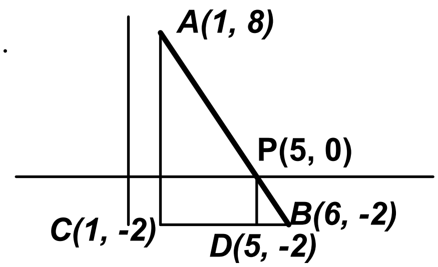
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006 SOLUTION KEY

Round 3

A) Method 1:



Since the equation of $\overleftrightarrow{AB}$ is $y = -2x + 10$, the x-intercept is at (5, 0). From the diagram, it is clear that $\overline{PD} \parallel \overline{AC}$ and the required ratio is the same as $BD : CD =$ **<u>1 : 4</u>**.

Method 2:
Alternately, after finding the x-intercept P, using the distance formula, you could compute the distance between P and B ($\sqrt{5}$) and the distance between A and P ($\sqrt{80} = 4\sqrt{5}$) → **<u>1 : 4</u>**.

Method 3 (**<u>without</u>** finding the equation or x-intercept of $\overleftrightarrow{AB}$):
Let X denote the x-intercept of the <u>vertical</u> line $\overleftrightarrow{AC}$. Clearly, the coordinates of X are (1, 0) and $CX : AX = 1 : 4$ → $BP : AP =$ **<u>1 : 4</u>** (since $\Delta BPD \sim \Delta BAC$).

B) The slope of $\overline{AB}$ is $\dfrac{-2006}{4250} = \dfrac{-2(17)(59)}{2(17)(125)} = \dfrac{-59}{125}$

Points with integer coordinates (i.e. lattice points), may be determined by starting at A and increasing the x-coordinate by 125 and decreasing the y-coordinate by 59 or alternately, starting at B and decreasing the x-coordinate by 125 and increasing the y-coordinate by 59.

Both strategies produce: (0, 2006) **(125, 1947)**, (250, 1888) … **(4125, 59)**, (4250, 0)
125 + 1947 + 4125 + 59 = **<u>6256</u>**.

In fact, suppose the slope of $\overline{AB}$ were $\dfrac{-a}{b}$, where a and b are positive integers.
Then $C(b, 2006 - a)$ and $D(4250 - b, a)$ → $p + q + r + s = 4250 + 2006 = 6256$ and it wasn't even necessary to find the slope of $\overline{AB}$. In the worst case scenario, if the slope fraction could not be reduced, point C would coincide with point B and D would coincide with point A.

C) Method 1:
Point S is the intersection of the perpendicular bisectors of the sides of ΔPQR.
The perpendicular bisector of $\overline{PQ}$ is the vertical line $\underline{x = 1}$.
The perpendicular bisector of $\overline{PR}$ is $x + 3y = 58$.
→ $3y = 57$ → $\underline{y = 19}$.

Method 2:
Shifting each vertex of ΔPQR left 1 unit. $P'(-13, 0)$, $Q'(13, 0)$ and $R'(1, 42)$
Clearly, point $S'(0, y)$, a point on the perpendicular bisector of $\overline{P'Q'}$, is equidistant from P' and Q'. To insure that it is equidistant from all three vertices, we require $(S'Q')^2 = (S'R')^2$ → $13^2 + y^2 = 1^2 + (42 - y)^2$
→ $169 + y^2 = 1 + 1764 - 84y + y^2$ → $84y = 1596$ → $y = 19$
→ $S'(0, 19)$ → $S(1, 19)$ → $x = \underline{1}$, $y = \underline{19}$.