MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2016 SOLUTION KEY

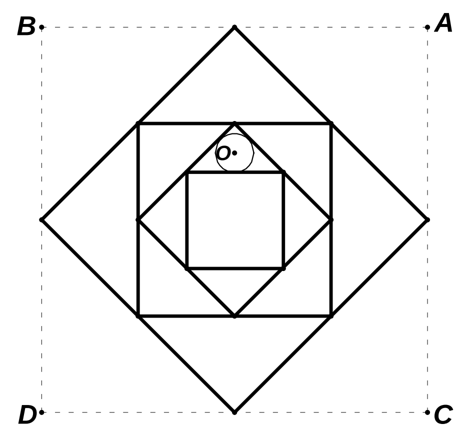
Round 6

A) $a_1 = 1 \Rightarrow a_2 = 1 \cdot 2 = 2 \Rightarrow a_3 = 2 \cdot 3 = 6 \Rightarrow a_4 = 6 \cdot 7 = 42 \Rightarrow a_5 = 42 \cdot 43 > 100$
Thus, only **4** terms are less than 100.

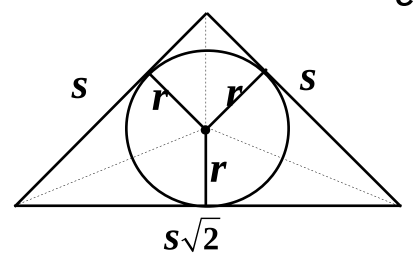
B) If the AP is $x - d$, x, $x + d$, then $3x = 30$ and $x = 10$.
Therefore, the GP is: $8 - d$, 6, $5 + d$ and $\frac{5+d}{6} = \frac{6}{8-d} \Rightarrow 40 + 3d - d^2 = 36$
$d^2 - 3d - 4 = (d-4)(d+1) = 0$
$d = -1 \Rightarrow$ AP: 11, 10, 9
$d = 4 \Rightarrow$ AP: **6, 10, 14**

C) Since the vertices of the each square (after the first) are midpoints of the proceeding square, we have a series of isosceles right triangles whose legs have lengths $50, 25\sqrt{2}, 25, \ldots$, i.e. a geometric progression with $a_1 = 50$ and $r = \frac{\sqrt{2}}{2}$. Thus, the legs of the isosceles right triangle framed by the 8th and 9th squares have length

$$ar^{n-1} = 50\left(\frac{\sqrt{2}}{2}\right)^7 = \frac{50 \cdot 2^3\sqrt{2}}{2^7} = \frac{25\sqrt{2}}{8}$$



To the right is a blowup of the isosceles right triangle and the circle whose radius must be found.



As an isosceles right triangle, its area is given by $\frac{1}{2}s^2$.

As the sum of three isosceles triangles with the same height, the area is given by $2\left(\frac{1}{2}rs\right) + \frac{1}{2}rs\sqrt{2} = \frac{rs}{2}\left(2+\sqrt{2}\right)$.

Equating, we have $s = r\left(2+\sqrt{2}\right) \Rightarrow r = \frac{s}{2+\sqrt{2}} = \frac{s\left(2-\sqrt{2}\right)}{2}$.

Thus, $r = \frac{25\sqrt{2}}{8} \cdot \frac{2-\sqrt{2}}{2} = \frac{25\left(2\sqrt{2}-2\right)}{16} = \underline{\boldsymbol{\frac{25\left(\sqrt{2}-1\right)}{8}}}$ or equivalent.
(Denominator must be rationalized.)