MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014 SOLUTION KEY

Team Round

B) Squaring both sides and cross multiplying,

$16=(4x-1)^2(4x+1)=(4x-1)\big((4x-1)(4x+1)\big)=(4x-1)\big(16x^2-1\big)=64x^3-16x^2-4x+1$

Thus, $64x^3-16x^2-4x-15=0$

Using synthetic substitution, $\begin{array}{c|cccc} & 64 & -16 & -4 & -15 \\ & & 48 & 24 & 15 \\ \hline \frac{3}{4} & 64 & 32 & 20 & 0 \end{array}$, we have $(4x-3)(4)\left(4x^2+8x+5\right)$

and the quadratic factor has no additional real roots $\left(8^2-4\cdot 20<0\right)$. The only real root is $\underline{\mathbf{\frac{3}{4}}}$.

C) As an identity or considering $\cos(3x)$ as $\cos(x+2x)$ and expanding using double angle identities, we have $\cos(3x)=4\cos^3 x-3\cos x$. Then:

$$\cos^4 x+6\cos^2 x+7\cos x+\cos 3x=-\frac{15}{16}\Leftrightarrow \cos^4 x+6\cos^2 x+7\cos x+\left(4\cos^3 x-3\cos x\right)+1=\frac{1}{16}$$

Rearranging terms, $\cos^4 x+4\cos^3 x+6\cos^2 x+4\cos x+1=\frac{1}{16}$

and recognizing that the coefficients on the left side are terms in Pascal's triangle, we realize this is equivalent to

		1		
	1		1	
	1	2	1	
1	3		3	1
1	4	6	4	1

$(\cos x+1)^4=\frac{1}{16}$.

$\Rightarrow \cos x+1=\pm\frac{1}{2}\Rightarrow \cos x=-\frac{1}{2},\cancel{-\frac{3}{2}}\Rightarrow x=\pm\frac{2\pi}{3}+2n\pi$ (reference value $\frac{\pi}{3}$ - quadrants 2 and 3)

$n=0\Rightarrow x=\underline{\mathbf{-\frac{2\pi}{3}}}$ $\quad n=-1\Rightarrow x=\underline{\mathbf{-\frac{4\pi}{3}}}$.

D) $x=\dfrac{-k\pm\sqrt{k^2-4k-44}}{2}=\dfrac{-k\pm\sqrt{(k-2)^2-48}}{2}\Rightarrow (k-2)^2-48$ must be a perfect square p.

Thus, $(k-2)^2-48=p^2\Rightarrow (k-2)^2-p^2=(k-2+p)(k-2-p)=48$

Since these two factors must be integers and have the same parity, we ignore (1)(48) and (3)(16) and restrict our attention to (2)(24), (4)(12) and (6)(8).

(2)(24): $(k+p)=26$ and $k-p=4\Rightarrow k=\underline{\mathbf{15}}$

(4)(12): $(k+p)=14$ and $k-p=6\Rightarrow k=\underline{\mathbf{10}}$

(6)(8): $(k+p)=10$ and $k-p=8\Rightarrow k=\underline{\mathbf{9}}$