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ESTIMATES FOR THE ERRORS IN EIGENVALUE AND EIGENVECTOR APPROXIMATION BY GALERKIN METHODS, WITH PARTICULAR ATTENTION TO THE CASE OF MULTIPLE EIGENVALUES*

I. BABUŠKA† AND J. E. OSBORN‡

Dedicated to Werner C. Rheinboldt on the occasion of his 60th birthday.

Abstract. Refined estimates for the errors in eigenvalue and eigenvector approximation by finite element, or, more generally, Galerkin methods, as they apply to selfadjoint problems, are presented. Particular attention is given to the case of multiple eigenvalues. The results are new in this case. The proof is based on a novel approach which yields the known results for simple eigenvalues in a simple way. Numerical computations are presented and analyzed in light of the theoretical results.

Key words. approximation of eigenvalues, approximation of eigenvectors, multiple eigenvalues, Galerkin methods, finite element methods

AMS(MOS) subject classifications. Primary 65N15, 65N25, 65N30, 35P15

1. Introduction. It is the purpose of this paper to derive some refined estimates for the errors in eigenvalue and eigenvector approximation by finite element, or, more generally, Galerkin methods, as they apply to selfadjoint problems. The results are new in the case of multiple eigenvalues. The proof is based on a novel approach which yields the known results for simple eigenvalues in a simple way.

Suppose λ_k is an eigenvalue of multiplicity q of a selfadjoint problem and let $M(\lambda_k)$ denote the space of eigenvectors corresponding to λ_k. Denote by $\|\cdot\|_{B_0}$ the energy norm for our problem. Let S be the finite-dimensional approximation space employed in the Galerkin method. Then λ_k will be approximated from above by q of the Galerkin approximate eigenvalues:

$$\lambda_k \leqq \lambda_{S,k} \leqq \cdots \leqq \lambda_{S,k+q-1}$$

and

$$\lambda_k \simeq \lambda_{S,k}, \lambda_{S,k+1}, \cdots, \lambda_{S,k+q-1}.$$

Our main estimate for the error in eigenvalue approximation is

$$\lambda_{S,k} - \lambda_k \leqq C\left(\inf_{\substack{u \in M(\lambda_k) \\ \|u\|_{B_0}=1}} \inf_{\chi \in S} \|u-\chi\|_{B_0} \right)^2 := C\varepsilon_{\lambda_k}(S)^2, \tag{1.1}$$

which shows that the error between λ_k and $\lambda_{S,k}$, the approximate eigenvalue closest to λ_k, is bounded by a constant times the square of the minimal energy norm distance between exact eigenvectors $u \in M(\lambda_k)$ with $\|u\|_{B_0}=1$ and S, i.e., the square of the energy norm distance between S and the eigenvector $u \in M(\lambda_k)$ with $\|u\|_{B_0}=1$ that

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† Institute for Physical Science and Technology and Department of Mathematics, University of Maryland, College Park, Maryland 20742. The work of this author was partially supported by the Office of Naval Research under contract N00014-85-K-0169 and by the National Science Foundation under grant DMS-85-16191.

‡ Department of Mathematics, University of Maryland, College Park, Maryland 20742. The work of this author was partially supported by the National Science Foundation under grant DMS-84-10324.

can be best approximated by S. For $\lambda_{S,k+q-1}-\lambda_k$, the error between λ_k and $\lambda_{S,k+q-1}$, the approximate eigenvalue farthest from λ_k, we prove that

$$(1.2)\qquad \lambda_{S,k+q-1}-\lambda_k \leqq C\Bigg(\sup_{\substack{u\in M(\lambda_k)\\ \|u\|_{B_0}=1}} \inf_{\chi\in S}\|u-\chi\|_{B_0}\Bigg)^2 := C\bar{\varepsilon}_{\lambda_k}(S)^2,$$

and for the errors $\lambda_{S,k+i}-\lambda_k$, $i=1,\cdots,q-2$, we obtain bounds in terms of quantities intermediate in size between $C\varepsilon_{\lambda_k}(S)^2$ and $C\bar{\varepsilon}_{\lambda_k}(S)^2$.

These results should be contrasted with those in the literature. In Babuška and Aziz [1], Fix [4] and Kolata [6], the estimates

$$(1.3)\qquad \lambda_{S,k+i}-\lambda_k \leqq C\bar{\varepsilon}_{\lambda_k}(S)^2, \qquad i=0,\cdots,q-1,$$

are proved. For $i=0,\cdots,q-2$, (1.3) is weaker than the estimates stated above ((1.1) for $i=0$ and those mentioned after (1.2) for $i=1,\cdots,q-2$); for $i=q-1$, (1.3) is the same as (1.2). In Birkhoff, de Boor, Swartz and Wendroff [2], which presents the earliest results of the general type we are discussing, the eigenvalue estimates depend on the sum of the squares of the energy norm distances between S and the unit eigenvectors associated with all the eigenvalues λ_j not exceeding λ_k. The feature of (1.1) that is new is the dependence of the error bound $C\varepsilon_{\lambda_k}(S)^2$ on only one eigenvector $u\in M(\lambda_k)$, namely the one best approximated by S.

Regarding the errors in the approximate eigenvectors, we show that if $u_{S,k}$ is the Galerkin approximate eigenvector corresponding to $\lambda_{S,k}$, then there is a $u_k=u_k(S)\in M(\lambda_k)$ with $\|u_k\|_{B_0}=1$ such that

$$(1.4)\qquad \|u_{S,k}-u_k\|_{B_0}\leqq C\varepsilon_{\lambda_k}(S).$$

The error $\|u_{S,k+q-1}-u_{k+q-1}\|_{B_0}$ is bounded by $C\bar{\varepsilon}_{\lambda_k}(S)$ and the errors $\|u_{S,k+i}-u_{k+i}\|_{B_0}$, $i=1,\cdots,q-2$, are bounded by quantities intermediate in size between $C\varepsilon_{\lambda_k}(S)$ and $C\bar{\varepsilon}_{\lambda_k}(S)$. The best previously known result is

$$(1.5)\qquad \|u_{S,k+i}-u_{k+i}\|_{B_0}\leqq C\bar{\varepsilon}_{\lambda_k}(S), \qquad i=0,\cdots,q-1.$$

In § 2 we introduce the class of variationally formulated selfadjoint eigenvalue problems considered in the paper, define the Galerkin approximations to these problems, and in Lemmas 2.1–2.3 give the preliminary results which are used in the sequel. The main theoretical result of the paper is presented and proved in § 3. The treatment is direct and self-contained, relying on a minimal amount of functional analysis background. In § 4 we present numerical computations for a finite element approximation of a problem with double eigenvalues for which each double eigenvalue has associated eigenvectors of strikingly different approximation properties. The quantities

$$\varepsilon_{\lambda_k}(S)=\inf_{\substack{u\in M(\lambda_k)\\ \|u\|_{B_0}=1}}\inf_{\chi\in S}\|u-\chi\|^2_{B_0}, \qquad \bar{\varepsilon}_{\lambda_k}(S)=\sup_{\substack{u\in M(\lambda_k)\\ \|u\|_{B_0}=1}}\inf_{\chi\in S}\|u-\chi\|^2_{B_0}$$

are thus of different sizes and we would therefore expect $\lambda_{S,k}-\lambda_k$ and $\lambda_{S,k+1}-\lambda_k$ to be of different sizes. This is clearly shown by the computations. The computations also show that $u_{S,k}$ (the approximate eigenvector belonging to the approximate eigenvalue closest to λ_k) converges to an exact eigenvector with good approximation properties, while $u_{S,k+1}$, the approximate eigenvector belonging to the approximate eigenvalue farthest from λ_k, converges to an exact eigenvector with poor approximation properties.

The literature on eigenvalue problems is extensive, with many papers bearing, at least tangentially, on the problem addressed in this paper. We have, however, mentioned only those papers that bear directly on the central theme of our results; namely, the

Galerkin approximation of eigenpairs corresponding to multiple eigenvalues. For a general treatment of eigenvalue problems and their literature, we refer to the excellent and comprehensive monograph of Chatelin [3].

2. Preliminaries. Suppose H is a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, respectively, and suppose we are given two symmetric bilinear forms $B_0(u, v)$ and $D(u, v)$ on $H\times H$. $B_0(u, v)$ is assumed to satisfy

$$|B_0(u, v)| \leqq C_1\|u\|\,\|v\| \quad \forall u, v\in H \tag{2.1}$$

and

$$C_2\|u\|^2 \leqq B_0(u, u) \quad \forall u\in H \quad \text{with } C_2>0. \tag{2.2}$$

It follows from (2.1) and (2.2) that $B_0(u, v)$ and $\|u\|_{B_0}=(B_0(u, u))^{1/2}$ are equivalent to (u, v) and $\|u\|$, respectively. Regarding D, we suppose that

$$0<D(u, u) \quad \forall 0\neq u\in H \tag{2.3}$$

and that

$$\|u\|_D := (D(u, u))^{1/2} \tag{2.4}$$

is compact with respect to $\|\cdot\|$, i.e., it has the property that from any subsequence which is bounded in $\|\cdot\|$, one can extract a subsequence which is Cauchy in $\|\cdot\|_D$.

We then consider the variationally formulated, selfadjoint eigenvalue problem:

$$\text{Seek } \lambda \text{ (real) and } 0\neq u\in H \text{ such that } B_0(u, v)=\lambda D(u, v) \quad \forall v\in H. \tag{2.5}$$

Under the assumptions we have made, there is a sequence of eigenvalues

$$0<\lambda_1\leqq\lambda_2\leqq\cdots\nearrow+\infty$$

and corresponding eigenvectors

$$u_1, u_2, \cdots$$

which can be chosen to satisfy

$$B_0(u_i, u_j)=\lambda_i D(u_i, u_j)=\delta_{i,j}, \tag{2.6}$$

where $\delta_{i,j}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$. Furthermore, any $u\in H$ can be written as

$$u=\sum_{j=1}^{\infty} a_j u_j \quad \text{with } a_j=B_0(u, u_j), \tag{2.7}$$

where (2.7) converges in $\|\cdot\|_{B_0}$. The eigenvalues λ_j satisfy the following well-known variational principles:

$$\lambda_1=\min_{u\in H}\frac{B_0(u, u)}{D(u, u)}=\frac{B_0(u_1, u_1)}{D(u_1, u_1)},$$

$$\lambda_k=\min_{\substack{u\in H\\ B_0(u,u_i)=0,\\ i=1,2,\cdots,k-1}}\frac{B_0(u, u)}{D(u, u)}=\frac{B_0(u_k, u_k)}{D(u_k, u_k)}, \qquad k=2, 3, \cdots \tag{2.8}$$

(the minimum principle),

and

$$\lambda_k = \min_{\substack{V_k \subset H \\ \dim V_k = k}} \max_{u \in V_k} \frac{B_0(u, u)}{D(u, u)} = \max_{u \in U_k = \mathrm{sp}\,(u_1, \cdots, u_k)} \frac{B_0(u, u)}{D(u, u)}, \qquad k = 1, 2, \cdots \tag{2.9}$$

(the minimum-maximum principle).

For any λ_k we let

$$M(\lambda_k) = \{u\colon u \text{ is an eigenvector of (2.5) corresponding to } \lambda_k\}. \tag{2.10}$$

We shall be interested in approximating the eigenpairs of (2.5) by finite element, or, more generally, Galerkin methods. Toward this end we suppose we are given a (one parameter) family $\{S_h\}_{0<h\leqq 1}$ of finite-dimensional subspaces $S_h \subset H$ and we consider the eigenvalue problem

$$\text{Seek } \lambda_h \text{ (real)}, 0 \neq u_h \in S_h \quad \text{such that } B_0(u_h, v) = \lambda_h D(u_h, v) \quad \forall v \in S_h. \tag{2.11}$$

The eigenpairs (λ_h, u_h) of (2.11) are then viewed as approximations to the eigenpairs (λ, u) of (2.5). The problem (2.11) is called the Galerkin method determined by the subspaces S_h for the approximation of the eigenvalues and eigenvectors of (2.5). We will also sometimes refer to problem (2.11) as the Galerkin approximation of the problem (2.5). The problem (2.11) has a sequence of eigenvalues

$$0 < \lambda_{h,1} \leqq \lambda_{h,2} \leqq \cdots \leqq \lambda_{h,N}, \qquad N = \dim S_h,$$

and corresponding eigenvectors

$$u_{h,1}, u_{h,2}, \cdots, u_{h,N}$$

which can be chosen to satisfy

$$B_0(u_{h,i}, u_{h,j}) = \lambda_{h,i} D(u_{h,i}, u_{h,j}) = \delta_{i,j}, \qquad i, j = 1, \cdots, N. \tag{2.12}$$

Minimum and minimum-maximum principles analogous to (2.8) and (2.9) hold for the problem (2.11); they are obtained by replacing H by S_h and restricting k to be less than or equal to N. We will refer to them by (2.8^h) and (2.9^h), respectively. Using (2.8) and (2.9) together with (2.8^h) and (2.9^h) we see immediately that

$$\lambda_k \leqq \lambda_{h,k}, \qquad k = 1, 2, \cdots, N = \dim S_h. \tag{2.13}$$

For every $\lambda_{h,k}$ we let

$$M(\lambda_{h,k}) = \{u\colon u \text{ is an eigenvector of (2.11) corresponding to } \lambda_{h,k}\}. \tag{2.10^h}$$

Because of (2.1)-(2.4), 0 is an eigenvalue of neither (2.5) nor (2.11). It will be convenient, however, to introduce the notation $\lambda_0 = \lambda_{h,0} = 0$.

