MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2013 SOLUTION KEY

Round 1

A) $f\left(\frac{8}{3}\right)=\sqrt{3\left(\frac{8}{3}\right)-4}=\sqrt{4}=2$.

Let $a=f^{-1}\left(\sqrt{5}\right)$. Then: $f(a)=\sqrt{5} \Leftrightarrow \sqrt{3a-4}=\sqrt{5} \Leftrightarrow 3a-4=5 \Leftrightarrow a=3$.

Thus, $f^{-1}\left(\sqrt{5}\right)+f\left(\frac{8}{3}\right)=3+2=\underline{\mathbf{5}}$.

B) $\begin{cases} f(x)=3x-2 \\ g(x)=(x-2)(x+3)+A, \text{ where } A<0 \end{cases} \Rightarrow g\circ f(x)=(3x-4)(3x+1)+A=9x^2-9x+(A-4)$

To find the zeros, we use the quadratic formula to solve $9x^2-9x+(A-4)=0$:

$$x=\frac{9\pm\sqrt{81-36(A-4)}}{18}=\frac{9\pm\sqrt{225-36A}}{18}$$

Examining the discriminant, we want the largest possible negative integer for which $225-36A$ is a perfect square. For $A=-1,\ldots-5$, we get non-perfect squares 261, 297, 333, 369, and 405; but $A=-6$ gives us $441=21^2$

Therefore, $x=\frac{9\pm 21}{18}\Rightarrow -\frac{12}{18},-\frac{30}{18}\Rightarrow -\frac{2}{3},\frac{5}{3}$

Since $r_1<r_2$, we have $\left(A,r_1,r_2\right)=\underline{\left(\mathbf{-6,-\frac{2}{3},\frac{5}{3}}\right)}$

The question alluded to several other values for which the composite function had rational zeros. Some other values of A are: –14, –24, –36, –50, –66, …
Do you see a pattern? Investigate. Can you prove a conjecture?

C) $\begin{cases} f(x)=x^4+ax^3+bx^2+cx+d \\ f(1)=f(2)=f(3)=f(4)=6 \end{cases} \Rightarrow$

We do <u>not</u> have to determine the actual zeros, since the coefficient a is the opposite of the sum of the zeros.

Subtracting, $\begin{cases} 175+37a+7b+c=0 \\ 65+19a+5b+c=0 \\ 15+7a+3b+c=0 \end{cases} \Rightarrow \begin{cases} 110+18a+2b=0 \\ 50+12a+2b=0 \end{cases} \Rightarrow a=-10$

Thus, the sum of the roots is $\underline{\mathbf{10}}$.
<u>FYI</u>:
Backtracking with $a=-60$ in the remaining equations,
$f(x)=x^4-10x^3+35x^2-50x+30$.

Alternative: Let $f(x)=(x-1)(x-2)(x-3)(x-4)+6$ and the given requirements are satisfied and the sum of the zeros is the opposite of the coefficient of x^3, i.e. $\underline{\mathbf{+10}}$.