Team Round

A) $\begin{cases} x^2 + xy + y^2 = 8 \\ x + y = xy + 2 \end{cases}$ → $\begin{cases} (1) \; x^2 + 2xy + y^2 = (x+y)^2 = 8 + xy \\ (2) \; x + y = xy + 2 \end{cases}$

Subtracting (2) from (1), $(x+y)^2 - (x+y) - 6 = 0$ → $((x+y)-3)((x+y)+2) = 0$

→ Case 1: $(x + y) = 3$ (and $xy = 1$) or Case 2: $(x + y) = -2$ (and $xy = -4$)
Since it is given that x and y have the same sign, case 2 is eliminated.
[An aside: Case 2 generates solutions of $(-1+\sqrt{5}, -1-\sqrt{5})$ and $(-1-\sqrt{5}, -1+\sqrt{5})$ which satisfy the
   system of equations, but are rejected because the x-coordinate and y-coordinates have opposite signs.]

Substituting $y = \dfrac{1}{x}$ in equation (2), $x + \dfrac{1}{x} = 3$ → $x^2 - 3x + 1 = 0$ → $x = \dfrac{3 \pm \sqrt{5}}{2}$ and

$y = \dfrac{2}{3 \pm \sqrt{5}} \cdot \dfrac{3 \mp \sqrt{5}}{3 \mp \sqrt{5}} = \dfrac{2(3 \mp \sqrt{5})}{9 - 5} = \dfrac{3 \mp \sqrt{5}}{2}$

→ two ordered pairs $\underline{\left(\dfrac{3+\sqrt{5}}{2}, \dfrac{3-\sqrt{5}}{2}\right), \left(\dfrac{3-\sqrt{5}}{2}, \dfrac{3+\sqrt{5}}{2}\right)}$

Alternate Solution (using the Method of Symmetric Equations):
Note: Replacing x by y and y by x does not change the system of equations.
This technique eliminates the xy- term.

Let $x = u + v$ and $y = u - v$. Then:

$\begin{cases} (u+v)^2 + (u+v)(u-v) + (u-v)^2 = 8 \\ u+v - (u+v)(u-v) + u - v = 2 \end{cases}$ → $\begin{cases} 3u^2 + v^2 = 8 \\ -u^2 + v^2 + 2u = 2 \end{cases}$

Subtracting, $4u^2 - 2u = 6$ → $2u^2 - u - 3 = (2u - 3)(u + 1) = 0$ → $u = \dfrac{3}{2}, -1$

Substituting, $3\left(\dfrac{3}{2}\right)^2 + v^2 = 8$ → $v^2 = 8 - \dfrac{27}{4} = \dfrac{5}{4}$ and $3(-1)^2 + v^2 = 8$ → $v^2 = 5$

$(u, v) = \left(\dfrac{3}{2}, +\dfrac{\sqrt{5}}{2}\right)$ → $(x, y) = \left(\dfrac{3+\sqrt{5}}{2}, \dfrac{3-\sqrt{5}}{2}\right)$

$(u, v) = \left(\dfrac{3}{2}, -\dfrac{\sqrt{5}}{2}\right)$ → $(x, y) = \left(\dfrac{3-\sqrt{5}}{2}, \dfrac{3+\sqrt{5}}{2}\right)$ (both are ok since x and y are each positive)

$(u, v) = \left(-1, +\sqrt{5}\right)$ → $(x, y) = \left(-1+\sqrt{5}, -1-\sqrt{5}\right)$

$(u, v) = \left(-1, -\sqrt{5}\right)$ → $(x, y) = \left(-1-\sqrt{5}, -1+\sqrt{5}\right)$ (both rejected since x and y have opposite signs)