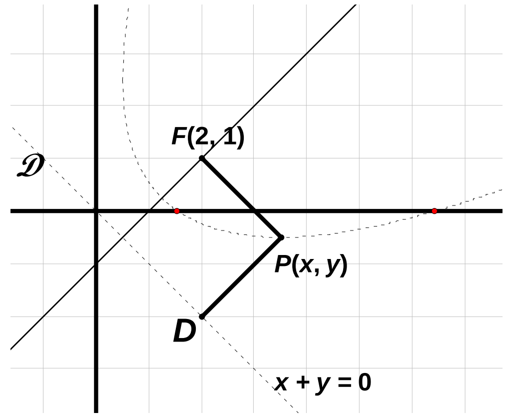
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013 SOLUTION KEY

Team Round



A) The set of points satisfying the given conditions is a parabola, but the axis of symmetry is neither vertical nor horizontal.

The directrix $\mathcal{D}$ is $y = -x$ and the axis of symmetry is $y = x$. The graph is shown at the right, but the sketch is only included to reinforce the algebra. Applying the point-to-point and point-to-line distance formulas gets us the equation we need.

$PF = PD \Rightarrow \sqrt{(x-2)^2+(y-1)^2} = \dfrac{|x+y+0|}{\sqrt{1^2+1^2}}$

Squaring both sides, $2\left((x-2)^2+(y-1)^2\right) = |x+y|^2 = (x+y)^2$

$\Rightarrow\ 2x^2-8x+2y^2-4y+10 = x^2+2xy+y^2$

$\Rightarrow\ \left(x^2-8x\right)+\left(y^2-4y\right) = 2xy-10$ Completing the square, we have

$\left(x^2-8x+16\right)+\left(y^2-4y+4\right) = 2xy-10+16+4$

$\Rightarrow (x-4)^2+(y-2)^2 = 2(xy+5)$

Letting $x = 0 \Rightarrow 16+(y-2)^2 = 10$ confirms what is clear from the sketch, namely that the graph does not cross the y-axis. Letting $y = 0 \Rightarrow (x-4)^2+4 = 10 \Rightarrow x = 4\pm\sqrt{6}$ and the coordinates of the x-intercepts are $\underline{\left(\mathbf{4\pm\sqrt{6},0}\right)}$.

Extra challenge:
You might want to try solving for y in terms of x.
You should get $y = x+2\pm\sqrt{6-(x-4)^2+(x+2)^2}$.
Aaaargh!! For the contest director, this was necessary in order to draw the sketch above, since the available software did not have the capability of plotting implicit functions.