The general equations are:

$$\dot{\delta} = -(1+w)\left(k\,v - 3\,\dot{\phi}\right) - 3\,H\left({c_s}^2 - w\right)\delta \tag{1}$$

$$\dot{v} = -H(1-3\,w)\,v - \frac{\dot{w}}{1+w}\,v + \frac{{c_s}^2}{1+w}\,k\,\delta + \mathrm{k}\phi \tag{2}$$

The individual fluids have

$$\dot{\delta}_i = -(1+w_i)\left(k\,v_i - 3\,\dot{\phi}\right) \tag{3}$$

$$\dot{v}_i = -H(1-3\,w_i)\,v_i + \frac{w_i}{1+w_i}\,k\,\delta_i + k\,\dot{\phi} \tag{4}$$

The GDM parameters are defined as follows (using the new v_T):

$$\rho_T\,\delta_T = \sum \delta_i\,\rho_i \tag{5}$$

$$v_T = \frac{\sum(1+w_i)\,\rho_i\,v_i}{(1+w_T)\sum\rho_i} \tag{6}$$

$${c_s}^2 = \frac{\sum\delta_i\,w_i\,\rho_i}{\sum\rho_i\,\delta_T} \tag{7}$$

Taking the derivative of (5), we have:

$$\left(\sum\dot{\rho}_i\right)\delta_T + \left(\sum\rho_i\right)\dot{\delta}_T = \sum\left(\dot{\delta}_i\,\rho_i + \delta_i\,\dot{\rho}_i\right) = \sum -(1+w_i)\left(k\,v_i - 3\,\dot{\phi}\right)\rho_i + \delta_i\,\dot{\rho}_i \tag{8}$$

$$\dot{\delta}_T = \frac{\sum -(1+w_i)\left(k\,v_i - 3\,\dot{\phi}\right)\rho_i + \delta_i\,\dot{\rho}_i - \dot{\rho}_i\,\delta_T}{\sum\rho_i} = \frac{\sum -(1+w_i)\left(k\,v_i - 3\,\dot{\phi}\right)\rho_i + (\delta_i - \delta_T)\,\dot{\rho}_i}{\sum\rho_i} \tag{9}$$

At this point we use the background density evolution equation:

$$\dot{\rho}_i = -3\,H(1+w_i)\,\rho_i \tag{10}$$

$$\dot{\delta}_T = \frac{1}{\sum\rho_i}\left(\sum -(1+w_i)\left(k\,v_i - 3\,\dot{\phi}\right)\rho_i - (\delta_i - \delta_T)\,3\,H(1+w_i)\,\rho_i\right) \tag{11}$$

Just looking at the first term, we have:

$$\frac{1}{\sum\rho_i}\left(\sum(1+w_i)\,\rho_i\,v_i + 3\,\dot{\phi}\sum\rho_i \;\; + 3\,\dot{\phi}\,\sum w_i\,\rho_i\right) \tag{12}$$

$$= (1+w_T)\left(-k\,\frac{\sum(1+w_i)\,\rho_i\,v_i}{(1+w_T)\sum\rho_i}\right) + 3\,\dot{\phi} + 3\,\dot{\phi}\,\frac{\sum w_i\,\rho_i}{\sum\rho_i} \tag{13}$$

$$= (1+w_T)\left(-k\,v_T\right) + 3\,\dot{\phi} + 3\,\dot{\phi}\,w_T = -(1+w_T)\left(k\,v_T - 3\,\dot{\phi}\right) \tag{14}$$

And for the second term, we have:

$$\dot{\delta}_T = -3\,H\left(\left(\sum\delta_i(1+w_i)\,\rho_i - \delta_T\sum(1+w_i)\,\rho_i\right)\Big/\left(\quad\sum\rho_i\right)\right) \tag{15}$$

$$= -3\,H\left(\left(\sum\delta_i(1+w_i)\,\rho_i - \delta_T\left(\sum w_i\,\rho_i + \rho_i\right)\right)\Big/\left(\quad\sum\rho_i\right)\right) \tag{16}$$

$$= -3\,H\,\frac{\sum\delta_i(1+w_i)\,\rho_i}{\sum\rho_i} - \delta_T(1+w_T) \tag{17}$$

$$= -3\,H\Big(\big(\sum \delta_i (1+w_i)\,\rho_i\big)\big/\big(\sum \rho_i\,\delta_T\big) - 1 - w_T\Big)\Big)\,\delta_T \tag{18}$$

$$= -3\,H\Big(\big(\sum \delta_i (1+w_i)\,\rho_i - \delta_T \sum \rho_i\big)\big/\big(\sum \rho_i\,\delta_T\big) - w_T\Big)\,\delta_T \tag{19}$$

$$= -3\,H\left(\frac{\sum \delta_i (1+w_i)\,\rho_i - \sum \delta_i\,\rho_i}{\sum \rho_i\,\delta_T} - w_T\right)\delta_T = -3\,H\left(\frac{\sum \delta_i\,w_i\,\rho_i}{\sum \rho_i\,\delta_T} - w_T\right)\delta_T \tag{20}$$

$$-3\,H\left(c_s{}^2 - w_T\right)\delta_T \tag{21}$$

And so altogether we have

$$\dot{\delta} = -(1+w_T)\left(k\,v_T - 3\,\dot{\phi}\right) - 3\,H\left(c_s{}^2 - w_T\right)\delta_T \tag{22}$$

