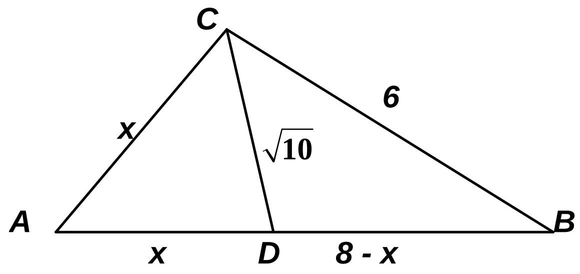
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014 SOLUTION KEY

Team Round



A) In ΔABC, $\cos A = \dfrac{8^2 + x^2 - 6^2}{2 \cdot 8 \cdot x} = \dfrac{28 + x^2}{16x}$.

In ΔACD, $\cos A = \dfrac{x^2 + x^2 - 10}{2 \cdot x \cdot x} = \dfrac{x^2 - 5}{x^2}$.

Equating (since $x \neq 0$),

$\dfrac{28 + x^2}{16x} = \dfrac{x^2 - 5}{x^2} \Leftrightarrow \dfrac{28 + x^2}{16} = \dfrac{x^2 - 5}{x}$

Cross multiplying, $28x + x^3 = 16x^2 - 80$ or $x^3 - 16x^2 + 28x + 80 = 0$.

By synthetic division, $\begin{array}{r|rrrr} 4 & 1 & -16 & 28 & 80 \\ & & 4 & -48 & -80 \\ \hline & 1 & -12 & -20 & 0 \end{array}$, we determine that $x = 4$ is a root and, if there are additional roots, then $x^2 - 12x - 20 = 0$. Applying the quadratic formula,

$x = \dfrac{12 \pm \sqrt{144 + 80}}{2} = \dfrac{12 \pm \sqrt{224}}{2} = \dfrac{12 \pm 4\sqrt{14}}{2} = 6 \pm 2\sqrt{14}$.

$x = 6 + 2\sqrt{14} > 8 \Rightarrow BD < 0$ and must be rejected.

$x = 6 - 2\sqrt{14} < 0$ and must be rejected.

Therefore, the only answer is $AC = \underline{\mathbf{4}}$.

Note that $x = 4$ implies that D was, in fact, a midpoint of $\overline{AB}$, and $\overline{CD}$ was a median.

You could also have verified this result using Stewart's Theorem.

B) Since 19 is prime it is not possible to find two fractions with denominators smaller than 19 that add to $\dfrac{5}{19}$, as it would if the sum were $\dfrac{8}{15}\left(\dfrac{1}{3} + \dfrac{1}{5} = \dfrac{8}{15}\right)$. If the first denominator is k, we have: $\dfrac{5}{19} = \dfrac{a}{k} + \dfrac{m}{n}$ or $\dfrac{5}{19} - \dfrac{a}{k} = \dfrac{m}{n}$. So, $n = 19k$, and the sum of the denominators is $20k$.

Now, to minimize the first denominator, we guesstimate.

$\dfrac{5}{19}$ is smaller than $\dfrac{1}{3}$, but it is just slightly larger than $\dfrac{1}{4}$, so we try $\dfrac{a}{k} = \dfrac{1}{4}$. Then:

$\dfrac{5}{19} - \dfrac{1}{4} = \dfrac{1}{76}$ which gives us $\dfrac{5}{19} = \dfrac{1}{4} + \dfrac{1}{76}$, so the minimum sum of the denominators is $\underline{\mathbf{80}}$.