MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2017 SOLUTION KEY

Team Round

A) The center of the hyperbola is $(-2,4)$, but the hyperbola could be vertical or horizontal.

Case 1: Vertical $(a,b)=(3,4)$

$$\frac{(y-4)^2}{9}-\frac{(x+2)^2}{16}=1$$

For $x=-7$, we have

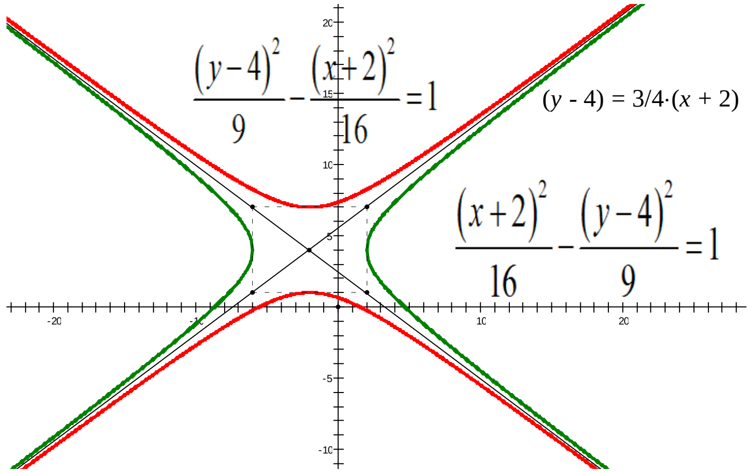
$$(y-4)^2=\frac{9\cdot 41}{16}\Rightarrow y=\underline{\mathbf{\frac{16\pm 3\sqrt{41}}{4}}}.$$

Case 2: Horizontal $(a,b)=(4,3)$

$$\frac{(x+2)^2}{16}-\frac{(y-4)^2}{9}=1$$

For $x=-7$, we have

$$(y-4)^2=9\left(\frac{25}{16}-1\right)=\frac{81}{16}\Rightarrow y=4\pm\frac{9}{4}=\underline{\mathbf{\frac{25}{4},\frac{7}{4}}}.$$



B) $4x^6-13x^4-13x^2+4=4\left(x^6+1\right)-13x^2\left(x^2+1\right)$

Using the sum of perfect cubes, $x^6+1=\left(x^2\right)^3+1^3=\left(x^2+1\right)\left(x^4-x^2+1\right)$.

$\Leftrightarrow\left(x^2+1\right)\left(4\left(x^4-x^2+1\right)-13x^2\right)=\left(x^2+1\right)\left(4x^4-17x^2+4\right)\Leftrightarrow$

$\left(x^2+1\right)\left(4x^2-1\right)\left(x^2-4\right)=\underline{\mathbf{\left(x^2+1\right)(x-2)(x+2)(2x-1)(2x+1)}}$

	4	0	−13	0	−13	0	4
2		8	16	6	12	−2	−4
	4	8	3	6	−1	−2	0
−2		−8	0	−6	0	2	
	4	0	3	0	−1	0	

Alternately, dividing synthetically by $x-2$ and then by $x+2$, we get $4x^4+3x^2-1$ which factors as $\left(4x^2-1\right)\left(x^2+1\right)$ and the solution follows.

C) To insure that sin(x) is being assigned a real value, the discriminant must be nonnegative, i.e. $N\le\frac{5}{4}$.

$$\sin(x)=\frac{1\pm\sqrt{1-4(N-1)}}{2}\Rightarrow\left(2\sin(x)-1\right)^2=\left(\pm\sqrt{1-4(N-1)}\right)^2\Rightarrow$$

$4\sin^2x-4\sin x+1=5-4N$. Transposing terms and dividing thru by 4

$\Rightarrow\ N=1+\sin x-\sin^2x$ or $N=\cos^2x+\sin x$

Now it was given that $x=(30k)°$ for $0\le k<12$.

$k=0,3,6\Rightarrow N=\underline{\mathbf{1}}$ $\qquad$ $k=1,5\Rightarrow N=\underline{\mathbf{\frac{5}{4}}}$

$k=2,4,8,10\Rightarrow N$ would be irrational $\qquad$ $k=9\Rightarrow N=\underline{\mathbf{-1}}$ $\qquad$ $k=7,11\Rightarrow N=\underline{\mathbf{\frac{1}{4}}}$.