MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2009 SOLUTION KEY

Round 1

A) Computing the determinant on the left side of the equation:

$$\begin{matrix} 1 & -3 & 5 & 1 & -3 \\ 0 & 2 & 1 & 0 & 2 \\ x & 2 & 8 & x & 2 \end{matrix} = (1\cdot 2\cdot 8 - 3\cdot 1\cdot x + 5\cdot 0\cdot 2) - (x\cdot 2\cdot 5 + 2\cdot 1\cdot 1 + 8\cdot 0\cdot -3)$$

$= 16 - 3x - 10x - 2 = 14 - 13x$

Therefore, $14 - 13x = 38 - 5x \rightarrow -8x = 24 \rightarrow x =$ **$\underline{-3}$**

B) If the 3rd equation is a linear combination of the 1st two equations (or any two equations equal a linear combination of the remaining equation) then there are an infinite number of solutions. (Each equation represents a plane and the planes intersect along a common line.)
Find constants p and q so that (3rd) = p(1st) + q(2nd).

Equating coefficients of x and y, $\begin{cases} 7 = p + 2q \\ 8 = 2p + q \end{cases} \rightarrow (p, q) = (3, 2)$

Note that the coefficients of z are also equal. $-3p + 2q = -5$
Equating the constant terms, $5p + 7q = a \rightarrow a =$ **$\underline{29}$**

Alternate solution:

The system $\begin{cases} (1) \quad x + 2y - 3z = 5 \\ (2) \quad 2x + y + 2z = 7 \\ (3) \quad 7x + 8y - 5z = a \end{cases}$ can be easily be solved using matrix algebra.

Here's the background. This technique allows the development of algorithms for solving systems of any number of linear equations mechanically on a computer.

Let M denote the matrix of coefficients, i.e. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 2 \\ 7 & 8 & -5 \end{bmatrix}$ and $\|M\|$ the determinant of M.

(M_x, M_y and M_z) denote related matrices, where the x-, y- and z-coefficients have been replaced by the constants on the right hand side of the equations, i.e.

$\begin{bmatrix} 5 & 2 & -3 \\ 7 & 1 & 2 \\ a & 8 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 5 & -3 \\ 2 & 7 & 2 \\ 7 & a & -5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 7 \\ 7 & 8 & a \end{bmatrix}$ and $\left(\|M_x\|, \|M_y\|, \|M_z\|\right)$ the respective determinants.

The solution of the system is $\left(\dfrac{\|M_x\|}{\|M\|}, \dfrac{\|M_y\|}{\|M\|}, \dfrac{\|M_z\|}{\|M\|}\right)$, provided $M \neq 0$.

If $M = 0$ and $M_x \neq 0$ (M_y or $M_z \neq 0$), then the system has <u>no</u> solution.
If $M = 0$ and $M_x = 0$ (M_y or $M_z = 0$), then the system has an <u>infinite</u> number of solutions.