MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2014 SOLUTION KEY

Round 6

A) Appealing strictly to the definition, $\binom{7}{1}=\binom{7}{6}$, $\binom{7}{2}=\binom{7}{5}$ and $\binom{7}{3}=\binom{7}{5}$, so only three combinations need be evaluated.

$$\binom{7}{1}=\frac{7!}{1!\cdot 6!}=\frac{7\cdot\cancel{6!}}{\cancel{1}\cdot\cancel{6!}}=7 \qquad \binom{7}{2}=\frac{7!}{2!\cdot 5!}=\frac{7\cdot 6\cdot\cancel{5!}}{2\cdot\cancel{5!}}=21 \qquad \binom{7}{3}=\frac{7!}{3!\cdot 4!}=\frac{7\cdot\cancel{6}\cdot 5\cdot\cancel{4!}}{\cancel{6}\cdot\cancel{4!}}=35$$

Thus, the required sum is $2(7+21+35)=\underline{\mathbf{126}}$.

Alternate solution #2:
These are the numbers from the 7th row of Pascal's triangle, excluding the first and last, both of which are 1s. All the numbers in the 7th row add up to $2^7=128$. Thus, our total is $128-2=\underline{\mathbf{126}}$.

Alternate Solution #3
According to the binomial Theorem,

$$(a+b)^7=\binom{7}{0}a^7+\binom{7}{1}a^6b^1+\binom{7}{2}a^5b^2+\ldots+\binom{7}{6}a^1b^6\binom{7}{7}b^7$$

The exponents of *a* go down by 1. The exponents of *b* go up by 1.
The exponent of the *b*-term matches the bottom number in the combination.
If we let $a=b=1$, none of this matters.

We have simply $2^7=\binom{7}{0}+\boxed{\binom{7}{1}+\ldots+\binom{7}{6}}+\binom{7}{7}$.

The boxed quantity is the required sum and we have the same result.

B) Since there are 11 terms in the expansion, the middle term is the 6th term.

$$\Rightarrow\binom{10}{5}(2A)^5\left(-\frac{k}{A}\right)^5=-\frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot 32A^5\cdot\frac{k^5}{A^5}=$$

$$-\frac{\overset{2}{\cancel{10}}\cdot 9\cdot\overset{2}{\cancel{8}}\cdot 7\cdot\cancel{6}}{\cancel{1\cdot 2\cdot 3}\cdot\cancel{4}\cdot\cancel{5}}\cdot 32\cancel{A^5}\cdot\frac{k^5}{\cancel{A^5}}=-252(32)k^5=\underline{\mathbf{-8064k^5}}$$

C) There are 3^5 = <u>243</u> sequence with no As. There are $\binom{5}{2}=10$ ways to position 2 As.

(_1_ _2_ _3_ _4_ _5_) $\Rightarrow$ Positions: 12, 13, 14, 15, 23, 24, 25, 34, 35, 45
Now fill in the remaining positions with any of the other letters.

Pick two letters and place them in the remaining 3 positions: $\binom{5}{2}3^3=10(27)=\underline{270}$

There are $\binom{5}{4}=5$ ways to arrange 4 As and 3 choices for the 5th position $\Rightarrow$ <u>15</u>

Thus, an even number of As may occur 243 + 270 + 15 = **<u>528</u>** ways.