MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2010 SOLUTION KEY

Round 6 - continued

A) Either by formula $a_n = \dfrac{(2n-1)(2n)}{(2n+1)(2n+2)} = \dfrac{n(2n-1)}{(n+1)(2n+1)}$ or by brute force we have

$a_{12} \cdot a_{13} = \dfrac{12\cdot 23}{13\cdot 25}\cdot\dfrac{13\cdot 25}{14\cdot 27} = \dfrac{12\cdot 23}{14\cdot 27} = \underline{\mathbf{\dfrac{46}{63}}}$

B) Given $a_1 = \left(3\cdot 5,\ 2^1\cdot 3^2\right)$, $a_2 = \left(5\cdot 7,\ 2^2\cdot 3^1\right), a_3 = \left(7\cdot 9,\ 2^3\cdot 3^0\right), a_4 = \left(9\cdot 11,\ 2^4\cdot 3^{-1}\right)$, ... ,

the general term is $a_n = \left((2n+1)(2n+3), 2^n\cdot 3^{3-n}\right)$.

Thus, $a_{15} = \left(31\cdot 33, 2^{15}\cdot 3^{-12}\right)$ → (A, B, x, y) = **<u>(31, 33, 15, –12)</u>**.

C) $\dfrac{AB}{3AB} = \dfrac{3AB}{18A}$ → $A, B \neq 0$. Canceling, $\dfrac{1}{3} = \dfrac{B}{6}$ → $B = 2$

$A^3 - A - 2 - 1 = A + 3 - 2$ → $A^3 - 2A - 4 = (A-2)(A^2 + 2A + 2) = (A-2)\left((A+1)^2+1\right) = 0$

The only real solution is $A = 2$.

$(2 + 2i)^3 = 2^3(1 + i)^3 = 8(1 + i)^2(1 + i) = 8(2i)(1 + i) = -16 + 16i = C + Di$ → $\dfrac{C}{D} = \dfrac{-16}{16} = \underline{\mathbf{-1}}$

Alternate solution:
Once $A = B = 2$, $A + Bi$ in polar form is $(2, 45°)$. Cubing this produces $(8, 135°)$, so the real and imaginary parts are equal, but opposite in sign and the results above follow.