In what follows we shall assume that the family $\{S_h\}$ satisfies the approximability assumption

$$\inf_{\chi \in S_h} \|u - \chi\|_{B_0} \to 0 \quad \text{as } h \to 0, \quad \text{for each } u \in H. \tag{2.14}$$

It follows from the variational principles (2.8), (2.9), (2.8^h), (2.9^h) and assumption (2.14) that $\lambda_{h,k} \to \lambda_k$ as $h \to 0$ for each k.

Our analysis employs two functions $\Phi(\lambda)$ and $\Phi_h(\lambda)$ of the nonnegative real variable λ which are associated with the eigenvalues of (2.5) and (2.11), respectively. We define

$$\Phi(\lambda)=\inf_{j=1,2,\cdots}\left|1-\frac{\lambda}{\lambda_j}\right| \tag{2.15}$$

and

$$\Phi_h(\lambda)=\min_{j=1,\cdots,N}\left|1-\frac{\lambda}{\lambda_{h,j}}\right|. \tag{2.16}$$

It is immediate that the functions are nonnegative and continuous in λ and that

$$\Phi(\lambda)=0 \quad \text{if and only if } \lambda=\lambda_j \quad \text{for some } j$$

and

$$\Phi_h(\lambda)=0 \quad \text{if and only if } \lambda=\lambda_{h,j} \quad \text{for some } j.$$

In the following lemmas we give characterizations of Φ and Φ_h which do not involve the eigenvalues λ_j and $\lambda_{h,j}$, respectively. For $0\leqq\lambda<\infty$ and $u, v\in H$ define

$$B(\lambda, u, v)=B_0(u, v)-\lambda D(u, v). \tag{2.17}$$

We now have the following.

LEMMA 2.1. *For all* $0\leqq\lambda<\infty$,

$$\Phi(\lambda)=\inf_{\substack{u\in H\\ \|u\|_{B_0}=1}}\ \sup_{\substack{v\in H\\ \|v\|_{B_0}=1}}|B(\lambda, u, v)|. \tag{2.18}$$

Suppose λ_k *has multiplicity* q, *i.e., suppose*

$$\lambda_{k-1}<\lambda_k=\lambda_{k+1}=\cdots=\lambda_{k+(q-1)}<\lambda_{k+q}.$$

Then, for

$$\bar\lambda_k:=\left(\frac{\lambda_{k-1}^{-1}+\lambda_k^{-1}}{2}\right)^{-1}\leqq\lambda\leqq\bar\lambda_{k+q}:=\left(\frac{\lambda_{k+q-1}^{-1}+\lambda_{k+q}^{-1}}{2}\right)^{-1}, \tag{2.19}$$

we have

$$\Phi(\lambda)=\left|1-\frac{\lambda}{\lambda_k}\right| \tag{2.20}$$

and

$$\Phi(\lambda)=\sup_{\substack{v\in H\\ \|v\|_{B_0}=1}}|B(\lambda, u, v)|=|B(\lambda, u, u)| \quad \forall u\in M(\lambda_k) \quad \textit{with } \|u\|_{B_0}=1. \tag{2.21}$$

Proof. For $u, v\in H$, write

$$u=\sum_{j=1}^{\infty}a_ju_j, \qquad v=\sum_{j=1}^{\infty}b_ju_j.$$

Then

$$B(\lambda, u, v)=\sum_{j=1}^{\infty}a_jb_j\left(1-\frac{\lambda}{\lambda_j}\right). \tag{2.22}$$

Thus

$$\sup_{\substack{v\in H\\ \|v\|_{B_0}=1}} |B(\lambda, u, v)| = \left[\sum_{j=1}^{\infty} a_j^2\left(1-\frac{\lambda}{\lambda_j}\right)^2\right]^{1/2}, \tag{2.23}$$

from which we get

$$\inf_{\substack{u\in H\\ \|u\|_{B_0}=1}} \sup_{\substack{v\in H\\ \|v\|_{B_0}=1}} |B(\lambda, u, v)| = \inf_{j=1,2,\cdots} \left|1-\frac{\lambda}{\lambda_j}\right| = \Phi(\lambda).$$

This is (2.18). Equation (2.20) follows from the definition of $\Phi(\lambda)$ and an examination of the graphs of $|1-\lambda/\lambda_j|$ for $j=1, 2, \cdots$. Equation (2.21) follows from (2.20), (2.22) and (2.23). □

In a similar way we have the following.

LEMMA 2.2. *With H replaced by S_h, λ_k by $\lambda_{h,k}$, and u_h by $u_{h,k}$, Lemma* 2.1 *holds for $\Phi_h(\lambda)$. (Relationships analogous to those of Lemma* 2.1 *will be indicated by a superscript h.)* □

If λ_k is an eigenvalue of multiplicity q, then $\lambda_{h,k}, \cdots, \lambda_{h,k+q-1}$ could be multiple or simple. The graphs of $\Phi(\lambda)$ and $\Phi_h(\lambda)$ are given in Fig. 2.1.

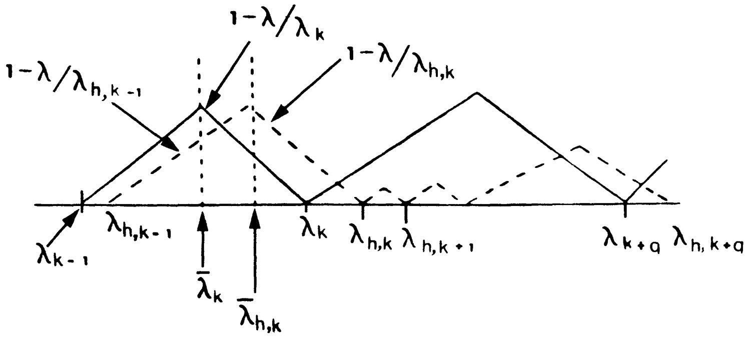

FIG. 2.1. *The graphs of* $\Phi(\lambda)$ (——) *and* $\Phi_h(\lambda)$ (---).

We end the section with a lemma that expresses a fundamental property of eigenvalue and eigenvector approximation.

LEMMA 2.3. *Suppose (λ, u) is an eigenpair of* (2.5) *with $\|u\|_D=1$, suppose w is any vector in H with $\|w\|_D=1$, and let $\tilde\lambda = B_0(w, w)$. Then*

$$\tilde\lambda - \lambda = \|w-u\|_{B_0}^2 - \lambda\|w-u\|_D^2. \tag{2.24}$$

(Note that we have assumed u and w are normalized with respect to $\|\cdot\|_D$ here, whereas in (2.6) and (2.12) we assumed u_i and $u_{h,i}$ are normalized with respect to $\|\cdot\|_{B_0}$.)

Proof. By an easy calculation,

$$\|w-u\|_{B_0}^2 - \lambda\|w-u\|_D^2 = \|w\|_{B_0}^2 - 2B_0(w,u) + \|u\|_{B_0}^2 - \lambda\|w\|_D^2 + 2\lambda D(w,u) - \lambda\|u\|_D^2.$$

Then, since

$$\|w\|_D = \|u\|_D = 1, \quad \|w\|_{B_0}^2 = \tilde\lambda, \quad \|u\|_{B_0}^2 = \lambda,$$

and

$$B_0(w, u) = \lambda D(w, u),$$

we get the desired result. □

3. The main result. For $i=1,2,\cdots$, suppose λ_{k_i} is an eigenvalue of (2.5) of multiplicity q_i, i.e., suppose

$$\lambda_{k_i-1}<\lambda_{k_i}=\lambda_{k_i+1}=\cdots=\lambda_{k_i+q_i-1}<\lambda_{k_i+q_i}=\lambda_{k_{i+1}}.$$

Here $k_1=1$, k_2 is the lowest index of the second distinct eigenvalue, k_3 is the lowest index of the third distinct eigenvalue, etc. Let

$$\varepsilon_{i,j}(h)=: \inf_{\substack{u\in M(\lambda_{k_i})\\ \|u\|_{B_0}=1\\ B_0(u,u_{h,k_i})=\cdots=B_0(u,u_{h,k_i+j-2})=0}} \inf_{\chi\in S_h}\|u-\chi\|_{B_0}, \qquad j=1,\cdots,q_i. \tag{3.1}$$

The restrictions $B_0(u,u_{h,k_i})=\cdots=B_0(u,u_{h,k_i+j-2})=0$ are considered vacuous if $j=1$. Note that $\varepsilon_{i,1}=\varepsilon_{\lambda_i}$ and $\varepsilon_{i,q_i}\leqq\bar{\varepsilon}_{\lambda_i}$, where ε_{λ_i} and $\bar{\varepsilon}_{\lambda_i}$ are the quantities introduced in § 1. It is the purpose of this section to estimate the eigenvalue and eigenvector errors for the Galerkin method (2.11) in terms of the approximability quantities $\varepsilon_{i,j}(h)$.

THEOREM 3.1. *There are constants C and h_0 such that*

$$\lambda_{h,k_i+j-1}-\lambda_{k_i+j-1}\leqq C\varepsilon_{i,j}^2(h) \quad \forall 0<h\leqq h_0,\quad j=1,\cdots,q_i,\quad i=1,2,\cdots, \tag{3.2}$$

and such that the eigenvectors $u_1,u_2,\cdots$, of (2.5) *can be chosen so that*

$$\|u_{h,k_i+j-1}-u_{k_i+j-1}\|_{B_0}\leqq C\varepsilon_{i,j}(h) \quad \forall 0<h\leqq h_0,\quad j=1,\cdots,q_i,\quad i=1,2,\cdots, \tag{3.3}$$

and so that (2.6) *holds.*

Proof. Overview of the proof. The complete details of the proof, which proceeds by induction, are given below. Here we provide an overview. In Step A we give the proof for $i=1$. The proof is very simple in this case and rests entirely on the minimum principle (2.8^h) and Lemma 2.3.

The central part of the proof is given in Step B. There we prove the theorem for $i=2$, proving first the eigenvalue estimate (3.2) and then the eigenvector estimate (3.3). In particular, in Steps B.1 and B.2 estimates (3.2) and (3.3), respectively, are proved for $j=1$. We further note that the argument used in Step B proves the main inductive step in our proof, yielding the result for $i=\underline{i}+1$ on the assumption that it is true for $i\leqq\underline{i}$. To be somewhat more specific, the argument in Step B.1 proves (3.2) directly for any $i\geqq 2$ (and $j=1$) and that in B.2 proves (3.3) for $i=\underline{i}+1$ (and $j=1$) under the assumption that $\|u_{h,l}-u_l\|_{B_0}\to 0$ as $h\to 0$ for $l\leqq k_{\underline{i}+1}-1$ (cf. (3.36)).

Details of the proof. Throughout the proof we use the fact that $\varepsilon_{i,j}(h)$ can also be expressed as

$$\varepsilon_{i,j}(h)=\inf_{\substack{u\in M(\lambda_{k_i})\\ \|u\|_{B_0}=1\\ B_0(u,u_{h,k_i})=\cdots=B_0(u,u_{h,k_i+j-2})=0}} \inf_{\substack{\chi\in S_h\\ B_0(\chi,u_{h,k_i})=\cdots=B_0(\chi,u_{h,k_i+j-2})=0}} \|u-\chi\|_{B_0}. \tag{3.1'}$$

Step A. Here we prove the theorem for $i=1$.

Step A.1. Suppose $\lambda_{k_1}(k_1=1)$ is an eigenvalue of (2.5) with multiplicity q_1, i.e., suppose $\lambda_1=\lambda_2=\cdots=\lambda_{q_1}<\lambda_{q_1+1}$. In this step we estimate $\lambda_{h,1}-\lambda_1$, the error between λ_1 and the approximate eigenvalue among $\lambda_{h,1},\cdots,\lambda_{h,q_1}$ that is closest to λ_1, i.e., we prove (3.2) for $i=j=1$. Note that

$$\varepsilon_{1,1}(h)=\inf_{\substack{u\in M(\lambda_1)\\ \|u\|_{B_0}=1}}\inf_{\chi\in S_h}\|u-\chi\|_{B_0}$$

is the error in the approximation by elements of S_h of the most easily approximated eigenvector associated with λ_1.

