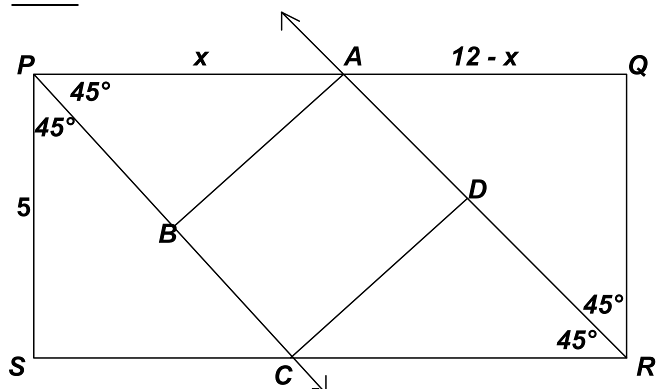
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2007 SOLUTION KEY

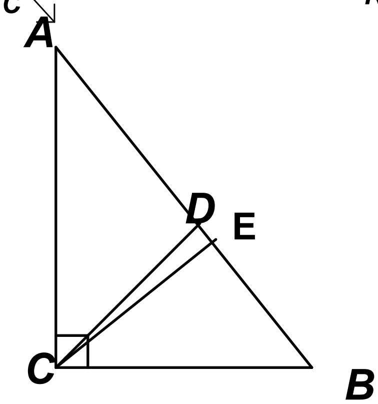
Round 2

A) $b^2 = (11\sqrt{3})^2 - (7\sqrt{7})^2 = 363 - 343 = 20 \rightarrow b = \underline{\mathbf{2\sqrt{5}}}$

B) Let $\overline{AB}$ (drawn $\perp \overrightarrow{PC}$) represent the distance between the angle bisectors.
Clearly, Δs PSC and RQA are both isosceles and congruent. Thus, x must be 7 and since $\overline{AB}$ is the leg in an isosceles right triangle with hypotenuse 7, $AB = CD = \underline{\mathbf{\frac{7\sqrt{2}}{2}}}$



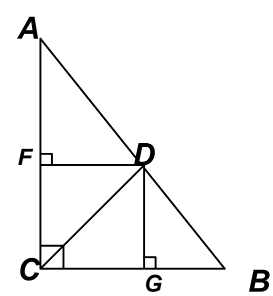
C) Since ΔCDA and ΔCDB have a common altitude and areas in a 3 : 4 ratio, their bases must be in a 3 : 4 ratio also.
Apply the Pythagorean theorem to ΔABC or note
$(CB, CA, AB) = (21, 28, ?) = 7\underline{(3, 4, ?)}$
Since the missing number in the triple is 5, $AB = 35$
$\rightarrow AD = 20$ and $BD = 15$



Drop a perpendicular from C to $\overline{AB}$ (Call it $\overline{CE}$.)

Then $(BC)(AC) = (AB)(CE)$, since each gives twice the area of ΔABC.
Thus, $21(28) = 35(CE) \rightarrow CE = 16.8$

Applying the Pythagorean Theorem to right triangle ACE, $AE = 22.4 \rightarrow DE = 2.4$
and to right triangle CDE, $CD^2 = 16.8^2 + 2.4^2 = 288 \rightarrow CD = \underline{\mathbf{12\sqrt{2}}}$



Alternative: Drop perpendiculars from D to $\overline{AC}$ and $\overline{BC}$.
Let $AD = 4x$ and $DB = 3x$. $AB = 7x = 35 \rightarrow x = 5$, $BD = 15$, $AD = 20$
Since $\Delta DGB \sim \Delta AFD \sim \Delta ACB \sim$ 3-4-5 right triangle,
$(DG, GB) = (12, 9)$ and $(DF, FA) = (12, 16)$ and $FDGC$ is a <u>12 x 12 square</u>!
Therefore, $CD = \underline{\mathbf{12\sqrt{2}}}$.