \text{ for some } y \in Y\}.$$

    Conjecture. Let A, B be sets, and $R \subseteq A \times B$. Then $R^{-1}[R[X]] = X$ for all $X \subseteq A$.

    (a) Refute this conjecture.

    (b) Propose a necessary and sufficient condition on R for the conjecture above to hold. Prove the necessity and the sufficiency of your condition.