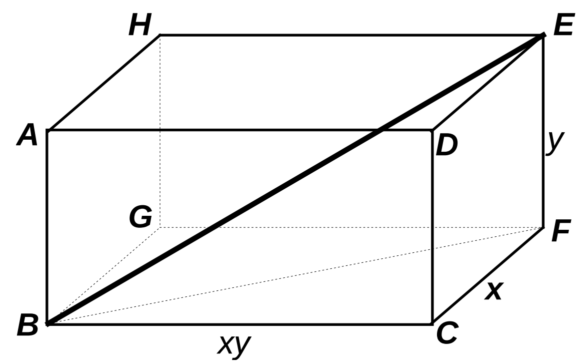
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016 SOLUTION KEY

Round 1

A) Since a cube has 12 edges, each edge has length 5.
Thus, each face of the cube is a square of side 5 and the surface area is $6\left(5^2\right) = \underline{\mathbf{150}}$.

B) In ΔBFC, $BF^2 = (xy)^2 + x^2$.



In ΔBEF, $BE^2 = BF^2 + FE^2 = x^2y^2 + x^2 + y^2$.

Equating, $x^2y^2 + x^2 + y^2 = (xy+1)^2$

$\Rightarrow \cancel{x^2y^2} + x^2 + y^2 = \cancel{x^2y^2} + 2xy + 1$

$\Rightarrow x^2 - 2xy + y^2 = (x-y)^2 = 1$

$\Rightarrow x - y = \pm 1 \Rightarrow y = \underline{\mathbf{x+1}}$ or $y = \underline{\mathbf{x-1}}$

Relationship discovered by Grant Landon (Miles River Middle School in Hamilton, MA)

<u>FYI</u>: Generalization - If the dimensions of a rectangular solid are x, y and $\dfrac{xy}{k}$ and the interior diagonal has length $\dfrac{xy}{k} + k$ then $y = x + k$ or $y = x - k$.

Proof of the generalization is left to you. Here is a numerical check.

1) Suppose a solid has dimensions 9, 12 and 36. How long is the interior diagonal?

$\dfrac{9 \cdot 12}{3} = 36$ and $12 = 9 + 3 \Rightarrow$ interior diagonal $= 36 + 3 = 39$.

2) If a solid has dimensions 5, 8, and $\dfrac{40}{3}$, then $(x, y, k) = (5, 8, 3)$ and the interior diagonal has length $\dfrac{40}{3} + 3 = \dfrac{49}{3}$.

C) Assume the dimensions of the solid are L, W and H. Then: $\begin{cases} (1)\ \ LW = 864 \\ (2)\ \ LH = 1440 \\ (3)\ \ WH = 2160 \end{cases}$

Dividing (3) and (2), $\dfrac{W}{L} = \dfrac{216}{144} = \dfrac{72(3)}{72(2)} \Rightarrow W = \dfrac{3}{2}L$.

Substituting in (1), $\dfrac{3}{2}L^2 = 864 \Rightarrow \dfrac{L^2}{2} = 288 \Rightarrow L^2 = 2^2(144) \Rightarrow L = 24 \Rightarrow W = 36$, $H = 60$

Since the GCF(24,36,60) is 12, the largest cube that can be packed inside the original solid has edge 12. Thus, we will need $2 \cdot 3 \cdot 5 = 30$ of these cubes and $(k, E) = (30, 12)$.

The diameter of each sphere will be the diagonal of the cube (which has length $12\sqrt{3}$).
Thus, the total volume of the 30 spheres is

$30 \cdot \dfrac{4}{3}\pi\left(6\sqrt{3}\right)^3 = 40\pi \cdot 216\left(3\sqrt{3}\right) = 25920\pi\sqrt{3} \Rightarrow (A, B) = \underline{(\mathbf{25920, 3})}$.