MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2017 SOLUTION KEY

Round 4

A) With integer coefficients, the roots must be conjugates of each other, i.e. the roots are $\frac{5 \pm i\sqrt{3}}{2}$ and the sum of the roots is 5 and the product of the roots is $\frac{25 - i^2 \cdot 3}{4} = \frac{25+3}{4} = 7$
Therefore, the equation is $x^2 - 5x + 7 = 0$ and $(A, B, C) = \underline{\mathbf{(1, -5, 7)}}$.

B) $x + 2 = y^2 - 7 \Leftrightarrow y^2 = x + 9$
$x = 7$ is the smallest positive value of *x* for which *y* is also an integer.
The first few (*J*, *K*) are: (<u>7</u>, 4), (16, 5), (27, 6), (40, 7), …
Note that, as the *y*-values increase by 1, the <u>gap</u> between the *x*-values increases by 2.
Therefore, subsequent pairs are: (40+15, 7+1) = (55,8), (55+17,8+1) = (<u>72</u>, 9), (72+19, 9+1) = (91, 10), …
We stop here, since $J + K > 100$.
The required sum is 7 + 72 = <u>**79**</u>.

C) $r_1 = \sqrt{\frac{40}{9} + \frac{41}{4}} = \sqrt{\frac{4(40) + 9(41)}{4 \cdot 9}} = \sqrt{\frac{160 + 369}{36}} = \sqrt{\frac{529}{36}} = \sqrt{\frac{23^2}{6^2}} = \frac{23}{6}$
Thus, one factor is $6x - 23$. Let's assume the other factor is $x - r_2$.
$(6x - 23)(x - r_2) = 6x^2 - (23 + 6r_2)x + 23r_2$ and $B = -23 - 6r_2 = r_2 - 2 \Rightarrow r_2 = -3$
$\Rightarrow (A, B, C) = \underline{\mathbf{(6, -5, -69)}}$