MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2014 SOLUTION KEY

Round 1

A) Notice the first term requires no evaluation and the arithmetic challenge in the second term may be avoided, since the function *f* returns –7, regardless of its argument.

$f(x) = -7,\ g(x) = x^2 + 4$ and $h(x) = -2x$

$f\left(g\left(h(-3)\right)\right) + g\left(f\left(h(2014)\right)\right) + h\left(g(-3)\right) = -7 + g\left(-7\right) + h\left(13\right) = -7 + 53 - 26 = \underline{\mathbf{20}}$

B) Let $f(x) = mx + b$. Then: $f\left(f(x)\right) = m(mx+b) + b = m^2x + b(m+1) = 4x + 15$.

Thus, $m^2 = 4$ and $b\left(m+1\right) = 15$

$m = 2 \Rightarrow 3b = 15 \Rightarrow b = 5 \Rightarrow y = 2x + 5 \Rightarrow f\left(2\right) = \underline{\mathbf{9}}$.

$m = -2 \Rightarrow b = -15 \Rightarrow y = -2x - 15 \Rightarrow f\left(2\right) = \underline{\mathbf{-19}}$.

C) $f(x+h) = 2\left(x+h\right)^3 - 3\left(x+h\right)^2 + 8\left(x+h\right) - 1$

$= 2\left(x^3 + 3x^2h + 3xh^2 + h^3\right) - 3\left(x^2 + 2xh + h^2\right) + 8x + 8h - 1$

Regrouping the terms, we have

$2x^3 + \left(6h-3\right)x^2 + \left(6h^2 - 6h + 8\right)x + \left(2h^3 - 3h^2 + 8h - 1\right)$. We require that $6h - 3 = 0 \Rightarrow h = \frac{1}{2}$.

For $h = \frac{1}{2}$, $B = 6h^2 - 6h + 8 = \frac{6}{4} - 3 + 8 = 6.5$, $C = 2h^3 - 3h^2 + 8h - 1 = \frac{1}{4} - \frac{3}{4} + 4 - 1 = -\frac{1}{2} + 3 = 2.5$

Thus, $\left(h, B, C\right) = \underline{\left(\mathbf{\frac{1}{2}, \frac{13}{2}, \frac{5}{2}}\right)}$ or $\underline{\mathbf{(0.5, 6.5, 2.5)}}$.

Alternate Solution (Norm Swanson – HW)

Based only on the inspection of the coefficients, the sum of the roots of $f(x) = 2x^3 - 3x^2 + 8x - 1$ is $\frac{3}{2}$. Thus, for the sum of the roots of the new cubic polynomial to be zero, each root will have to be reduced by $\frac{1}{3}$ of that amount, namely $h = \frac{1}{2}$. Then:

$\left(x - .5\right)^3 - 3\left(x - .5\right)^2 + 8\left(x - .5\right) - 1 = 2x^3 + Bx + C$ Applying synthetic division,

$$\begin{array}{l}
\quad\ \underline{2\ \ -3\ \ \ 8\ \ -1} \\
.5\,|\,2\ \ -2\ \ \ 7\ \ \ \mathbf{2.5} = C \\
.5\,|\,2\ \ -1\ \ \ \mathbf{6.5} = B \\
.5\,|\,2\ \ \ \mathbf{0} = \text{sum of roots} \\
.5\,|\,\mathbf{2}\ = \text{lead coefficient}
\end{array}$$

Thus, the new polynomial is $2x^3 + 0x^2 + 6.5x + 2.5$ and the same result follows. You should think about why this shortcut works and avoids multiplying out the left-hand side.