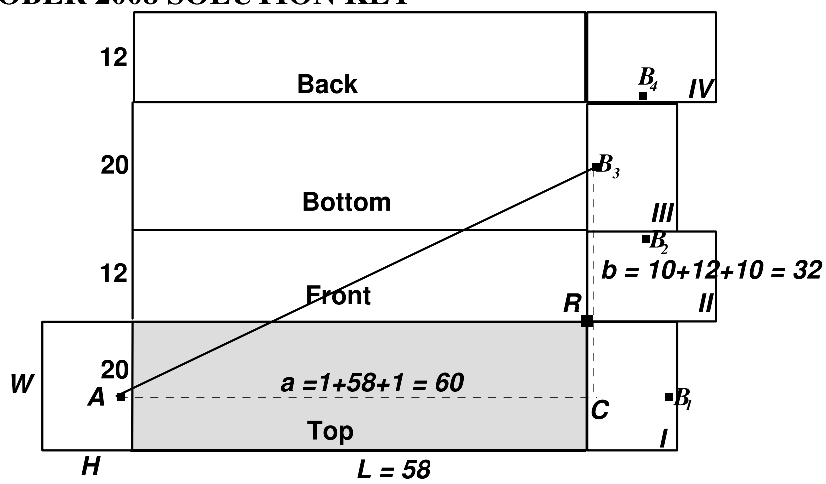
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008 SOLUTION KEY

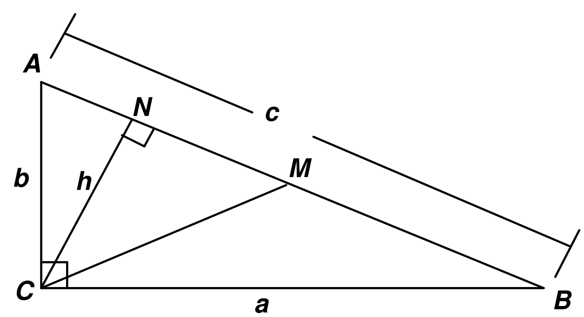
Team Round

A) Consider the foldout at the right, where the left wall "flap" is fixed and the possible positions of the right wall "flap" are illustrated, along with the positions of point B, which must be 1 unit from the floor midway between the front and back walls. Note that Flap II (and the position of B) is obtained from flap I by rotating I CCW about the point R. Similarly, II maps to III and III to IV.



$AB_1 = 1 + 58 + 11 = 70$ $\quad AB_2 = \sqrt{69^2+21^2} = 72.124\cdots$ $\quad AB_4 = \sqrt{69^2+43^2} = 81.301\cdots$
However, ΔAB_3C has legs of 60 and 32. Factoring out a 4, we notice the 8 – 15 – 17 triple and $AB = 4(17) =$ **68**.

B) $CM = \dfrac{c}{2}$. Let $(BC, AC) = (a, b)$. Area(ΔABC) =



$\dfrac{1}{2}ab = \dfrac{1}{2}hc \rightarrow c = \dfrac{ab}{h}$ In ΔCNM, $\left(\dfrac{c}{2}\right)^2 = NM^2 + h^2$

$\rightarrow NM = \dfrac{1}{2}\sqrt{c^2-4h^2}$. Thus, the ratio is

$$\frac{\frac{1}{2}ab}{\frac{1}{2}\cdot\frac{1}{2}\sqrt{c^2-4h^2}\cdot h} = \frac{2c}{\sqrt{c^2-4h^2}} \text{ and } c = 10 \rightarrow$$

$$\frac{20}{\sqrt{100-4h^2}} = \frac{10}{\sqrt{25-h^2}} \text{ must be rational.}$$

For $h = 5$, $CM = CN$ and ΔCNM collapses and we must avoid division by zero. Thus, examining integer h over [1, 4], we obtain rational values of **5/2** for $h = 3$ and **10/3** for $h = 4$.

Alternate solution: [using altitude to the hypotenuse/geometric means]

$$\frac{P}{Q} = \frac{(1/2)\cdot 10\cdot h}{(1/2)\cdot(5-x)\cdot h} = \frac{10}{5-x} \text{ must be rational.}$$

Clearly, $h = CN < 5$ and we need to examine $h = 1, 2, 3$ and 4.
$(AN)(BN) = CN^2 \rightarrow x(10-x) = h^2 = 1, 4, 9$ or 16
Only $x^2 - 10x + 9 = 0$ and $x^2 - 10x + 16 = 0$ have rational solutions, namely $(h, x) = (3, 1)$ and $(4, 2)$ and the required ratios are $10/4 =$ **5/2** and **10/3**.