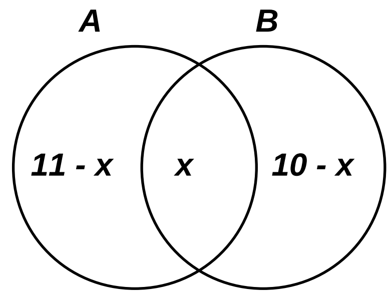
# MASSACHUSETTS MATHEMATICS LEAGUE
# CONTEST 6 – MARCH 2013 SOLUTION KEY

## Round 6

A) Let $n(X)$ denote the number of integers in set $X$.
Let $X$ U $Y$ denote the union of two sets, i.e. the integers in either $X$ or $Y$ or possibly both.
Consider two sets $A$ and $B$.
$n(A) = 11$, $n(B) = 10$ and $n(A \text{ U } B) = 15$




Let $x$ denote the number of integers in the overlap.
Then: $(11 - x) + x + (10 - x) = 21 - x = 15 \Rightarrow x = \underline{6}$.
Thus, there are 6 out of 15 integers which belong to both sets and the probability is $\frac{6}{15} = \frac{2}{5} = \underline{\mathbf{40\%}}$.

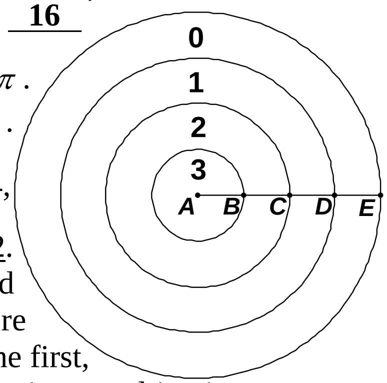
**Alternate interpretation: Appeal accepted**
The question required choosing an integer from one of the sets. This can be done by first deciding from which set ($A$ or $B$) to choose my number and then, choosing the number. For example, let $A$ be { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 } and $B$ be { 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 }. Under this interpretation, there are 21 elements that can be chosen (11 from $A$ and 10 from $B$) Each of the 6 integers, 6 thru 11, could be chosen from $A$ or from $B$. Thus, there are 12 possible successful draws from 21 possible, resulting in a percentage of $\frac{4}{7} \cdot 100 = \underline{\mathbf{\frac{400}{7}}} = \underline{\mathbf{57\frac{1}{7}}} = \underline{\mathbf{57.\overline{14285}}}$ %.

Any of the underlined answers are acceptable under this interpretation. **6/21 is rejected**, since this counts the 6 overlapping integers twice in the denominator and once only in the numerator.

B) Since the sum of the coefficients in the expansion of $(a+b)^n$ is $2^n$ and $8^4 = 2^{12}$, we know that $n = 12$. The expansion of $(a+b)^{12}$ has 13 terms, the middle term is the $7^{\text{th}}$ term, that is,

$$\binom{12}{6} a^6 b^6 = \frac{\cancel{12} \cdot 11 \cdot \cancel{10} \cdot 9 \cdot 8 \cdot 7}{1 \cdot \cancel{2} \cdot \cancel{3 \cdot 4} \cdot \cancel{5} \cdot 6} \cdot 4^6 \cdot \left(-\frac{1}{8}\right)^6 = 11 \cdot 3 \cdot 4 \cdot 7 \cdot 2^{12} \cdot 2^{-18} = \frac{3 \cdot 7 \cdot 11}{2^4} = \underline{\mathbf{\frac{231}{16}}}.$$

C) If $AB=BC=CD=DE=1$, the areas of the concentric circle are: $\pi, 4\pi, 9\pi, 16\pi$.
The areas of the central circle and the three outer rings are: $\pi, 3\pi, 5\pi, 7\pi$.




The probabilities of hitting the regions are: $\frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}$ To score a 4, the three darts must hit (in any order) 0, 1 and 3 or 0, 2 and 2 or 1, 1 and 2.
In the first case, the first dart could hit any one of the 3 regions, the second dart, any two of the remaining regions, resulting in 6 possible ways to score 4 points. In the last two cases, the singleton score could be the result of the first, second or third dart, but both of the remaining darts must hit the "other" region, resulting in only 3 possible ways to score 4 points. Thus, the probability is

$$6\left(\frac{7}{16} \cdot \frac{5}{16} \cdot \frac{1}{16}\right) + 3\left(\frac{7}{16} \cdot \frac{3}{16} \cdot \frac{3}{16}\right) + 3\left(\frac{5}{16} \cdot \frac{5}{16} \cdot \frac{3}{16}\right) = \frac{210 + 189 + 225}{16^3} = \frac{624}{4096} = \frac{2^4 \cdot 39}{2^{12}} = \frac{39}{256}$$

$\Rightarrow (P, Q) = \underline{\mathbf{(39, 256)}}$.