Now we obtain the equation for $\dot{v}_T$:

$$v_T(1+w_T)\sum \rho_i = \sum (1+w_i)\,\rho_i\,v_i \tag{23}$$

$$\dot{v}_T(1+w_T)\sum \rho_i + v_T\,\dot{w}_T \sum \rho_i + v_T(1+w_T)\sum \dot{\rho}_i = \sum (1+w_i)\,(\dot{\rho}_i\,v_i + \rho_i\,\dot{v}_i) \tag{24}$$

On the right we have

$$\sum (1+w_i)\left((-3\,H(1+w_i)\,\rho_i)\,v_i + \rho_i\left(-H(1-3\,w_i)\,v_i + \frac{w_i}{1+w_i}\,k\,\delta_i + k\,\dot{\phi}\right)\right) \tag{25}$$

$$= \sum -3\,H(1+w_i)\,(1+w_i)\,\rho_i\,v_i - H(1-3\,w_i)\,(1+w_i)\,\rho_i\,v_i + w_i\,\rho_i\,k\,\delta_i + (1+w_i)\,\rho_i\,k\,\dot{\phi} \tag{26}$$

$$= \sum -3\,H\left(1+2\,w_i+w_i{}^2\right)\rho_i\,v_i - H\left(1-2\,w_i-3\,w_i{}^2\right)\rho_i\,v_i + w_i\,\rho_i\,k\,\delta_i + (1+w_i)\,\rho_i\,k\,\dot{\phi} \tag{27}$$

$$= \sum -H\,\rho_i\,v_i\left(3+6\,w_i+3\,w_i{}^2+1-2\,w_i-3\,w_i{}^2\right) + w_i\,\rho_i\,k\,\delta_i = \sum -H\,\rho_i\,v_i(4+4\,w_i) + w_i\,\rho_i\,k\,\delta_i + (1+w_i)\,\rho_i\,k\,\dot{\phi} \tag{28}$$

$$= \sum -4\,H\,\rho_i\,v_i(1+w_i) + w_i\,\rho_i\,k\,\delta_i + (1+w_i)\,\rho_i\,k\,\dot{\phi} \tag{29}$$

Dividing through by $(1+w_T)\sum\rho_i$ gives

$$\dot{v}_T + \frac{v_T\,\dot{w}_T \sum \rho_i}{(1+w_T)\sum \rho_i} + \frac{v_T(1+w_T)\sum \dot{\rho}_i}{(1+w_T)\sum \rho_i} = \frac{\sum -4\,H\,\rho_i\,v_i(1+w_i) + w_i\,\rho_i\,k\,\delta_i + (1+w_i)\,\rho_i\,k\,\dot{\phi}}{(1+w_T)\sum \rho_i} \tag{30}$$

$$\begin{aligned}&\dot{v}_T\,\frac{\dot{w}_T}{1+w_T}\,v_T + \frac{v_T \sum \dot{\rho}_i}{\sum \rho_i} = \\ &\frac{-4\,H \sum \rho_i\,v_i(1+w_i)}{(1+w_T)\sum \rho_i} + \frac{\sum w_i\,\rho_i\,\delta_i}{(1+w_T)\sum \rho_i}\,k + \frac{\sum (1+w_i)\,\rho_i\,k\,\dot{\phi}}{(1+w_T)\sum \rho_i} = -4\,H\,v_T + \frac{\sum w_i\,\rho_i\,\delta_i}{(1+w_T)\sum \rho_i}\,k + \frac{\sum (1+w_i)\,\rho_i\,k\,\dot{\phi}}{(1+w_T)\sum \rho_i}\end{aligned} \tag{31}$$

For the second term on the right we get:

$$\frac{\sum w_i\,\rho_i\,\delta_i}{(1+w_T)\sum \rho_i}\,k = \frac{1}{(1+w_T)}\,\frac{\sum w_i\,\rho_i\,\delta_i}{\sum \rho_i\,\delta_T}\,k\,\delta_T = \frac{c_s{}^2}{1+w_T}\,k\,\delta_T \tag{32}$$

For the third term on the right we get:

$$\frac{\sum (1+w_i)\,\rho_i\,k\,\dot{\phi}}{(1+w_T)\sum \rho_i} = \frac{1}{(1+w_T)}\,\frac{\sum \rho_i + \sum \rho_i\,w_i}{\sum \rho_i}\,k\,\dot{\phi} = k\,\dot{\phi} \tag{33}$$

Rearranging,

$$\dot{v}_T = -4\,H\,v_T - \frac{v_T \sum \dot{\rho}_i}{\sum \rho_i} - \frac{\dot{w}_T}{1+w_T}\,v_T + \frac{c_s{}^2}{1+w_T}\,k\,\delta_T + k\,\dot{\phi} \tag{34}$$

For the second term we have:

$$-\frac{v_T \sum \dot{\rho}_i}{\sum \rho_i} = -\frac{1}{\sum \rho_i} v_T \sum (-3H(1+w_i)\rho_i) = \frac{3 H v_T \sum \rho_i}{\sum \rho_i} + \frac{3 H v_T \sum w_i \rho_i}{\sum \rho_i} = 3 H v_T + 3 H v_T w_T \tag{35}$$

So altogether, we have

$$\dot{v}_T = -4 H v_T + 3 H v_T + 3 H v_T w_T - \frac{\dot{w}_T}{1 + w_T} v_T + \frac{{c_s}^2}{1 + w_T} k \delta_T + k \dot{\phi} \tag{36}$$

$$\dot{v}_T = -H(1 - 3 w_T) - \frac{\dot{w}_T}{1 + w_T} v_T + \frac{{c_s}^2}{1 + w_T} k \delta_T + k \dot{\phi} \tag{37}$$