MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2011 SOLUTION KEY

Team Round
Proof of the conjectures

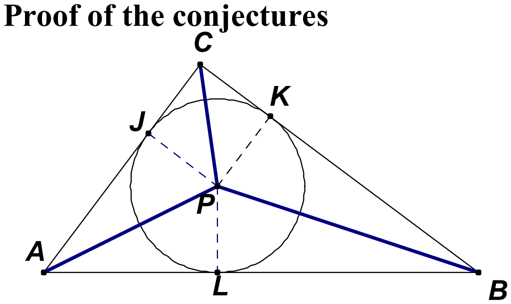

Diagram #1

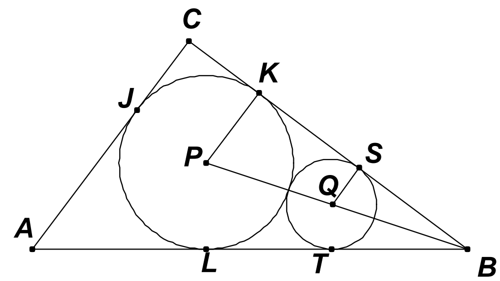

Diagram #2

Conjecture #1 (Diagram #1):
Let P denote the center of the larger circle with radii R in ΔABC with sides $BC = a$, $AC = b$ and $AB = c$.
The area of ΔABC equals the sum of the areas of Δs BPC, APC and APB.

Using $A(\Delta) = \frac{1}{2}bh$, $A(\Delta ABC) = \frac{1}{2}Ra + \frac{1}{2}Rb + \frac{1}{2}Rc = \left(\frac{a+b+c}{2}\right)R$.

Since ABC is a right triangle with hypotenuse $AB = c$ and legs $BC = a$ and $AC = b$,

we have $\frac{1}{2}ab = \left(\frac{a+b+c}{2}\right)R \rightarrow R = \boxed{\frac{ab}{a+b+c}}$.

The equivalent formula $\frac{a+b-c}{2}$ can be verified by showing the cross products are equal.

$\frac{ab}{a+b+c} = \frac{a+b-c}{2} \rightarrow (a + b + c)(a + b - c) = ((a + b) + c)((a + b) - c) = (a + b)^2 - c^2 = a^2 + 2ab + b^2 - c^2$

ΔABC is a right triangle $\rightarrow a^2 + b^2 = c^2$. Regrouping, $(a^2 + b^2 - c^2) + 2ab = 0 + 2ab = 2ab$.
Alternately, note that $CKPJ$ is a square, $R = CK$ and use argument similar to that used below to find BK.
Q.E.D

Conjecture #2 (Diagram #2):
As tangents to circle P from external points A, B and C, $AJ = AL$, $BK = BL$ and $CK = CJ$.
The perimeter of ΔABC may be expressed as $2AJ + 2CJ + 2BK = 2AC + 2BK$.

Thus, $a + b + c = 2b + 2BK$ or $BK = \frac{a+c-b}{2}$. Similarly, $AJ = \frac{b+c-a}{2}$ and $CK = \frac{a+b-c}{2}$

Now, since P and Q both lie on the bisector of $\angle ABC$, B, Q and P must be collinear.
In right triangle BPK, $PB^2 = PK^2 + BK^2$ or

$$PB^2 = R^2 + \left(\frac{a+c-b}{2}\right)^2 = \left(\frac{a+b-c}{2}\right)^2 + \left(\frac{a+c-b}{2}\right)^2 = \frac{a^2+b^2+c^2-2bc}{2} = \frac{2c^2-2bc}{2} = c(c-b)$$

Since $\Delta BQS \sim \Delta BPK$, $\frac{QB}{PB} = \frac{PB-(R+r)}{PB} = \frac{SQ}{KP} = \frac{r}{R} \rightarrow 1 - \frac{R+r}{PB} = \frac{r}{R} \rightarrow R(PB) - R(R+r) = rPB$

$\rightarrow rPB + rR = R(PB) - R^2 \rightarrow r\left(PB + R\right) = R\left(PB - R\right)$

$\rightarrow r = R\left(\frac{PB-R}{PB+R}\right)$

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