MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2007 SOLUTION KEY

Team Round

A) To find the points of intersection, substitute x^2 for y: $2x - 5x^2 + 3 = 0$
$\rightarrow 5x^2 - 2x - 3 = (5x + 3)(x - 1) = 0 \rightarrow x = -3/5$ or $+1$.

Thus, the points of intersection are: $A\left(\frac{-3}{5}, \frac{9}{25}\right)$ and $B(1, 1)$.

The slope of the segment $\overline{AB}$ is $\dfrac{\frac{9}{25}-1}{\frac{-3}{5}-1} = \dfrac{-16/25}{-8/5} = \dfrac{16}{40} = \dfrac{2}{5}$.

The slope of the perpendicular is $+\dfrac{-5}{2}$.

The equation of the perpendicular line is of the form $5x + 2y = c$, for some constant c.
Since the point $P(7, 2)$ is on this line, its coordinates must satisfy the equation and we have
$5(7) + 2(2) = c \rightarrow c = 39 \rightarrow$ equation: <u>**$5x + 2y = 39$**</u>

B) Assume $x^2 + kx + k + 11$ factors to $(x + a)(x + b)$, where a and b are integers.

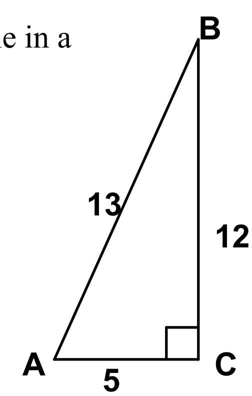
By the quadratic formula, $x = \dfrac{-k \pm \sqrt{k^2 - 4k - 44}}{2}$

To insure rational roots, $k^2 - 4k - 44 = (k - 2)^2 - 48$ must denote the perfect square of an integer, say t^2.
Thus, $(k - 2)^2 = t^2 + 48 = r^2 \rightarrow r^2 - t^2 = (r + t)(r - t) = 48$

$(r + t)$	=	48	24	16	12	8
$(r - t)$	=	1	2	3	4	6
Adding/dividing by 2 $\rightarrow$ r =		impossible	13	impossible	8	7 $\rightarrow t = 11, 4$ and 1

Thus, $(k - 2)^2 = 169, 64$ or $49 \rightarrow k = 2 \pm 13, 2 \pm 8$ or $2 \pm 7 \rightarrow k = 15, -11, 10, -6, 9, -5$
Adding these 6 possible values $\rightarrow$ <u>**12**</u>

C) $5\sin(x) + 12\cos(x) = 13(\frac{5}{13}\sin x + \frac{12}{13}\cos x)$. Let A denote the larger acute angle in a 5-12-13 triangle (see diagram). Using the $\sin(A + B)$ expansion, this simplifies to $13\sin(x + A)$ which has a maximum value of 13, when $x + A = 90°$ (or any coterminal value)



If $-13 \le k^2 - k + 1 \le 13$, there will be a solution.

To determine the maximum value of k, we need only solve
$k^2 - k + 1 \le 13$, since $-13 \le k^2 - k + 1 \rightarrow k^2 - k + 14 \ge 0 \rightarrow (k - ½)^2 + 55/4 \ge 0$
which is true for all real values of k.
$k^2 - k - 12 = (k + 3)(k - 4) \le 0 \rightarrow -3 \le k \le 4$ and the maximum value of k is <u>**4**</u>.