Lecture 6
Part II

RECURSION

Consider the following sequences:

- $2,\ 9,\ 16,\ 23,\ 30, \ldots$
- $1,\ 2,\ 4,\ 8,\ 16, \ldots$
- $1,\ 1,\ 2,\ 3,\ 5, \ldots$
- $2,\ 3,\ 5,\ 7,\ 11, \ldots$

Observe that if the sequence is denoted $a_1, a_2, a_3, \ldots$, then first three sequences above satisfies:

1. $a_{n+1} = a_n + 7$ for all $n \in \mathbb{Z}^+$;
2. $a_{n+1} = 2a_n$ for all $n \in \mathbb{Z}^+$;
3. $a_{n+1} = a_n + a_{n-1}$ for all $n \in \mathbb{Z}_{\geq 2}$.

These are **recursively defined** sequences, where, other than the first few terms, each successive term depends on the previous terms in such a sequence.

Formal Definition

Recall that sequences are functions from $\mathbb{Z}^+$ to a set A.

(In the earlier examples of sequences, $A = \mathbb{Z}$ (or $\mathbb{Z}^+$). In general, A can be any set.)

Definition

Let A be a set. A function $f : \mathbb{Z}^+ \to A$ is **recursively defined** if for each $n \in \mathbb{Z}_{\geq 2}$, there exists a function $g_n : A^{n-1} \to A$ such that

$$f(n) = g_n(f(1), f(2), \ldots, f(n-1)).$$

Existence and Uniqueness

Theorem

Let A be a set, and for each $n \in \mathbb{Z}_{\geq 2}$, let $g_n : A^{n-1} \to A$ be a function. Let $a \in A$. Then there exists a unique function $f : \mathbb{Z}^+ \to A$ such that $f(1) = a$, and $f(n) = g_n(f(1), f(2), \ldots, f(n-1))$ for all $n \in \mathbb{Z}_{\geq 2}$.

Sketch of proof.

Existence: Define $a_1, a_2, \ldots$ recursively as follows:

- $a_1 = a$.
- Having defined $a_1, \ldots, a_{n-1}$, let $a_n = g_n(a_1, \ldots, a_{n-1})$.

Let $f : \mathbb{Z}^+ \to A$; $n \mapsto a_n$. Then $f(1) = a$, and $f(n) = g_n(f(1), f(2), \ldots, f(n-1))$ for all $n \in \mathbb{Z}_{\geq 2}$ by construction.

Uniqueness: Let $h : \mathbb{Z}^+ \to A$ be another function such that $h(1) = a$, and $h(n) = g_n(h(1), h(2), \ldots, h(n-1))$ for all $n \in \mathbb{Z}_{\geq 2}$. An easy proof by induction on n shows that $f(n) = h(n)$ for all $n \in \mathbb{Z}^+$. (Exercise: fill in the details.) Thus $f = h$, i.e. f is unique.

$\square$

First-order Recurrence Relation

Theorem

Let $k, d \in \mathbb{R}$ with $k \neq 0$. Suppose that the sequence $a_1, a_2, \ldots$ of integers satisfies

$$a_{n+1} = ka_n + d$$

for all $n \in \mathbb{Z}^+$. Then

$$a_n = \begin{cases} k^{n-1}a_1 + \frac{k^{n-1}-1}{k-1}d, & \text{if } k \neq 1; \\ a_1 + (n-1)d, & \text{if } k = 1 \end{cases}$$

for all $n \in \mathbb{Z}^+$.

Proof.

1. Case 1: $k = 1$. Simple exercise about arithmetic progression.
2. Case 2: $k \neq 1$. [If $d = 0$, this is a simple exercise about geometric progression.] By the last theorem, we only need to verify that if $a_n = k^{n-1}a_1 + \frac{k^{n-1}-1}{k-1}d$ for all $n \in \mathbb{Z}^+$, then $a_{n+1} = ka_n + d$. Now,

$$\begin{aligned} a_{n+1} &= k^n a_1 + \tfrac{k^n-1}{k-1}d \\ &= k(k^{n-1}a_1 + \tfrac{k^{n-1}-1}{k-1}d) - k(\tfrac{k^{n-1}-1}{k-1})d + \tfrac{k^n-1}{k-1}d \\ &= ka_n + \tfrac{-(k^n-k)+(k^n-1)}{k-1}d \\ &= ka_n + d. \end{aligned}$$

$\square$

Example

Bank X offers a promotional savings account that pays both a 1% annual interest (compounded annually) and a special bonus of $50 at the end of each year (i.e. at 23:59:59 on 31 Dec). Determine how much John's account will have on 1 Jan 2030 if he puts $10000 into this savings account on 1 Jan 2020.

Solution:

1. Let $\$a_n$ be the amount in John's account on 1 Jan $(2020+n-1)$ (so that $a_1 = 10000$).
2. Then $a_{n+1} = 1.01a_n + 50$ for all $n \in \mathbb{Z}^+$.
3. Thus $a_n = (1.01)^{n-1}a_1 + \frac{1.01^{n-1}-1}{1.01-1}50$.
4. Thus $a_{11} = 11569.33$.
5. John's account will have $11569.33 on 1 Jan 2030.

