Mathematical Analysis II. Assignment 5

1. Random vector $(\xi; \eta)^T$ is uniformly distributed inside ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

   (a) Find marginal distributions of ξ and η;
   (b) determine if ξ and η are independent;
   (c) find covariation matrix of this random vector;
   (d) Find conditional expectations $E(\xi|\eta)$ and $E(\eta|\xi)$.

2. Random vector $(\xi; \eta)^T$ is uniformly distributed in a triangle with vertices $(-5; 0)$, $(5; 0)$ and $(0; 4)$. Find $E(\eta|\xi = 2)$, $\text{Var}(\eta|\xi = 2)$, $E(\xi|\eta = 2)$, $\text{Var}(\xi|\eta = 2)$.

3. Random vector $(\xi; \eta)^T$ is given by its cumulative distribution function:

$$F_{\xi,\eta}(x; y) = \left(1 - e^{-\lambda x} - e^{-\mu y} + e^{-\lambda x - \mu y}\right) I_{x>0} I_{y>0}.$$

   (a) Find marginal distributions of ξ and η;
   (b) determine if ξ and η are independent;
   (c) find covariation matrix of this random vector;
   (d) Find conditional expectations $E(\xi|\eta)$ and $E(\eta|\xi)$.

4. Probability density of random vector $(\xi; \eta)^T$ is given by $f(x; y) = \frac{a}{1+x^2+y^2+x^2y^2}$. Find a and marginal densities of ξ and η. Determine whether ξ and η are independent and correlated.

5. Let $\xi \sim N(0; 1)$ and $\eta = \xi^2$. Determine if ξ and η are (a) independent, (b) correlated.

6. The covariance matrix of random vector $(\xi; \eta)^T$ is equal to

$$\begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix}.$$

   Find $\text{Var}(-\xi - 2\eta)$ and $\text{Var}(3\xi - \eta + 2)$.

7. Let a, b, c, d be constants, and $ac \neq 0$. Show that $\rho(a\xi + b, c\eta + d) = \pm\rho(\xi, \eta)$. How do we determine the sign at the right side of the equality?

8. Let $\zeta \sim \text{Exp}(\lambda)$. Find the correlation coefficient between (a) $2\zeta + 3$ and $3\zeta - 1$; (b) ζ^2 and $\zeta^2 - \zeta$.

9. Let $\zeta \sim N(0; 1)$. Find the correlation coefficient between (a) 2ζ and ζ^3; (b) $3\zeta^2 - 2$ and $2\zeta^2 + 3$.

10. Let ξ and η be independent random variables with $N(0; 1)$ distribution. Find the probability that a point with coordinates $(\xi; \eta)$ is situated within a rectangle centered at the origin whose sides are equal to $2a$ and $2b$. (Note that the sides of s rectangle do not have to be parallel to coordinate axes.)

11. Let ξ and η be independent random variables with $N(0; 1)$ distribution. Find the probability that a point with coordinates $(\xi; \eta)$ (a) is situated within figure $|x + y| + |x - y| \leqslant 2$; (b) is situated within figure $|x| + |y| \leqslant 1$.

12. Random variables ξ_1 and ξ_2 are independent and uniformly distributed on $[0; 1]$. Find the distributions of $\eta = \xi_1 \xi_2$ and $\zeta = \frac{\xi_2}{\xi_1}$.

13. Random variables ξ_1 and ξ_2 are independent and exponentially distributed with parameter λ. Find the distributions of $\zeta = \frac{\xi_2}{\xi_1}$ and $\zeta = \frac{\xi_2}{\xi_1 + \xi_2}$.

14. Random variables ξ_1 and ξ_2 are independent and have a standard normal distribution. Find the distributions of $\zeta = \frac{\xi_2}{\xi_1}$, $\eta = \frac{|\xi_2|}{\xi_1}$ and $\gamma = \frac{\xi_2}{|\xi_1|}$.

15. Random vector $(\xi; \eta)^T$ is uniformly distributed within a circle given by $x^2 + y^2 \leqslant 25$. Find the distribution of $\zeta = \frac{\xi}{\eta}$.

16. Probability density of random vector $(\xi; \eta)$ is given by $f_{\xi,\eta}(x; y) = \frac{C \cdot I_{x>0} I_{y>0}}{(1+x+y)^3}$. Find probability density of $\zeta = \xi + \eta$.

17. Find probability density of $\xi + \eta$ where ξ and η are independent random variables uniformly distributed on $(0; 1)$.

18. The altitude of a cylinder η and the radius of its base ξ are independent random variables whose distributions are $U[a; b]$ and $\mathrm{Exp}(\lambda)$ respectively. Find expectation and variance of a volume of such a cylinder.

19. Prove that $\mathrm{Var}(\xi\eta) = \mathrm{Var}\,\xi \cdot \mathrm{Var}\,\eta + \mathrm{Var}\,\xi \cdot (E\eta)^2 + \mathrm{Var}\,\eta \cdot (E\xi)^2$ for arbitrary independent random variables ξ and η.