From the definitions of $\varepsilon_{1,1}(h)$ we see that there is a $\bar{u}_h \in M(\lambda_1)$ with $\|\bar{u}_h\|_{B_0}=1$ and an $s_h \in S_h$ such that

$$\|\bar{u}_h - s_h\|_{B_0} = \varepsilon_{1,1}(h). \tag{3.4}$$

Let

$$\tilde{\bar{u}} = \frac{\bar{u}_h}{\sqrt{D(\bar{u}_h, \bar{u}_h)}}, \qquad \tilde{s}_h = \frac{s_h}{\sqrt{D(s_h, s_h)}}.$$

By the minimum principle (2.8^h) we have

$$\lambda_{h,1} - \lambda_1 \leqq B_0(\tilde{s}_h, \tilde{s}_h) - \lambda_1. \tag{3.5}$$

Now apply Lemma 2.3 with $(\lambda, u) = (\lambda_1, \tilde{\bar{u}}_h)$, $w = \tilde{s}_h$, and $\tilde{\lambda} = B_0(\tilde{s}_h, \tilde{s}_h)$. This yields

$$\begin{aligned} B_0(\tilde{s}_h, \tilde{s}_h) - \lambda_1 &\leqq \|\tilde{s}_h - \tilde{\bar{u}}_h\|_{B_0}^2 - \lambda_1 \|\tilde{s}_h - \tilde{\bar{u}}_h\|_D^2 \\ &\leqq \|\tilde{s}_h - \tilde{\bar{u}}_h\|_{B_0}^2 \leqq C\|s_h - \bar{u}_h\|_{B_0}^2. \end{aligned} \tag{3.6}$$

Formulas (3.4)–(3.6) yield the desired result.

Step A.2. In this step we prove (3.3) for $i=j=1$. Let $u_1, u_2, \cdots$ be eigenvectors of (2.5) satisfying (2.6). Write

$$u_{h,1} = \sum_{j=1}^{\infty} a_j^{(1)} u_j. \tag{3.7}$$

From (2.12), (2.17) and (3.5)–(3.7) we have

$$\begin{aligned} \left(1 - \frac{\lambda_1}{\lambda_{q_1+1}}\right) \sum_{j=q_1+1}^{\infty} (a_j^{(1)})^2 &\leqq \left| \sum_{j=q_1+1}^{\infty} (a_j^{(1)})^2 \left(1 - \frac{\lambda_1}{\lambda_j}\right) \right| \\ &= \left| \sum_{j=1}^{\infty} (a_j^{(1)})^2 \left(1 - \frac{\lambda_1}{\lambda_j}\right) \right| \\ &= |B(\lambda_1, u_{h,1}, u_{h,1})| \\ &= |B(\lambda_{h,1}, u_{h,1}, u_{h,1}) + (\lambda_1 - \lambda_{h,1}) D(u_{h,1}, u_{h,1})| \\ &= (\lambda_{h,1} - \lambda_1)\lambda_{h,1}^{-1} \\ &\leqq C\varepsilon_{1,1}^2(h). \end{aligned} \tag{3.8}$$

Hence

$$\begin{aligned} \left\| u_{h,1} - \sum_{j=1}^{q_1} a_j^{(1)} u_j \right\|_{B_0} &= \left[\sum_{j=q_1+1}^{\infty} (a_j^{(1)})^2 \right]^{1/2} \\ &\leqq C(1 - \lambda_1/\lambda_{q_1+1})^{-1/2} \varepsilon_{1,1}(h). \end{aligned} \tag{3.9}$$

Redefining u_1 to be

$$\sum_{j=1}^{q_1} a_j^{(1)} u_j \Big/ \left\| \sum_{j=1}^{q_1} a_j^{(1)} u_j \right\|_{B_0},$$

we easily see that $\|u_1\|_{B_0} = 1$, so that (2.6) still holds, and

$$\|u_{h,1} - u_1\|_{B_0} \leqq C\varepsilon_{1,1}(h), \tag{3.10}$$

as desired. Note that u_1 may depend on h.

Step A.3. Suppose $q_1 \geqq 2$. From (3.1′) we see that

$$\varepsilon_{1,2}(h) = \inf_{\substack{u \in M(\lambda_1) \\ \|u\|_{B_0}=1 \\ B_0(u, u_{h,1})=0}} \inf_{\substack{\chi \in S_h \\ B_0(\chi, u_{h,1})=0}} \|u - \chi\|_{B_0}. \tag{3.11}$$

Choose $\bar{u}_h \in M(\lambda_1)$ with $\|\bar{u}_h\|_{B_0} = 1$, $B_0(\bar{u}_h, u_{h,1}) = 0$ and $s_h \in S_h$ with $B_0(s_h, u_{h,1}) = 0$ so that

$$\|\bar{u}_h - s_h\|_{B_0} = \varepsilon_{1,2}(h), \tag{3.12}$$

and let

$$\tilde{\bar{u}}_h = \frac{\bar{u}_h}{\sqrt{D(\bar{u}_h, \bar{u}_h)}}, \qquad \tilde{s}_h = \frac{s_h}{\sqrt{D(s_h, s_h)}}.$$

Since $B_0(s_h, u_{h,1}) = 0$, from the minimum principle (2.8^h), Lemma 2.3, and (3.12), we have

$$\lambda_{h,2} - \lambda_2 \leqq \|\tilde{s}_h - \tilde{\bar{u}}_h\|^2_{B_0} \leqq C\varepsilon^2_{1,2}(h). \tag{3.13}$$

This is (3.2) for $i = 1$ and $j = 2$.

Step A.4. In Step A.2 we redefined u_1. Now redefine $u_2, \cdots, u_{q_1}$ so that $u_1, \cdots, u_{q_1}$ are B_0-orthogonal. Write

$$u_{h,2} = \sum_{j=1}^{\infty} a_j^{(2)} u_j.$$

Now, proceeding as in Step A.2 and using (3.13), we have

$$\begin{aligned} (1 - \lambda_2/\lambda_{q_1+1}) \sum_{j=q_1+1}^{\infty} (a_j^{(2)})^2 &\leqq \left| \sum_{j=1}^{\infty} (a_j^{(2)})^2 (1 - \lambda_2/\lambda_j) \right| \\ &= |B(\lambda_2, u_{h,2}, u_{h,2})| \\ &= (\lambda_{h,2} - \lambda_2)\lambda_{h,2}^{-1} \\ &\leqq C\varepsilon^2_{1,2}(h). \end{aligned}$$

Thus

$$\left\| u_{h,2} - \sum_{j=1}^{q_1} a_j^{(2)} u_j \right\|_{B_0} \leqq C\varepsilon_{1,2}(h). \tag{3.14}$$

But by (3.10),

$$\begin{aligned} a_1^{(2)} &= B_0(u_{h,2}, u_1) \\ &= B_0(u_{h,2}, u_1 - u_{h,1}) \\ &\leqq \|u_{h,2}\|_{B_0} \|u_1 - u_{h,1}\|_{B_0} \\ &\leqq C\varepsilon_{1,1}(h) \\ &\leqq C\varepsilon_{1,2}(h). \end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15) we get

$$\begin{aligned} \left\| u_{h,2} - \sum_{j=2}^{q_1} a_j^{(2)} u_j \right\|_{B_0} &\leqq \left\| u_{h,2} - \sum_{j=1}^{q_1} a_j^{(2)} u_j \right\|_{B_0} + \|a_1^{(2)} u_1\|_{B_0} \\ &\leqq C\varepsilon_{1,2}(h). \end{aligned}$$

Redefining u_2 to be

$$\sum_{j=2}^{q_1} a_j^{(2)} u_j \Big/ \Big\| \sum_{j=2}^{q_1} a_j^{(2)} u_j \Big\|_{B_0},$$

we see that $\|u_2\|_{B_0} = 1$ and $B_0(u_1, u_2) = 0$, so that (2.6) holds and

$$\|u_{h,2} - u_2\|_{B_0} \leqq C\varepsilon_{1,2}(h), \tag{3.16}$$

which is (3.3) for $i = 1$, $j = 2$.

Step A.5. Continuing in the above manner we obtain the proof of (3.2) and (3.3) for $i = 1$ and $j = 1, \cdots, q_1$.

Step B. Here we prove Theorem 3.1 for $i = 2$.

Step B.1. Suppose λ_{k_2} ($k_2 = q_1 + 1$) is an eigenvalue of (2.5) of multiplicity q_2. In this step we estimate $\lambda_{h,k_2} - \lambda_{k_2}$, the error between λ_{k_2} and the approximate eigenvalue among $\lambda_{h,k_2}, \cdots, \lambda_{h,k_2+q_2-1}$ that is closest to λ_{k_2}. Note that

$$\varepsilon_{2,1}(h) = \inf_{\substack{u \in M(\lambda_{k_2}) \\ \|u\|_{B_0} = 1}} \inf_{\chi \in S_h} \|u - \chi\|_{B_0}. \tag{3.17}$$

Write $\lambda_{k_2-1} = \xi \lambda_{k_2}$, $\lambda_{h,k_2-1} = \xi_h \lambda_{h,k_2}$, and $\lambda_{h,k_2} = \varphi_h \lambda_{k_2}$. Then $0 < \xi < 1$ and $\varphi_h \geqq 1$. Let $\psi_h = 2\varphi_h/(1 + \xi^{-1})$. From (2.13) and the definitions of $\bar{\lambda}_k$ and $\bar{\lambda}_{h,k}$ in Lemmas 2.1 and 2.2, respectively, we see that

$$\bar{\lambda}_{k_2} \leqq \bar{\lambda}_{h,k_2}. \tag{3.18}$$

A simple calculation shows that

$$\bar{\lambda}_{k_2} = \frac{2}{1 + \xi^{-1}} \lambda_{k_2}$$

and

$$\bar{\lambda}_{h,k_2} = \frac{2\varphi_h}{1 + \xi_h^{-1}} \lambda_{k_2} = \psi_h \lambda_{k_2}.$$

Hence (3.18) shows that

$$\frac{2}{1 + \xi^{-1}} \leqq \psi_h.$$

Since $\lambda_{h,j} \to \lambda_j$ as $h \to 0$ (see § 2), we see that $\xi_h \to \xi$, $\varphi_n \to 1$, and $\psi_h \to 2/(1 + \xi^{-1})$ as $h \to 0$. Thus, noting that $2/(1 + \xi^{-1}) < 1$, we see that we can choose h_0 such that $0 < h < h_0$ implies that

$$\xi \leqq \frac{2}{1 + \xi^{-1}} \leqq \psi_h \leqq \frac{2}{1 + \xi^{-1}} + 1 \Big/ 2 = \frac{1 + 3\xi}{2 + 2\xi} < 1. \tag{3.19}$$

From (2.19), (2.20), (2.19^h) and (2.20^h) (see also Fig. 2.1) we get

$$0 \leqq \Phi_h(\bar{\lambda}_{h,k_2}) - \Phi(\bar{\lambda}_{h,k_2}) = \left(1 - \frac{\bar{\lambda}_{h,k_2}}{\varphi_h \lambda_{k_2}}\right) - \left(1 - \frac{\bar{\lambda}_{h,k_2}}{\lambda_{k_2}}\right)$$

$$= \psi_h(1 - \varphi_h^{-1}),$$

and hence, using (3.19), we get

$$
\begin{aligned}
\lambda_{h,k_2}-\lambda_{k_2}&=\lambda_{k_2}(\varphi_h-1)\\
&=\frac{\lambda_{k_2}[\Phi_h(\bar\lambda_{h,k_2})-\Phi(\bar\lambda_{h,k_2})]\psi_h^{-1}}{1-[\Phi_h(\bar\lambda_{h,k_2})-\Phi(\bar\lambda_{h,k_2})]\psi_h^{-1}}\\
&\leqq\frac{\lambda_{k_2}[\Phi_h(\bar\lambda_{h,k_2})-\Phi(\bar\lambda_{h,k_2})]\xi^{-1}}{1-[\Phi_h(\bar\lambda_{h,k_2})-\Phi(\bar\lambda_{h,k_2})]\xi^{-1}}\\
&\leqq 2\lambda_k[\Phi_h(\bar\lambda_{h,k_2})-\Phi(\bar\lambda_{h,k_2})]\xi^{-1},
\end{aligned}
\tag{3.20}
$$

provided that

$$[\Phi_h(\bar\lambda_{h,k_2})-\Phi(\bar\lambda_{h,k_2})]\xi^{-1}\leqq\tfrac{1}{2}. \tag{3.21}$$

