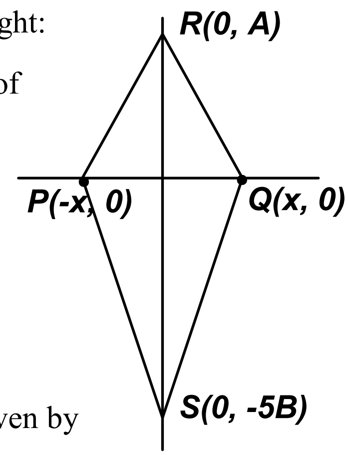
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2010 SOLUTION KEY

Team Round

A) The graph of the bounded region is represented in the diagram at the right:



$Q(x, 0)$ is $\left(\frac{A}{k},0\right)$ from f's point of view and $\left(\frac{5D}{4k},0\right)$ from g's point of view. Evaluating the determinants, equating and canceling out the common factors of k in the denominator,

$\frac{1+k^2}{k}=\frac{5\left(k^2-1\right)}{4k}$ ➔ $4 + 4k^2 = 5k^2 - 5$ ➔ $k^2 = 9$ ➔ $k = 3$.

$P\left(-\frac{10}{3},0\right)$, $Q\left(\frac{10}{3},0\right)$, $R(0,10)$ and $S(0,-40)$

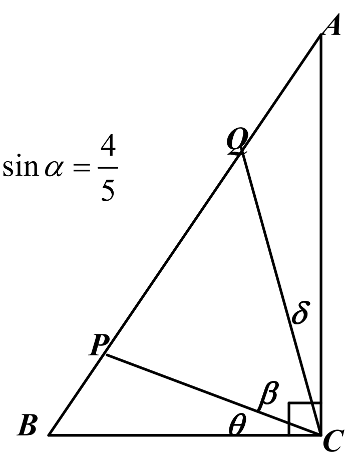
$PRQS$ is a quadrilateral w/perpendicular diagonals; hence its area is given by $\frac{1}{2}d_1d_2$ ➔ $\frac{1}{2}PQ\cdot RS = \frac{1}{2}\left(\frac{20}{3}\right)(10-(-40))=\frac{10}{3}\cdot 50 = \underline{\mathbf{\frac{500}{3}}}$.

B) $\frac{\sqrt[4]{2\cdot 3^a}\cdot\sqrt[6]{3\cdot 2^b}}{\sqrt[3]{12}}=\frac{\left(2\cdot 3^a\right)^{\frac{1}{4}}\left(3\cdot 2^b\right)^{\frac{1}{6}}}{\left(2^2\cdot 3\right)^{\frac{1}{3}}}=\frac{\left(2\cdot 3^a\right)^{\frac{3}{12}}\left(3\cdot 2^b\right)^{\frac{2}{12}}}{\left(2^2\cdot 3\right)^{\frac{4}{12}}}=\left(\frac{2^3 3^{3a}3^2 2^{2b}}{2^8 3^4}\right)^{\frac{1}{12}}=\left(2^{2b-5}3^{3a-2}\right)^{\frac{1}{12}}$

$=\sqrt[12]{2^{2b-5}\cdot 3^{3a-2}}$.

Since this expression must equal $\sqrt[r]{2^p 3^q}$, we know that $r = 12$, $p = 2b - 5 < 12$, $q = 3a - 2 < 12$. Since each of the original radicals was also in simplest form, $a = 1, 2, 3$ and $b = 1, 2, 3, 4, 5$. Thus, $p = 1, 3$ or 5 and $q = 1, 4, 7$.
Since $pqr = 12\,pq$, the minimum value occurs for $(p, q) = (1,1)$ and the maximum occurs for $(p, q) = (5, 7)$. (count, Max, min) = **<u>(9, 420, 12)</u>**.

C) Think θ: 3 – 4 – 5 and δ: 7 – 24 – 25
Let $\alpha = \theta + \delta$. Then



$\cos\alpha=\cos(\theta+\delta)=\cos\theta\cos\delta-\sin\theta\sin\delta=\frac{4}{5}\cdot\frac{24}{25}-\frac{3}{5}\cdot\frac{7}{25}=\frac{75}{125}=\frac{3}{5}$ ➔ $\sin\alpha=\frac{4}{5}$

$\sin(\beta+(\theta+\delta))=\sin(\beta+\alpha)=\sin\beta\cos\alpha+\cos\beta\sin\alpha=\sin(90°)=1$

If $\sin\beta=k$, then $\cos\beta=\sqrt{1-k^2}$ ➔ $k\left(\frac{3}{5}\right)+\sqrt{1-k^2}\left(\frac{4}{5}\right)=1$

➔ $4\sqrt{1-k^2}=5-3k$ ➔ $16\left(1-k^2\right)=25-30k+9k^2$ ➔

$25k^2-30k+9=(5k-3)^2=0$ ➔ $k=\sin\beta=\underline{\mathbf{\frac{3}{5}}}$.

Does this imply that $\overrightarrow{CP}$ actually bisects $\angle QCB$?