Question 5:

a) Base Step $P(1)$, $n=1$

$$(1)^3+2(1)=3$$

3 is divisible by 3 so $P(1)$ base step is true.

Inductive Step $P(n)$

$n^3+2n=3p$ $\quad$ p = positive integer

$P(n)$ is true, $P(n+1)$ is true.

$$(n+1)^3+2(n+1)=$$

$$= n^3+3n^2+3n+1+2n+2)$$

$$=(n^3+2n)+3(n^2+n+1)$$

$$= 3p+3(n^2+n+1)$$

$$= 3(p+n^2+n+1)$$

$(n+1)^3+2(n+1)=3(p+n^2+n+1)$ is divisible by 3 and $p+n^2+n+1$ is an integer

therefore $P(n+1)$ is also true and by inductive $P(n)$ is true for all n positive integer

5. (b) Base Case $n=2$

$2 = 2$ is product of prime

Induction Step:

We will show that for $k \geq 2$

if all integer smaller or equal to k and greater than 2 can be written as product of prime.

the $k+1$ can be written a product of primes

Case (a) if $k+1$ is a prime,

then we are done.

Case (b) if $k+1$ is not prime.

then $k+1 = a \cdot b$

where $2 \leq a, b \leq k$

both a, b are products of primes

then $k+1$ is product of primes

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Question 6:

a) $n = 3$

Left $= 1^2 + 2^2 + 3^2 = 14$

Right $= \dfrac{3(3+1)(3\times 2+1)}{6} = 14$

so Left = Right.

b) $P(k)$

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

c) $P(k+1)$

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

d) Base case:

$n = 1$

We must prove

$$(1)^2 = \frac{1(1+1)(2+1)}{6}$$

e) Inductive Step:

We will show for any integer $k \geq 1$

if $\displaystyle\sum_{j=1}^{k} j^2 = \frac{k(k+1)\cdot(2k+1)}{6}$ $\Rightarrow$

then
$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

g) Base case:

$n=1$

Left $=1$

Right $= \frac{1(1+1)(2+1)}{6} = 1$

left = Right

Inductive Step:

Suppose $k \geq 1$

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

then
$$\sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k+1)^2 =$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 =$$

$$= \frac{(k+1)(k(2k+1)+6(k+1))}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6} \Rightarrow$$

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Therefore

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

6 (b) Base case: $n=1$

Left $=1$

Right $= 2-1=1$

So left $\leq$ Right

Inductive Case: We will prove that for $n \geq 1$

if $\displaystyle\sum_{j=1}^{k} \frac{1}{j^2} \leq 2-\frac{1}{k}$ then

$$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2-\frac{1}{k+1}$$

$$\sum_{j=1}^{k+1} \frac{1}{j^2} = \sum_{j=1}^{k} \frac{1}{2} + \frac{1}{(k+1)^2} \leq 2-\frac{1}{k^2}+\frac{1}{(k+1)^2}$$

$$\leq 2-\frac{1}{k}+\frac{1}{k(k+1)}$$

$$= 2-\frac{1}{k}+\frac{1}{k}-\frac{1}{k+1}$$

$$= 2-\frac{1}{k+1}$$

So $\displaystyle\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2-\frac{1}{k+1}$

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