We will now show that

$$\Phi_n(\bar\lambda_{h,k_2})-\Phi(\bar\lambda_{h,k_2})\leqq C\varepsilon_{2,1}^2(h)\quad\text{for } h\leqq h_0, \tag{3.22}$$

where C depends only on λ_1, λ_{k_2-1}, and λ_{k_2}, and, in particular, is independent of h, and h_0 depends only on λ_1, λ_{k_2-1}, λ_{k_2}, and the approximability of the eigenvectors in $M(\lambda_{k_2})$ by S_h. As in Step A.1, choose $\bar u_h\in M(\lambda_{k_2})$ with $\|\bar u_h\|_{B_0}=1$ and $s_h\in S_h$ such that

$$\|\bar u_h-s_h\|_{B_0}=\varepsilon_{2,1}(h). \tag{3.23}$$

We see that s_h is the B_0-orthogonal projection of $\bar u_h$ onto S_h, i.e., that

$$B_0(\bar u_h-s_h,v)=0\quad\forall v\in S_h. \tag{3.24}$$

Let $w_h=\bar u_h-s_h$. Then

$$\|w_h\|_{B_0}=\varepsilon_{2,1}(h) \tag{3.25}$$

and

$$\|s_h\|_{B_0}^2\leqq\|s_h\|_{B_0}^2+\|w_h\|_{B_0}^2=\|\bar u_h\|_{B_0}^2=1. \tag{3.26}$$

Next we write

$$s_h=c_h\bar u_h+e_h,$$

where

$$B_0(\bar u_h,e_h)=0,$$

i.e., we let $c_h\bar u_h$ be the B_0-orthogonal projection of s_h onto span $(\bar u_h)$. Let $r_h=(1-c_h)\bar u_h$. Then

$$w_h=r_h-e_h,\qquad B(e_h,r_h)=0,$$

which implies

$$\|e_h\|_{B_0}\leqq\|w_h\|_{B_0}. \tag{3.27}$$

Furthermore, using (3.24), (3.25) and (3.26) we have

$$c_h=B_0(s_h,\bar u_h)\leqq\|s_h\|_{B_0}\|\bar u_h\|_{B_0}\leqq 1$$

and

$$
\begin{aligned}
c_h=B_0(s_h,\bar u_h)&=B_0(\bar u_h-w_h,\bar u_h)\\
&=\|\bar u_h\|_{B_0}^2-B_0(w_h,\bar u_h)\\
&\geqq 1-\|w_h\|_{B_0}\\
&=1-\varepsilon_{2,1}(h)\\
&>0\quad\text{for } h<h_0
\end{aligned}
$$

for h_0 sufficiently small. Thus we can assume that

$$0 < c_h \leqq 1.$$

Also,

$$\|r_h\|_{B_0} = \|w_h\|^2_{B_0}. \tag{3.28}$$

To see this we refer to Fig. 3.1 and note that

$$\|r_h\|_{B_0} = \|w_h\|_{B_0} \cos \alpha$$

and

$$\|w_h\|_{B_0} = \|\bar{u}_h\|_{B_0} \sin \beta = \cos \alpha.$$

From (3.28) and the definition of r_h we have

$$c_h = 1 - \|w_h\|^2_{B_0}. \tag{3.29}$$

Assume now that $u_{k_2} = \bar{u}_h$ (redefining u_{k_2}, if necessary). Write

$$e_h = \sum_{j=1}^{\infty} q_j u_j, \qquad v = \sum_{j=1}^{\infty} b_j u_j.$$

Then

$$\begin{aligned} B(\bar{\lambda}_{h,k_2}, s_h, v) &= c_h B_0(\bar{u}_h, v) + B_0(e_h, v) - \bar{\lambda}_{h,k_2} c_h D(\bar{u}_h, v) - \bar{\lambda}_{h,k_2} D(e_h, v) \\ &= c_h b_{k_2} + \sum_{\substack{j=1 \\ j \neq k_2}}^{\infty} q_j b_j - \bar{\lambda}_{h,k_2} \lambda_{k_2}^{-1} c_h b_{k_2} - \bar{\lambda}_{h,k_2} \sum_{\substack{j=1 \\ j \neq k_2}}^{\infty} q_j b_j \lambda_j^{-1} \\ &= c_h b_{k_2}(1 - \bar{\lambda}_{h,k_2} \lambda_{k_2}^{-1}) + \sum_{\substack{j=1 \\ j \neq k_2}}^{\infty} q_j b_j (1 - \bar{\lambda}_{h,k_2} \lambda_j^{-1}), \end{aligned}$$

and hence

$$\begin{aligned} \sup_{\substack{v \in S_h \\ \|v\|_{B_0}=1}} |B(\bar{\lambda}_{h,k_2}, s_h, v)| &\leqq \sup_{\substack{v \in H \\ \|v\|_{B_0}=1}} |B(\bar{\lambda}_{h,k_2}, s_h, v)| \\ &= \left[c_h^2 (1 - \bar{\lambda}_{h,k_2} \lambda_{k_2}^{-1})^2 + \sum_{\substack{j=1 \\ j \neq k_2}}^{\infty} q_j^2 (1 - \bar{\lambda}_{h,k_2} \lambda_j^{-1})^2 \right]^{1/2}. \end{aligned} \tag{3.30}$$

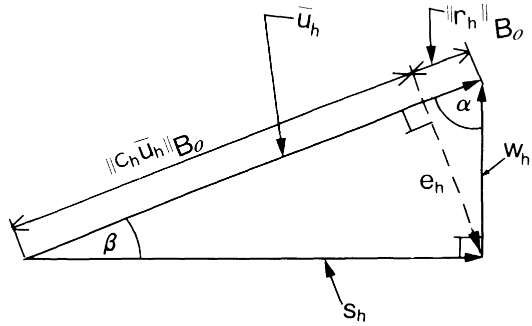

FIG. 3.1. *Configuration of r_h, w_h, $\bar{u}_h$, α and β.*

Combining (3.27), (3.29) and (3.30), we get

$$\begin{aligned}
\sup_{\substack{v\in S_h\\ \|v\|_{B_0}=1}} |B(\bar\lambda_{h,k_2}, s_h, v)| &\leqq \Big[(1-\|w_h\|_{B_0}^2)^2(1-\bar\lambda_{h,k_2}\lambda_{k_2}^{-1})^2 \\
&\qquad + \sup_{\substack{j=1,2,\cdots\\ j\neq k_2}} (1-\bar\lambda_{h,k_2}\lambda_j^{-1})^2 \sum_{j=1}^{\infty} q_j^2\Big] \\
&\leqq \Big[(1-\|w_h\|_{B_0}^2)^2(1-\bar\lambda_{h,k_2}\lambda_{k_2}^{-1})^2 \\
&\qquad + \sup_{\substack{j=1,2,\cdots\\ j\neq k_2}} (1-\bar\lambda_{h,k_2}\lambda_j^{-1})^2 \|e_h\|_{B_0}^2\Big]^{1/2} \\
&= \Big[(1-\|w_h\|_{B_0}^2)^2\Phi^2(\bar\lambda_{h,k_2}) \\
&\qquad + \sup_{\substack{j=1,2,\cdots\\ j\neq k_2}} (1-\bar\lambda_{h,k_2}\lambda_j^{-1})^2 \|w_h\|_{B_0}^2\Big]^{1/2} \\
&:= Q.
\end{aligned}$$

Thus, using (2.18^h), (3.25) and (3.26), we have

$$\begin{aligned}
\Phi_h(\bar\lambda_{h,k_2}) &\leqq \frac{Q}{\|s_h\|_{B_0}} \\
&\leqq \Big[(1-\|w_h\|_{B_0}^2)\Phi^2(\bar\lambda_{h,k_2}) \\
&\qquad + \sup_{\substack{j=1,2,\cdots\\ j\neq k_2}} (1-\bar\lambda_{h,k_2}\lambda_j^{-1})^2 \|w_h\|_{B_0}^2(1-\|w_h\|_{B_0}^2)^{-1}\Big]^{1/2} \\
&\leqq [\Phi^2(\bar\lambda_{h,k_2}) + R\varepsilon_{2,1}^2(h)]^{1/2},
\end{aligned}$$

and hence

$$\Phi_h(\bar\lambda_{h,k_2}) - \Phi(\bar\lambda_{h,k_2}) \leqq \frac{R\varepsilon_{2,1}^2(h)}{2\Phi(\bar\lambda_{h,k_2})}, \tag{3.31}$$

where

$$R =: \sup_{\substack{j=1,2,\cdots\\ j\neq k_2}} (1-\bar\lambda_{h,k_2}\lambda_j^{-1})(1-\varepsilon_{2,1}^2(h))^{-1} - \Phi^2(\bar\lambda_{h,k_2}).$$

One can easily show that $R/2\Phi(\bar\lambda_{h,k_2})$ is bounded independent of h; in fact, using (2.20) and (3.19), one can show that

$$\frac{R}{2\Phi(\bar\lambda_{h,k_2})} \leqq \frac{2\lambda_{k_2}}{\lambda_1(1-\xi)[1-\varepsilon_{2,1}^2(h)]} \leqq \frac{4\lambda_{k_2}}{\lambda_1(1-\xi)}, \tag{3.32}$$

provided $\varepsilon_{2,1}^2(h) < \frac{1}{2}$, i.e., provided $h < h_0$ for h_0 sufficiently small. Hence, combining (3.31) and (3.32) we get

$$\Phi_h(\bar\lambda_{h,k_2}) - \Phi(\bar\lambda_{h,k_2}) \leqq \frac{4\lambda_{k_2}}{\lambda_1(1-\xi)} \varepsilon_{2,1}^2(h), \tag{3.33}$$

which is (3.22). Combining (3.20), (3.21), and (3.33) we have

$$\lambda_{h,k_2}-\lambda_{k_2} \leqq C\varepsilon_{2,1}^2(h) \quad \text{for } h \leqq h_0, \tag{3.34}$$

where

$$C=\frac{8\lambda_{k_2}^2}{\lambda_1\xi(1-\xi)}. \tag{3.35}$$

This is (3.2) for $i=2$, $j=1$.

Comment on (3.34). C in (3.34) clearly depends on λ_1, λ_{k_2-1} and λ_{k_2}, but is independent of h. Note that if we were considering a family of problems depending on a parameter τ, we could bound $C=C(\tau)$ above, independent of τ, provided $\lambda_{k_2}=\lambda_{k_2}(\tau)$ was bounded above, $\lambda_1=\lambda_1(\tau)$ was bounded away from 0, and $\xi=\xi(\tau)$ was bounded away from 0 and 1. It follows from (3.31)-(3.33) that (3.34) is valid for $h \leqq h_0$, where h_0 depends only on λ_1, λ_{k_2-1}, λ_{k_2}, and the approximability of the eigenvectors u_j, $j=k_2,\cdots,k_2+q_2-1$, by S_h. For a family of problems, $h_0(\tau)$ could be bounded away from 0 if $\lambda_1(\tau)$ was bounded away from 0, $\lambda_{k_2}(\tau)$ was bounded above, $\xi(\tau)$ was bounded away from 0 and 1, and the eigenvectors $u_j=u_j(\tau)$, $j=k_2,\cdots,k_2+q_2-1$, could be approximated by S_h, uniformly in τ.

Step B.2. Suppose, as in Step B.1, that λ_{k_2} has multiplicity q_2. We have shown in Step A.5 that we can choose the eigenvectors $u_1, u_2, \cdots$ of (2.5) so that (2.6) holds and so that

$$\|u_{h,j}-u_j\|_{B_0} \leqq C\varepsilon_{1,j}(h), \qquad j=1,\cdots,q_1=k_2-1. \tag{3.36}$$

Write

$$u_{h,k_2}=\sum_{j=1}^{\infty} a_j^{(k_2)}u_j. \tag{3.37}$$

From (3.37) we have

$$\begin{aligned}\left|\sum_{j=1}^{\infty}[a^{(k_2)}]^2(1-\lambda_{k_2}/\lambda_j)\right| &= |B(\lambda_{k_2},u_{h,k_2},u_{h,k_2})| \\ &= |B(\lambda_{h,k_2},u_{h,k_2},u_{h,k_2})+(\lambda_{k_2}-\lambda_{h,k_2})D(u_{h,k_2},u_{h,k_2})| \\ &= (\lambda_{h,k_2}-\lambda_{k_2})\lambda_{h,k_2}^{-1},\end{aligned}$$

which, together with (3.34), yields

$$\left|\sum_{j=1}^{k_2-1}[a^{(k_2)}]^2(1-\lambda_{k_2}/\lambda_j)+\sum_{j=k_2+q_2}^{\infty}[a^{(k_2)}]^2(1-\lambda_{k_2}/\lambda_j)\right| \leqq C\varepsilon_{2,1}^2(h). \tag{3.38}$$

Note that the first term inside the absolute value is negative and the second is positive. In addition

$$C_1 \leqq |1-\lambda_{k_2}/\lambda_j| \leqq C_2 \quad \forall j \neq k_2, k_2+1,\cdots,k_2+q_2-1,$$

with C_1, C_2 positive numbers. Hence from (3.38) we obtain

$$\sum_{j=1}^{k_2-1}[a^{(k_2)}]^2 \leqq D_1\varepsilon_{2,1}^2(h)+D_2\sum_{j=k_2+q_2}^{\infty}[a^{(k_2)}]^2 \tag{3.39}$$

and

$$\sum_{j=k_2+q_2}^{\infty} [a^{(k_2)}]^2 \leqq D_3 \varepsilon_{2,1}^2(h) + D_4 \sum_{j=1}^{k_2-1} [a^{(k_2)}]^2. \tag{3.40}$$

