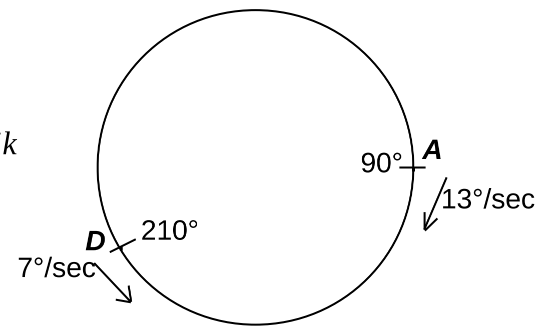
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017 SOLUTION KEY

Team Round - continued

B) $90+13k=210-7k \Rightarrow k=6$ [$90+13\cdot 6=210-7\cdot 6=168$ verifies the smallest solution.]

In general, we require that $\begin{cases} 90+13k \equiv n \mod 360 \\ 210-7k \equiv n \mod 360 \end{cases}$



or, $\begin{cases} 90+13k-n=360A \\ 210-7k-n=360B \end{cases} \Leftrightarrow -n=360A-90-13k=360B-210+7k$

$\Leftrightarrow 120-20k=360B-360A$

$\Leftrightarrow k-6=18(A-B)$

$B=A \Rightarrow k=6$

$A=B+1 \Rightarrow k=24$

$A=B+2 \Rightarrow k=42$

The sequence 6, 24, 42, … is generated by $6(3n-2)$ for $n=1,2,3,\ldots$.

Thus, our meeting points are generated by

$90+13(6(3n-2))=90+78(3n-2)=78(3n)-66=\boxed{6(39n-11) \mod 360}$

The sequence generated by this formula is

168, 42, 267, 150, 24, 258, 132, 6, 240, 114, 348, 222, 96, 330, 204, 78, 312, 186, 60, 294, <u>168</u>

$n=$ **<u>20</u>** distinct values before the cycle repeats.

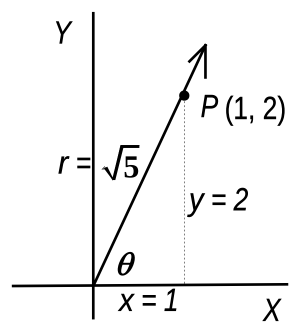
Is there an easy/easier way to evaluate this sequence or verify the number of distinct solutions without first substituting successive values of n, finding the solution, then subtracting 360s until the result is between 0° and 360° ? Inquiring minds want to know! Send your ideas to olson.re@gmail.com

Is it a coincidence that 13 + 7 = 20?

C) Think of the left hand side as sin(A + B) + sin(A – B) = 2sin$A$cos$B$.

$Sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$ denotes a quadrant 1 value ("angle") as indicated in the diagram to the right and the tangent of this value is clearly 2.

Using reduction formulas, cos(270° + x) = sin(x). Therefore,



$\sin 3x + \sin x = \tan\left(Sin^{-1}\frac{2}{\sqrt{5}}\right)\cdot\cos(270°+x)$

$\Rightarrow 2\sin 2x\cos x = 2\sin x$

$\Rightarrow 4\sin x\cos^2 x - 2\sin x = 2\sin x\left(2\cos^2 x - 1\right) = 0$

$\Rightarrow \sin x = 0,\ \cos x = \pm\frac{\sqrt{2}}{2}$

$\Rightarrow x =$ **<u>0, 180, 45, 135, 225, 315</u>**

(The degree symbols may be included.)