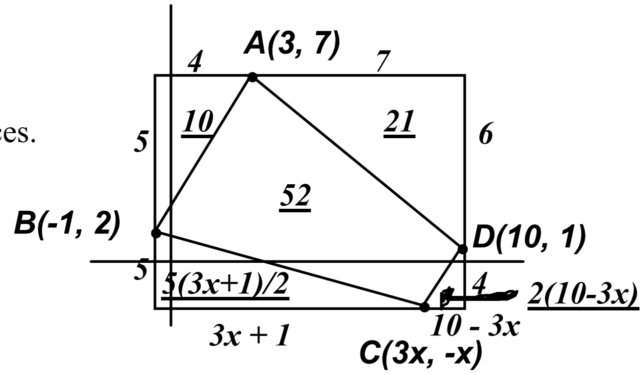
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2010 SOLUTION KEY

Round 1 - continued

Alternative #2 (Utilizing only basic area formulas)
Study the diagram at the right. Draw a pair of verticals and a pair of horizontals through opposite pairs of vertices.



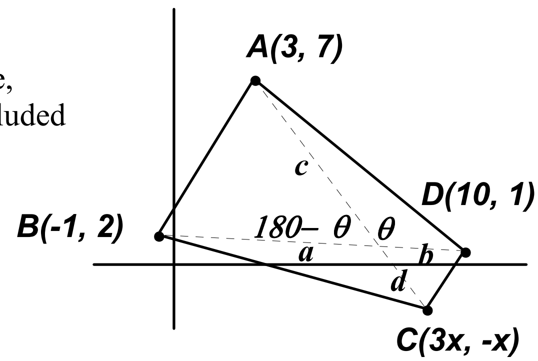
$$10+21+\frac{5(3x+1)}{2}+2(10-3x)+52=10(11)$$

➔ $\dfrac{5(3x+1)}{2}+2(10-3x)=110-83=27$

➔ $15x+5+40-12x=54$ ➔ $3x=9$ ➔ $x=3$ ➔ **(9, –3)**.

Alternate #3

Here is a (more involved) solution which utilizes the triangle area formula for any quadrilateral, convex or concave, namely half the product of the diagonals times the sine of the included angle. We start with an area formula for the triangle $\boxed{\dfrac{1}{2}ab\sin\theta}$, where θ denotes the included angle.



Since $\sin(180-\theta)=\sin\theta$, we have Area($ABCD$) =

$$\frac{1}{2}\sin\theta\left(ac+ad+bd+bc\right)=\frac{1}{2}\sin\theta\left(a(c+d)+b(c+d)\right)=$$

$$\frac{1}{2}\sin\theta\left((a+b)(c+d)\right)=\boldsymbol{\frac{1}{2}\sin\theta\cdot AC\cdot BD}$$. Let's show that (9, –3) produces an area of 52.

Using the distance formula, $AC=\sqrt{(-6)^2+10^2}=\sqrt{136}=2\sqrt{34}$ and $BD=\sqrt{11^2+(-1)^2}=\sqrt{122}$

The slope of $\overleftrightarrow{AC}=-\dfrac{5}{3}$ and the slope of $\overleftrightarrow{BD}=-\dfrac{1}{11}$.

We need the angle between the two lines. If θ denotes the acute angle, then $\tan\theta=\left|\dfrac{m_2-m_1}{1+m_1m_2}\right|$.

In this case, $\tan\theta=\dfrac{-\frac{1}{11}+\frac{5}{3}}{1+\frac{1}{11}\cdot\frac{5}{3}}=\dfrac{-3+55}{33+5}=\dfrac{52}{38}=\dfrac{26}{19}$ ➔ $\sin\theta=\dfrac{26}{\sqrt{1037}}$ and the area is

$\dfrac{1}{2}\cdot\dfrac{26}{\sqrt{1037}}\cdot2\sqrt{34}\cdot\sqrt{122}$ But does this equal 52 ?????

$\dfrac{1}{2}\cdot\dfrac{26}{\sqrt{1037}}\cdot2\sqrt{34}\cdot\sqrt{122}=\dfrac{1}{2}\cdot\dfrac{26}{\sqrt{61}\cdot\sqrt{17}}\cdot2\left(\sqrt{2}\cdot\sqrt{17}\right)\cdot\left(\sqrt{2}\cdot\sqrt{61}\right)=13(4)=$ **52**.

Amazing (albeit tedious) solution. The basic area formulas above are worth remembering.