MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2013 SOLUTION KEY

Team Round

A) $\left(\dfrac{4+4i}{5}\right)^{4k} = \dfrac{4^{4k}\left((1+i)^2\right)^{2k}}{5^{4k}} = \dfrac{2^{8k}(2i)^{2k}}{5^{4k}} = \dfrac{2^{8k}2^{2k}\left(i^2\right)^k}{5^{4k}} = \dfrac{(-1)^k 2^{10k}}{5^{4k}}$

Thus, k must be even to insure that the quotient is positive.

For even values of k, we ignore $(-1)^k$.

We require that $\dfrac{2^{10k}}{5^{4k}} > 8 \Leftrightarrow 2^{10k} > 8\left(5^{4k}\right)$ Taking $\log_{10}$ of both sides, we have

$10k\left(\log_{10} 2\right) > 3\log_{10} 2 + 4k\left(\log_{10} 5\right)$

$\log_{10} 2 \approx 0.3 \Rightarrow \log_{10} 5 = \log_{10}\left(\dfrac{10}{2}\right) = \log_{10} 10 - \log_{10} 2 = 1 - \log_{10} 2 \Rightarrow \log_{10} 5 \approx 0.7$

Substituting, $10k\left(\log_{10} 2\right) > 3\log_{10} 2 + 4k\left(\log_{10} 5\right) \Rightarrow 3k > 0.9 + 2.8k \Rightarrow 2k > 9 \Rightarrow k > 4.5$

Therefore, the minimum value of k is **<u>6</u>**.

Alternate Solution:

$\left(\dfrac{4+4i}{5}\right)^{4k} = \left(\left(\dfrac{4(1+i)}{5}\right)^4\right)^k = \left(\dfrac{2^8\left((1+i)^2\right)^2}{5^4}\right)^k = \left(\dfrac{2^8(2i)^2}{5^4}\right)^k = \left(\dfrac{-1024}{625}\right)^k$

Clearly k must be even if the inequality is to be satisfied, since odd powers will produce a negative real number.

$\dfrac{1024}{625} \approx 1.64^-$

Since $1.6^4 = 2.56^2 < 2.6^2 = 6.76$ and $1.65^4 < 2.73^2 < 2.8^2 = 7.84$

Thus, k = 4 is too small, since the value of the expression lies between 6.76 and 7.84 .

Suspect k = 6, but let's check k = 5 (ignoring the minus sign) just to be sure.

Using an underestimate, $1.6^5 = \left(\dfrac{16}{10}\right)^5 = \dfrac{2^{20}}{10^5} = \dfrac{2^{15}}{5^5} = \dfrac{2^{10}2^5}{5^5} = \dfrac{1024(32)}{3125} = \dfrac{32768}{3125} > 10 > 8$.

So, clearly, k = **<u>6</u>** produces an even larger value and is the required minimum value of k.