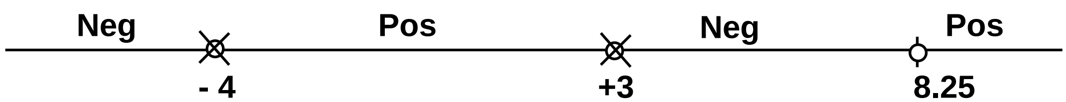
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2014 SOLUTION KEY

Team Round - continued

D) $\dfrac{7}{x+4}-\dfrac{3}{x-3} = \dfrac{7(x-3)-3(x+4)}{(x-3)(x+4)}=\dfrac{(4x-33)}{(x-3)(x+4)}$

The sign of the difference equals the sign of this equivalent quotient.
Whereas the sign of a difference can be hard to determine, the sign of a quotient is easy to determine, especially when both the numerator and denominators have been factored.
We simply count the number of negative factors!
As x increases, each of the parenthesized expressions increases and becomes positive.
The critical values are -4, 3 and 33/4 (= 8.25).



At the extreme left, all three expressions are negative (hence the quotient is also).
At the extreme right, all three expressions are positive (hence the quotient is also).

Testing between -4 and 3, of the three expressions only $(x + 4)$ becomes positive and the sign of the quotient is determined by two negatives and one positive; hence the quotient is positive.

Testing between 3 and 8.25, of the three expressions only $(4x - 33)$ remains negative and the sign of the quotient is determined by one negative and two positives; hence the quotient is negative.

Thus, as x increases from left to right along the number line, the sign of the quotient changes as we pass each critical point. In this case NEG – POS – NEG – POS.
Thus, $x = -3$ and substituting $y = \dfrac{4(-3)-33}{(-3-3)(-3+4)}=\dfrac{-45}{-6}=\dfrac{15}{2}$ or 7.5 $\Rightarrow$ **<u>(–3, 7.5)</u>** or equivalent

E) By long division, $\dfrac{n^3-32}{n^2+30}=n-2\left(\dfrac{15n+16}{n^2+30}\right)$

Since we know n is even, let $n = 2k$, where k is an integer.
Then $\left(\dfrac{15n+16}{n^2+30}\right)=\left(\dfrac{30k+16}{4k^2+30}\right)=\dfrac{15k+8}{2k^2+15}$. For this last fraction to be an integer, the numerator must be greater than or equal to the denominator.

$15k+8\ge 2k^2+15 \Rightarrow\ 2k^2-15k+7\le 0 \Rightarrow\ (2k-1)(k-7)\le 0 \Rightarrow\ \dfrac{1}{2}\le k\le 7 \Rightarrow n=2,4,...,14$.

Since we are looking for the *largest* possible even integer, we now resort to brute force, starting with $k = 7 \Rightarrow \dfrac{105+8}{98+15}=1$. Bingo! Thus, $n =$ **<u>14</u>**.
(In fact, 14 is the only even value of n, for which the given quotient is an integer.)