MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2015 SOLUTION KEY

Round 5

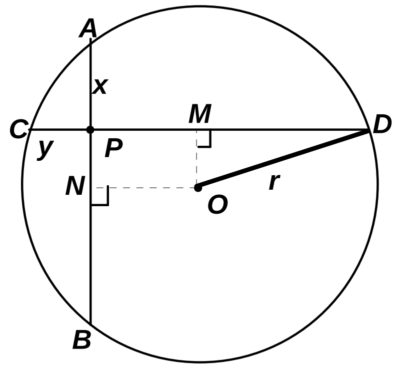
A) Since $d = \frac{n(n-3)}{2} = 135$, by trial and error, we have $\frac{18 \cdot 15}{2} = 9 \cdot 15 = 135$ and $n = 18$.
Thus, there are 18 sides and 15 diagonals at each vertex.
The 18 vertices divide the circle into 18 congruent arcs, each measuring 20°.
The 1st and 15th diagonal at any vertex form an inscribed angle consisting of 14 of these 18 arcs. Its measure is $\frac{1}{2}(14 \cdot 20) = \underline{\mathbf{140}}°$.

B) If x is the longest side, $x < 10 + 15 = 25$.
If it were a right triangle, x would be the hypotenuse and $10^2 + 15^2 = x^2$. In an obtuse triangle, $x^2 > 10^2 + 15^2 = 325$. So, we have a lower limit for x; specifically, since $18^2 = 324$, $x \geq 19$.
In this case $19 \leq x \leq 24$ (6 possibilities)

If 15 is the longest side, $x + 10 > 15 \Rightarrow x \geq 6$ and $x^2 + 100 < 225$.
$x^2 < 125 \Rightarrow x \leq 11$.
In this case, $6 \leq x \leq 11$ (6 possibilities)
Total: **<u>12</u>**

C) Given: $(AB, CD) = (16, 19)$

Let x and y denote the lengths of $\overline{PA}$ and $\overline{PC}$ respectively.
According to the product chord theorem, $x(16-x) = y(19-y)$ where x and y are integers. Without this restriction there are infinitely many solutions. Examine the products of the lengths of each chord.



on $\overline{AB}$: (1, 15) - 15, (2, 14) - 28, (3,13) - 39, (4, 12) - **48**,
(5, 11) - 55, (6, 10) - **60**, (7, 9) - 63, (8, 8) - 64

on $\overline{CD}$: (1, 18) - 18, (2, 17) - 34, (3, 16) – **48**, (4, 15) – **60**,
(5, 14) – 70, (6,13) – 78, (7,12) – 84, (8,11) – 88
(9,10) - 90

$\overline{ON}$ bisects $\overline{AB}$ and $\overline{OM}$ bisects $\overline{CD}$.
$x = 4 \Rightarrow NP = OM = 8 - 4 = 4$

$MD = 9.5$. In ΔMOD, $r^2 = 4^2 + 9.5^2 = 16 + \frac{19^2}{4} = \frac{64+361}{4} = \frac{425}{4} = \frac{25 \cdot 17}{4} \Rightarrow r = \underline{\mathbf{\frac{5}{2}\sqrt{17}}}$.

For $x = 6$, we get $r^2 = 2^2 + 9.5^2$ and clearly this r-value will be smaller.

See additional comments on the <u>original</u> 5C at the end of the solution key.