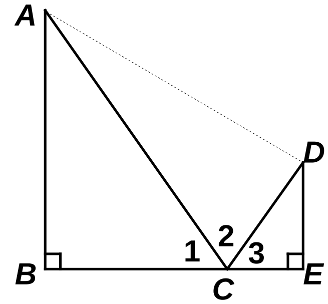
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015 SOLUTION KEY

Round 1

A) ΔDEF must be a 3-4-5 triangle. Since $141 = 47 \cdot 3$,
$(b,c) = (47\cdot 4, 47\cdot 5) = \underline{(\mathbf{188,\ 235})}$.

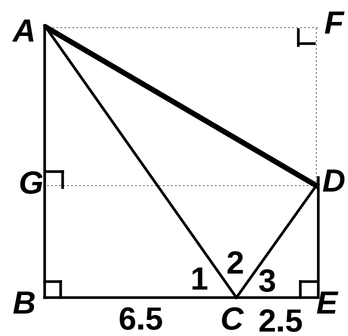


B) $m\angle 1 = m\angle 2 = m\angle 3 = 60°$,
$BC + CE = 2CE + 4 = 9 \Rightarrow CE = 2.5, BC = 6.5$ Both of these sides are opposite a 30° angle in a 30-60-90 right triangle. Thus, the hypotenuses are 5 and 13. Applying the Law of Cosines to ΔACD,

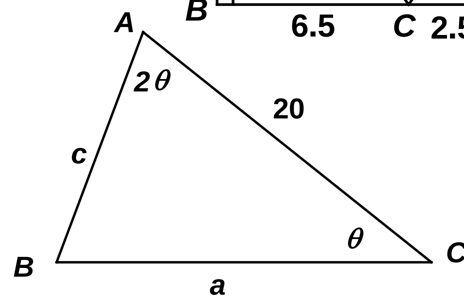
$$AD^2 = 5^2 + 13^2 - 2\cdot 5\cdot 13\cos 60° = 25 + 169 - 130\cdot\frac{1}{2} = 194 - 65 = 129 \Rightarrow AD = \underline{\sqrt{\mathbf{129}}}.$$

Solution #2 (Norm Swanson – Hamilton Wenham – retired)
Construct $\overline{DG}$ and point F so that $\overline{DG} \perp \overline{AB}$, $\overline{AF} \perp \overline{FDE}$.
$AF = 9$, ΔDEC and ΔABC are 30-60-90 right triangles, $FE = AB$.
$FD = AB - GB = FE - DE = 6.5\sqrt{3} - 2.5\sqrt{3} = 4\sqrt{3}$ and
Applying the Pythagorean Theorem to ΔFAD,

$$AD^2 = 9^2 + \left(4\sqrt{3}\right)^2 = 81 + 48 = 129 \Rightarrow AD = \underline{\sqrt{\mathbf{129}}}$$



C) Since both θ and 2θ are angles in ΔABC, $\theta < 90°$.



$$\sin\theta = \frac{\sqrt{7}}{4} \Rightarrow \cos\theta = +\sqrt{1 - \left(\frac{\sqrt{7}}{4}\right)^2} = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

$$\sin A = \sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{\sqrt{7}}{4}\right)\cdot\frac{3}{4} = \frac{3\sqrt{7}}{8}$$

$$\cos B = \cos^2 C \Rightarrow \cos B = 1 - \sin^2\theta = 1 - \frac{7}{16} = \frac{9}{16}$$

$$\sin B = +\sqrt{1 - \left(\frac{9}{16}\right)^2} = \sqrt{\frac{256-81}{16^2}} = \sqrt{\frac{175}{16^2}} = \frac{5}{16}\sqrt{7}$$. Applying the Law of Sines,

$$\frac{\sin A}{BC} = \frac{\sin B}{AC} = \frac{\sin C}{AB} \Rightarrow \frac{3\sqrt{7}}{8BC} = \frac{5\sqrt{7}}{16\cdot 20} = \frac{3}{4AB} \Rightarrow (AB, BC) = (16, 24)$$

Thus, the area of ΔABC is $\frac{1}{2}ac\sin B = \frac{1}{2}\cdot 24\cdot 16\cdot\frac{5\sqrt{7}}{16} = 60\sqrt{7} \Rightarrow (K, L) = \underline{(\mathbf{60, 7})}$.

Solution #2 (Applying the lesser known formula for the area of a triangle: $\frac{b^2}{2}\cdot\frac{\sin A\sin C}{\sin B}$)

$$\frac{b^2}{2}\cdot\frac{\sin\theta\sin 2\theta}{\sin B} = \frac{b^2\sin^2\theta\cos\theta}{\sin B} = \frac{400\left(\frac{7}{\cancel{16}}\right)\left(\frac{3}{4}\right)}{\frac{5}{\cancel{16}}\sqrt{7}} = \frac{20\cdot 3\cdot 7}{\sqrt{7}} = 60\sqrt{7}.$$