Second-order Recurrence Relation

Theorem

Let $s, p \in \mathbb{R}$ with $p \neq 0$ and $s^2 \geq -4p$. Suppose that the sequence $a_1, a_2, \ldots$ of integers satisfies

$$a_{n+2} = sa_{n+1} + pa_n$$

for all $n \in \mathbb{Z}^+$. Let α and β be the (real) roots of the quadratic equation $x^2 - sx - p = 0$. Then

$$a_n = \begin{cases} A\alpha^n + B\beta^n, & \text{if } \alpha \neq \beta; \\ (Cn + D)\alpha^n, & \text{if } \alpha = \beta \end{cases}$$

for all $n \in \mathbb{Z}^+$, where $A, B, C, D \in \mathbb{R}$ satisfy

$$\begin{cases} A\alpha + B\beta = a_1 \\ A\alpha^2 + B\beta^2 = a_2 \end{cases} \quad \text{and} \quad \begin{cases} (C + D)\alpha = a_1 \\ (2C + D)\alpha^2 = a_2 \end{cases}.$$

Proof.

Easy verification, after noting that $\alpha + \beta = s$ and $\alpha\beta = -p$, that the asserted close formula for a_n satisfies the given recurrence relation. We leave the details to you as an exercise. $\square$

Example

Determine the closed formula for the n-th term in the Fibonacci sequence, i.e. the sequence $(a_n)_{n\in\mathbb{Z}^+}$ satisfying $a_1 = 1 = a_2$ and $a_{n+2} = a_{n+1} + a_n$ for all $n \in \mathbb{Z}^+$.

Solution:

1. The equation $x^2 - x - 1 = 0$ has distinct roots $\frac{1\pm\sqrt{5}}{2}$.
2. Thus $a_n = A(\frac{1+\sqrt{5}}{2})^n + B(\frac{1-\sqrt{5}}{2})^n$, where
$$A\tfrac{1+\sqrt{5}}{2} + B\tfrac{1-\sqrt{5}}{2} = 1;$$
$$A(\tfrac{1+\sqrt{5}}{2})^2 + B(\tfrac{1-\sqrt{5}}{2})^2 = 1.$$
3. Solving the simultaneous equations in (3), we get $A = \frac{1}{\sqrt{5}}$, $B = -\frac{1}{\sqrt{5}}$.
4. Hence $a_n = \frac{1}{\sqrt{5}}\big((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n\big)$.

Recursively Defined Sets

A **recursively defined set** consists of the following components:

Base: A statement that certain objects belong to the set.

Recursion: A collection of rules indicating how to form new objects that belong the set from those already known to be in the set.

Restriction: A statement that no object belong to the set other than those coming from the base and recursion.

Example

Let S be the set defined recursively as follows:

Base: $3 \in S$;

Recursion: for all $x, y \in S$, $x + y \in S$;

Restriction: No integer belongs to S other than those coming from the base and recursion.

Then $S = \{3, 6, 9, \dots\} = \{3k \mid k \in \mathbb{Z}^+\}$.

Existence and Uniqueness

Theorem

Let A be a set. Let I be an indexing set, and for each $i \in I$, let $n_i \in \mathbb{Z}^+$ and $f_i : \mathcal{U}^{n_i} \to \mathcal{U}$ be a function. There exists a unique set S such that

- *$A \subseteq S \subseteq \mathcal{U}$;*
- *$f_i(S^{n_i}) \subseteq S$ for all $i \in I$;*
- *if T is another set such that $A \subseteq T \subseteq \mathcal{U}$ and $f_i(T^{n_i}) \subseteq T$ for all $i \in I$, then $S \subseteq T$.*

Idea of proof when $n_i = 1$ for all $i \in I$.

Let $S = \{f_{i_1}(f_{i_2}(\cdots(f_{i_k}(a))\cdots)) \mid a \in A,\ k \in \mathbb{Z}_{\geq 0},\ i_1, i_2, \ldots, i_k \in I\}$ (where, when $k = 0$, $f_{i_1}(f_{i_2}(\cdots(f_{i_k}(a))\cdots))$ is to be read as a). Then S can be easily verified to satisfy the first two properties. Furthermore, if T also satisfies the first two properties, then $f_{i_1}(f_{i_2}(\cdots(f_{i_k}(a))\cdots)) \in T$ for all $a \in A$, $k \in \mathbb{Z}^+$ and $i_1, i_2, \ldots, i_k \in I$, so that $S \subseteq T$.
Uniqueness of S is left as a simple exercise. $\square$

Corollary

Recursively defined sets are well-defined.

Structural Induction

Let S be a recursively defined set, and suppose that we want to prove that $\forall x \in S\ p(x)$ (where $p(x)$ is a predicate). We may do this by **structural induction**, as follows:

1. Verify $p(b)$ for all $b \in B$, where B is the base of S;
2. Show that $p(y)$ is true if y is obtained from $x_1, x_2, \ldots$ by applying a rule in the recursion of S and $p(x_1), p(x_2), \ldots$ are true.

Example

Let S be the set defined recursively as follows:

Base: $1 \in S$;

Recursion: For all $x \in S$, $2x \in S$;

Restriction: No integer belongs to S other than those coming from the base and recursion.

Prove that $3 \nmid s$ for all $s \in S$.

Solution:

1. Base case: $3 \nmid 1$.
2. Inductive step: If $x \in S$ and $3 \nmid x$, then since $3 \nmid 2$ and 3 is prime, we have $3 \nmid 2x$.
3. By structural induction, $3 \nmid s$ for all $s \in S$.

Summary

We have covered:

- Recursively defined sequences, existence and uniqueness
- Closed formulae for some first-order and second-order recurrence relations.
- Recursively defined sets and structural induction