MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - NOVEMBER 2011 SOLUTION KEY

Team Round - continued

D) $k^2 - 96$ must be positive, so $k \geq 10$ for starters.

$k^2 - 96$ must also be a perfect square, i.e. there is an integer N such that $k^2 - 96 = N^2$ or $k^2 - N^2 = 96$

Let's examine a list of consecutive perfect squares and look for gaps of 96.

1 4 9 16 25 36 49 64 81 100 121 144 169 196 ….

The gap between consecutive entries grows uniformly by 2, so eventually the gap can be any odd integer. Consecutive perfect squares always differ by an odd amount, so we are looking for non-consecutive perfect squares.

We are looking for perfect squares which sandwich an odd number of consecutive perfect squares. (Perfect squares sandwiching an even number of perfect squares would differ by an odd number.)

The minimum difference if there are 9 intermediate numbers is 121-1 = 120, so 7 is the maximum number of intermediate perfect squares. There are 4 possible answers.

$100 - 4 = 96 \Rightarrow k =$ **10**

5 intermediate numbers $\Rightarrow$ 6 gaps $\Rightarrow a, a + 2, \ldots a + 10 \Rightarrow 6a + 30 = 96 \Rightarrow a = 11$

25 (36 49 64 81 100) 121 $\Rightarrow k^2 = 121 \Rightarrow k =$ **11**

3 intermediate numbers $\Rightarrow$ 4 gaps $\Rightarrow a, a + 2, \ldots a + 6 \Rightarrow 4a + 12 = 96 \Rightarrow a = 21$

100 (121 144 169) 196 $\Rightarrow k^2 = 196 \Rightarrow k =$ **14**

1 intermediate number $\Rightarrow a + (a + 2) = 96 \Rightarrow a = 47$ which occurs between ….. hmm?

Continuing the list of perfect squares …, 400, 441, 484, 529, (576), 625 $\Rightarrow k^2 = 625 \Rightarrow k =$ **25**

Alternate solution: (**Aaargh!** - Why didn't I go down this road first!!!)

$k^2 - 96 = N^2 \Leftrightarrow k^2 - N^2 = 96 \Leftrightarrow (k + N)(k - N) = 96$

So just look at factor pairs of 96, namely (1, 92), (2, 48), (3, 32), (4, 24), (6, 16), (8, 12)

Equating $k - N$ with the smaller factor and $k + N$ with the larger, we have:

$$\begin{cases} k - N = & 1 \quad 2 \quad 3 \quad 4 \quad 6 \quad 8 \\ k + N = & 92 \ 48 \ 32 \ 24 \ 16 \ 12 \end{cases}$$

Adding and ignoring odd sums, we have $2k = 50, 28, 22, 20 \Rightarrow k =$ **25, 14, 11** and **10** (in any order).