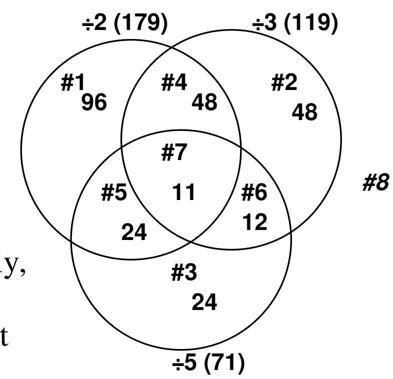
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2009 SOLUTION KEY

Team Round

A) $y = 1 + \dfrac{1}{a^t}$, but $x = 1 - a^t \rightarrow a^t = 1 - x$

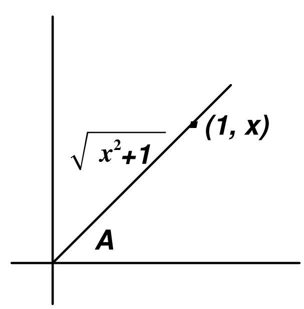
Thus, $y = 1 + \dfrac{1}{1-x} = \dfrac{1-x+1}{1-x} = \dfrac{2-x}{1-x} \rightarrow \dfrac{2-\frac{2}{3}}{1-\frac{2}{3}} = \dfrac{6-2}{3-2} = \underline{\mathbf{4}}$

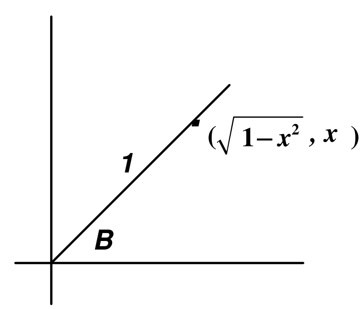
B) Since $360 = 2^3 \cdot 3^2 \cdot 5^1$, the fraction can be reduced whenever A is a multiple of 2, 3 or 5.
Since multiples of 2, 3 and 5 overlap, counting each integer from 1 through 359 inclusive exactly once is made easier with the use of a Venn Diagram.
The upper left circle contains multiples of 2, the upper right multiples of 3 and the lower circle multiples of 5.
Section #7 contains integers divisible by 2, 3 <u>and</u> 5 (i.e. 30)
Sections (#4, #7), (#5, #7) and (#6, #7) contains integers divisible by 6, 10 and 15 respectively.
Section #4 contains integers divisible by 6, but <u>not</u> 5.
Section #5 contains integers divisible by 10, but <u>not</u> 3.
Section #6 contains integers divisible by 15, but <u>not</u> 2.
Sections #1, #2 and #3 respectively contain integers divisible by 2 only, 3 only and 5 only.



Totaling the number of integers in sections 1 – 7, we have <u>**263**</u> distinct integer values of A for which $\dfrac{A}{360}$ can be reduced, since numbers in each of these regions has at least one factor of 2, 3 or 5. (In region #8, outside all three circles, there are $359 - 263 = 96$ numbers, so $\dfrac{A}{360}$ is already simplified for 96 values of A.)

C) Let $A = \text{Tan}^{-1}(x)$ and $B = \text{Sin}^{-1}(x)$
$x > 0 \rightarrow$ both A and B are in quadrant 1.
Taking the cosine of both sides,
$\cos(A + B) = \cos(\pi/2)$
$\cos A \cos B - \sin A \sin B = 0$





$$\left(\frac{1}{\sqrt{x^2+1}}\right)\left(\sqrt{1-x^2}\right) - \left(\frac{x}{\sqrt{x^2+1}}\right)x = 0$$

$$\rightarrow \frac{\sqrt{1-x^2} - x^2}{\sqrt{x^2+1}} = 0$$

This is only possible if the numerator is equal to zero.

$\sqrt{1-x^2} - x^2 = 0 \rightarrow 1 - x^2 = x^4 \rightarrow x^4 + x^2 - 1 = 0$

Applying the quadratic formula and rejecting the negative result, we have $x^2 = \underline{\dfrac{\sqrt{5}-1}{2}}$