MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2016 SOLUTION KEY

Round 2

A) The prime digits are 2, 3, 5 and 7.
Clearly, if the units digit is 2 or 5, the 3-digit integer will not be prime.
For a units digit of 3, we have 213, 423, 633, and 843 and all of these are divisible by 3.
For a units digit of 7, we have 217, 427, 637 and 847 and all of these are divisible by 7.
Thus, **<u>none</u>** of these 3-digit integer will be prime. Also accept 0 or zero.

B) Over $100 \le n \le 999$, there are 900 3-digit integers. There are $8 \cdot 9 \cdot 9 = 648$ numbers with no 9s.
Thus, $900 - 648 = \underline{\mathbf{252}}$ with at least one nine.

Solution #2: There is only 1 number with exactly 3 - 9s. In both of the following cases, we must account for the fact that 0 cannot be the leftmost digit.

For exactly two 9s, we have $\binom{8}{1}\binom{2}{2} + \binom{9}{1}\binom{2}{1} = 8 + 18 = 26$

For exactly one 9, we have $\binom{8}{1}\binom{9}{1}\binom{2}{1} + \binom{9}{1}\binom{9}{1} = 144 + 81 = 225$

Thus, the total number is **<u>252</u>**.

Solution #3:

Case 1 - There is one 9:
in the first position: 1 x 9 x 9 = 81
in the second position: 8 x 1 x 9 = 72
in the third position: 8 x 9 x 1 = 72

Case 2 - There are two 9s:
when the non-9 is in the first position: 8
when the non-9 is in the second position: 9
when the non-9 is in the first position: 9

Case 3 – There are three 9s: 1

Total: 225 + 26 + 1 = **<u>252</u>**

C) $\dfrac{(n!)(n+1)!}{2016^3} = \dfrac{(n!)^2}{2016^2} \cdot \dfrac{n+1}{2016}$. Since the first factor is a perfect square, $\dfrac{n+1}{2016} = \dfrac{n+1}{2^5 3^2 7}$ must be a perfect square $\Rightarrow \dfrac{n+1}{2 \cdot 7}$ must be a perfect square and $n = \underline{\mathbf{13}}$ is the smallest integer which does this. However, it still remains to be shown that $\dfrac{(n!)(n+1)!}{2016^2}$ will be an integer for n = 13.

Factoring the denominator, we have $\dfrac{n!(n+1)!}{2016^3} = \dfrac{(n!)^2(n+1)}{2^{15}3^6 7^3}$. For $n = 13$,

$\dfrac{(n!)^2}{2016^2} \cdot \dfrac{n+1}{2016} = \dfrac{(13!)^2(14)}{2^{15}3^6 7^3} = \dfrac{\frac{(13!)^2}{7^2}}{2^{14}3^6}$. Clearly, the numerator is an integer, but we must show that this integer is divisible by 14 factors of 2 and 6 factors of 3.

$\dfrac{13!}{7} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 = 2^{1+2+1+3+1+2} \cdot 3^{1+1+2+1} \cdot 5^{1+1} \cdot 11 \cdot 13 = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 11 \cdot 13$

$\left(\dfrac{13!}{7}\right)^2$ contains 20 factors of 2 and 10 factors of 3. Thus, $\dfrac{(n!)(n+1)!}{2016^2}$ is an integer, as required.