MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2008 SOLUTION KEY

Team Round

A) $\dfrac{x^2}{81} = \sec^2 t$ and $\dfrac{y^2}{49} = \tan^2 t$

Since $1 + \tan^2 x = \sec^2 x$, $\dfrac{x^2}{81} = \dfrac{y^2}{49} + 1 \rightarrow x^2 = \dfrac{81}{49}\left(y^2 + 49\right) \rightarrow x = \pm\dfrac{9}{7}\sqrt{y^2+49}$

However, since $90° < t < 180°$, $\cos(t) < 0 \rightarrow \sec(t) < 0 \rightarrow x < 0 \rightarrow x =$ $\underline{\boldsymbol{-\dfrac{9}{7}\sqrt{y^2+49}}}$ only

B) $x^3 - x^2y - xy^2 + y^3 = x^2(x-y) - y^2(x-y) = (x^2 - y^2)(x-y) = (x-y)^2(x+y)$ and
Here's a list of factors of 1024, where the first factor is a perfect square.

1(1024), 4(256), 16(64), 64(16), 256(4), 1024(1)

Since $x + y$ and $x - y$ have the same parity (both even or both odd),
only the middle 4 case are considered.
Thus, $x - y = 2, 4, 8$ or 16 and the corresponding values of $x + y = 256, 64, 16$ or 4 respectively.
Adding, $2x = 258, 68, 24$ or $20 \rightarrow (x, y) =$ **(129, 127), (34, 30), (12, 4)**
[(10, -6) is rejected since both coordinates were required to be positive.]

C) The original equation is equivalent to: $\tan^2(2x) - \sec(2x) - 1 = 0$

$\rightarrow \sec^2(2x) - 1 - \sec(2x) - 1 = \sec^2(x) - \sec(2x) - 2 = (\sec(2x) - 2)(\sec(2x) + 1) = 0$
$\sec(2x) = 2 \rightarrow \cos(2x) = ½ \rightarrow 2x = \pm 60° + 360n \rightarrow x = \pm 30 + 180n \rightarrow 30, 210, 150, 330$
$\sec(2x) = -1 \rightarrow \cos(2x) = -1 \rightarrow 2x = 180 + 360n \rightarrow x = 90 + 180n \rightarrow x = 90, 270$
The required sum is 30 + 90 + 150 + 210 + 270 + 330 = **1080°**

D) Expanding, $(3 + 4 + 5 + \ldots + n)x + 3(n - 3 + 1) = 45$

$\rightarrow \left(\dfrac{n(n+1)}{2} - 3\right)x + 3n - 6 = 45 \rightarrow \left(\dfrac{n^2+n-6}{2}\right)x = 51 - 3n \rightarrow (n+3)(n-2)x = 6(17-n)$

$\rightarrow x = \dfrac{6(17-n)}{(n+3)(n-2)}$

A list provides us with integer solutions **(5, 3)** and **(17, 0)**.
Here is a graph of this function – the graph has an open point at (3, 14), intersects the horizontal axis at (17, 0), drops slightly below the axis and then becomes asymptotic to the axis for $n > 17$.