MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014 SOLUTION KEY

Round 4

A) $\log_2 \sqrt{x^2 + x^2 + x^2 + x^2} = \log_2 \sqrt{4x^2} = 5 \Rightarrow \sqrt{4x^2} = 2^5 \Rightarrow 4x^2 = 2^{10} \Rightarrow x = \sqrt{\frac{2^{10}}{4}} = \frac{2^5}{2} = \underline{\mathbf{16}}$.

B) Factoring $125^x + 4(25)^x - 13(5)^x - 52 = 0$, we have
$(25)^x\left[5^x + 4\right] - 13\left[5^x + 4\right] = \left[25^x - 13\right]\left[5^x + 4\right] = 0 \Rightarrow 25^x = 13$ ($5^x + 4 > 0$ for all x).
Thus, $5^{2x} = 13 \Rightarrow x = \underline{\dfrac{\mathbf{\log_5 13}}{\mathbf{2}}}$ or $\underline{\mathbf{\log_5 \sqrt{13}}}$.

C) Given: $\dfrac{x}{4} = 2^{\log_x 8}$
Taking $\log_2$ of both sides, $\log_2 x - \log_2 4 = \log_x 8 = 3\log_x 2$.
$\Leftrightarrow \log_2 x - 2 = \dfrac{3}{\log_2 x}$
$\Leftrightarrow (\log_2 x)^2 - 2(\log_2 x) - 3 = 0$
$\Leftrightarrow (\log_2 x - 3)(\log_2 x + 1) = 0$
$\Leftrightarrow x = 2^3,\ 2^{-1}$
Thus, $x = \underline{\mathbf{8}, \dfrac{\mathbf{1}}{\mathbf{2}}}$.

Alternate Solution (Norm Swanson – Hamilton Wenham)
Let $p = \log_2 x$. Then $x = 2^p$.
Converting the original equation, $\dfrac{x}{4} = 2^{\log_x 8} = 2^{3\log_x 2} = 2^{\frac{3}{\log_2 x}} \Leftrightarrow \dfrac{2^p}{4} = 2^{p-2} = 2^{3/p}$.
Equating exponents, $p - 2 = \dfrac{3}{p} \Rightarrow p^2 - 2p - 3 = 0 \Leftrightarrow (p-3)(p+1) = 0$.
Therefore, $p = 3, -1$ and the solution follows.

What is <u>WRONG</u> with the following "solution"?

$$x = 4\left(2^{\log_x 8}\right) = 4\left(2^{3\log_x 2}\right) = 4\left(2^{\frac{1}{\log_2 x}}\right)^3 = 4\left(2^{(\log_2 x)^{-1}}\right)^3 = 4\left(2^{\log_2 x}\right)^{-3} = 4x^{-3} \Rightarrow x^4 = 4 \Rightarrow x = +\sqrt{2}$$