MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016 SOLUTION KEY

Team Round - continued

C) $\tan^2 x+\cot^2 x=14 \Leftrightarrow \dfrac{\sin^2 x}{\cos^2 x}+\dfrac{\cos^2 x}{\sin^2 x}=14 \Leftrightarrow \sin^4 x+\cos^4 x=14\sin^2 x\cos^2 x$

$\Leftrightarrow \sin^4 x+2\sin^2 x\cos^2 x+\cos^4 x=16\sin^2 x\cos^2 x \Leftrightarrow \left(\sin^2 x+\cos^2 x\right)^2=16\sin^2 x\cos^2 x$

$\Leftrightarrow 16\sin^2 x\cos^2 x=1$. Dividing through by 4 and applying the double angle identity, $\sin 2\theta=2\sin\theta\cos\theta$, $4\sin^2 x\cos^2 x=\left(2\sin x\cos x\right)^2=\left(\sin 2x\right)^2=\dfrac{1}{4} \Leftrightarrow \sin 2x=\pm\dfrac{1}{2}$.

Thinking the 30°-family of related angles in all 4 quadrants , i.e. $\dfrac{\pi}{6}$ radians, we have

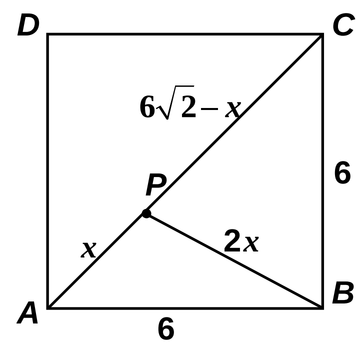
$$2x=\begin{cases}\dfrac{\pi}{6}+n\pi \ (\text{quadrants } 1,3)\\ \dfrac{5\pi}{6}+n\pi \ (\text{quadrants } 2,4)\end{cases} \Rightarrow x=\begin{cases}\dfrac{\pi}{12}+\dfrac{n\pi}{2}\\ \dfrac{5\pi}{12}+\dfrac{n\pi}{2}\end{cases} \Rightarrow x=\dfrac{\pi}{12}\begin{cases}1+6n\\ 5+6n\end{cases}.$$

For $n=0,1$, we have $x=\underline{\boldsymbol{\dfrac{\pi}{12},\dfrac{5\pi}{12},\dfrac{7\pi}{12},\dfrac{11\pi}{12}}}$.

Alternately, $\tan^2 x+\cot^2 x=14 \Leftrightarrow \tan^2 x+\cot^2 x+2=16$

$$\Leftrightarrow \left(\tan x+\frac{1}{\tan x}\right)^2=\frac{\left(\tan^2 x+1\right)^2}{\tan^2 x}=\frac{\sec^4 x}{\tan^2 x}=\frac{1}{\cos^4 x}\cdot\frac{\cos^2 x}{\sin^2 x}=\frac{1}{\left(\sin x\cdot\cos x\right)^2}=16$$

$\Leftrightarrow \dfrac{4}{\left(2\sin x\cos x\right)^2}=\dfrac{4}{\sin^2 2x}=16$. Thus, $\sin^2 2x=\dfrac{1}{4}$ and the same result follows.



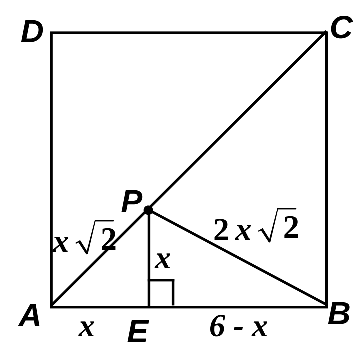
D) Using the dimensions in the diagram at the right, apply Stewart's Theorem. $6^2\cdot x+6^2\cdot\left(6\sqrt{2}-x\right)=\left(2x\right)^2\cdot 6\sqrt{2}+x\cdot\left(6\sqrt{2}-x\right)\cdot 6\sqrt{2}$

$\Rightarrow 36x+216\sqrt{2}-36x=24\sqrt{2}x^2+72x-6\sqrt{2}x^2$

$\Rightarrow 18\sqrt{2}x^2+72x-216\sqrt{2}=0$

$\Rightarrow \sqrt{2}x^2+4x-12\sqrt{2}=0$. Applying the quadratic formula,

$$x=AP=\frac{-4+\sqrt{16+96}}{2\sqrt{2}}=\frac{-4+4\sqrt{7}}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\left(-1+\sqrt{7}\right)\sqrt{2} \Rightarrow N=\underline{7}.$$



Alternately, without resorting to Stewart's Theorem, drop a perpendicular from P to $\overline{AB}$. Let $AE=x$ and mark the other sides as indicated. In ΔBEP, $x^2+\left(6-x\right)^2=\left(2x\sqrt{2}\right)^2$

$\Rightarrow 2x^2-12x+36=8x^2 \Rightarrow x^2+2x-6=0 \Rightarrow x=-1+\sqrt{7}$

$\Rightarrow AP=\left(-1+\sqrt{7}\right)\sqrt{2} \Rightarrow N=\underline{7}$.