**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 1 - NOVEMBER 2011 SOLUTION KEY**

## Team Round

A) Let $P(x_1, y_1)$ denote $z + (i-2) = (a-2) + (b+1)i$. Then: $x_1 = a-2,\ y_1 = b+1$

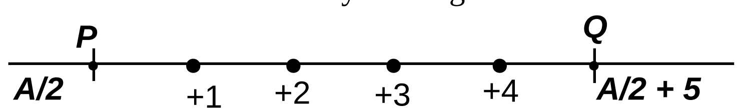
Let $Q(x_2, y_2)$ denote $\bar{z} + (1-i) = (a+1) + (-b-1)i$. Then: $x_2 = a+1,\ y_2 = -b-1$

$PQ = 5 \Rightarrow ((a-2)-(a+1))^2 + ((b+1)-(-b-1))^2 = 25 \Rightarrow (-3)^2 + (2b+2)^2 = 25$

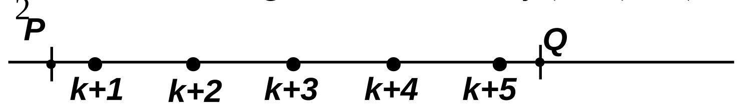
$\Rightarrow 4(b+1)^2 = 16 \Rightarrow b = -1 \pm 2 = 1, -3$

|z| = $\Rightarrow$ $a^2 + b^2 = 49 \Rightarrow a^2 = 49 - 1$ or $49 - 9 \Rightarrow a =$ $\underline{\boldsymbol{\pm 4\sqrt{3}, \pm 2\sqrt{10}}}$

B) If $0 < 2x - A < 10$, then the solutions in $x$ lie strictly between $A/2$ and $A/2 + 5$.
If $A$ is even, then so are the coordinates of the endpoints $P$ and $Q$. Since this a strict inequality, the endpoints are excluded and there are always 4 integer solutions.

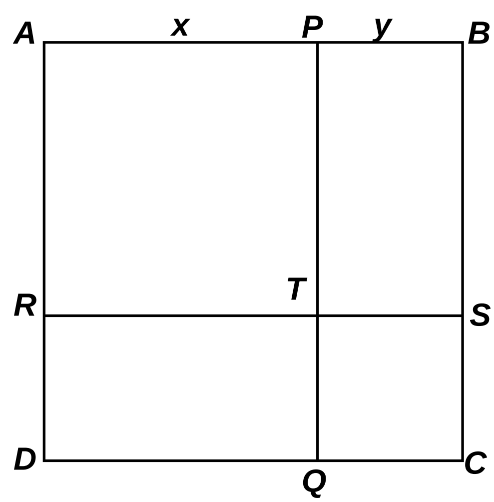



If $A$ is odd (say $2k + 1$ for some integer $k$), then the coordinates of the endpoints $P$ and $Q$ are $k + \frac{1}{2}$ and $k + 5\frac{1}{2}$. There are 5 integer solutions, namely $(k + 1) \ldots (k + 5)$




Thus, only D and E are true statements.

C) $x^2 + y^2 = k(2xy)$ Moving all terms to the left side and dividing both sides by $y^2$, we get a quadratic equation in $\frac{x}{y}$:




$$\left(\frac{x}{y}\right)^2 - 2k\left(\frac{x}{y}\right) + 1 = 0$$

Using the quadratic formula,

$$\frac{x}{y} = \frac{2k \pm \sqrt{(2k)^2 - 4}}{2} = k \pm \sqrt{k^2 - 1}.$$

Since $x > y$, the required ratio is greater than 1 and

$$\frac{x}{y} = \underline{\boldsymbol{k + \sqrt{k^2 - 1}}}$$

The same result is obtained if $x^2 - (2ky)x + y^2 = 0$ is treated as a quadratic in $x$ and solved for $x$ in terms of $k$ and $y$. Verify that you get $x = ky \pm y\sqrt{k^2 - 1}$ and the same result follows.