MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2008 SOLUTION KEY

Round 3

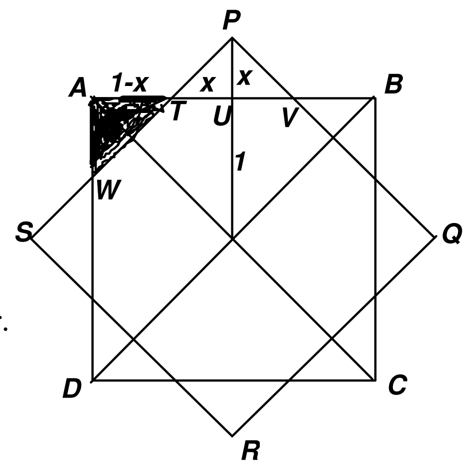
A) If the side of the square has length 1, then the diagonal and the altitude of the equilateral triangle will have lengths $\sqrt{2}$. Dividing by $\sqrt{3}$ and multiplying by 2, the side of the equilateral triangle is $\frac{\sqrt{2}}{\sqrt{3}}\cdot 2=\frac{2\sqrt{6}}{3}$. Thus, the areas are $1^2 = 1$ and $\frac{1}{2}\cdot\frac{2\sqrt{6}}{3}\cdot\sqrt{2}=\frac{2\sqrt{3}}{3}$

$1:\frac{2\sqrt{3}}{3}=3:2\sqrt{3}=3\sqrt{3}:6=\underline{\mathbf{\sqrt{3}:2}}$ (or $\frac{\sqrt{3}}{2}:1$)

B) $m\angle TPU = 45°$ and $PU = \sqrt{2}-1 \rightarrow PT = 2-\sqrt{2}$

$\rightarrow$ area(ΔTPV) = $\frac{1}{2}\left(2-\sqrt{2}\right)^2 = 3-2\sqrt{2}$

$\rightarrow$ area of overlap = $4 - 4(3-2\sqrt{2}) = \underline{\mathbf{8\left(\sqrt{2}-1\right)}}$



Alternate solution

A(regular octagon) = $\frac{1}{2}ap$, where a = apothem (perpendicular from center to side) and p = its perimeter.

$a = 1$, $x = PU = \sqrt{2}-1$, $TV = 2x = 2\sqrt{2}-2$, $TW = (1-x)\sqrt{2} = \left(2-\sqrt{2}\right)\sqrt{2} = 2\sqrt{2}-2$

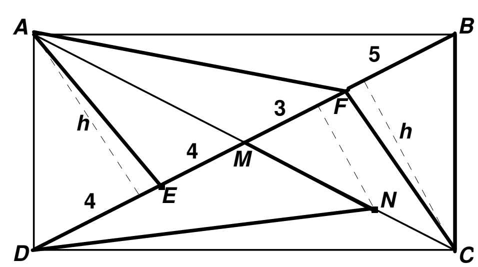
So the resulting octagon is in fact regular.

$p = 8\left(2\sqrt{2}-2\right)$ and the area of the overlap is $\underline{\mathbf{8\left(\sqrt{2}-1\right)}}$.

C) $DE : EB = 1 : 3 = 4 : 12$, $DF : FB = 11 : 5$

Each of these ratios divides segment $\overline{BD}$ into 16 parts.
Thus, without loss of generality, let $BD = 16$.
Then: $DE = 4$, $BF = 5$, $EF = 7$ and $DM = MC = 8$

The altitude from A to $\overline{BD}$ has the same length as the altitude from C to $\overline{BD}$ - call it h.



$MN : NC = 3 : 1 \rightarrow MN = \frac{3}{4}MC$ and the altitude from N to $\overline{BD}$ has length $\frac{3}{4}h$

$|\Delta AEF| : |\Delta CBF| : |\Delta DMN| = \frac{1}{2}\cdot 7\cdot h : \frac{1}{2}\cdot 5\cdot h : \frac{1}{2}\cdot 8\cdot\frac{3}{4}h = \underline{\mathbf{7:5:6}}$

Note: $|\Delta AEF|$ denotes the <u>area</u> of ΔAEF.