MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014 SOLUTION KEY

Team Round – continued

D) If R lies on $y = x$, then $4^{2P-Q} = 2^{4P-2Q} = \log_2(P+2Q)$. Taking $\log_2$ of both sides, $4P-2Q = \log_2\left(\log_2\left(P+2Q\right)\right)$. Each expression represents the same positive integer.
$P + 2Q$ must be a power of 2 to insure that $\log_2\left(P+2Q\right)$ is an integer.
$\log_2\left(P+2Q\right)$ must be a power of 2 to insure that $\log_2\left(\log_2\left(P+2Q\right)\right)$ is an integer.
Thus, $P + 2Q$ could be: $2^4 = 16$, $2^8 = 256$, $2^{16} = 65536$, …

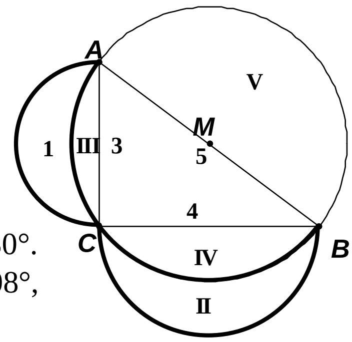
$\begin{cases} P+2Q = 16 \quad 256 \quad 65536 \\ 4P-2Q = \underline{\ 2\ } \quad \underline{\ 3\ } \quad \underline{\quad 4\ } \end{cases} \cdots$ Adding, $5P = \cancel{18}, \cancel{259}, 65540 \Rightarrow (P, Q) = (13108, 26214) \Rightarrow$ **<u>39322</u>**.

E) Note that the Pythagorean Theorem is true for semi-circles (as well as squares), i.e. the area of the semi-circle on $\overline{AB}$ equals the sum of the areas of the semi-circles on $\overline{AC}$ and on $\overline{BC}$.

$\frac{1}{2}\pi\left(\frac{5}{2}\right)^2 = \frac{1}{2}\pi\left(\frac{3}{2}\right)^2 + \frac{1}{2}\pi\left(\frac{4}{2}\right)^2 \Leftrightarrow \frac{\pi}{8}\left(5^2 = 3^2 + 4^2\right)$

Thus, $\left(I + \cancel{III}\right) + \left(II + \cancel{IV}\right) = V = \cancel{III} + \cancel{IV} + \Delta$

$\Rightarrow I + II = \Delta$ and the required ratio is **<u>1 : 1</u>**.



F) As the number of sides of a regular polygon increases, the measure of the interior angles increases with a maximum measure less than 180°. The rate of increase is decelerating. (3,4,5,6, … sides ⇒ 60°, 90°, 108°, 120°, …, differences of 30, 18, 12, … .)

Can we avoid an algebraic blizzard, solving $\begin{cases} \dfrac{180(m-2)}{m} = x° \\ \dfrac{180(n-2)}{n} = (x+2)° \end{cases}$ for ordered pairs (m,n)? Yes!

For a regular polygon with k sides and interior angles of j°, we have $\dfrac{180(k-2)}{k} = j$ or $j = 180 - \dfrac{360}{k}$.
k must be a factor of 360. Examining the factors of 360 in <u>decreasing</u> order produces the largest possible values of j. $360 = 2^3 \cdot 3^2 \cdot 5^1 \Rightarrow (3+1)(2+1)(1+1) = 24 \Rightarrow$ 12 <u>pairs</u> of factors.
Think of pairing the largest factor with the smallest factor, the next largest with the next smallest, etc., namely, $(360,1), (180,2), ..., (20,18) \Rightarrow$

$(k, j) = (360,179),(180,178),(120,177),(90,176),(72,175),(60,174),(45,172),$
$(40,171),(36,170),(30,168),(24,165),(20,162),(18,160),(15,156),(12,150),...$

Search for j-values which differ by 2 and save the corresponding k-values. By inspection, the possible ordered pairs (m,n) are

$(120,360),(90,180),(72,120),(60,90),(45,60),(36,45),(30,36),(18,20)$ which produces differences of **<u>240, 90, 48, 30, 15, 9, 6</u> and <u>2</u>**.