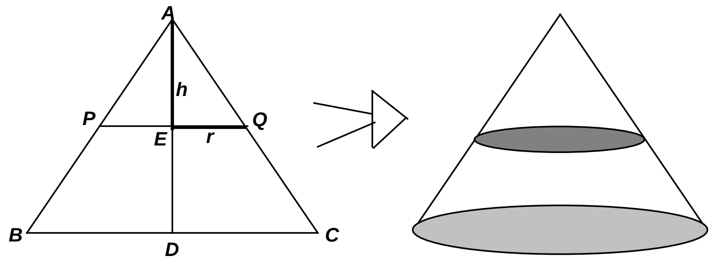
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015 SOLUTION KEY

Team Round



A) $AC = 35$ and $BD = CD = 21$
$\Rightarrow$ $AD = 28$ [7(3 - 4 - 5)].
Let $(AE, QE) = (h, r)$.

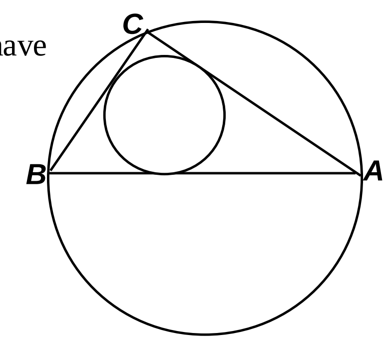
By similar triangles, $\frac{h}{r} = \frac{28}{21} = \frac{4}{3} \Rightarrow r = \frac{3}{4}h$.

The volume of the frustum is just the difference in the volumes of two cones.

$$\frac{\frac{1}{3}\pi\left(\frac{3}{4}h\right)^2 h}{\frac{1}{3}\pi(21)^2(28) - \frac{1}{3}\pi\left(\frac{3}{4}h\right)^2 h} = \frac{\cancel{\frac{\pi}{3}}\left(\frac{9}{16}\right)h^3}{\cancel{\frac{\pi}{3}}\left((21)^2(28) - \left(\frac{9}{16}\right)h^3\right)} = \frac{9h^3}{K - 9h^3} = \frac{a}{b} = \frac{27}{98}$$, where $K = 21^2 \cdot 28 \cdot 16$.

Cross multiplying, $9bh^3 = aK - 9ah^3 \Rightarrow h^3 = \frac{aK}{9(a+b)} = \frac{27\left(21^2 \cdot 28 \cdot 16\right)}{9(125)} = \frac{2^6 3^3 7^3}{5^3} \Rightarrow h = AE = \underline{\mathbf{\frac{84}{5}}}$.

B) Given: $AC = x + 3y + 1$, $BC = 2x + y - 3$, $r_{ic} = x - y$, and $d_{cc} = x + 5y$



Since the hypotenuse $\overline{AB}$ is the diameter of the <u>circumscribed</u> circle, we have $AB = x + 5y$.
*** The diameter of the <u>inscribed</u> circle equals $\underline{\underline{BC + AC - AB}} = 2(x - y)$
Thus, $(2x + y - 3) + (x + 3y + 1) - (x + 5y) = 2(x - y)$ or
$2x - y - 2 = 2x - 2y \Rightarrow y = 2$
Substituting into expressions for the sides,
$AC = x + 7$, $BC = 2x - 1$ and $AB = x + 10$
Applying the Pythagorean Theorem,
$(x+7)^2 + (2x-1)^2 = (x+10)^2 \Leftrightarrow 5x^2 + 10x + 50 = x^2 + 20x + 100 \Leftrightarrow 4x^2 - 10x - 50 = 0$
$\Rightarrow$ $2x^2 - 5x - 25 = 0 \Leftrightarrow (2x+5)(x-5) = 0$, so $x = 5$.
$\Rightarrow$ $AC = 12$, $BC = 9$ and $AB = 15$, producing a perimeter of <u>**36**</u>.
Alternately, we could add the expressions for AC, BC and AB to get $4x + 9y - 2$,
producing 20 + 18 – 2 = <u>**36**</u>.

Alternately, after finding $y = 2$, I conjecture a 3-4-5 right triangle or a multiple thereof,
where $\overline{AC}$ is actually longer than $\overline{BC}$! $\frac{2x-1}{x+7} = \frac{3}{4} \Rightarrow 8x - 4 = 3x + 21 \Rightarrow x = 5$.
$(x, y) = (5, 2) \Rightarrow 9 - 12 - 15$ Bingo!

***** Challenge: Justify the double underlined result above.**
The proof is included at the end of this solution key.