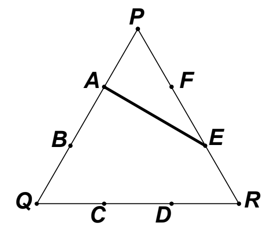
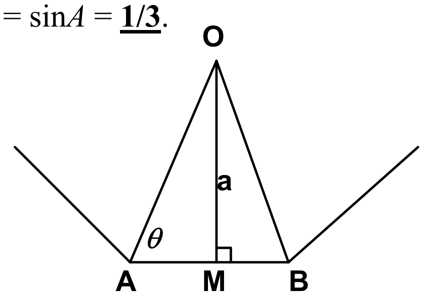
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006 SOLUTION KEY



Round 1

A) Using the law of cosine, $AE^2 = 2^2 + 4^2 - 2(2)(4)\cos 60°$
$= 20 - 16(1/2) = 12 \rightarrow PQ = \underline{\mathbf{2\sqrt{3}}}$.

B) Using the law of sine, $\dfrac{\sin A}{10} = \dfrac{\sin 150°}{15} \rightarrow \sin A = \dfrac{2}{3}\cdot\dfrac{1}{2} = \dfrac{1}{3}$
The given information (2 sides and the non-included angle) is the ambiguous case, but since $\angle B$ is obtuse, there is exactly one triangle satisfying the given conditions.
Since $A + B + C = 180°$, $B + C = 180 - A$ and $\sin(B + C) = \sin(180 - A) = \sin A = \underline{\mathbf{1/3}}$.



C) $m\angle BOA = (360/n)° \rightarrow \theta = 90 - 180/n$ and $AM = ½(p/n) = p/(2n)$
$\tan(\theta) = (OM)/(AM) = (2na)/p \rightarrow a = p\tan(\theta)/(2n)$
Replacing the angle θ by its complement and the trig function by its cofunction, $\rightarrow \dfrac{p\cot(\frac{180}{n})}{2n} \rightarrow (X, Y) = \underline{\mathbf{(180, 2)}}$.

Round 2

A) The rightmost digit of positive integer powers of 4 are alternately 4 and 6.
The rightmost digit of positive integer powers of 9 are alternately 9 and 1
2 → 2,4,8,6 ***3*** → 3,9,7,1 ***7*** → 7,9,3,1 ***8*** → 8,4,2,6 ***0*** → 0 ***1*** → 1 ***5*** → 5 ***6*** → 6
Thus, *d* may be **<u>4 or 9</u>**.

B) $77 = 7 \cdot 11$ and $119 = 7 \cdot 17$

If $N = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdot \ldots \cdot p_k^{e_k}$, then the number of factors of N is given by $(e_1+1)(e_2+1)\cdot\ldots\cdot(e_k+1)$.
Note: The number of positive factors of *N* does not depend on what its prime factors are, only how many of each there are.

Thus, $7 \cdot 11 \cdot 17 = 7^1 \cdot 11^1 \cdot 17^1 = \underline{\mathbf{1309}}$ has $(1 + 1)(1 + 1)(1 + 1) = 8$ positive factors.
(The factors are: 1, 7, 11, 17, $7 \cdot 11 = 77$, $7 \cdot 17 = 119$, $11 \cdot 17 = 187$, $7 \cdot 11 \cdot 17 = 1309$)

C) Since $12 = 3 \cdot 4$, a number divisible by 12 is divisible by 3 and 4 and vice versa.
Divisibility Rules
÷ by 3: check the sum of the digits – it must be divisible by 3
÷ by 4: check the number formed by the rightmost 2 digits – it must be divisible by 4

The digit sum $2(A + B)$ must be divisible by 3 → $(A + B)$ must be divisible by 3
Thus, for some integer k, $A + B = 3k$ or $B = 3k - A$

The positive two-digit number $10B + A = 10(3k - A) + A = 30k - 9A = 3(10k - 3A)$ must be a multiple of 4, so $(10k - 3A)$ must be a multiple of 4 and $k \geq 1$.
Remember A and B denote digits in base 10 and, therefore, are restricted to 0, 1, …, 9.
$k = 1 \rightarrow B = 3 - A$ and $10 - 3A = 4j \rightarrow A = 2, B = 1 \rightarrow$ **2112**
$k = 2 \rightarrow B = 6 - A$ and $20 - 3A = 4j \rightarrow A = 4, B = 2 \rightarrow$ **4224**
$k = 3 \rightarrow B = 9 - A$ and $30 - 3A = 4j \rightarrow A = 2$ or 6 and $B = 7$ or 3 → **2772** or **6336**
$k = 4 \rightarrow B = 12 - A$ and $40 - 3A = 4j \rightarrow A = 4$ or 8 and $B = 8$ or 4 → **4884** or **8448**
$k = 5 \rightarrow B = 15 - A$ and $50 - 3A = 4j \rightarrow A = 2$ or 6
and only $A = 6$ produces a legal value for B → **6996**
$k = 6 \rightarrow B = 18 - A$ and $60 - 3A = 4j \rightarrow A = 4$ or 8 and neither produces a legal value for B
and the search stops. 2112 + 2772 + 4224 + 4884 + 6336 + 6996 + 8448 = **<u>35772</u>**.