Write

$$u_{h,i} - u_i = \sum_{j=1}^{\infty} b_{i,j} u_j, \qquad i = 1, \cdots, k_2 - 1 = q_1. \tag{3.41}$$

Then, by (3.36),

$$\sum_{j=1}^{\infty} b_{i,j}^2 = \|u_{h,i} - u_i\|_{B_0}^2 \leqq C\varepsilon_{1,i}^2(h), \qquad i = 1, \cdots, k_2 - 1. \tag{3.42}$$

Next we wish to find constants $\alpha_1, \cdots, \alpha_{k_2-1}$ so that

$$B_0\left(u_i, \sum_{j=1}^{k_2-1} \alpha_j u_{h,j}\right) = a_i^{(k_2)}, \qquad i = 1, \cdots, k_2 - 1. \tag{3.43}$$

Using (3.41), we can write these equations as

$$\begin{aligned} B_0\left(u_i, \sum_{j=1}^{k_2-1} \left(\alpha_j u_j + \alpha_j \sum_{l=1}^{\infty} b_{j,l}, u_l\right)\right) &= \alpha_i + \sum_{j=1}^{k_2-1} b_{j,i}\alpha_j \\ &= a_i^{(k_2)}, \qquad i = 1, \cdots, k_2 - 1. \end{aligned} \tag{3.44}$$

Since (2.14) implies $\varepsilon_{1,k_2-1}^2(h) \to 0$ as $h \to 0$, from (3.42) we see that the $b_{j,i}$ are small for $h \leqq h_0$, with h_0 sufficiently small, and hence the system (3.44) is uniquely solvable, and, moreover, there is a constant L, depending only on k_2, such that

$$\left(\sum_{j=1}^{k_2-1} \alpha_i^2\right)^{1/2} \leqq L\left[\sum_{j=1}^{k_2-1} (a_i^{(k_2)})^2\right]^{1/2}. \tag{3.45}$$

Now, from (3.36) we obtain

$$\begin{aligned} |a_j^{(k_2)}| &= |B_0(u_{h,k_2}, u_j)| \\ &= |B_0(u_{h,k_2}, u_j - u_{h,j})| \\ &\leqq \|u_{h,k_2}\|_{B_0} \|u_j - u_{h,j}\|_{B_0} \\ &= \|u_j - u_{h,j}\|_{B_0} \\ &\leqq C\varepsilon_{1,j}(h), \qquad j = 1, \cdots, k_2 - 1. \end{aligned}$$

Letting

$$\rho_{k_2}^2(h) = \sum_{j=1}^{k_2-1} \varepsilon_{1,j}^2(h), \tag{3.46}$$

we see that

$$\left[\sum_{j=1}^{k_2-1} (a_j^{(k_2)})^2\right]^{1/2} \leqq C\rho_{k_2}(h), \tag{3.47}$$

and thus, from (3.45)

$$\left(\sum_{j=1}^{k_2-1} \alpha_i^2\right)^{1/2} \leqq LC\rho_{k_2}(h) \leqq C\rho_{k_2}(h). \tag{3.48}$$

Now let

$$\psi = u_{h,k_2} - \sum_{j=1}^{k_2-1} \alpha_j u_{h,j}. \tag{3.49}$$

Then $\psi \in S_h$. Furthermore, from (3.41) and (3.43) we get

$$B_0(u_i, \psi) = \begin{cases} 0, & i \leqq k_2 - 1, \\ a_i^{(k_2)} - \sum_{j=1}^{k_2-1} \alpha_j b_{j,i}, & i \geqq k_2. \end{cases} \tag{3.50}$$

From (3.48) and (3.49),

$$\begin{aligned} \left| \|\psi\|_{B_0} - 1 \right| &= \left| \|\psi\|_{B_0} - \|u_{h,k_2}\|_{B_0} \right| \\ &\leqq \|\psi - u_{h,k_2}\|_{B_0} \\ &\leqq \left(\sum_{j=1}^{k_2-1} \alpha_j^2\right)^{1/2} \\ &\leqq C\rho_{k_2}(h). \end{aligned} \tag{3.51}$$

Using (3.34), (2.21^h), (3.50) and (3.51), and the fact that $\rho_{k_2}(h) \to 0$ as $h \to 0$, we get

$$\begin{aligned} C\varepsilon_{2,1}^2(h) &\geqq \frac{\lambda_{h,k_2} - \lambda_{k_2}}{\lambda_{h,k_2}} \\ &= \Phi_h(\lambda_{k_2}) \\ &\geqq B\left(\lambda_{k_2}, u_{h,k_2}, \frac{\psi}{\|\psi\|_{B_0}}\right) \\ &= C'\left[\sum_{l=k_2+q_2}^{\infty} a_l^{(k_2)}\left(a_l^{(k_2)} - \sum_{i=1}^{k_2-1} \alpha_i b_{i,l}\right)\left(1 - \frac{\lambda_{k_2}}{\lambda_l}\right)\right], \end{aligned} \tag{3.52}$$

where $C' > 0$ and is independent of h. Combining (3.42), (3.45), (3.46), and (3.52) we obtain

$$\begin{aligned} \sum_{l=k_2+q_2}^{\infty} (a_l^{(k_2)})^2 &\leqq C\left[\varepsilon_{2,1}^2(h) + \sum_{l=k_2+q_2}^{\infty} |a_l^{(k_2)}| \sum_{i=1}^{k_2-1} |\alpha_i||b_{i,l}|\right] \\ &\leqq C\left[\varepsilon_{2,1}^2(h) + \sum_{i=1}^{k_2-1} |\alpha_i| \sum_{l=k_2+q_2}^{\infty} |a_l^{(k_2)}||b_{i,l}|\right] \\ &\leqq C\left[\varepsilon_{2,1}^2(h) + \sum_{i=1}^{k_2-1} |\alpha_i| \left(\sum_{l=k_2+q_2}^{\infty} |a_l^{(k_2)}|^2\right)^{1/2} \times \left(\sum_{l=k_2+q_2}^{\infty} |b_{i,l}|^2\right)^{1/2}\right] \\ &\leqq C\left[\varepsilon_{2,1}^2(h) + \sum_{i=1}^{k_2-1} |\alpha_i| \left(\sum_{l=k_2+q_2}^{\infty} |a_l^{(k_2)}|^2\right)^{1/2}\right. \end{aligned}$$

$$\times \max_{i=1,\cdots k_2-1} \varepsilon_{1,i}(h)\Bigg]$$

$$(3.53)\qquad \leqq C\Bigg[\varepsilon_{2,1}^2(h)+\varepsilon_{1,k_2-1}(h)\sqrt{k_2-1}\Bigg(\sum_{i=1}^{k_2-1}|\alpha_i|^2\Bigg)^{1/2} \times\Bigg(\sum_{l=k_2+q_2}^{\infty}|a_l^{(k_2)}|^2\Bigg)^{1/2}\Bigg]$$

$$\leqq C\Bigg[\varepsilon_{2,1}^2(h)+\varepsilon_{1,k_2-1}(h)\sqrt{k_2-1}\,L\Bigg(\sum_{i=1}^{k_2-1}[a_i^{(k_2)}]^2\Bigg)^{1/2} \times\Bigg(\sum_{l=k_2+q_2}^{\infty}|a_l^{(k_2)}|^2\Bigg)^{1/2}\Bigg]$$

$$\leqq C\Bigg[\varepsilon_{2,1}^2(h)+\varepsilon_{1,k_2-1}(h)\Bigg(\sum_{i=1}^{k_2-1}[a_i^{(k_2)}]^2\Bigg)^{1/2} \times\Bigg(\sum_{l=k_2+q_2}^{\infty}|a_l^{(k_2)}|^2\Bigg)^{1/2}\Bigg].$$

Inequality (3.53) is a quadratic inequality in

$$\Bigg(\sum_{l=k_2+q_2}^{\infty}[a_l^{(k_2)}]^2\Bigg)^{1/2}$$

whose solution yields

$$(3.54)\qquad \sum_{l=k_2+q_2}^{\infty}[a_l^{(k_2)}]^2 \leqq C\varepsilon_{1,k_2-1}^2(h)\sum_{i=1}^{k_2-1}[a_i^{(k_2)}]^2 + C\varepsilon_{2,1}^2(h).$$

Combining (3.39) and (3.54) we get

$$\sum_{i=1}^{k_2-1}(a_i^{(k_2)})^2 \leqq D_1\varepsilon_{2,1}^2(h)+D_2C\varepsilon_{1,k_2-1}^2(h)\sum_{i=1}^{k_2-1}(a_i^{(k_2)})^2+D_2C\varepsilon_{2,1}^2(h),$$

and thus, since $\varepsilon_{1,k_2-1}(h)$ is small for h small,

$$(3.55)\qquad \sum_{i=1}^{k_2-1}(a_i^{(k_2)})^2 \leqq D_5\varepsilon_{2,1}^2(h).$$

Next, combining (3.40) and (3.55), we get

$$(3.56)\qquad \sum_{l=k_2+q_2}^{\infty}(a_l^{(k_2)})^2 \leqq D_6\varepsilon_{2,1}^2(h).$$

Finally, from (3.37), (3.55), and (3.56), we have

$$\Bigg\|u_{h,k_2}-\sum_{j=k_2}^{k_2+q_2-1}a_j^{(k_2)}u_j\Bigg\|_{B_0} = \Bigg[\sum_{j=1}^{k_2-1}(a_j^{(k_2)})^2+\sum_{j=k_2+q_2}^{\infty}(a_j^{(k_2)})^2\Bigg]^{1/2} \leqq C\varepsilon_{2,1}(h).$$

Redefining u_{k_2} to be

$$\sum_{j=k_2}^{k_2+q_2-1}a_j^{(k_2)}u_j \Bigg/ \Bigg\|\sum_{j=k_2}^{k_2+q_2-1}a_j^{(k_2)}u_j\Bigg\|_{B_0},$$

we see that $\|u_{k_2}\|_{B_0}=1$, so that (2.6) holds, and

$$(3.57)\qquad \|u_{h,k_2}-u_{k_2}\|_{B_0} \leqq C\varepsilon_{2,1}(h).$$

This is (3.3) for $i=2$, $j=1$.

Comment on estimate (3.57). In the proof of (3.57) we used (3.36), which was proved in Step A. A careful examination of the proof of (3.57) shows that we did not use the full strength of (3.36), but only the weaker fact that $\|u_{h,j}-u_j\|_{B_0}\to 0$ as $h\to 0$ for $j\leqq k_2-1$ (cf. the Overview of the Proof).

Step B.3. Suppose $q_2\geqq 2$. In Step B.1 we estimated $\lambda_{h,k_2}-\lambda_{k_2}$. In this step we estimate $\lambda_{h,k_2+1}-\lambda_{k_2+1}$.

We proceed by modifying problems (2.5) and (2.11) by restricting them to the space

$$H^{h,k_2}=\{u\in H\colon B_0(u,u_{h,k_2})=0\}$$

and

$$S_h^{h,k_2}=\{u\in S_h\colon B_0(u,u_{h,k_2})=0\},$$

respectively, i.e., we consider the problems $(2.5^{h,k_2})$ and $(2.11^{h,k_2})$ obtained when H and S_h are replaced by H^{h,k_2} and S_h^{h,k_2} in (2.5) and (2.11), respectively. Problem $(2.11^{h,k_2})$ has the same eigenpairs $(\lambda_{h,j},u_{h,j})$ as does (2.11) except that the pair $(\lambda_{h,k_2},u_{h,k_2})$ is eliminated. Problem $(2.5^{h,k_2})$ has eigenpairs $(\lambda_j^{h,k_2},u_j^{h,k_2})$ which in general depend on h. Nevertheless,

$$\lambda_{k_2+l}^{h,k_2}=\lambda_{k_2+l},\qquad l=0,\cdots,q_2-2, \tag{3.58}$$

i.e., λ_{k_2}, the eigenvalue under consideration, is an eigenvalue of multiplicity q_2-1 for problem $(2.5^{h,k_2})$. Its eigenspace is $M^{h,k_2}(\lambda_{k_2})=\{u\in M(\lambda_{k_2})\colon B_0(u,u_{h,k_2})=0\}$.

