MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2009 SOLUTION KEY

Round 1 - continued

B) continued

If you are unfamiliar with how to take the determinant of a 3 x 3 matrix, ask a teammate or your coach to explain the technique using the space below:

Computing the determinant of M: $\begin{array}{|ccccc|} 1 & 2 & -3 & \mathbf{1} & \mathbf{2} \\ 2 & 1 & 2 & \mathbf{2} & \mathbf{1} \\ 7 & 8 & -5 & \mathbf{7} & \mathbf{8} \end{array}$

$M = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 2 \\ 7 & 8 & -5 \end{bmatrix} \rightarrow \|M\| = (-5 + 28 - 48) - (-21 + 16 - 20) = -25 + 25 = 0$

Thus, if $\|M_z\| = 0$, the system has an infinite number of solutions.

$M_z = \begin{bmatrix} 1 & 2 & \mathbf{5} \\ 2 & 1 & \mathbf{7} \\ 7 & 8 & \boldsymbol{a} \end{bmatrix} \rightarrow \|M_z\| = (a + 98 + 80) - (35 + 56 + 4a) = 0 \rightarrow 3a = 87 \rightarrow a = \underline{\mathbf{29}}$

C) Let b (base 10) denote the Cylon base. Then

$$\begin{cases} 1)\ r_1 + r_2 = -2b - 1 \\ 2)\ r_1 r_2 = 9b + 9 \\ 3)\ r_1 - r_2 = b + 2 \end{cases}$$

Adding 1) and 3) and dividing by 2, $r_1 = \dfrac{1-b}{2}$. Substituting, $r_2 = -\dfrac{3b+3}{2}$

Substituting in 2), $\left(\dfrac{1-b}{2}\right)\left(-\dfrac{3b+3}{2}\right) = 9b + 9$

$\rightarrow (b-1)(3b+3) = 4(9b+9) \rightarrow (b-1)(b+1) = 12(b+1)$

Canceling ($b > 0 \rightarrow b + 1 \neq 0$), $b - 1 = 12 \rightarrow b = \underline{\mathbf{13}}$

Alternate solution

Since the digit 9 is used in the equation, b must be greater than 9.

Consider the equation factored as $(x + r)(x + s) = x^2 + 21x + 99$, for integer roots r and s.

$r + s = 21_b = 2b + 1$ (always odd) and $r - s = 12_b = b + 2$ (even or odd, depending on b)

Since the sum and the difference of the integer roots must have the same parity, b must be odd.

So try $b = 11$. Adding, $2r = 33_b = 3b + 3 = 36$ so $r = 18$. Substituting, $18 + s = 21_b = 23 \rightarrow s = 5$.

But $rs = 18(5) = 90$ and $rs = 99_b = 9(11) + 9 = 108$ Oops!

Next try $b = 13$ and, as indicated below, it works.

Adding, $2r = 33_b = 42 \rightarrow r = 21$ Substituting, $21 + s = 3(13)+1 = 27 \rightarrow s = 6$

$rs = 21(6) = 126$ and $rs = 99_b = 9(13) + 9 = 126$ Bingo!

How do you argue that there are no other solutions larger than 13?