MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016 SOLUTION KEY

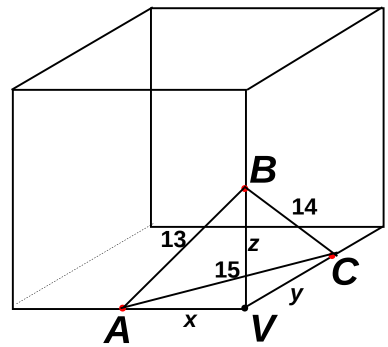
Team Round

A) The area of ΔABC may be determined by $\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$, where *a*, *b*, and *c* denote side-lengths and *s* denotes the semi-perimeter (Heron's Formula).

$$s = \frac{13+14+15}{2} = 21 \Rightarrow \text{Area} = \sqrt{21\cdot 8\cdot 7\cdot 6} = \sqrt{2^4 3^2 7^2} = 84$$

Let *h* denote the distance from *V* to the plane *ABC*. Then:

$$\text{Vol} = \frac{1}{3}h(84) = 28h.$$



But, using ΔBAV as the base and $\overline{VC}$ as the height,

$$\text{Vol} = \frac{1}{3}\left(\frac{1}{2}xy\right)z = \frac{xyz}{6}$$

$$\begin{cases} x^2+y^2=225 \\ x^2+z^2=169 \\ y^2+z^2=196 \end{cases} \Rightarrow (x,y,z) = \left(3\sqrt{11}, 3\sqrt{14}, \sqrt{70}\right)$$

Thus, $28h = \dfrac{3\sqrt{11}\cdot 3\sqrt{14}\cdot\sqrt{70}}{6} = \dfrac{\cancel{2\cdot 3}\cdot 3\cdot 7\sqrt{5\cdot 11}}{\cancel{6}} = 21\sqrt{55} \Rightarrow h = \dfrac{3}{4}\sqrt{55} \Rightarrow m+n+r = \underline{\mathbf{62}}$

Do ***integers*** *x*, *y* and *z* exist so that triangle *ABC* has sides of integer length, i.e. *a*, *b* and *c* are also integers?

If yes, then $\begin{cases} x^2+y^2=a^2 \\ y^2+z^2=b^2 \\ x^2+z^2=c^2 \end{cases}$ $\Rightarrow x^2 - z^2 = a^2 - b^2 \Rightarrow x^2 = \dfrac{a^2-b^2+c^2}{2}$.

What do you think?