A Brief Demonstration of Retrieving Coterminal and Reference Angles

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1 Coterminal Angles

Two angles with the same initial and terminal sides, but with different rotations, are called coterminal angles [1]. Two or more angles is a coterminal if it has coincided with the terminal sides while the position of its vertex is located in the origin of the coordinate system and the positive x-axis is with the initial side. Every angle θ that is expressed in degrees will be in the form of $\theta + 360°k$, where k is an integer and θ are the coterminal.

Example 1. Retrieve the least positive and greatest negative cotermiinal of the given angle measures.

(a) $754°$

(b) $-15°$

(c) $-\frac{3\Pi}{4}°$

Solution. To find the least positive and greatest negative coterminal angles of the given angle measures, the standard formula of $\theta \pm 360°k$ is utilized to provide an answer for the specific problem.

(a) $754°$

Notice that when we plot the values in the standard formula, in this case, 1 revolution (k), the result would be $1114°$. However, that is not the least positive since $1114° \div 360° = 3.094$, which means that $1114°$ is the third least positive coterminal of $754°$. So, to find the first least positive coterminal, we can subtract $360°$ as many times as we need from $754°$ to obtain the coterminal angle that we are trying to find.

Additionally, a time-saving and efficient method would be to find how many times $754°$ divides into $360°$. We need to reduce the given angle, then solve the equation using the standard formula. Since $360°$ divides into $754°$ two times (2 complete rotations), the remainder is the angle we are looking for.

$754°$ is beyond $360°$. Therefore, $754°$ must be divided by $360°$ then the remainder, which must not be rounded off and leave it as it is, is the number of revolution the angle had taken.

$$754° \div 360° = 2.09°$$

$$754° - 360(2)° = 754° - 720° = 34°$$

Proof that 2 is the remainder that retrieves the least positive angle.

$$754° - 360(-2)° = 754° - (-720)° = 1474°$$

$$754° - 360(-1)° = 754° - (-360)° = 1114°$$

$$754° - 360(0)° = 754° - 0° = 754°$$

$$754° - 360(1)° = 754° - 360° = 394°$$

$$754° - 360(2)° = 754° - 720° = 34°$$

The angle between $0°$ and $360°$ that is coterminal with the angle $754°$ is $34°$ which is located in the first quadrant.

If the integers (k) that is ≥ 2 will obtain the positive angles, then the integers (k) that is < 2, but $k \neq 0$, must obtain the negative angles.

$$754° - 360(3)° = 754° - 1080° = -326°$$

$$754° - 360(4)° = 754° - 1440° = -686°$$

$$754° - 360(5)° = 754° - 1800° = -1046°$$

$$754° - 360(6)° = 754° - 2160° = -1406°$$

$$754° - 360(7)° = 754° - 2520° = -1766°$$

The least positive coterminal angle of $754°$ is $34°$ and the greatest negative coterminal angle of $754°$ is $-326°$.

(b) $-15°$

To find the least positive and greatest negative coterminal angle of the angle $-15°$, we can start by directly using the standard formula, $\theta + 360°k$, since $-15°$ is $> 0°$ and $< 360°$

$$-15° + 360(-2)° = -15° + (-720)° = -735°$$

$$-15° + 360(-1)° = -15° + (-360)° = -375°$$

$$-15° + 360(0)° = -15° + 0° = -15°$$

$$-15° + 360(1)° = -15° + 360° = 345°$$

$$-15° + 360(2)° = -15° + 720° = 705°$$

$$-15° + 360(3)° = -15° + 1080° = 1065°$$

Therefore, the least positive coterminal angle of $-15°$ is $345°$ and the greatest negative is $-375°$.

(c) $-\frac{3\Pi}{4}°$

To find the least positive and greatest negative coterminal angle of $-\frac{3\Pi}{4}°$, it must be converted from radians to degrees to make it easier.

$$-\frac{3\Pi}{4} radians = -\frac{3(180°)}{4} radians = -\frac{540°}{4} = 135°$$

$$-\frac{3\Pi}{4} = 135°$$

$$-135° + 360(-3)° = -135° + (-1080)° = -1215°$$

$$-135° + 360(-2)° = -135° + (-720)° = -855°$$

$$-135° + 360(-1)° = -135° + (-360)° = -495°$$

$$-135° + 360(1)° = -135° + 360° = 225°$$

$$-135° + 360(2)° = -135° + 720° = 585°$$

$$-135° + 360(3)° = -135° + 1080° = 945°$$

The least positive coterminal angle of $-\frac{3\Pi}{4}$ is $225°$ and the greatest negative is $-495°$.

