MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2014 SOLUTION KEY

Round 6

A) By trial and error, starting with $\left(A,B,C\right)=\left(9,17,25\right)$,

$x=1 \Rightarrow \left(9,16,25\right)$ is not a geometric sequence.

$\frac{16}{9}\neq\frac{25}{16}$, since, cross multiplying, $16\cdot 16=256\neq 9\cdot 25=225$

$x=2 \Rightarrow \left(9,15,25\right)$ is a geometric sequence, since $R=\frac{15}{9}=\frac{25}{15}=\underline{\mathbf{\frac{5}{3}}}$.

Algebraic solution

$\frac{17-x}{9}=\frac{25}{17-x}\Leftrightarrow 289-34x+x^2=225\Leftrightarrow x^2-34x+64=0$

$\Rightarrow (x-2)(x-32)=0\Rightarrow x=2,32$

The second solution is rejected, since $B=-32<0$ and $R=+\frac{5}{3}$ only.

B) The new sum is $\frac{2n(2n+1)(2n+2)}{6}$. The old sum was $\frac{n(n+1)(n+2)}{6}$.

New = 7(Old) $\Rightarrow$ $2n(2n+1)(2n+2)=7\left(n(n+1)(n+2)\right)$

$\Rightarrow$ $4n(2n+1)\cancel{(n+1)}=7\left(n\cancel{(n+1)}(n+2)\right)$ $\Rightarrow$ $8n^2+4n=7n^2+14n$ $\Rightarrow$ $n^2=10n$

$n\neq 0\Rightarrow n$ = $\underline{\mathbf{10}}$

C) Let m denote the mean and M the median. We require that $m=M$.

The mean of the 5 numbers is $m=\frac{40+A}{5}$.

This expression can only be an integer if A is a multiple of 5.
Sorting the known values, we have 4, 7, 13, 16 and we consider possible multiples of 5.

$A<\underline{\mathbf{-5}}\Rightarrow m<7$ and M = 7 (rejected)
$A=-5\Rightarrow m=7$ and M = 7 Bingo!
$A=0\Rightarrow m=8$ and M = 7 (rejected)
$A=5\Rightarrow m=9$ and M = 7 (rejected)
$A=\underline{\mathbf{10}}\Rightarrow m=10$ and M = 10 Bingo!
$A=15\Rightarrow m=11$ and M = 13 (rejected)
$A=20\Rightarrow m=12$ and M = 13 (rejected)
$A=\underline{\mathbf{25}}\Rightarrow m=13$ and M = 13 Bingo!
$A>25\Rightarrow m>13$ and M = 13 (rejected)
Notice that, regardless of the value of A, 4 and 16 cannot be the median value.