MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014 SOLUTION KEY

Round 3

A) $\left(x^2-16x+\quad\right)+\left(y^2+10y+\quad\right)=11 \Rightarrow$

$\left(x^2-16x+8^2\right)+\left(y^2+10y+5^2\right)=11+64+25=100$

$\Rightarrow \left(x-8\right)^2+\left(y^2+5\right)^2=10^2 \Rightarrow$ Center: **$\underline{(8,-5)}$** Radius: **$\underline{10}$**

B) Given $A(-3,1)$ and a slope of $-\dfrac{3}{2}$, in point-slope form, $L_1:(y-1)=-\dfrac{3}{2}(x+3) \Leftrightarrow 3x+2y=-7$.

Given $B(4,3)$ and $C(0,7)$, the midpoint of $\overline{BC}$ is $(2,5)$ and the slope of $\overline{BC}$ is $\dfrac{7-3}{0-4}=-1$.

Thus, L_2, the perpendicular bisector of $\overline{BC}$, has a slope of +1, and

$L_2:(y-5)=+1(x-2) \Leftrightarrow x-y=-3$.

Solving $\begin{array}{l} L_1:\ 3x+2y=-7 \\ L_2:\ x-y=-3 \end{array}$ simultaneously, (x, y) = $\underline{\left(-\mathbf{\dfrac{13}{5}},\mathbf{\dfrac{2}{5}}\right)}$.

C) Given: $A(-2, 3)$, $B(6, 5)$, and $C(8,1)$ For $P(x, 0)$,

$PA^2+PB^2+PC^2=(x+2)^2+9+(x-6)^2+25+(x-8)^2+1=3x^2-24x+139$

Factoring out a 3 and completing the square,

$3\left(x^2-8x\ \underline{+16}\right)+139\ \underline{-48}=3(x-4)^2+91$

Since, for $x \neq 4$, $3(x-4)^2>0$, $x = 4$ produces the minimum value of $91 \Rightarrow (x,N)$ = **$\underline{(4, 91)}$**.

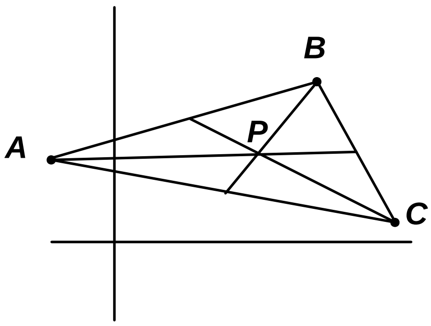
This problem generalizes nicely for an arbitrary point $P(x, y)$.

Show that $PA^2+PB^2+PC^2=3\left(\left(x-4\right)^2+\left(y-3\right)^2\right)+64$, so the minimum value is 64 and it occurs for the point $P(4, 3)$.

This point is the centroid of ΔABC, the point of intersection of the three medians of ΔABC.

This result generalizes to <u>any</u> triangle!

In general, how is the minimum value determined?



Specifically, for ΔABC with vertices $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$, the minimum value of

$$PA^2+PB^2+PC^2=3\left(\left(x-\frac{x_1+x_2+x_3}{3}\right)^2+\left(y-\frac{y_1+y_2+y_3}{3}\right)^2\right)+k$$

Describe how to compute k in terms of x_1, x_2 and x_3.