2 Reference Angle

An angle that is $< 90°$ that is formed by the terminal side of a given angle, in which the vertex is located in the origin of the coordinate system and the positive x-axis, is called the reference angle.

Angles can be positive, negative, and can have multiple rotations. To draw these angles, we use the Cartesian plane, x-y plane. Each quarter of this plane is called a quadrant and is represented by the Roman numerals, I, II, III, IV [1].

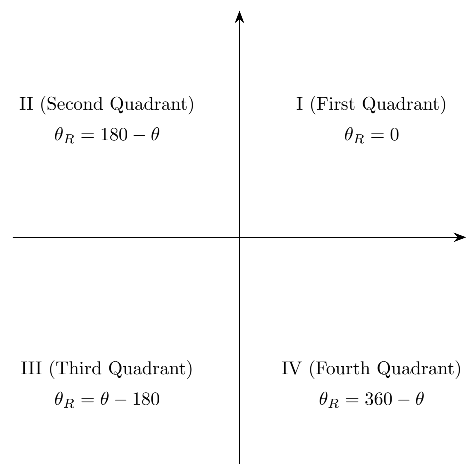
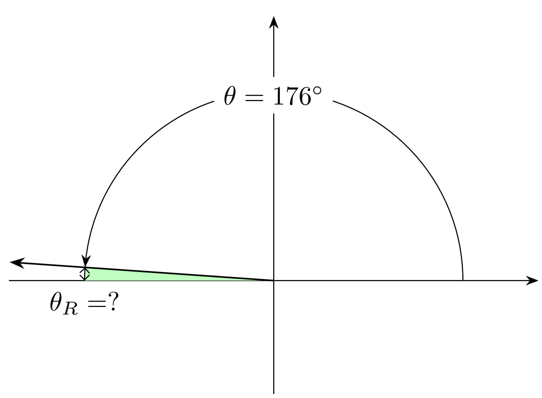


Figure 1: Positions of the four quadrants

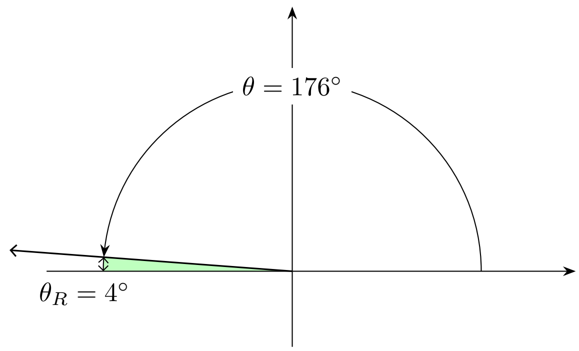
Quadrants	**Reference Angle Values**
First	$\theta_R = 0$
Second	$\theta_R = 180 - \theta$
Third	$\theta_R = \theta - 180$
Fourth	$\theta_R = 360 - \theta$

Table 1: Statements used for solving the reference angles [2]

Example 1. Find the reference angle of 176°



The formula $\theta_R = 180 - \theta$ will be used to find its reference angle since the initial angle lies in 2nd quadrant as $176°$ is $> 90°$ and $< 180°$.

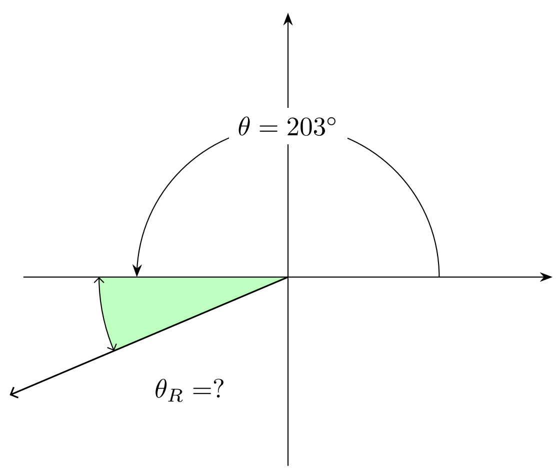


$$\theta_R = 180 - \theta$$

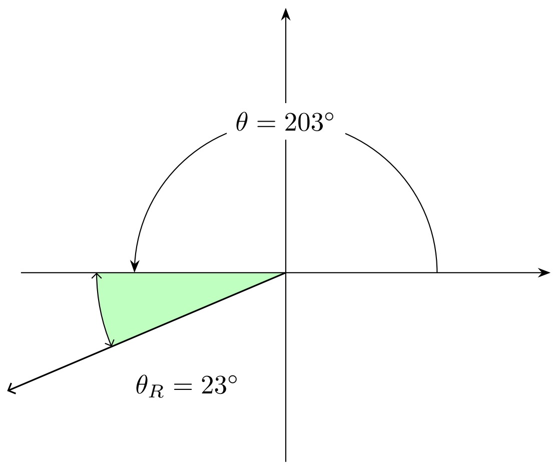
$$\theta_R = 180 - 176°$$

$$\theta_R = 4°$$

Example 2. Find the reference angle of $203°$



The formula $\theta_R = \theta - 180$ will be used to find its reference angle since the initial angle lies in 3rd quadrant as $203°$ is $> 180°$ and $< 270°$.

