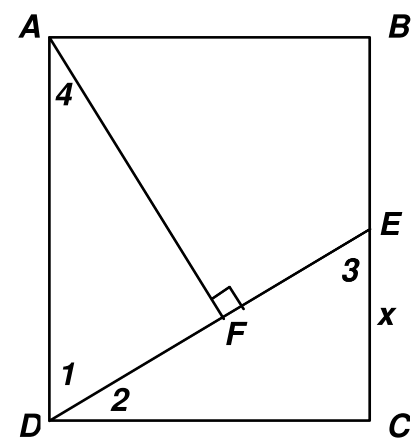
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009 SOLUTION KEY

Round 5

A) The area of the original rectangle J is 18. Since $(2x - 3, 6 - 2y) = \left(2(x+1.5), -2(y-3)\right)$
The original rectangle has been translated 1.5 units to the right and 3 units down. This has no effect on the area. However, the original rectangle has been dilated (stretched) by a factor of 2 in both the x – direction and y – direction (as well as reflected across the x – axis). So the new rectangle is similar to the original rectangle with 4 times the area → **72**

B) $m\angle 1 = m\angle 3$ and $m\angle 2 = m\angle 4$ → $\Delta ECD \sim \Delta DFA$ →



$$\frac{x}{DF} = \frac{13}{AF} = \left(\frac{DE}{15} = \frac{\sqrt{10}}{3}\right)$$

Cross multiplying, $DE = 5\sqrt{10}$ Using the Pythagorean theorem on ΔECD, $\left(5\sqrt{10}\right)^2 = 250 = x^2 + 169$ → $x = \mathbf{9}$

The following is an alternate algebraic solution:
From the proportion, $AF = \dfrac{13 \cdot 15}{DE}$ and $DF = \dfrac{15x}{DE}$ Then:

$$\frac{\frac{1}{2} \cdot 13x}{\frac{1}{2} DF \cdot AF} = \frac{10}{9} \rightarrow \frac{13x}{\frac{13 \cdot 15^2 \cdot x}{DE^2}} = \frac{DE^2}{15^2} = \frac{10}{9} \rightarrow DE^2 = 250$$

But, applying the Pythagorean Theorem to ΔDEC, $DE^2 = x^2 + 13^2$ → $x^2 = 81$ → $x = \mathbf{9}$

The following is a trigonometric solution: Let $\theta = m\angle 2 = m\angle 4$.
Then: $AF = 15\cos(\theta)$, $DF = 15\sin(\theta)$ and $x = EC = 13\tan(\theta)$

$$\frac{\frac{1}{2} \cdot 13 \cdot (13\tan\theta)}{\frac{1}{2} \cdot 15\sin\theta \cdot 15\cos\theta} = \frac{10}{9} \rightarrow \frac{13^2}{10 \cdot 5^2} = \cos^2\theta$$

Finally, $\tan^2\theta = \sec^2\theta - 1$ → $\tan^2\theta = \dfrac{10 \cdot 5^2 - 13^2}{13^2} = \dfrac{81}{13^2}$ → $EC = \mathbf{9}$

C) The altitude and median to the base of an isosceles triangle are one and the same. The centroid divides the median into segments whose lengths are in a 2 : 1 ratio. $\overline{DE} \parallel \overline{BC}$ → $\Delta ADE \sim \Delta ABC$ → $DE : BC = 2 : 3$.
Then lengths are represented in the diagram at the right:
The bases of the trapezoid $DECG$ are $DE = 4x$ and $GC = 5x$

$$\frac{A(\Delta ADE)}{A(DECG)} = \frac{\frac{1}{2} \cdot 4x \cdot 2h}{\frac{1}{2} \cdot h \cdot (4x + 5x)} = \frac{8xh}{9xh} = \mathbf{8 : 9}$$