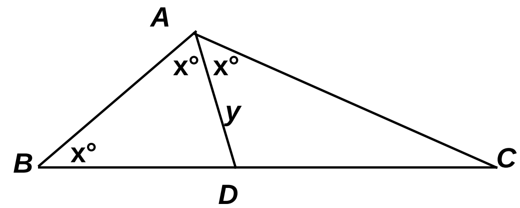
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015 SOLUTION KEY

Team Round



A) ΔABD is isosceles, so let $DB = DA = y$ and $DC = 2y$.
Using the angle bisector theorem on ΔABC,
$\frac{AC}{AB} = \frac{DC}{DB} = \frac{2y}{y} = 2$. $BC = 6 \Rightarrow y = 2$.
Let $AB = k$ and $AC = 2k$. Using the Law of Cosines,
on $\angle ABD$ in ΔABD: $2^2 = k^2 + 2^2 - 4k\cos(x°) \Rightarrow 4k\cos(x°) = k^2 \Rightarrow \cos(x°) = \frac{k}{4}$ $(k \neq 0)$
on $\angle DAC$ in ΔDAC: $4^2 = 2^2 + 4k^2 - 8k\cos(x°)$. Substituting for $\cos(x°)$,
$12 = 4k^2 - 8k\left(\frac{k}{4}\right) = 2k^2 \Rightarrow k = \sqrt{6}$. The perimeter of ΔABC is $3(y + k) \Rightarrow$ $\underline{\mathbf{3\left(2+\sqrt{6}\right)}}$.

B) Let $x = 5k + 1$ and $x + 2 = 7j - 1$ so that $x - 1$ and $x + 3$ will be multiples of 5 and 7 respectively. Then: $(x+3)-(x-1) = 4 = 7j - 5k$
Construct a table of (k, j) values satisfying this relation.
Since the linear relation $4 = 7j - 5k$ or $k = \frac{7j-4}{5}$ has a <u>slope</u> of 7/5, once we find an initial pair of (k, j) values, subsequent pairs are easily determined. $(k, j) = (2, 2)$ is our initial pair.

	X	k	j	$x + 2$
1	11	2	2	13
2	46	9	7	48
3	81	16	12	83
4	116	23	17	118

Since both x and $x + 2$ are even in even rows (and therefore not prime), we consider only odd rows. The x-values in odd rows are of the form $70n + 11$, where n = 0, 1, 2, … .
$n = 0$ gives us the first pair of primes $(x, x+2) = (11, 13)$
We try successive values of n until both x and $x + 2$ are again prime.
$n = 1 \Rightarrow$ (81, 83) rejected (81 is not prime)
$n = 2 \Rightarrow$ (151, 153) rejected (153 is divisible by 3)
$n = 3 \Rightarrow$ (221, 223) rejected ($221 = 13\cdot 17$)
$n = 4 \Rightarrow$ (291, 293) rejected (291 is divisible by 3)
$n = 5 \Rightarrow$ (361, 363) rejected ($361 = 19^2$)
$n = 6 \Rightarrow$ (<u>**431**</u>, 433) Bingo! – both are prime
Both numbers must be checked for divisibility by primes smaller than their square root.
Since $21^2 = 441$ is larger than both numbers, we need check for divisibility by only 8 primes - 2, 3, 5, 7, 11, 13, 17 and 19.
There are well-known rules for 2, 3, 5 and 11.
Brute force suffices for 7, 13, 17 and 19.
The details of the divisibility check are left to you.