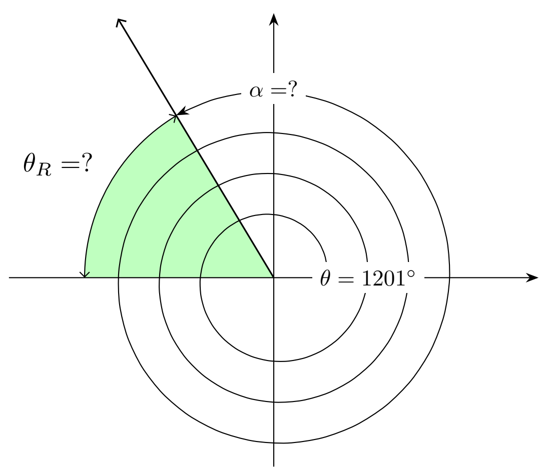


$$\theta_R = \theta - 180$$

$$\theta_R = 203° - 180$$

$$\theta_R = 23°$$

Example 3. Find the reference angle of 1201°

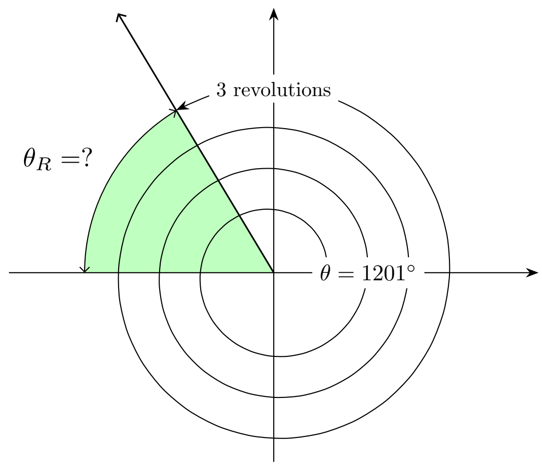


First, to make things simple, we must find the number of revolutions (α) taken by the given angle 1201°. The formula $\alpha = \theta \div 360°$ will be used to obtain the number of revolutions, which is the remainder of the result.

$$\alpha = \theta \div 360°$$

$$\alpha = 1201° \div 360°$$

$$\alpha = 3.34 \approx 3$$



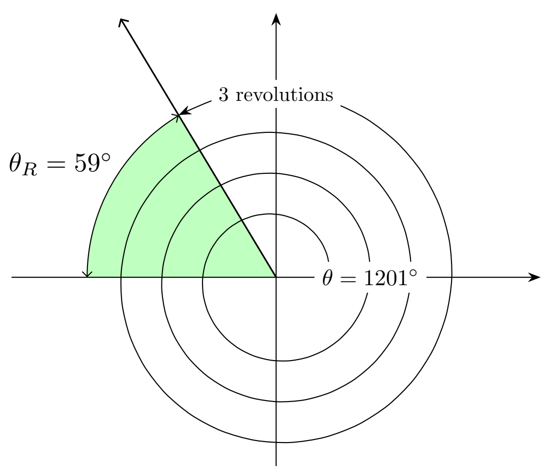
Next, the number of revolutions will be multiplied by 360° and then subtract it to θ, which is 1201°. The result is the simplified angle (θ) of the given angle.

$$\alpha \times 360°$$

$$3 \times 360° = 1080°$$

$$1201° - 1080° = 121°$$

The formula $\theta_R = 180 - \theta$ will be used to find its reference angle since the initial angle lies in 2nd quadrant as 121° is $> 90°$ and $< 180°$.



$$\theta_R = 180 - 121°$$

$$\theta_R = 180 - 121°$$

$$\theta_R = 59°$$

References

[1] King, C., Tam Evelyn, F. Y., and Carvajal, B. E., "Trigonometry: A Brief Conversation," *CUNY Academic Works*, 2020, pp. 108.

[2] Andres, J., "Cor Jesu College G12 Precalculus Module," *Cor Jesu College Inc Basic Education Department Academic Module*, 2020.