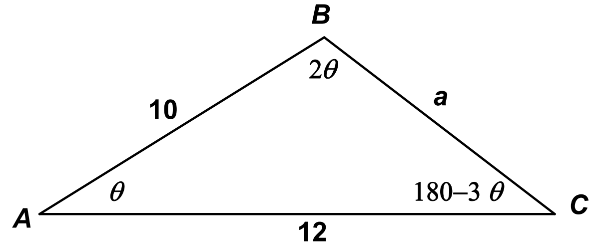
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010 SOLUTION KEY

Team Round



A) Let m$\angle A = \theta$. Then m$\angle B = 2\theta$ and m$\angle C = 180 - 3\theta$.

Using the law of Cosines,
$a^2 = 10^2 + 12^2 - 2(10)(12)\cos\theta = 244 - 240\cos\theta$ (#1).
Using the law of Sines,

$$\frac{\sin\theta}{a} = \frac{\sin 2\theta}{12} = \frac{\sin(180-3\theta)}{10} \rightarrow \frac{\sin\theta}{a} = \frac{2\sin\theta\cos\theta}{12} = \frac{\sin(3\theta)}{10} = \frac{\sin\theta\left(3-4\sin^2\theta\right)}{10}$$

Since $\sin\theta \neq 0$, we have $\frac{1}{a} = \frac{\cos\theta}{6} = \frac{3-4\sin^2\theta}{10}$ (#2).

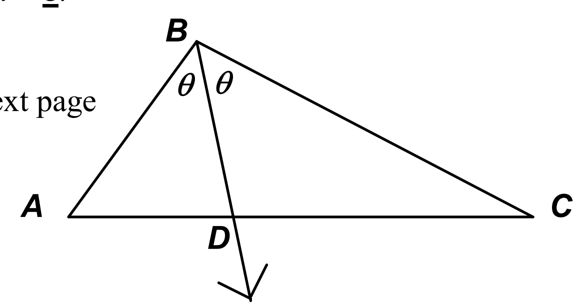
Method #1: Substituting $\cos\theta = \frac{6}{a}$, we have $a^2 = 244 - 240\left(\frac{6}{a}\right) \rightarrow a^3 - 244a + 1440 = 0$

As the smallest side, $a < 10$. Since $a = 7 \rightarrow +75$ and $a = 9 \rightarrow -27$, we try $a =$ **8** and hit paydirt!

Method #2: Using the last two ratios in #2, $5\cos\theta = 3\left(3-4\sin^2\theta\right) = 3\left(4\cos^2\theta - 1\right) \rightarrow$

$12\cos^2\theta - 5\cos\theta - 3 = (4\cos\theta - 3)(3\cos\theta + 1) = 0 \rightarrow \cos\theta = +\frac{3}{4}$

($\cos\theta = -\frac{1}{3}$ would imply θ was obtuse which is impossible for the smallest angle in ΔABC.)
Substituting in #1, $a^2 = 244 - 240(3/4) = 244 - 180 = 64 \rightarrow a =$ **8**.

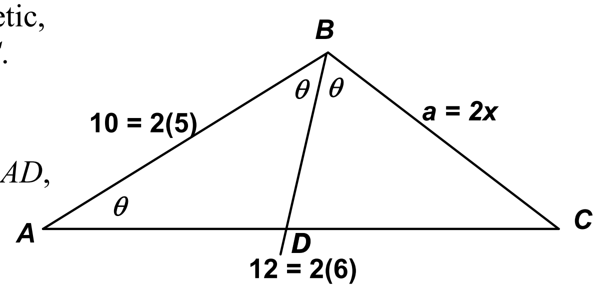


Alternate Solution (Norm Swanson)
Requisite Notions (using diagram at right) - proved on the next page

Angle Bisector Theorem #1: $\frac{AD}{CD} = \frac{AB}{CB}$

Angle Bisector Theorem #2: $BD^2 = (AB)(BC) - (AD)(DC)$

Draw the angle bisector of $\angle B$, intersecting $\overline{AC}$ at point D.
ΔBAD is isosceles with $AD = BD$. To simplify the arithmetic,
let $AB = 5$, $AC = 6$ and then double our answer for $x = BC$.



Using angle bisector theorem #1, $AD = BD = \frac{30}{x+5}$.

Using angle bisector theorem #2, substituting for BD and AD,

$$\frac{900}{(x+5)^2} = 5x - \frac{180x}{(x+5)^2}$$

$\rightarrow x(x+5)^2 - 36x = 180 \rightarrow x^3 + 10x^2 - 11x = 180$
The left side of this equation factors to $x(x-1)(x+11)$; the right side factors as $2^2 \cdot 3^2 \cdot 5$.
By inspection 4(3)15) = 180, so $x = 4$ and $a = BC = 8$.
Also the cubic $x^3 + 10x^2 - 11x = 180$ factors as $(x-4)(x+5)(x+9)$, so again $x = 4$.