MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008 SOLUTION KEY

Team Round

A) $x + 1$ can not be the longest side and therefore, can not be opposite the largest angle of 120°.

Case 1: $(x + 3)^2 = (x + 1)^2 + (11 - 2x)^2 - 2(x + 1)(11 - 2x)(-1/2)$
$\rightarrow x^2 + 6x + 9 = 5x^2 - 42x^2 + 122 + 11 + 9x - 2x^2$
$\rightarrow 2x^2 - 39x + 124 = (2x - 31)(x - 4) = 0 \rightarrow 4$ only

Case 2: $(11 - 2x)^2 = (x + 1)^2 + (x + 3)^2 - 2(x + 1)(x + 3)(-1/2)$
$\rightarrow 121 - 44x + 4x^2 = 2x^2 + 8x + 10 + x^2 + 4x + 3$
$\rightarrow x^2 - 56x + 108 = (x - 2)(x - 54) = 0 \rightarrow 2$ only

Note: In both cases, the triangle has sides of lengths 3, 5 and 7.

Alternate solution:
Case 1: $x + 3 > 11 - 2x \rightarrow x > 2\ 2/3$, but $11 - 2x > 0 \rightarrow x < 5\ ½$
Thus, $2.4 < x < 5.5$
By trial and error, 3 → 4, 5, 6 rejected **<u>4</u>** → 3, 5, 7 OK 5 → 1, 6, 8 rejected

Case 2: $11 - 2x > x + 3$ and $x + 1 > 0 \rightarrow -1 < x < 2\ 2/3$
By trial and error, 0 → 1, 3, 11 rejected 1 → 2, 4, 9 rejected **<u>2</u>** → 3, 5, 7 OK

B) The leftmost three digits must sum to 9. They must also be a multiple of 13; otherwise, there would be a positive remainder r and the three-digit integer r13 is never a multiple of 13.

Notice the pattern of remainders?

Remainders upon division by 13

r	r13	remainder	r	r13	remainder
1	113	9	7	713	11
2	213	5	8	813	7
3	313	1	9	913	3
4	413	10	10	1013	12
5	513	6	11	1113	8
6	613	2	12	1213	4

Thus, only 013 is a multiple of 13.
Examining the three-digits multiples of 13, the smallest is 104 (but it is not a multiple of 9)
117 is the smallest with a digit sum of 9 → $N_{\min} = 11713$

Finding $N_{\max}$:
Three-digit multiples of 13 must be of the form $104 + 13k$.
$9xy$: Only 900 could work and it is not a multiple of 13
$8xy$: The smallest are 806, 819, 832, …. and clearly none of these will have a digit-sum of 9.
$7xy$: The smallest are 702, 715, 728, …
702 is a multiple of 13 and clearly none of the other $7xy$ integers will have a digit-sum of 9
→ $N_{\max} = 70213$ and the required sum is **<u>81926</u>**