We can now apply the argument used in Step B.1 to problems $(2.5^{h,k_2})$ and $(2.11^{h,k_2})$ and, with the aid of (3.1′), we obtain (cf. (3.34))

$$\lambda_{h,k_2+1}-\lambda_{k_2+1}\leqq C\varepsilon_{2,2}^2(h)\quad\text{for } h<h_0. \tag{3.59}$$

Since u_{h,k_2} depends on h, the problems $(2.5^{h,k_2})$ and $(2.11^{h,k_2})$ depend on h. It follows from the Comment on Inequality (3.34) with $\tau=h$ that we can apply the argument in Step B.1 obtaining C and h_0 that are independent of h. To see this, note that $\lambda_{k_2}^{h,k_2}=\lambda_{k_2}$, by (3.58), $\lambda_1^{h,k_2}\geqq\lambda_1$, by the minimum principle, and

$$\xi^{h,k_2}=\lambda_{k_2-1}^{h,k_2}/\lambda_{k_2}^{h,k_2}\to\lambda_{k_2-1}/\lambda_{k_2}=\xi\quad\text{as } h\to 0,$$

by the minimum-maximum principle and the fact that $u_{h,k_2}\to u_{k_2}$ (cf. (3.57)), and hence $\lambda_{k_2}^{h,k_2}$ is bounded from above, λ_1^{h,k_2} is bounded from below, and ξ^{h,k_2} is bounded away from 0 and 1. Then observe that the eigenvectors in $M^{h,k_2}(\lambda_{k_2})$ can be approximated by S_h, uniformly in h. Note that in Step B.1 we also used the fact that $\lambda_{h,j}\to\lambda_j$ as $h\to 0$ for $j=k_2-1$ and k_2. It is easy to see that the corresponding fact is true in the present context.

Step B.4. Suppose $q_2\geqq 2$ as in Step B.3. Here we show that u_{k_2+1} can be chosen so that $\|u_{h,k_2+1}-u_{k_2+1}\|_{B_0}\leqq C\varepsilon_{2,2}(h)$. We know that

$$\|u_{h,j}-u_j\|_{B_0}\leqq\begin{cases}C\varepsilon_{1,j}(h), & j=1,\cdots,q_1,\\ C\varepsilon_{2,1}(h), & j=q_1+1=k_2\end{cases} \tag{3.60}$$

(cf. (3.10), (3.16), and (3.57)). Assume that $u_{k_2+1},\cdots,u_{k_2+q_2-1}$ have been redefined so that (2.6) holds. Write

$$u_{h,k_2+1}=\sum_{j=1}^{\infty}a_j^{(k_2+1)}u_j. \tag{3.61}$$

If we apply the argument used in Step B.2 to u_{h,k_2+1}, i.e., if we let k_2 be replaced by k_2+1 and use (3.59) instead of (3.34), we obtain

$$\left\|u_{h,k_2+1}-\sum_{j=k_2}^{k_2+q_2-1}a_j^{(k_2+1)}u_j\right\|_{B_0}\leqq C\varepsilon_{2,2}(h). \tag{3.62}$$

But, by (3.60),

$$\begin{aligned}|a_{k_2}^{(k_2+1)}| &= |B_0(u_{h,k_2+1}, u_{k_2})| \\ &= |B_0(u_{h,k_2+1}, u_{k_2} - u_{h,k_2})| \\ &\leqq \|u_{k_2} - u_{h,k_2}\|_{B_0} \\ &\leqq C\varepsilon_{2,1}(h) \\ &\leqq C\varepsilon_{2,2}(h)\end{aligned}$$

and hence

$$\left\| u_{h,k_2+1} - \sum_{j=k_2+1}^{k_2+q_2-1} a_j^{(k_2)} u_j \right\|_{B_0} \leqq C\varepsilon_{2,2}(h).$$

Redefining u_{k_2+1} to be

$$\sum_{j=k_2+1}^{k_2+q_2-1} a_j^{(k_2)} u_j \Big/ \left\| \sum_{j=k_2+1}^{k_2+q_2-1} a_j^{(k_2)} u_j \right\|_{B_0},$$

we see that $\|u_{k_2+1}\|_{B_0} = 1$, $B_0(u_{k_2+1}, u_j) = 0$, $j = 1, \cdots, k_2$, so that (2.6) holds, and

$$\|u_{h,k_2+1} - u_{k_2+1}\|_{B_0} \leqq C\varepsilon_{2,2}(h), \tag{3.63}$$

which is (3.3) for $i = j = 2$.

Step B.5. Continuing in this manner we prove (3.2) and (3.3) for $i = 2$ and $j = 1, \cdots, q_2$.

Step C. Repeating the argument in Step B we get (3.2) and (3.3) for $i = 3, 4, \cdots$. This completes the proof. □

Remark 3.1. It is possible to use an alternate argument in Step B.1 if we introduce the so-called Riesz formulas for the spectral projections associated with an operator. We suppose the space H and the bilinear forms B_0 and D have been complexified in the usual manner. Let P_{λ_2} and P_{h,λ_2} be the B_0-orthogonal projections of H onto $M(\lambda_{k_2})$ and $\sum_{i=0}^{q_2-1} M(\lambda_{h,k_2+i})$, the direct sum of the eigenspaces $M(\lambda_{h,k_2+i})$, $i = 0, \cdots, q_2 - 1$, respectively. Introduce next the operators T, $T_h : H \to H$ defined by

$$Tf \in H,$$
$$B_0(Tf, v) = D(f, v) \quad \forall v \in H$$

and

$$T_h f \in S_h,$$
$$B_0(T_h f, v) = D(f, v) \quad \forall v \in S_h.$$

It follows from (2.1)-(2.4) that T and T_h are defined and compact on H. Furthermore,

$$\|(T - T_h)f\|_{B_0} \leqq C \inf_{\chi \in S_h} \|Tf - \chi\|_{B_0}. \tag{3.64}$$

It is immediate that (λ, u) is an eigenpair of (2.5) if and only if $(\mu = \lambda^{-1}, u)$ is an eigenpair of T. Likewise (λ_h, u_h) is an eigenpair of (2.11) if and only if $(\mu_h = \lambda_h^{-1}, u_h)$ is an eigenpair of T_h. As a consequence of (2.14), $T_h \to T$ in the operator norm associated with $\|\cdot\|_{B_0}$. Let Γ be a circle in the complex plane centered at $\mu_{k_2} = \lambda_{k_2}^{-1}$, enclosing no other eigenvalues of T. Then for h sufficiently small, Γ will contain the eigenvalues $\mu_{h,k_2+i} = \lambda_{h,k_2+i}^{-1}$, $i = 0, \cdots, q_2 - 1$, of T_h. Also, P_{λ_2} and P_{h,λ_2}, the spectral projections associated with T and μ_{λ_2} and T_h and μ_{h,k_2+i}, $i=0,\cdots,q_2-1$, respectively, can be written as

$$P_{\lambda_2}=\frac{1}{2\pi i}\int_\Gamma (z-T)^{-1}\,dz, \tag{3.65}$$

$$P_{h,\lambda_2}=\frac{1}{2\pi i}\int_\Gamma (z-T_h)^{-1}\,dz. \tag{3.66}$$

These are the Riesz formulas. With these formulas we can derive an eigenvector error estimate which will lead to the eigenvalue estimate (3.34).

Let $u\in M(\lambda_2)$ with $\|u\|_D=1$. Then $v_h=P_{h,\lambda_2}u\in\sum_{i=0}^{q_2-1}M(\lambda_{h,k_2+i})$ and from the formulas (3.65) and (3.66) we obtain

$$\begin{aligned}\|u-v_h\|_{B_0}&=\|(P_{\lambda_{k_2}}-P_{h,\lambda_{k_2}})u\|_{B_0}\\&=\left\|\frac{1}{2\pi i}\int_\Gamma (z-T_h)^{-1}(T-T_h)(z-T)^{-1}u\,dz\right\|\\&=\left\|\frac{1}{2\pi i}\int_\Gamma (z-T_h)^{-1}(T-T_h)\frac{u}{z-\mu_{k_2}}\,dz\right\|\\&\leqq\frac{1}{2\pi}[2\pi\operatorname{rad}(\Gamma)]\sup_{z\in\Gamma}\|(z-T_h)^{-1}\|\times\frac{1}{\operatorname{rad}(\Gamma)}\|(T-T_h)u\|_{B_0}\\&=(-\mu_{k_2}+\operatorname{rad}(\Gamma)+\mu_{h,k_2+q_2-1})^{-1}\|(T-T_h)u\|_{B_0}\\&\leqq C\|(T-T_h)u\|_{B_0}.\end{aligned} \tag{3.67}$$

Formulas (3.64) and (3.67) yield

$$\begin{aligned}\|u-v_h\|_{B_0}&\leqq C\inf_{\chi\in S_h}\|Tu-\chi\|_{B_0}\\&=C\inf_{\chi\in S_h}\|\mu_{k_2}u-\chi\|_{B_0}\\&\leqq C\inf_{\chi\in S_h}\|u-\chi\|_{B_0}.\end{aligned} \tag{3.68}$$

This is an eigenvector estimate; it shows that starting from any $u\in M(\lambda_2)$ with $\|u\|_D=1$ we can construct a $v_h=v_h(u)\in\sum_{i=0}^{q_2-1}M(\lambda_{h,k_2+i})$ that is close to u. We now use (3.68) to prove (3.34).

By the minimum principle (2.8^h) we have

$$\lambda_{h,k_2}-\lambda_{k_2}=\inf_{\substack{v\in S_h\\ \|v\|_D=1\\ B_0(v,u_{h,i})=0,\\ i=1,\cdots,k_2-1}} B_0(v,v)-\lambda_{k_2}. \tag{3.69}$$

Since $v_h\in\sum_{i=1}^{q_2-1}M(\lambda_{h,k_2+i})$ we know that $B_0(v_h,u_{h,i})=0$, $i=1,\cdots,k_2-1$. Thus, from (3.69) we find

$$\lambda_{h,k_2}-\lambda_{k_2}\leqq B_0\left(\frac{v_h}{\|v_h\|_D},\frac{v_h}{\|v_h\|_D}\right)-\lambda_{k_2}.$$

Combining this with Lemma 2.3 and (3.68) we obtain

$$\begin{aligned}\lambda_{h,k_2}-\lambda_{k_2}&\leqq\left\|\frac{v_h}{\|v_h\|_D}-\frac{u}{\|u\|_D}\right\|_{B_0}^2-\lambda_{k_2}\left\|\frac{v_h}{\|v_h\|_D}-\frac{u}{\|u\|_D}\right\|_D\\&\leqq C\|v_h-u\|_{B_0}^2\\&\leqq C\inf_{\chi\in S_h}\|u-\chi\|_{B_0}^2.\end{aligned}$$

Since this is valid for any $u \in M(\lambda_{k_2})$ with $\|u\|_D = 1$, we see that

$$\lambda_{h,k_2} - \lambda_{k_2} \leqq C \inf_{\substack{u \in M(\lambda_{k_2}) \\ \|u\|_{B_0}=1}} \inf_{\chi \in S_h} \|u - \chi\|_{B_0}^2$$
$$= C\varepsilon_{2,1}^2(h),$$

which is (3.34). We note that the proof given here rests on (2.24) in Lemma 2.3 and employs formulas (3.65) and (3.66) to construct $v_h = v_h(u)$ that is B_0-orthogonal to $u_{h,i}$, $i = 0, \cdots, k_2 - 1$, and that satisfies

$$\|u - v_h\|_{B_0} \leqq C\varepsilon_{2,1}(h).$$

We have already seen that the eigenvalue estimates in Steps A.1, A.3, $\cdots$, can be based on Lemma 2.3. Proceeding as we have here, we see that all of the eigenvalue estimates (3.2) can be based on Lemma 2.3.

4. Numerical computations. In the previous sections we have analyzed the errors in the Galerkin approximation of an eigenvalue problem, concentrating especially on the case of multiple eigenvalues. In this section we consider a finite element-Galerkin method for the approximation of a model, one-dimensional problem with multiple eigenvalues, presenting numerical results and their analysis in terms of the results of § 3.

Consider the eigenvalue problem

$$(4.1)\qquad \begin{aligned} &-\left(\frac{1}{\varphi'(x)} u'(x)\right)' = \lambda \varphi'(x) u, \qquad x \in I = (-\pi, \pi), \\ &u(-\pi) = u(\pi), \\ &\left(\frac{1}{\varphi'} u'\right)(-\pi) = \left(\frac{1}{\varphi'} u'\right)(\pi), \end{aligned}$$

where

$$\varphi(x) = \pi^{-\alpha} |x|^{1+\alpha} \operatorname{sgn} x, \qquad 0 < \alpha < 1.$$

It is easy to check that the eigenvalues and eigenfunctions are as shown in Table 4.1. We see that $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$, etc.

We cast this problem into the variational form (2.5) by choosing

$$H = \left\{ u(x) \colon \|u\| = \left(\int_{-\pi}^{\pi} \frac{(u')^2}{\varphi'} \, dx \right)^{1/2} < \infty, \ u(-\pi) = u(\pi), \ \int_{-\pi}^{\pi} \varphi' u \, dx = 0 \right\},$$

$$B_0(u, v) = \int_{-\pi}^{\pi} u' v' \frac{1}{\varphi'} \, dx,$$

TABLE 4.1
Eigenvalues and eigenfunctions of the eigenvalue problem (4.1).

i	λ_i	u_i
0	0.0	1
1	1.0	$\cos \varphi(x)$
2	1.0	$\sin \varphi(x)$
3	4.0	$\cos 2\varphi(x)$
4	4.0	$\sin 2\varphi(x)$
$\vdots$	$\vdots$	$\vdots$

and

$$D(u, v) = \int_{-\pi}^{\pi} uv\varphi' \, dx.$$

With these choices (2.5) is equivalent to (4.1), with the understanding that the eigenpair $(\lambda_0, u_0) = (0, 1)$ of (4.1) is not present in (2.5). Note that $\|u\| = \|u\|_{B_0}$. Assumptions (2.1)-(2.4) are clearly satisfied.

