MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014 SOLUTION KEY

Team Round

A) - **continued**

Solution #5 (Using the calculus)

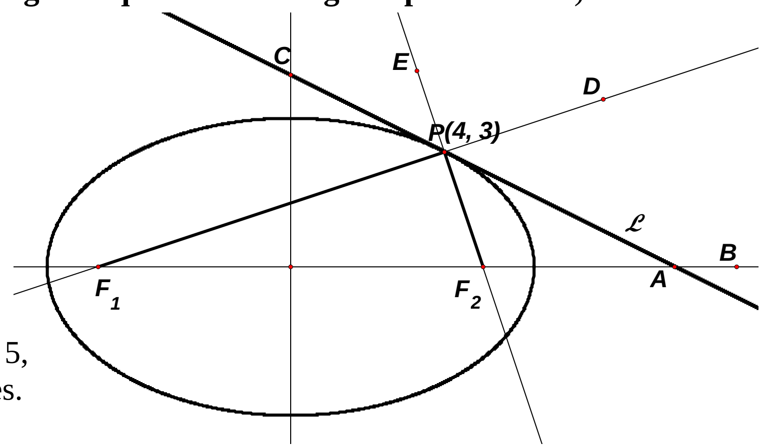
The equation of the ellipse has the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $b^2x^2+a^2y^2=a^2b^2$.

Taking the implicit derivative, we can find the slope of a tangent line at any point.

$2b^2x+2a^2yy'=0 \Rightarrow y'=\frac{-b^2x}{a^2y}$. Using the strategy in solution #4 to find *a* and *b*, we have the slope of the tangent at *P*(4, 3) is $\frac{-15(4)}{40(3)}=-\frac{1}{2}$ and the same result follows.

Solution #6 (**The equations of the angle bisector of the vertical angles formed by perpendicular lines can be found by adding and subtracting the equations of the given pair of lines.**)

Draw lines $\overleftrightarrow{PF_2}$ and $\overleftrightarrow{PF_1}$.
We have two pairs of vertical angles.
Line $\mathcal{L}$ bisects $\angle F_2PD$.
The equation of $\overleftrightarrow{PF_2}$ is $3x+y=15$.
The equation of $\overleftrightarrow{PF_1}$ is $x-3y=-5$.
Subtracting the equations,
we have $2x+4y=20$ or <u>**x + 2y = 10**</u>.
Note: Adding the equations, we get $2x - y = 5$, the bisector of the other pair of vertical angles.



Here is the proof of this assertion on which this solution is based.

Assume the equations are $Ax+By+C=0$ and $Bx-Ay+D=0$.

Since the respective slopes are negative reciprocals, namely $-\frac{A}{B}$ and $\frac{B}{A}$, the lines are parallel. If *A* or *B* is 0, then the lines are vertical and horizontal and still perpendicular.
From geometry, we know that points on the angle bisector of an angle are equidistant from the sides of the angle. Let (*x*, *y*) be an arbitrary point of the angle bisector. Using the point-to-line distance formula, we have $r=\frac{|Ax+By+C|}{\sqrt{A^2+B^2}}$ and $r=\frac{|Bx-Ay+D|}{\sqrt{B^2+(-A)^2}}$.

Since the denominators are equal, we can ignore them. Equating the numerators,

$|Ax+By+C|=|Bx-Ay+D| \Leftrightarrow Ax+By+C=\pm(Bx-Ay+D)$

$\Leftrightarrow \begin{cases}(A-B)x+(B+A)y+(C-D)=0\\(A+B)x+(B+A)y+(C+D)=0\end{cases}$, the required result. Notice that there is no requirement that *A*, *B*, *C* and *D* be integers. So working with equations in slope intercept form, namely $y=mx+b_1$ and $y=-\frac{x}{m}+b_2$, the same results would follow.