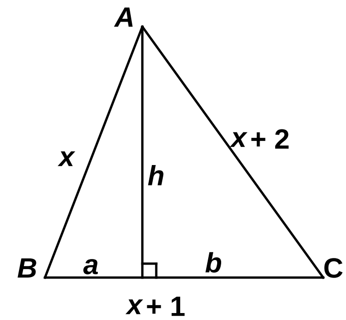
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2014 SOLUTION KEY



E) Assume the triangle we seek has sides as indicated at the right.

Solution #1: (using Heron's Formula)

$s=\dfrac{3x+3}{2}\quad A=\sqrt{\dfrac{3(x+1)}{2}\cdot\dfrac{x+3}{2}\cdot\dfrac{x+1}{2}\cdot\dfrac{x-1}{2}}$

$\Rightarrow\ 4A=(x+1)\sqrt{3(x+3)(x-1)}$

By trial and error, $x=3\ \Rightarrow\ 4A=4\sqrt{3\cdot 6\cdot 2}=24\Rightarrow A=6$ (This is the 3-4-5 triangle.)

$x=13\ \Rightarrow\ 4A=14\sqrt{3\cdot 16\cdot 12}=14\cdot 24\Rightarrow A=84$ (This is the 13-14-15 triangle above.)

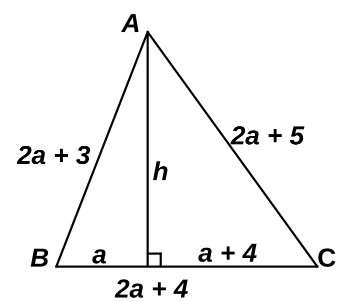
$x=51\ \Rightarrow\ 4A=52\sqrt{3\cdot 54\cdot 50}=52\cdot\sqrt{81\cdot 100}=52\cdot 90\Rightarrow A=1170$

We leave it to you to check that integer values of *x* between 13 and 51 do not yield any solutions.

Thus, the next non-right triangle is a 51-52-53 triangle and $(P,A)=$ **(156,1170)**

Solution #2 (Pythagorean Theorem, Quadratic Formula and Recognition of Perfect squares)

$h^2=x^2-a^2=(x+2)^2-b^2=(x+2)^2-(x+1-a)^2$



Expanding,

$x^2-a^2=x^2+4x+4-\left(x^2+1+a^2+2x-2ax-2a\right)$

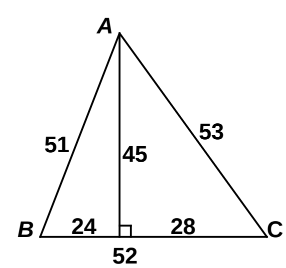
$x^2-a^2=(2+2a)x-\left(a^2-2a-3\right)$

$x^2-(2+2a)x-(2a+3)=0$ Applying the quadratic formula, we have

$$x=\frac{2(a+1)\pm\sqrt{4(a+1)^2+4(2a+3)}}{2}=(a+1)\pm\sqrt{a^2+4a+4}$$

$=(a+1)\pm(a+2)=2a+3$ or $\cancel{-1}$

Substituting for *x*, the diagram has dimensions strictly in terms of *a*.
Equating the expressions for *h*,



$h=\sqrt{(2a+3)^2-a^2}=\sqrt{3a^2+12a+9}$

We can look at the radicand as $3(a+3)(a+1)$ or $3\left((a+2)^2-1\right)$

Either way it must be perfect square.

Taking the second view, $(a+2)^2-1$ must be three times a perfect square, so

$(a+2)^2-1=3k^2$ or $a=-2+\sqrt{3k^2+1}$

Choosing *k* so that $3k^2+1$ is a perfect square (greater than 4), we have solutions.

$k=4\Rightarrow a=-2+7=5$ and we have the given solution.

$k=5...15\Rightarrow$ 76, 104, 148, 193, 244, 301, 364, 433, 508, 589, 676 = 26^2

Here are the relevant perfect squares (11^2 thru 25^2) for comparison:

121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625

Thus, $5\le k\le 14$ fail.