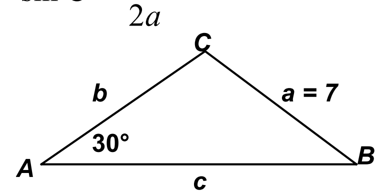
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2007 SOLUTION KEY

Team Round - continued

C) Since the given information represents two sides and a **non**-included angle, ΔABC might not exist or there could be exactly 1 or 2 non-congruent triangles that satisfy the stated conditions.

Let $(BC, AB) = (a, c)$. Then using the law of Sine, $\frac{\sin 30^\circ}{a} = \frac{\sin C}{c} \rightarrow \sin C = \frac{c}{2a}$



Case 1: $\frac{c}{2a} > 1 \rightarrow$ no solution ($\sin C$ can not exceed 1)

Case 2: $\frac{c}{2a} = 1 \rightarrow \Delta ABC$ is a right triangle (in fact 30-60-90)

Case 3: $\frac{1}{2} < \frac{c}{2a} < 1 \rightarrow a < c < 2a \rightarrow$ 2 solutions 1 acute and 1 obtuse

(Ex: $C = 45^\circ$ and 135° or any pair of supplementary angles θ and $180 - \theta$, where $\theta > 30^\circ$)

Case 4: $\frac{c}{2a} = \frac{1}{2} \rightarrow c = a \rightarrow m\angle A = m\angle C = 30^\circ \rightarrow m\angle B = 120^\circ \rightarrow \Delta ABC$ is obtuse

Case 5: $0 < \frac{c}{2a} < \frac{1}{2} \rightarrow c < a$, $m\angle C < 30^\circ$ or $m\angle C > 150^\circ$. The latter is impossible, but if $m\angle C < 30^\circ$, then $m\angle B > 120^\circ$ and ΔABC is obtuse.

Thus, all the acute triangles arise from case 3. We have $7 < c < 14 \rightarrow 8 \le c \le 13$.
The number of integer values of c is $13 - 8 + 1 =$ **<u>6</u>**

Note, in general, the solution is $(2a - 1) - (a + 1) + 1 = a - 1$

D) Let building #1 have $(N + 3)$ floors w/ ceilings H feet high and building #2 have N floors with ceilings $(H - 0.5)$ feet high.

$12 + (N + 3)H = 2(12 + N(H - 0.5)) \rightarrow 12 + NH + 3H = 24 + 2NH - N$

$3H - NH = 12 - N \rightarrow H = \frac{12 - N}{3 - N}$

$N = 1 \rightarrow H = 11/2$ (rejected – ceilings not at least 8 feet high)

$N = 2 \rightarrow H = 10/1$

Thus, building #1 is $12 + 5(10) =$ **<u>62</u>** feet tall.