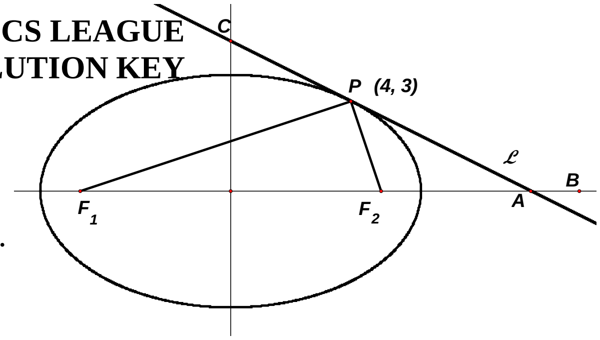
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014 SOLUTION KEY



Team Round

A) Solution #1: (Angle of incidence equals angle of reflection.)
Let $F_1 = (-c, 0)$ and $F_2 = (c, 0)$. Let the slope of $\mathscr{L}$ be m_2.

Then: $\overline{PF_1} \perp \overline{PF_2} \Rightarrow \dfrac{3-0}{4+c} \cdot \dfrac{3-0}{4-c} = -1 \Rightarrow c^2 - 16 = 9 \Rightarrow c = 5,\ F_1(-5,0),\ F_2(5,0)$

Since $m\angle APF_2 = m\angle CPF_1$, these angles both measure 45°. The slope of $\overline{F_2P} = \dfrac{0-3}{5-4} = -3$.

The angle θ between two lines can be determined on the basis of the slopes of the two lines, namely $\left(\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}\right)$. Thus, we have $1 = \dfrac{m_2 + 3}{1 - 3m_2} \Rightarrow 1 - 3m_2 = m_2 + 3 \Rightarrow m_2 = -\dfrac{1}{2}$ and the equation of $\mathscr{L}$ is $y - 3 = -\dfrac{1}{2}(x-4) \Rightarrow \underline{\boldsymbol{x + 2y = 10}}$.

Solution #2:
As an exterior angle of ΔPF_2A, $m\angle APF_2 + m\angle PF_2A = m\angle PAB$.
The slope *m* of any non-vertical line equals the tangent of its angle of inclination.
Therefore,

$$\tan(\angle PAB) = \tan(\angle APF_2 + \angle PF_2A) = \frac{\tan\angle APF_2 + \tan\angle PF_2A}{1 - \tan\angle APF_2 \cdot \tan\angle PF_2A} = \frac{1 + (-3)}{1 - (1)(-3)} = -\frac{1}{2}$$

and the same result follows.

Solution #3: (**In any <u>non-right</u> triangle *ABC*,** $\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$**.**)
Applying the theorem, in ΔAPF_2, $\tan F_2 = -3, \tan P = \tan 45° = 1$.

If $\tan A = x$, we have $x + (-3) + 1 = -3x \Rightarrow x = \dfrac{1}{2}$.

As supplementary angles, $\tan\angle PAF_2 = \dfrac{1}{2} \Rightarrow \tan\angle PAB = m_2 = -\dfrac{1}{2}$ and the same result follows.

Solution #4:
(**The tangent line to ellipse** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ **(or** $b^2x^2 + a^2y^2 = a^2b^2$**) at *P*(*m*, *n*) is** $b^2mx + a^2ny = a^2b^2$**.**)
To get the equation of the ellipse with vertices V_1 and V_2, first note that $2a = V_1V_2 = PF_1 + PF_2$.
$PF_1^2 = 9 + 81 = 90, PF_2^2 = 9 + 1 = 10$, so $2a = 3\sqrt{10} + \sqrt{10} \Rightarrow a = 2\sqrt{10}$.
Substituting, $b^2 = a^2 - c^2 = \left(2\sqrt{10}\right)^2 - 5^2 = 15 \Rightarrow b = \sqrt{15}$
Thus, the equation of the ellipse is $15x^2 + 40y^2 = 600$, or reducing $3x^2 + 8y^2 = 120$
The equation of $\mathscr{L}$ is $3(4)x + 8(3)y = 120$ and the result follows.