MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2013 SOLUTION KEY

Round 2

A) Positive integers leaving a remainder of 1 when divided by 4 are of the form $N = 4k + 1$, for $k = 3, 4, 5, \ldots$ Substituting $\Rightarrow (k, N) = (3, 13), (4, 17), (7, 29), (9, 37), (10, 41), (14, 53), (15, 61), (18, 73), (22, 89)$ and $(24, 97) \Rightarrow$ **<u>10</u>** two-digit primes

B) The only numbers with an odd number of divisors are perfect squares.
$2^2 = 4$ (4, 5) fails, since 5 is prime
$3^2 = 9$ (9, 10) fails, since 10 has 4 divisors
$5^2 = 25$ (25, 26) fails, since 26 has 4 divisors
$7^2 = 49$, $50 = 2 \cdot 5^2 \Rightarrow 6$ factors
Thus, N = **<u>49</u>**.

C) $2^{30} - 2^{16} + 1 = 1 \cdot 2^{30} - 2 \cdot 2^{15} + 1 = \left(2^{15} - 1\right)^2$
$2^{15} = 2^{10} \cdot 2^5 = 1024(32) = 32768$
We require the smallest prime factor of 32767.
Obviously 2, 3 and 5 fail, but 32767 = 7(4681).
4681 may not be prime, but it does not have a smaller factor than **<u>7</u>**.
FYI: In fact, 4681 is composite, since (31)(151) = 4681.
Also, you may have noticed that $2^{15} - 1 = \left(2^3 - 1\right)\left(2^{12} + 2^9 + 2^6 + 2^3 + 1\right)$ and the same result follows. Verify that $A^5 - 1 = (A-1)\left(A^4 + A^3 + A^2 + A + 1\right)$ and substitute $A = 2^3$.
[In the world of computers, it's very useful to remember (<u>memorize</u>) the fact that **1 K (kilobyte) = 2^{10} = 1024 bytes**. K means thousand and 2^{10} is the closest power of 2.]
1 megabyte (Mb) = 2^{20} (1,048,576 bytes). M means million and 2^{20} is the closest power of 2.
1 gigabyte (Gb) = 2^{30}(1,073,741,824 bytes). G means billion and 2^{30} is the closest power of 2.
1 terabyte (Tb) = 2^{40} (1,099,511,627,776 bytes). T means trillion and 2^{40} is the closest power of 2.