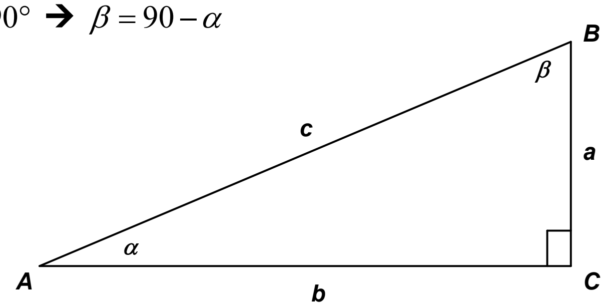
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2011 SOLUTION KEY

Team Round

C) Let $\alpha = Tan^{-1}\left(\frac{a}{b}\right)$ and $\beta = Sin^{-1}\left(\frac{a}{b}\right)$. Then: $\alpha + \beta = 90°$ → $\beta = 90 - \alpha$



$a, b > 0$ → $0 < \alpha, \beta \le 90$

$\sin\beta = \frac{a}{b} = \sin(90-\alpha) = \cos\alpha = \frac{b}{c} = \frac{b}{\sqrt{a^2+b^2}}$

→ $b^2 = a\sqrt{a^2+b^2}$ → $b^4 = a^2(a^2+b^2)$

→ $b^4 - a^2b^2 - a^4 = 0$

→ $b^2 = \dfrac{a^2 \pm \sqrt{a^4+4a^4}}{2} = a^2\left(\dfrac{1\pm\sqrt{5}}{2}\right)$ →

$\dfrac{b^2}{a^2} = \dfrac{1+\sqrt{5}}{2}$ $\left(\cancel{\dfrac{1-\sqrt{5}}{2}} < 0 \text{ is rejected.}\right)$

Inverting, $\dfrac{a^2}{b^2} = \dfrac{2}{1+\sqrt{5}} = \boldsymbol{\dfrac{\sqrt{5}-1}{2}}$

Note: Using a calculator, $\dfrac{1+\sqrt{5}}{2} \approx 1.6180339887...$ and $\dfrac{\sqrt{5}-1}{2} \approx 0.6180339887...$.

The first constant is called ϕ, the golden ratio and the second is $\phi - 1$.

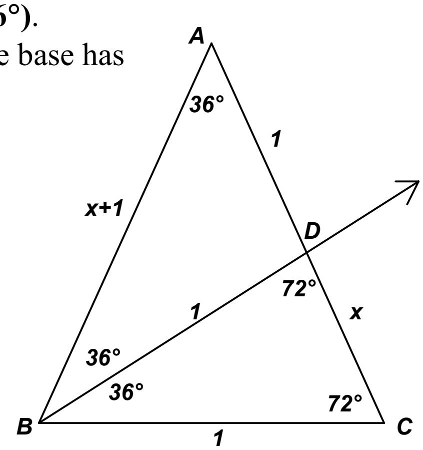
Check: $\frac{a}{b} \approx 0.7861513778$ → $\alpha \approx 38.17270763°$, $\beta \approx 51.82729238°$ and $\alpha + \beta \approx 90.00000001 \to 90°$.

<u>An aside</u>:

Actually, did you know that besides the 30°, 45° and 60° families of angles, it is also possible to compute an <u>exact</u> value for the trig functions of 36° ? In fact, $\cos(36°) = \phi/2 = \dfrac{1+\sqrt{5}}{4}$

Here's how you can determine a closed (exact) expressions for cos(36°).

Start with an isosceles triangle *ABC* whose vertex angle is 36° and whose base has length 1. Bisect a base angle. Let $CD = x$ and mark the remaining sides accordingly. Then: $\Delta BAC : \Delta CBD$ → $\dfrac{BA}{CB} = \dfrac{BC}{CD}$ → $\dfrac{x+1}{1} = \dfrac{1}{x}$



Cross multiplying and using the quadratic formula, $x = \dfrac{\sqrt{5}-1}{2}$.

Using the law of cosines on ΔCBD, $x^2 = 1^2 + 1^2 - 2\cdot1\cdot1\cdot\cos36°$

Substituting for *x* and solving for cos36°, we have

$\cos36° = 1 - \dfrac{x^2}{2} = 1 - \dfrac{6-2\sqrt{5}}{8} = \dfrac{1+\sqrt{5}}{4} = \dfrac{\phi}{2}$.

Q.E.D

Euclid ended many of his proofs with these 3 letters, an abbreviation for the Latin phrase "quod erat demonstratum" (which was to be proven).

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