MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2015 SOLUTION KEY

Team Round

A) If $f(2x) = 2ax^2 + 6bx + c$, we can evaluate $f(1)$ and $f(8)$ by letting $x = \frac{1}{2}$ and $x = 4$ respectively.

Thus, $\begin{cases} f(1) = \frac{a}{2} + 3b + c = 15 \\ f(8) = 32a + 24b + c = 36 \end{cases}$. Subtracting $f(8)$ from $64\left(f(1)\right)$, we have $168b + 63c = 924$.

Since $168b + 63c = 924 \Leftrightarrow 84(2)b + 63c = 84(11)$, the right hand side of the equation is divisible by 84 and, therefore the left hand side must also be divisible by 84. This forces $63c$ to be a multiple of 84. Since the gcd(63, 84) = 21, c must be a multiple of 4.
Therefore, let $c = 4k$ and $168b + 63c = 924$ becomes

$168b + 252k = 924 \Leftrightarrow 2b + 3k = 11 \Leftrightarrow b = \frac{11-3k}{2}$

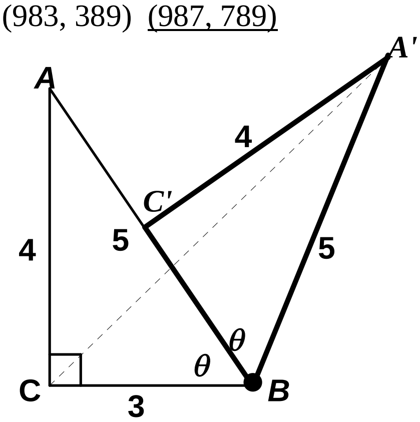
Since $2b$ is always even, $3k$ must be odd which forces k to be odd.

From $f(1)$, we have $a = 30 - 6b - 2c = 30 - 6\left(\frac{11-3k}{2}\right) - 2(4k) = 30 - 33 + 9k - 8k = k - 3$

Thus, minimizing $a + b + c$ is equivalent to minimizing $k - 3 + \frac{11-3k}{2} + 4k = \frac{5+7k}{2}$ for odd positive values of k. However, we must also verify that $a \geq -\frac{3b}{4}$.

$k = 1 \Rightarrow (a, b, c) = (-2, 4, 4)$ and $-2 \geq -\frac{3 \cdot 4}{4} = -3$. Thus, the minimum value of $a + b + c$ is **<u>6</u>**.

B) To be checked: <u>(381, 183)</u>, (781, 187), <u>(783, 387)</u>, <u>(981, 189)</u>, (983, 389) <u>(987, 789)</u>
The underlined pairs fail because each is divisible by 3.
The second pair fails since 781 is divisible by 11.
Only the 5th pair needs be exhaustively checked:
389 must be checked for divisibility by primes up to $\sqrt{389} < 20$
$\Rightarrow$ 7, 13, 17, 19 (all fail - it's prime)
983 must be checked for divisibility by primes up to $\sqrt{983} < 32$
$\Rightarrow$ 7, 13, 17, 19, 23, 29, 31 (all fail - it's prime)
Thus, the only ordered pair is **<u>(9, 3)</u>**.



C) Using the Law of Cosines, we have

$\left(A'C\right)^2 = 3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cos 2\theta = 34 - 30\cos 2\theta$. In ΔABC, $\cos\theta = \frac{3}{5} \Rightarrow \theta = Cos^{-1}\left(\frac{3}{5}\right)$.

Using the double-angle identity, $\cos 2\theta = 2\cos^2\theta - 1$, $\cos\left(2Cos^{-1}\left(\frac{3}{5}\right)\right) = 2\left(\frac{3}{5}\right)^2 - 1 = -\frac{7}{25}$

Substituting, $\left(A'C\right)^2 = 34 + 30 \cdot \frac{7}{25} = 34 + \frac{42}{5} = \underline{\mathbf{\frac{212}{5}}}$ or **<u>42.4</u>** .