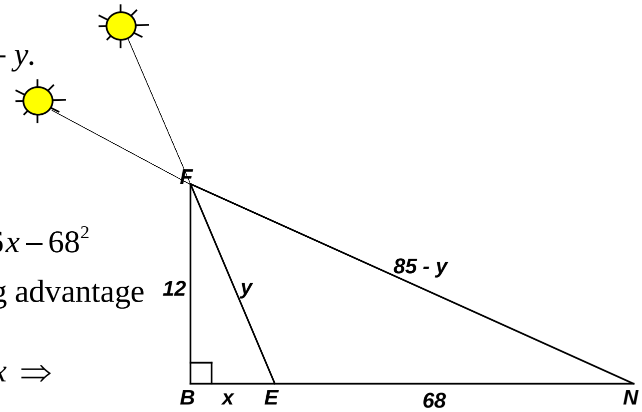
**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 1 - OCTOBER 2012 SOLUTION KEY**

**Team Round**




B) Let ($BE$, $FE$) = ($x$, $y$). Then $FN$ = 153 – ($y$ + 68) = 85 – $y$.

$$\begin{cases} \Delta FBE : y^2 - x^2 = 12^2 \\ \Delta FBN : 12^2 + (x+68)^2 = (85-y)^2 \end{cases}$$

Substituting for $y^2 - x^2$,

$$\left(y^2 - x^2\right) = (85-y)^2 - (x+68)^2 = 85^2 - 170y + y^2 - x^2 - 136x - 68^2$$

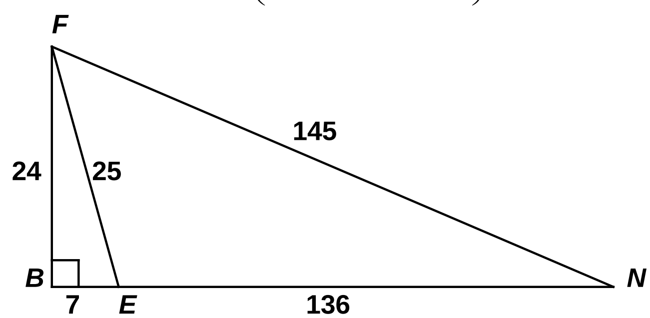
Re-arranging terms, pulling out common factors and taking advantage of the difference of perfect squares

$$\Rightarrow\ 0 = 85^2 - 68^2 - 170y - 136x = (153)(17) - 17\cdot 10y - 17\cdot 8x\ \Rightarrow$$

$$10y + 8x = 153 \Rightarrow\ y = \frac{153-8x}{10}\quad \text{Substituting, } \left(\frac{153-8x}{10}\right)^2 - x^2 = 12^2$$

$$\Rightarrow\ \left(153-8x\right)^2 - 100x^2 = 100\cdot 12^2 \Rightarrow 153^2 - 16\cdot 153x - 36x^2 = 100\cdot 12^2$$

Dividing through by $3^2$ = 9, $51^2 - 16\cdot 17x - 4x^2 = 100\cdot 16 = 40^2\ \Rightarrow\ 4x^2 + 16\cdot 17x + \overbrace{\left(40^2 - 51^2\right)}^{(91)(-11)=-7\cdot 11\cdot 13} = 0$




$$\Rightarrow\ \left(2x-7\right)\left(2x+11\cdot 13\right) = 4x^2 + \overbrace{\left(2\cdot 11\cdot 13 - 2\cdot 7\right)}^{2(143-7)=2\cdot 8\cdot 17}x - 7\cdot 11\cdot 13 = 0$$

$\Rightarrow\ x$ = <u>**3.5**</u>.

Note the familiar 7-24-25 right triangle embedded in the diagram. Dividing the dimensions by 2 gives us the conditions and answer we derived above.

Alternate solution (Norm Swanson – Hamilton Wenham)

In right $\Delta FBN$, $\cos N = \dfrac{x+68}{85-y}$. Using the law of Cosines in $\Delta FEN$,

$\cos N = \dfrac{(85-y)^2 + 68^2 - y^2}{2\cdot 68\cdot (85-y)} = \dfrac{85^2 + 68^2 - 170y}{2\cdot 68\cdot (85-y)}$. Noting that each term in the trinomial in the numerator is a multiple of 17, we simplify this to $\dfrac{697-109y}{8(85-y)}$. Equating, we have 8$x$ + 10$y$ = 153. Since the coefficients 8 and 10 are even and the constant 153 is odd, $x$ and $y$ must be of the form $\dfrac{x'}{2}$ and $\dfrac{y'}{2}$ respectively. Substituting, $4x' + 5y' = 153 = 9(17)$.

Clearly, $x' = y' = 17$ is a solution. We are looking for $P(x', y')$ which produces a Pythagorean Triple form the sides of $\Delta FBE$ doubled. $\left(FB, BE, FE\right) = \left(12, \dfrac{x'}{2}, \dfrac{y'}{2}\right) = (24, 17, 17)$ fails. Using the slope of $\dfrac{4}{-5}$, we move <u>4 up and 5 left</u> until we find the required triple $\cancel{(24,12,21)}$, <u>$\mathbf{(24,7,25)}$</u> - Bingo!

But remember we have doubled the sides of $\Delta FBE$, so $BE$ = $\dfrac{7}{2}$ = <u>**3.5**</u>.