Our approximation is defined by (2.11) with

$$S_h = \{u \in H: u \text{ linear on } (-\pi + jh, -\pi + (j+1)h), j = 0, 1, \cdots, n-1\},$$

where $h = 2\pi/n$ and n is an even integer.

With this choice for $\{S_h\}$ it is easily seen that

$$\inf_{\chi \in S_h} \|\cos \varphi(x) - \chi\|^2_{B_0} \simeq Ch^2, \tag{4.2}$$

$$\inf_{\chi \in S_h} \|\sin \varphi(x) - \chi\|^2_{B_0} \simeq Ch^{1+\alpha}. \tag{4.3}$$

Hence from Theorem 3.1 we would expect $\lambda_{h,1}$ and $\lambda_{h,2}$, the two approximate eigenvalues that converge to the double eigenvalue $\lambda_1 = \lambda_2$, to have different convergence rates.

From Tables 4.1 and 4.2 we can find the errors in $\lambda_{h,i}$, $i = 1, 2, 3, 4$, for $\alpha = .4$. These errors are plotted in Fig. 4.1 in log-log scale. We clearly see the different rates of convergence, specifically seeing the rates h^2 and $h^{1+\alpha} = h^{1.4}$ for the errors in $\lambda_{h,i}$ for $i = 1, 3$ and $i = 2, 4$, respectively, as suggested by (4.2) and (4.3). It should be noted that the estimates presented in Theorem 3.1 are of an asymptotic nature in that they provide information only for small h (or large n), i.e., for h (or n) in the asymptotic range. From Fig. 4.1 we see that for $\alpha = .4$ we are in the asymptotic range quite quickly, say for $n \geqq 16$.

We computed $u_{h,1}$ and $u_{h,2}$, the approximate eigenfunctions corresponding to $\lambda_{h,1}$ and $\lambda_{h,2}$, respectively, normalized by $\|\cdot\|_D = 1$. The results of § 3 suggest that $u_{h,1}$ should be close to $C \cos \varphi(x)$ and $u_{h,2}$ close to $C \sin \varphi(x)$ (cf. (4.2) and (4.3)), where C is such that $C \sin \varphi(x)$ and $C \cos \varphi(x)$ are normalized by $\|\cdot\|_D = 1$, i.e., $C = \pi^{-1/2}$. To illustrate this point we computed $C_1^{(i)}$ and $C_2^{(i)}$, $i = 1, 2, 3, 4$, so that

$$K(i) = \begin{cases} \|u_{i,h} - C_1^{(i)} \cos \varphi(x) - C_2^{(i)} \sin \varphi(x)\|_{B_0}, & i = 1, 2, \\ \|u_{i,h} - C_1^{(i)} \cos 2\varphi(x) - C_2^{(i)} \sin 2\varphi(x)\|_{B_0}, & i = 3, 4 \end{cases} \tag{4.4}$$

is minimal. We would expect that

$$C_1^{(2)}, C_1^{(4)}, C_2^{(1)}, C_2^{(3)} \simeq 0, \tag{4.5}$$

and

$$C_1^{(1)} = C_2^{(2)} = C_1^{(3)} = C_2^{(4)} \simeq C = .564189583 \cdots. \tag{4.6}$$

Table 4.2 shows some of the results for $\alpha = .4$. We see clearly the results predicted in (4.5) and (4.6). The increase in $C_1^{(2)}$, $C_1^{(4)}$, $C_2^{(1)}$ and $C_2^{(3)}$ with increasing n is due to the eigenvalue solver we used. Table 4.2 also shows that $K(1) < K(2)$ and $K(3) < K(4)$, as we would expect.

The last columns in Table 4.2 and Fig. 4.1 show that the ratios

$$\frac{\lambda_{h,i+1} - \lambda_{i+1}}{\lambda_{h,i} - \lambda_i}, \qquad i = 1, 3$$

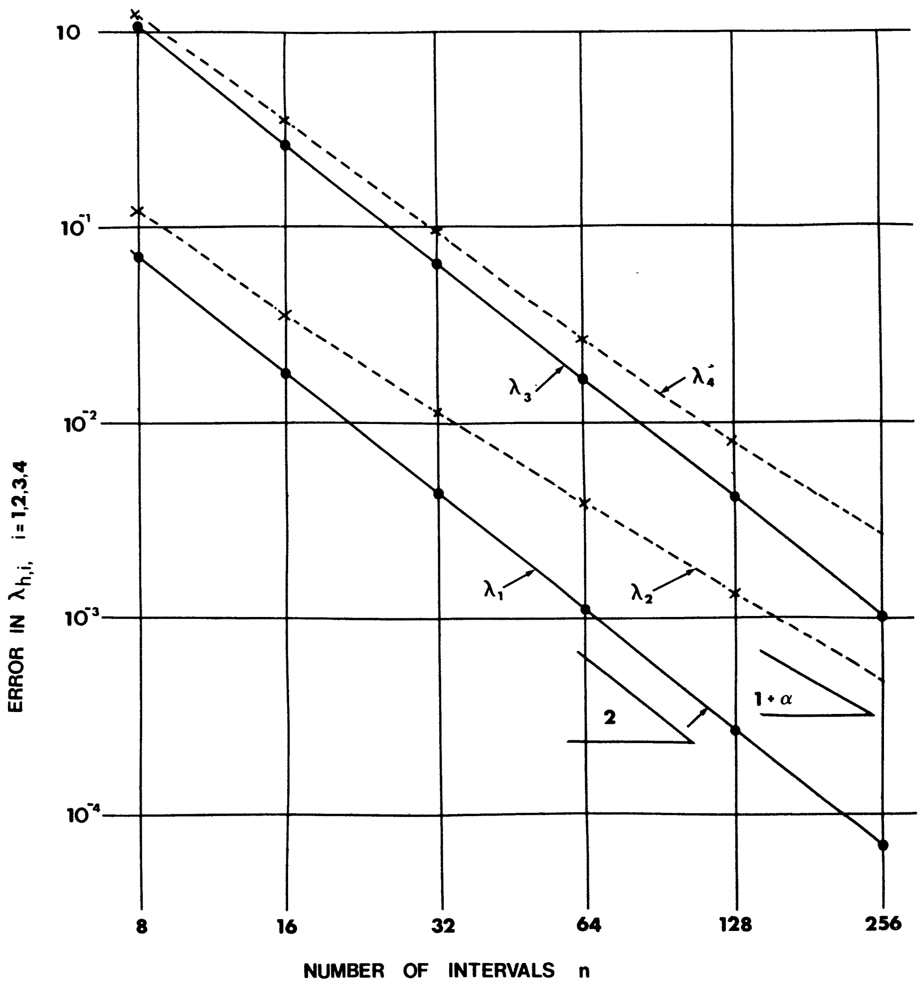

FIG. 4.1. *The error in the eigenvalues* $\lambda_{h,1}$, $\lambda_{h,2}$ *and* $\lambda_{h,3}$, $\lambda_{h,4}$ *for* $\alpha = .4$ *in dependence on the number of intervals n.*

increase as $h \to 0$. This shows that in the whole h-range we considered, the approximate eigenvalues converging to a multiple eigenvalue are well separated.

We next consider the case when $\alpha = .01$. Table 4.3 presents the same results for $\alpha = .01$ as Table 4.2 does for $\alpha = .4$. Figure 4.2 shows the graph of

$$\log \frac{\lambda_{h,i+1} - \lambda_{i+1}}{\lambda_{h,i} - \lambda_i}, \qquad i = 1, 3$$

as a function of the number of intervals n in a semi-logarithmic scale. The computed values are indicated by ●'s and ×'s. The graphs are formed by interpolation and extrapolation. We note three related phenomena that did not occur with $\alpha = .4$. For small n the approximate eigenfunction associated with $\lambda_{h,1}$ is $u_{h,1} \simeq \pi^{-1/2} \sin \varphi(x)$, in contrast to $u_{h,1} \simeq \pi^{-1/2} \cos \varphi(x)$ when $\alpha = .4$. We remark that $\pi^{-1/2} \cos \varphi(x)$ is more easily approximated by S_h than is $\pi^{-1/2} \sin \varphi(x)$ for all $0 < \alpha < 1$. This anomaly is present for $n \leqq 64$ but for $n \geqq 128$ we get results which are in agreement with the (asymptotic) results in § 3. For $\lambda_{h,3}$ and $\lambda_{h,4}$ we have to take $n \geqq 256$ to get results which agree with the asymptotic theory.

TABLE 4.2
Numerical solution of the eigenvalue problem (4.1) *for* $\alpha = .4$.

n	i	$\lambda_{h,i}$	$K(i)$	$C_1^{(i)}$	$C_2^{(i)}$	$\dfrac{\lambda_{h,i+1}-\lambda_{i+1}}{\lambda_{h,i}-\lambda_i}$
8	1	1.0716754	.2704 0	.5637791 0	−.1124891 −16	1.5562955
	2	1.1115481	.3423 0	−.4151973 −13	.5636998 0	
	3	5.0394692	.1075 +1	.5558919 0	.1317809 −12	1.1943249
	4	5.2414639	.1191 +1	.5022638 −13	.5516234 0	
16	1	1.0175850	.1329 0	.5641633 0	.1596754 −12	2.0041570
	2	1.0352431	.1881 0	−.8916589 −12	.5641519 0	
	3	4.2691915	.5259 0	.5636643 0	.1124328 −13	1.2575063
	4	4.3385100	.5869 0	−.2689727 −12	.5637697 0	
32	1	1.0043740	.6618 −1	.5641879 0	.6411454 −11	2.6003887
	2	1.0113741	.1067 0	.1323421 −10	.5641830 0	
	3	4.0666055	.2589 0	.5641561 0	.1970954 −10	1.4067517
	4	4.0936974	.3067 0	−.7375504 −10	.5641613 0	
64	1	1.0010921	.3305 −1	.5641895 0	.7729760 −9	3.5190001
	2	1.0038431	.6202 −1	.8670648 −9	.5641883 0	
	3	4.0166006	.1289 0	.5641875 0	.3641341 −10	1.6437659
	4	4.0272875	.1653 0	.1415775 −8	.5641858 0	
128	1	1.0002729	.1651 −1	.5641895 0	.4535626 −7	4.9215830
	2	1.0013431	.3665 −1	.3251219 −7	.5641893 0	
	3	4.0041468	.6440 −1	.5641895 0	.4409247 −7	2.0107071
	4	4.0083380	.9135 −1	−.9705611 −8	.5641890 0	
256	1	1.0000682	.8255 −2	.5641896 0	.8070959 −5	7.0542522
	2	1.0004811	.2193 −1	.7269570 −6	.5641895 0	
	3	4.0010365	.3217 −1	.5641896 0	.6435344 −6	2.5706705
	4	4.0026645	.5162 −1	−.2601000 −6	.5641895 0	

For $\alpha=.01$ we see that $K(2)<K(1)$ for small $n (n\leqq 64)$ and $K(2)>K(1)$ for large n and $K(4)<K(3)$ for small $n(n\leqq 128)$ and $K(4)>K(3)$ for large n. Recall that $K(2)>K(1)$ and $K(4)>K(3)$ for all n when $\alpha=.4$.

Finally we note that when $\alpha=.01$ the ratio

$$\frac{\lambda_{h,i+1}-\lambda_{i+1}}{\lambda_{h,i}-\lambda_i}, \qquad i=1,3$$

first decreases as n increases, then for some n the two eigenvalue errors become equal, and then the ratio increases again. This is in contrast to the case for $\alpha=.4$, in which the ratio increased over the whole range of n values. We further note that the value $\bar{n}$ for which the eigenvalue errors are equal—$\bar{n}\simeq 70$ for $i=1$ and $\bar{n}\simeq 160$ for $i=2$ (see Fig. 4.2)—marks a transition in each of these situations from $u_{h,1}\simeq\pi^{-1/2}\sin\varphi(x)$ to $u_{h,1}\simeq\pi^{-1/2}\cos\varphi(x)$ and $u_{h,3}\simeq\pi^{-1/2}\sin 2\varphi(x)$ to $u_{h,3}\simeq\pi^{-1/2}\cos 2\varphi(x)$, from $K(2)<K(1)$ and $K(4)<K(3)$ to $K(2)>K(1)$ and $K(4)>K(3)$, and from $(\lambda_{h,i+1}-\lambda_{i+1})/(\lambda_{h,i}-\lambda_i)$, $i=1,3$, decreasing to increasing.

