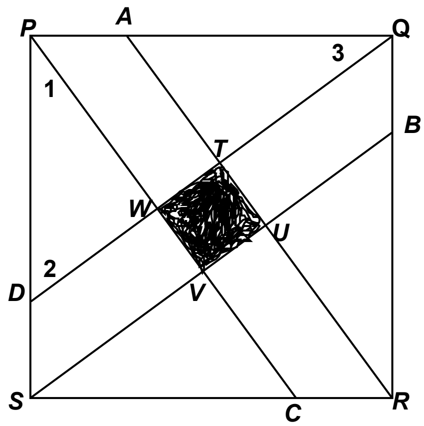
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2011 SOLUTION KEY

Team Round – continued



E) $PA = QB = RC = CD = 2$,
$AQ = BR = CS = DP = 6$
→ $PC = AR = DQ = SB = 10$
Also $\overline{PC} \parallel \overline{AR}$ and $\overline{DQ} \parallel \overline{SB}$.

$\Delta QPD \cong \Delta PSC$ (SAS) → $\angle 3 \cong \angle 1$.
Since ΔQPD is a right triangle, $m\angle 2 + m\angle 3 = 90°$.
Substituting for $m\angle 3$, $m\angle 2 + m\angle 1 = 90°$.
Therefore, $\angle PWD$ and $\angle TWV$ are right angles.
Similarly, all angles with vertices at W, T, U and V are right angles.
$\Delta PWD \cong \Delta QTA \cong \Delta RUB \cong \Delta SVC$ (ASA)
→ $PW = QT = RU = SV$ and $WD = TA = UB = VC$
By subtraction of equals, $TU = UV = VW = WT$.
Thus, $TUVW$ is equiangular and equilateral and must be a square.

Continuing, Solution #1:

$\Delta VSC \sim \Delta RSB$ → $\dfrac{VS}{RS} = \dfrac{SC}{SB}$ → $\dfrac{VS}{8} = \dfrac{6}{10}$ → $VS = \dfrac{24}{5}$

$\Delta RUB \sim \Delta SRB$ → $\dfrac{BU}{BR} = \dfrac{RB}{SB}$ → $\dfrac{BU}{6} = \dfrac{6}{10}$ → $BU = \dfrac{18}{5}$

Finally, $VU = 10 - \dfrac{24}{5} - \dfrac{18}{5} = \dfrac{50-24}{5} = \dfrac{8}{5}$ and the area of square $TUVW$ = $\underline{\mathbf{\dfrac{64}{25}}}$.

Note ΔVSC, ΔRSB and ΔURB are scaled versions of a 3 – 4 – 5 right triangle!

Continuing, Solution #2:

Drop a perpendicular from A to $\overline{PC}$, intersecting $\overline{PC}$ at point N.

$\Delta PAN \sim \Delta DQP$ → $\dfrac{PA}{DQ} = \dfrac{AN}{QP}$ → $\dfrac{2}{10} = \dfrac{AN}{8}$ → $AN = \dfrac{8}{5}$ → $TW = \dfrac{8}{5}$ → area = $\underline{\mathbf{\dfrac{64}{25}}}$.

Note also that ΔPAN is a scaled version of a 3 – 4 – 5 right triangle!

Solution #3: (Norm Swanson)

Set up a coordinate system where $S(0, 0)$, $\overline{SP}$ lies on the Y-axis and $\overline{SR}$ along the X-axis.
Then: $\overleftrightarrow{DQ}: 3x - 4y = -8$ and $\overleftrightarrow{SB}: 3x - 4y = 0$ The distance between these parallel lines is 8/5.

Verify that the distance between $Ax + By = C$ and $Ax + By = D$ is $\dfrac{|C-D|}{\sqrt{A^2+B^2}}$.

$\overleftrightarrow{PC}: 4x + 3y = 24$ and $\overleftrightarrow{AR}: 4x + 3y = 32$ These lines are parallel and perpendicular to the above pair and also 8/5 apart. Thus, $TUVW$ is a square with area **<u>64/25</u>**. What if $PQRS$ were a rectangle, or a rhombus or a parallelogram? Can you generalize?

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