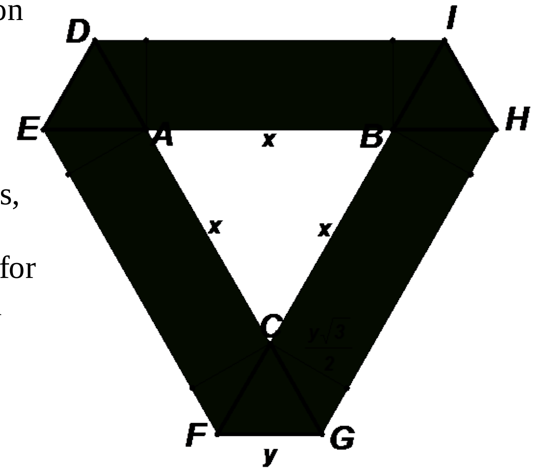
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016 SOLUTION KEY

Team Round

C) From the diagram at the right, we see that the shaded region is comprised of 3 congruent equilateral triangles (ADE, BHI and CFG), 6 congruent 30-60-90 right triangles and 3 congruent rectangles. Any pair of 30-60-90 right triangles can be combined to form an equilateral triangle congruent to the named equilateral triangles. Thus, the area of the shaded region is equivalent to 6 equilateral triangles and 3 rectangles. The numbers are atrocious, so for now, forget the numerical values and assume $AB = x$ and $FG = y$. The required area is



$$6\left(\frac{y^2\sqrt{3}}{4}\right)+3\left(x\cdot\frac{y\sqrt{3}}{2}\right)=\boxed{\frac{3\sqrt{3}}{2}\left(y^2+xy\right)}$$

Now we find x, substitute and simplify.

$$\frac{x^2\sqrt{3}}{4}=1\Rightarrow x^2=\frac{4}{\sqrt{3}}=\frac{4\sqrt{3}}{3}=\frac{4\cdot 3\sqrt{3}}{9}=\frac{4\sqrt{27}}{9}\Rightarrow x=\frac{2}{3}\sqrt[4]{27}.$$

For $y=\frac{1}{2}$, the required area is $\frac{3\sqrt{3}}{2}\left(\frac{1}{4}+\frac{x}{2}\right)=\frac{3\sqrt{3}}{8}(1+2x)=\frac{3\sqrt{3}}{8}\left(1+\frac{4\sqrt[4]{27}}{3}\right)$

$$=\frac{\sqrt{3}}{8}\left(3+4\sqrt[4]{27}\right)=\frac{3\sqrt{3}+4\sqrt{3}\cdot\sqrt[4]{27}}{8}=\frac{3\sqrt{3}+4\sqrt[4]{9}\cdot\sqrt[4]{27}}{8}=\frac{3\sqrt{3}+4\sqrt[4]{3^5}}{8}=\frac{3\left(\sqrt{3}+4\sqrt[4]{3}\right)}{8}\Rightarrow \underline{\mathbf{(3,4,8)}}.$$

D) Solution #1: Quadratic Formula

But the QF works for an equation in a single variable??? Treat y as a constant!

$$36x^2-3xy-60y^2+18x+38y-4=36x^2+(18-3y)x-\left(60y^2-38y+4\right)$$

$$x=\frac{3y-18\pm\sqrt{(18-3y)^2+4(36)\left(60y^2-38y+4\right)}}{72}$$. Factoring a 9 out of the radicand allows us to eliminate a factor of 3 in the numerator and denominator.

$$\boxed{x=\frac{y-6\pm\sqrt{(6-y)^2+16\left(60y^2-38y+4\right)}}{24}}$$. Expanding the radicand, we are looking for a perfect square trinomial.

$$(6-y)^2+16\left(60y^2-38y+4\right)$$
$$36-12y+y^2+960y^2-608y+64=961y^2-620y+100$$

Voila! Both the lead coefficient and the constant term are perfect squares. We have $(31y-10)^2$.

Simplifying the boxed equation, $x=\frac{y-6\pm(31y-10)}{24}=\frac{32y-16}{24},\ \frac{-30y+4}{24}$.

Reducing the fractions, $x=\frac{4y-2}{3},\ \frac{-15y+2}{12}$. Clearing the fractions and transposing terms,

$3x-4y+2=0$, $12x+15y-2=0$, and these are our two factors$\Rightarrow$ $\underline{\mathbf{(3, -4, 2, 12, 15, -2)}}$.