MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2015 SOLUTION KEY

Round 3

A) $\cos\left(6\cdot\left(Sin^{-1}\left(-\frac{1}{2}\right)+Tan^{-1}\left(-\sqrt{3}\right)\right)\right)=\cos 6\cdot\left(-\frac{\pi}{6}-\frac{\pi}{3}\right)=\cos 6\cdot -\frac{\pi}{2}=\cos\left(-3\pi\right)=\cos\pi=\underline{\mathbf{-1}}$

B) Given: $\sqrt{1-\cos^2(2x)}=\tan x$

Note that the left hand side of the equation always returns a nonnegative value.

Using the identities, $\begin{cases}\sin^2\theta+\cos^2\theta=1\\ \tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}\end{cases}$, and squaring both sides, we have

$1-\cos^2(2x)=\tan^2 x$

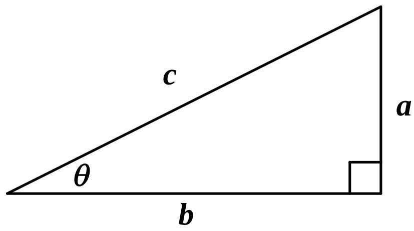
$\Rightarrow 1-\cos^2(2x)=\frac{(1-\cos 2x)^2}{\sin^2(2x)}=\frac{(1-\cos 2x)^2}{1-\cos^2(2x)}=\frac{(1-\cos 2x)^2}{(1-\cos 2x)(1+\cos 2x)}=\frac{1-\cos 2x}{1+\cos 2x}$

$\Rightarrow (1-\cos 2x)(1+\cos 2x)=\frac{1-\cos 2x}{1+\cos 2x}\Rightarrow 1-\cos 2x=0$ or $(1+\cos 2x)^2=1$

$\Rightarrow$ $\cos 2x=1\Rightarrow \underline{\mathbf{0,\ \pi}}$ or $1+\cos 2x=\pm 1\Rightarrow \cos 2x=0,\ \cancel{-2}$

$\cos 2x=0\Rightarrow 2x=\frac{\pi}{2}+n\pi\Rightarrow x=\frac{\pi}{4}+\frac{n\pi}{2}=\frac{\pi(2n+1)}{4}\Rightarrow \underline{\boldsymbol{\frac{\pi}{4}}},\ \cancel{\frac{3\pi}{4}},\ \boldsymbol{\frac{5\pi}{4}},\ \cancel{\frac{7\pi}{4}}$

C) $\theta=Cos^{-1}(k)=Tan^{-1}(k)\Rightarrow \sin\theta=\tan\theta=k\ \Rightarrow \frac{b}{c}=\frac{a}{b}\Rightarrow b^2=ac$



$a^2+b^2=c^2\Rightarrow a^2+ac=c^2$

$\Rightarrow a^2+ac-c^2=0\Leftrightarrow\left(\frac{a}{c}\right)^2+\left(\frac{a}{c}\right)-1=0$, provided $c\neq 0$

$\Leftrightarrow \sin^2\theta+\sin\theta-1=0$

Applying the quadratic formula, we have $\sin\theta=\frac{-1\pm\sqrt{5}}{2}$.

Since we know that θ is in the first quadrant, $\sin\theta=\underline{\boldsymbol{\frac{\sqrt{5}-1}{2}}}$ (only).

$\frac{-1-\sqrt{5}}{2}<-1$ is extraneous.

FYI: The approximate value of θ is $38.17270763°$.

For this value of θ, $\begin{cases}\cos\theta=0.7861513777\\ \tan\theta=0.7861513778\end{cases}$