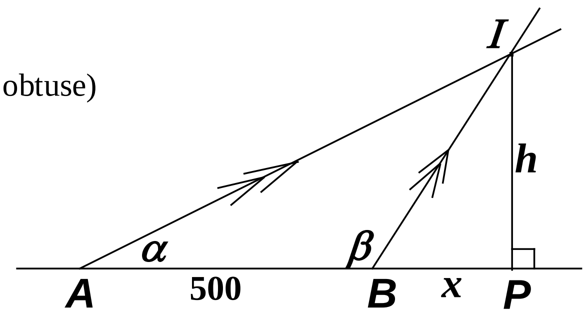
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2016 SOLUTION KEY

Team Round - continued



C) $\tan\alpha = \dfrac{h}{500+x}$, $\tan\beta = \tan 2\alpha = -\dfrac{h}{x}$ (since β is obtuse)

$$\tan 2\alpha = \frac{2\cancel{\tan\alpha}}{1-\tan^2\alpha} = \frac{-(500+x)\cancel{\tan\alpha}}{x}.$$

Cross multiplying, $2x = (500+x)(\tan^2\alpha - 1)$.

Transposing terms,

$$2x - x(\tan^2\alpha - 1) = 500(\tan^2\alpha - 1) \Leftrightarrow x(3-\tan^2\alpha) = 500(\tan^2\alpha - 1).$$

$$x = \frac{500(\tan^2\alpha - 1)}{(3-\tan^2\alpha)} = \frac{500\left(\frac{4}{3}-1\right)}{3-\frac{4}{3}} = 500\cdot\frac{1}{3}\cdot\frac{3}{5} = 100.$$

Substituting, $\tan\alpha = \dfrac{h}{500+x} \Rightarrow \dfrac{2\sqrt{3}}{3} = \dfrac{h}{600} \Rightarrow h = 400\sqrt{3} \Rightarrow (x,h) = \underline{\mathbf{(100, 400\sqrt{3})}}$.

D) The area of the 6 stamps is $6\cdot 8x\cdot 5x = 240x^2$ and the area of the sheet is $L(L+12)$.

Therefore,

$$240x^2 = \frac{L(L+12)}{4} + 260 = \frac{L(L+12)+1040}{4} = \frac{(L^2+12L+144)+1040-144}{4} = \frac{(L+12)^2+896}{4}$$

$\Rightarrow (L+12)^2 = 960x^2 - 896 = 64(15x^2-14)$

Since x must be an integer, we resort to trial and error until we find the minimum possible value of x for which L is an integer.

$x = 1 \Rightarrow (15x^2-14) = 1 \Rightarrow L+12 = 8$ (rejected).

$x = 2 \Rightarrow (15x^2-14) = 46$ (rejected).

$x = 3 \Rightarrow (15x^2-14) = 135-14 = 121 = 11^2 \Rightarrow L+12 = 8\cdot 11 \Rightarrow L = 76$.

Thus, the minimum dimensions of the sheet are 76 by 88, resulting in an area of **<u>6688</u>** mm^2.