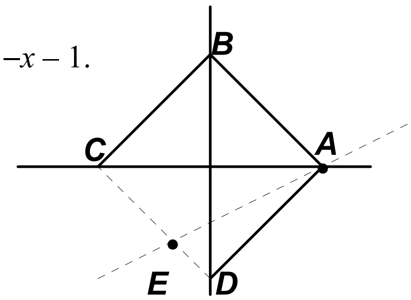
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2010 SOLUTION KEY

Round 1

A) If the three lines were parallel, then no points of intersection would be determined.
This is not possible, since the first two lines have different slopes and the third line is horizontal.
The (unique) minimum will occur when the horizontal line passes through the point of intersection of the first two lines. $17 - 2x = 3x + 2 \rightarrow x = 3 \rightarrow y = 11 \rightarrow k = \underline{\mathbf{11}}$.

B) The 1st condition describes a line $y = \dfrac{-x+1}{2}$ (slope of $-1/2$ and an x-intercept of $(1, 0)$).
The 2nd condition describes square connecting (in order): $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$.
Clearly, **<u>(1, 0)</u>** is a solution.
The equation of the side of the square in quadrant 3 lies on the line $y = -x - 1$.
$\dfrac{x-1}{2} = -x - 1 \rightarrow x - 1 = -2x - 2 \rightarrow 3x = -1 \rightarrow \underline{\left(-\mathbf{\dfrac{1}{3}}, -\mathbf{\dfrac{2}{3}}\right)}$.



C) $Area(ABCD) = Area(\Delta ABD) + Area(\Delta BCD)$
Using the determinant method,

$$Area(ABCD) = \frac{1}{2}\begin{vmatrix} 3 & 7 & 1 \\ -1 & 2 & 1 \\ 10 & 1 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} -1 & 2 & 1 \\ 3x & -x & 1 \\ 10 & 1 & 1 \end{vmatrix} =$$

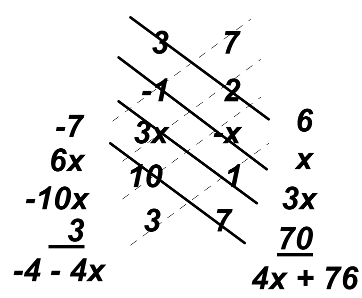
$$\frac{1}{2}\big((6 - 1 + 70) - (-7 + 20 + 3)\big) + \frac{1}{2}\big((x + 3x + 20) - (6x - 10x - 1)\big) = 52$$

$\rightarrow \quad \dfrac{1}{2}\big(59 + (8x + 21)\big) = 52 \rightarrow 8x + 80 = 104 \rightarrow x = 3 \rightarrow \underline{\mathbf{(9, -3)}}$.

Alternative #1:

The area is given by $\dfrac{1}{2}\begin{vmatrix} 3 & 7 \\ -1 & 2 \\ 3x & -x \\ 10 & 1 \\ 3 & 7 \end{vmatrix}$. This "determinant" is evaluated by the *lattice* method. Take the absolute value of the difference between

the sum of the diagonal down products and
the sum of the diagonal up products.



Thus, the area is $\dfrac{1}{2}\big(4x + 76 - (-4 - 4x)\big) = 52 \rightarrow x = 3 \rightarrow \underline{\mathbf{(9, -3)}}$.
This technique works for finding the area of any convex polygonal region, given
given a clockwise or counterclockwise listing of the coordinates
of its vertices, repeating the starting coordinates. See if it works for some concave regions!
Does it work for all polygonal regions – convex or concave?
The simplicity and generality of this method is astounding.