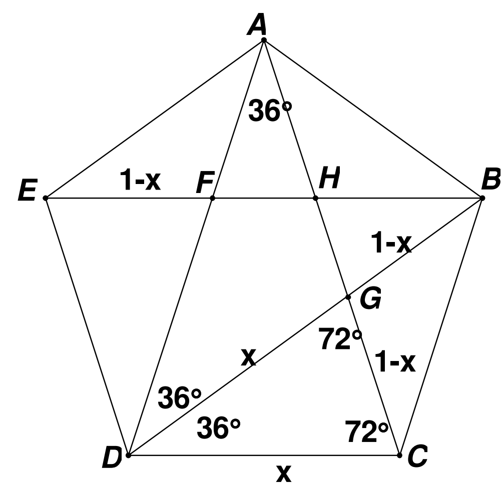
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009 SOLUTION KEY

Team Round – continued



E) (*) $\dfrac{x}{1}=\dfrac{1-x}{x}$ → (**) $x^2=1-x$ → $x^2+x-1=0$

Applying the quadratic formula, $x=\mathbf{\dfrac{-1+\sqrt{5}}{2}}$ and

$x^2 = 1 - x = CG = EF = 1-\left(\dfrac{-1+\sqrt{5}}{2}\right)=\dfrac{3-\sqrt{5}}{2}$

$FH = EB - (EF + HB) = 1 - 2(1 - x) = 2x - 1$

$\Delta AFE \sim \Delta FHG$ → $\dfrac{AE}{AF}=\dfrac{FG}{FH}$ → $\dfrac{x}{1-x}=\dfrac{FG}{2x-1}$

(*) → $\dfrac{1}{x}=\dfrac{FG}{2x-1}$

or $FG = \dfrac{2x-1}{x}=2-\dfrac{1}{x}=2-\left(\dfrac{\sqrt{5}+1}{2}\right)=\dfrac{3-\sqrt{5}}{2}=x^2$

Thus, the perimeter of the trapezoid $CGFE$ is $1 + 2(1 - x) + x^2$

(**) → $1 + 2(x^2) + x^2 = 3x^2 + 1 = 3\left(\dfrac{3-\sqrt{5}}{2}\right)+1=\dfrac{9-3\sqrt{5}+2}{2} = \underline{\mathbf{\dfrac{11-3\sqrt{5}}{2}}}$

Oops, I missed a much simpler way of showing that $FG = x^2$.

Note that $\Delta DFG \sim \Delta DAB$ → $\dfrac{DF}{DA}=\dfrac{FG}{AB}$ → $\dfrac{x}{1}=\dfrac{FG}{x}$ → $FG = x^2$.

[There are many occurrences of the constant ϕ (referred to as the golden ratio) in the regular pentagon. The value of ϕ is $\dfrac{1+\sqrt{5}}{2}\approx 1.61803\cdots$. The value of x above is $\dfrac{1}{\phi}$.
You can read about this amazing constant in many outstanding books on mathematical topics, e.g. chapter 15 of The Loom of God – Mathematical Tapestries at the Edge of Time by Clifford A. Pickover (a writer for both Discover and OMNI magazines).]