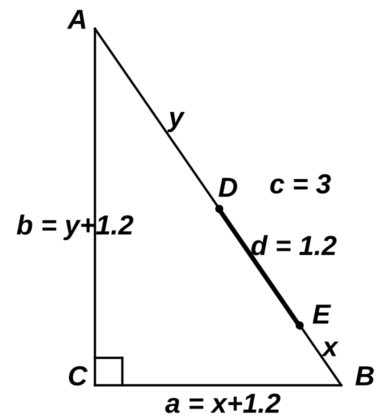
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2016 SOLUTION KEY

Team Round - continued

E) $\begin{cases} x+y=1.8=\dfrac{9}{5} \\ \left(x+\dfrac{6}{5}\right)^2+\left(y+\dfrac{6}{5}\right)^2=3^2=9 \end{cases}$



$$\Rightarrow \left(x+\frac{6}{5}\right)^2+\left(\frac{9}{5}-x+\frac{6}{5}\right)^2=3^2=9 \Rightarrow \left(x+\frac{6}{5}\right)^2+(3-x)^2=9$$

$$\Rightarrow x^2+\frac{12}{5}x+\frac{36}{25}-6x+x^2=0 \Rightarrow x^2-\frac{9}{5}x+\frac{18}{25}=0$$

$$\Rightarrow \left(x-\frac{3}{5}\right)\left(x-\frac{6}{5}\right)=0 \Rightarrow (x,y)=\left(\frac{3}{5},\frac{6}{5}\right) \text{ or } \left(\frac{6}{5},\frac{3}{5}\right)$$

Therefore, the area of ΔABC is $\frac{1}{2}(1.8)(2.4)=0.9(2.4)=$ **<u>2.16</u>** .

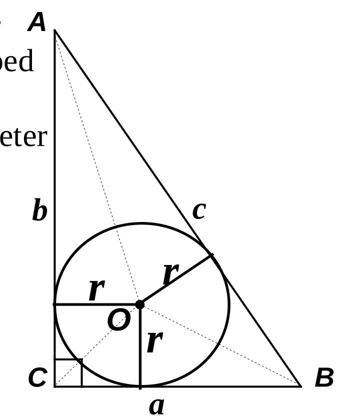
Generalization: Show that DE is equal to the diameter of the inscribed circle in ΔABC.

$DE=d \Rightarrow AD=b-d$, $BE=a-d$, $AB=(b-d)+d+(a-d)=c$. Therefore, $a+b-d=c \Leftrightarrow d=a+b-c$ which is the length of the diameter of the inscribed circle of right triangle ABC.

[If r denotes the radius of the inscribed circle in ΔABC and s, the semi-perimeter of ΔABC, then $r\cdot s=\frac{1}{2}ar+\frac{1}{2}br+\frac{1}{2}cr=\frac{1}{2}ab \Rightarrow r(a+b+c)=ab$

$\Rightarrow r=\dfrac{ab}{(a+b)+c}$. $(a+b)^2-c^2=2ab$, since $a^2+b^2=c^2$. Factoring,

$(a+b+c)(a+b-c)=2ab \Rightarrow a+b-c=\dfrac{2ab}{a+b+c}=2r=d_{(ic)}$.]



F) $$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^4=20.25 \Leftrightarrow \left(\sqrt{x}\right)^4+4\left(\sqrt{x}\right)^3\left(\frac{1}{\sqrt{x}}\right)+6\left(\sqrt{x}\right)^2\left(\frac{1}{\sqrt{x}}\right)^2+4\sqrt{x}\left(\frac{1}{\sqrt{x}}\right)^3+\left(\frac{1}{\sqrt{x}}\right)^4=20.25$$

$$\Rightarrow x^2+4x+6+\frac{4}{x}+\frac{1}{x^2}=20\frac{1}{4} \Rightarrow \left(x^2+\frac{1}{x^2}\right)+4\left(x+\frac{1}{x}\right)=20\frac{1}{4}-6=14\frac{1}{4}=\frac{57}{4}$$

Let $u=x+\frac{1}{x}$. Then: $u^2=x^2+2+\frac{1}{x^2}$ or $x^2+\frac{1}{x^2}=u^2-2$. Substituting, we have

$$\left(u^2-2\right)+4u-\frac{57}{4}=0 \Leftrightarrow u^2+4u-\frac{65}{4}=0 \Leftrightarrow 4u^2+16u-65=(2u-5)(2u+13)=0$$

$\Rightarrow u=\frac{5}{2}$ only. [$u=-\frac{13}{2}=x+\frac{1}{x} \Rightarrow 2x^2+13x+2=0$ which has only negative irrational roots.]

Re-substituting, $\left(x+\frac{1}{x}=\frac{5}{2}\right)\cdot 2x \Rightarrow 2x^2+2-5x=(2x-1)(x-2)=0 \Rightarrow x=$ <u>$\mathbf{2, \frac{1}{2}}$</u> .