MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2009 SOLUTION KEY

Team Round

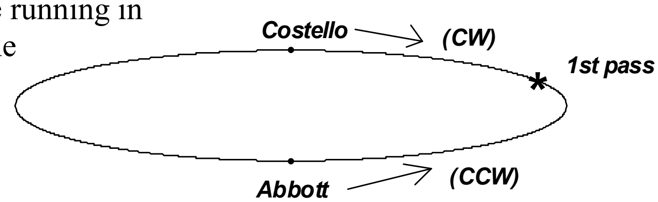
D) $f(x)=g(x) \Rightarrow \dfrac{2^x-2^{-x}}{2}=\dfrac{2}{2^x+2^{-x}} \Rightarrow \left(2^x\right)^2-\left(2^{-x}\right)^2=4 \Rightarrow 4^x-4^{-x}=4$

Let $Y=4^x$. Then: $Y-\dfrac{1}{Y}=4 \Rightarrow Y^2-4Y-1=0 \Rightarrow Y=\dfrac{4\pm\sqrt{16+4}}{2}=2\pm\sqrt{5}$

($2-\sqrt{5}$ is extraneous.) Thus, $4^x=2+\sqrt{5} \Rightarrow 2^{2x}=2+\sqrt{5} \Rightarrow 2x=\log_2\left(2+\sqrt{5}\right)$

$\Rightarrow x=\dfrac{1}{2}\log_2\left(2+\sqrt{5}\right)=\log_2\sqrt{2+\sqrt{5}} \Rightarrow N=\underline{\boldsymbol{\sqrt{2+\sqrt{5}}}}$

E) Since distance equals rate times time, the runners cover distances of $5t$ and $4t$ laps respectively in t units of time. Since they are running in opposite directions, they pass for the first time when $5t+4t=\dfrac{1}{2}$, for the second time when $5t+4t=1+\dfrac{1}{2}=\dfrac{3}{2}$, etc.



In general, they pass for the m^{th} time when $5t+4t=m-\dfrac{1}{2}$ or when $t=\dfrac{2m-1}{18}$.

For Abbott to complete n laps takes $t=\dfrac{n}{5}$ units of time. Therefore, the number of times the runners pass each other is the <u>largest</u> integer m for which $\dfrac{2m-1}{18}\le\dfrac{n}{5}$ or $m\le\dfrac{18n+5}{10}$

If k denotes the number of laps Abbott has completed when the runners have passed each other for the 100^{th} time, then $100\le\dfrac{18k+5}{10} \Rightarrow 18k\ge 995 \Rightarrow k\ge 55^{+} \Rightarrow k=\underline{\mathbf{56}}$