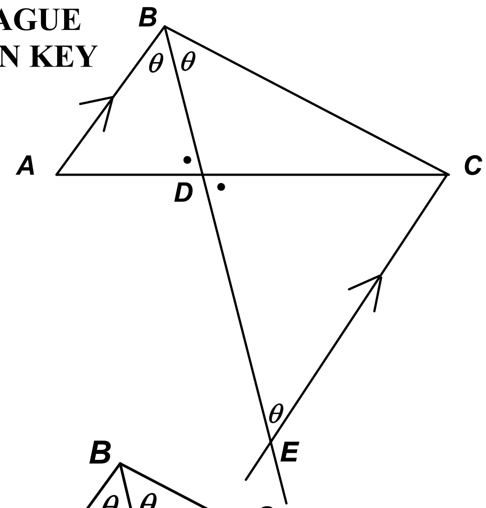
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010 SOLUTION KEY

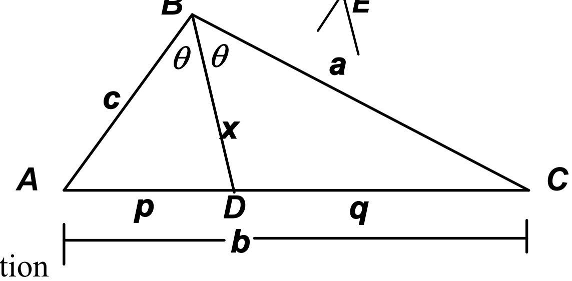


Proof of Angle Bisector Theorem #1:

Draw a line through C parallel to $\overline{AB}$, intersecting $\overrightarrow{BD}$ in E.

Note: As alternate interior angles of parallels, $\angle ABD \cong \angle CED$
ΔBCE is isosceles, with $BC = CE$.
$\Delta ABD \sim \Delta CED$. As corresponding sides of similar triangles,

$\dfrac{AD}{CD} = \dfrac{AB}{CE}$. Substituting BC for CE gives us the required result.



Proof of Angle Bisector Theorem #2:

In the diagram at the right, let $BD = x$, $AD = p$ and $CD = q$.
Using Angle Bisector theorem #1, note that

$\dfrac{p}{q} = \dfrac{p}{b-p} = \dfrac{a}{c} \rightarrow p = \dfrac{bc}{a+c}$. Similarly, show that $q = \dfrac{ab}{a+c}$.

Now a double application of the Law of Cosines, some substitution and rather impressive simplification!

ΔBAD: $x^2 = c^2 + p^2 - 2pc\cos A$ (***) ΔABC: $a^2 = b^2 + c^2 - 2bc\cos A$ or $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

Now substituting for p and cos$A$ in (***), $x^2 = c^2 + \left(\dfrac{bc}{a+c}\right)^2 - \dfrac{2bc^2}{a+c}\cdot\dfrac{b^2+c^2-a^2}{2bc}$

$$= c^2 + \frac{b^2c^2}{(a+c)^2} + \frac{a^2-b^2-c^2}{a+c} = \frac{c^2(a+c)^2 + b^2c^2 + (a+c)(a^2-b^2-c^2)}{(a+c)^2}$$

$$= \frac{a^2c^2 + 2ac^3 + c^4 + b^2c^2 + a^3c - ac^3 - ab^2c + a^2c^2 - c^4 - b^2c^2}{(a+c)^2} = \frac{2a^2c^2 + ac^3 + a^3c - ab^2c}{(a+c)^2}$$

$$= \frac{ac(2ac + c^2 + a^2 - b^2)}{(a+c)^2} = \frac{ac((a+c)^2 - b^2)}{(a+c)^2} = ac - \frac{ab}{a+c}\cdot\frac{bc}{a+c} = ac - pq$$

or $BD^2 = (AB)(BC) - (AD)(DC)$, as required. A truly remarkable result - worth remembering for future contests!

For those familiar with Stewart's Theorem, ($c^2q + a^2p = x^2b + bpq$ in terms of the diagram above), the algebraic manipulations are greatly simplified. Since $p = \dfrac{bc}{a+c}$ and $q = \dfrac{ab}{a+c}$ from Angle Bisector Theorem #1, we have

$$\frac{abc^2}{a+c} + \frac{a^2bc}{a+c} = x^2b + bpq \rightarrow \frac{ac^2}{a+c} + \frac{a^2c}{a+c} = \frac{ac\cancel{(c+a)}}{\cancel{a+c}} = x^2 + pq \rightarrow x^2 = ac - pq.$$