We have thus seen that for $\alpha=.4$ the numerical results are in concert with the (asymptotic) results in § 3 for the whole range of n considered, while for $\alpha=.01$ they are in disagreement for small n, but are in agreement for large n. We now make an

TABLE 4.3
Numerical solution of the eigenvalue problem (4.1) *for* $\alpha = .01$.

n	i	$\lambda_{h,i}$	$K(i)$	$C_1^{(i)}$	$C_2^{(i)}$	$\dfrac{\lambda_{h,i+1}-\lambda_{i+1}}{\lambda_{h,i}-\lambda_i}$
8	1	1.0520268	.2338 0	.8181940 −11	.5634386 0	1.0171143
	2	1.0529172	.2268 0	.5645965 0	−.2916448 −11	
	3	4.8576239	.9593 0	−.9346720 −13	.5597529 0	1.0164293
	4	4.8717141	.9615 0	.5604533 0	.1167277 −11	
16	1	1.0128661	.1223 0	.8717399 −10	.5635957 0	1.0111689
	2	1.0130098	.1052 0	.5647369 0	−.8480131 −9	
	3	4.2088367	.4650 0	.2507177 −10	.5636658 0	1.0087030
	4	4.2106542	.4577 0	.5642694 0	−.3101833 −10	
32	1	1.0032139	.7274 −1	−.9345818 −9	.5636031 0	1.0068764
	2	1.0032360	.3568 −1	.5647430 0	.1273043 −7	
	3	4.0515675	.2384 0	.3745461 −9	.5638178 0	1.0057284
	4	4.0518629	.2205 0	.5644172 0	−.4115544 −9	
64	1	1.0008063	.5369 −1	−.1311961 −5	.5636032 0	1.0017363
	2	1.0008077	.3398 −1	.5647430 0	.2462939 −7	
	3	4.0128623	.1343 0	.2743681 −7	.5638240 0	1.0035997
	4	4.0129086	.9792 −1	.5644235 0	.3196172 −8	
128	1	1.0002018	.4196 −1	.5647430 0	.3356056 −5	1.0064420
	2	1.0002031	.4775 −1	.7414162 −6	.5636032 0	
	3	4.0032196	.9166 −1	.2379072 −6	.5638239 0	1.0010560
	4	4.0032230	.9745 −2	.5644235 0	.1197135 −5	
256	1	1.0000504	.4372 −1	.5647429 0	.1061527 −4	1.0218254
	2	1.0000515	.4614 −1	−.1553659 −4	.5636031 0	
	3	4.0008054	.5011 −1	.5644234 0	−.2123278 −4	1.0031040
	4	4.0008079	.7741 −1	.11655012 −5	.5638238 0	

observation that further illuminates these two phases of error behavior—the pre-asymptotic and the asymptotic. Toward this end we note that if (λ_1, u_1), with $\|u_1\|_D=1$, and $(\lambda_{h,1}, u_{h,1})$, with $\|u_{h,1}\|_D=1$, are first eigenpairs of (2.5) and (2.11), respectively, then

$$
\begin{aligned}
0 \leqq \lambda_{h,1}-\lambda_1 &= \|u_{h,1}-u_1\|_{B_0}^2-\lambda_1\|u_{h,1}-u_1\|_D^2 \\
&= \inf_{\substack{\chi\in S_h\\ \|\chi\|_D=1}} [\|\chi-u_1\|_{B_0}^2-\lambda_1\|\chi-u_1\|_D^2].
\end{aligned} \tag{4.7}
$$

If λ_1 is a multiple eigenvalue, then the u_1 in (4.7) can be any corresponding eigenvector with $\|u_1\|_D=1$. (Note that we are here assuming u_1 and $u_{h,1}$ have $\|\cdot\|_D$-length equal to 1, whereas in (2.6) and (2.12) they are assumed to have $\|\cdot\|_{B_0}$-length equal to 1.) The first inequality in (4.7) follows from the minimum principle (2.8^h) and has already been stated in (2.13). The first equality in (4.7) follows immediately from Lemma 2.3 with $(\lambda, u)=(\lambda_1, u_1)$, $w=u_{h,1}$, and $\tilde{\lambda}=B_0(u_{h,1}, u_{h,1})=\lambda_{h,1}$. If $\chi\in S_h$ with $\|\chi\|_D=1$, then from the minimum principle (2.8^h),

$$\lambda_{h,1}-\lambda_1 \leqq B_0(\chi,\chi)-\lambda_1. \tag{4.8}$$

Again from Lemma 2.3, this time with $(\lambda, u)=(\lambda_1, u_1)$, $w=\chi$ and $\tilde{\lambda}=B_0(\chi,\chi)$, we have

$$B_0(\chi,\chi)-\lambda_1=\|\chi-u_1\|_{B_0}^2-\lambda_1\|\chi-u_1\|_D^2. \tag{4.9}$$

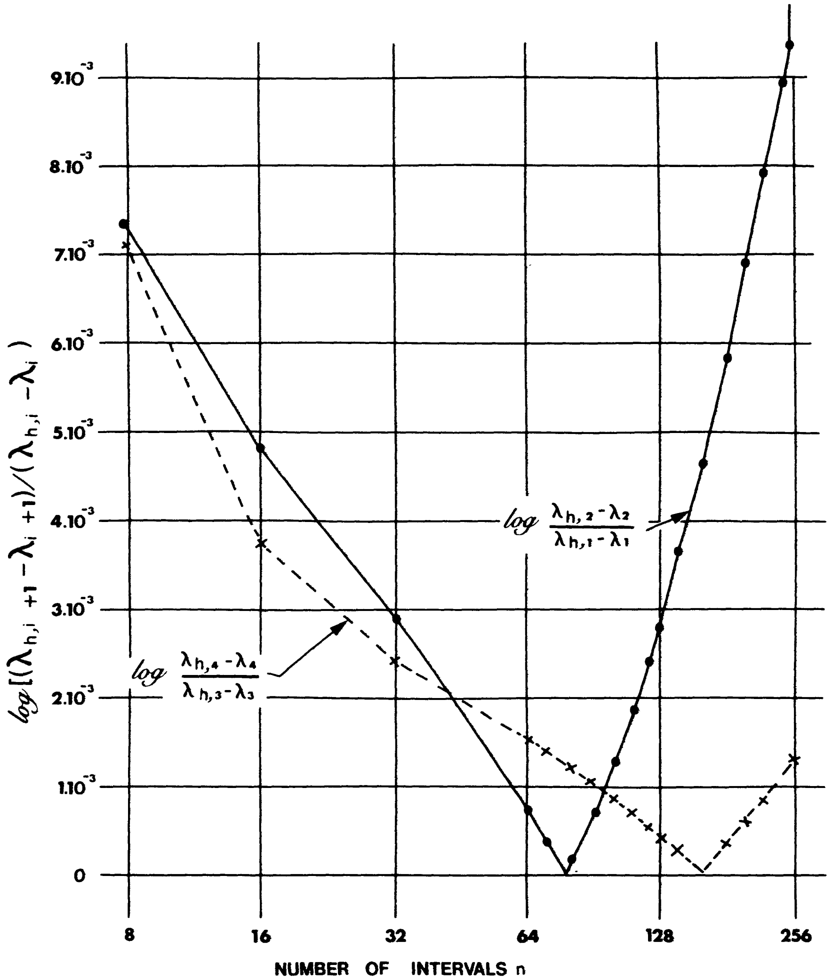

FIG. 4.2. *The graphs of* $\log(\lambda_{h,2}-\lambda_2)/(\lambda_{h,1}-\lambda_2)$ *and log* $(\lambda_{h,4}-\lambda_4)/(\lambda_{h,3}-\lambda_3)$ *for* $\alpha=.01$ *in dependence on the number of intervals n.*

The second equality in (4.7) follows from (4.8) and (4.9). It is clear from the above discussion that u_1 can be any eigenvector corresponding to λ_1.

From (4.7) we have

$$\lambda_{h,1}-\lambda_1 \leqq \|\chi-u_1\|^2_{B_0}-\lambda_1\|\chi-u_1\|^2_D \quad \forall\chi\in S_h \quad \text{with } \|\chi\|_D=1. \tag{4.10}$$

If χ is $\|\cdot\|_{B_0}$-close to u_1, to be more precise, if χ is taken to be the B_0-projection of u_1 onto S_h, then the second term on the right side of (4.10) is negligible with respect to the first term. This follows from the compactness assumption made on $\|\cdot\|_D$ in § 2. On the other hand, if $\|u_1-\chi\|_{B_0}$ is not small, $\lambda_{h,1}-\lambda_1$ may still be small because of cancellation between the two terms on the right side of (4.10). Regarding the case $\alpha=.01$, this explains why for h large (the pre-asymptotic phase), we can have $u_{h,1}\simeq \pi^{-1/2}\sin\varphi(x)$ and $K(1)>K(2)$, and yet have $\lambda_{h,1}$, the approximate eigenvalue associated with $u_{h,1}$, closer to λ_1 than is $\lambda_{h,2}$, the approximate eigenvalue associated with $u_{h,2}\simeq\pi^{-1/2}\cos\varphi(x)$, while for h small (the asymptotic phase), we have $u_{h,1}\simeq \pi^{-1/2}\cos\varphi(x)$, $K(1)<K(2)$, and $\lambda_{h,1}$ closer to λ_1 than is $\lambda_{h,2}$, showing that the eigenvalue error, $\lambda_{h,i}-\lambda_i$, is governed by $\inf_{\chi\in S_h}\|\chi-u_1\|^2_{B_0}$.

TABLE 4.4
Numerical solution of the eigenvalue problem (4.1) *for* $n=4$ *and for various* α.

α	i	$\lambda_{h,i}$	$K(i)$	$C_1^{(i)}$	$C_2^{(i)}$	$\dfrac{\lambda_{h,2}-\lambda_2}{\lambda_{h,1}-\lambda_1}$
.1	1	1.2030785	.4689 0	−.2438763 −13	.5594106 0	1.2293000
	2	1.2496444	.5161 0	.5594467 0	.000	
.2	1	1.2238488	.5017 0	−.8324194 −14	.5562571 0	1.1969235
	2	1.2679299	.5366 0	.5585072 0	.000	
.25	1	1.2474367	.5322 0	−.8324194 −14	.5562571 0	1.0967540
	2	1.2713772	.5414 0	.5585072 0	.000	
.275	1	1.2628455	.5506 0	−.1979358 −12	.5526908 0	1.0342787
	2	1.2718555	.5426 0	.5577904 0	.000	
.28750	1	1.2715137	.5605 0	−.7895498 −11	.5520158 0	1.0011045
	2	1.2718136	.5428 0	.55768520 0	.5301589 −11	
.28779	1	1.2717254	.5607 0	.2170355 −10	.5519997 0	1.0003132
	2	1.2718105	.5429 0	.5576828 0	−.2773092 −10	
.28794	1	1.2718089	.5429 0	.5576816 0	.1753561 −10	1.0000809
	2	1.2718309	.5608 0	.1125467 −10	.5519917 0	
.28809	1	1.2718072	.5429 0	.5576804 0	.1631249 −11	1.0004794
	2	1.2719375	.5610 0	−.2645923 −11	.5519836 −8	
.28868	1	1.2718005	.5429 0	.5576757 0	.2249782 −16	1.0020684
	2	1.2723627	.5614 0	.3464439 −12	.5519513 0	
.29	1	1.2717839	.5429 0	.5576649 0	.2249782 −16	1.0056780
	2	1.2733271	.5625 0	.8906887 −13	.5518783 0	
.29735	1	1.2717263	.5429 0	.5576289 0	.000 0	1.0160882
	2	1.2760979	.5661 0	−.2000056 −13	.5515258 0	
.30	1	1.2715965	.5430 0	.5575862 0	.000 0	1.0340803
	2	1.2808526	.5790 0	−.1073461 −13	.5513203 0	
.40	1	1.2646804	.5382 0	.5570759 0	.2249782 −16	1.4473652
	2	1.3830892	.6704 0	.1185635 −13	.5451205 0	
.60	1	1.2364746	.5093 0	.5576799 0	.000 0	3.4841353
	2	1.8239095	.9695 0	.2249822 −16	.5304168 0	

This situation is very similar to the situation with n fixed and α varying, as can be seen from Table 4.4 where computations for the case $n=4$ are shown. We see that the characteristics observed in Table 4.3 regarding dependence on n are present in Table 4.4 regarding dependence on α. These characteristics are the abrupt switch in the values of $C_1^{(i)}$, $C_2^{(i)}$, the abrupt switch from $K(2)<K(1)$ and $K(4)<K(3)$ to $K(2)>K(1)$ and $K(4)>K(3)$, and the abrupt switch from decreasing to increasing ratio of errors near the parameter value corresponding to $\lambda_{h,1}=\lambda_{h,2}$. We mention this situation—α varying and n fixed—since it is easier to understand in terms of perturbation theory (cf. Kato [5]) than is our original situation—n varying and α fixed.

Note added in proof. After completing this paper, the authors realized that part of their analysis could be simplified by using the techniques developed in § 6.6 of [3].

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