MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013
ROUND 7 TEAM QUESTIONS
ANSWERS

A) ____________________________ D) ____________________________

B) 1 2 3 4 5 6 7 E) ____________________________

C) ____________________________ F) ____________________________

A) The sides of a unique right triangle *ABC* are, in order of increasing magnitude, $\left[\frac{x}{8}\right]-1$, $x-1$ and x. The sides are integer lengths with no common factor (other than 1). Compute the perimeter of this triangle.
Note: $[N]$ denotes the greatest integer less than or equal to *N*.
For positive real numbers, the fractional part is truncated (dropped).

B) Definitions: $x \heartsuit y = \frac{x+y}{2}$ (arithmetic average) and $x \diamondsuit y = \frac{2xy}{x+y}$ (harmonic average).

Under which of the following condition(s) does the operation ♥ <u>distribute</u> over the operation ♦, i.e. $a \heartsuit (b \diamondsuit c) = (a \heartsuit b) \diamondsuit (a \heartsuit c)$, where *a*, *b* and *c* are non-negative integers for which <u>both</u> sides of the equality are defined. Circle your choice(s) above.

1) $a = 0$ 2) $b = 0$ 3) $c = 0$ 4) $a = b$ 5) $a = c$ 6) $b = c$ 7) none of the above

C) A line $\mathcal{L}_1$ through the point $Q(-2, 3)$ divides the circle *P* with equation $x^2 + y^2 - 8x + 10y - 23 = 0$ into two congruent regions. A line $\mathcal{L}_2$ passes through the center of the circle perpendicular to $\mathcal{L}_1$. Let *R* be the *x*-intercept of $\mathcal{L}_1$ and *S* be the *y*-intercept of $\mathcal{L}_2$ respectively. Compute the area of quadrilateral *PROS*, where *O* denotes the origin.

D) For $10 \le x \le 10000$, define the function $f(x) = x^{4-\log_{10} x}$.

Let *M* be the minimum value and *N* be the maximum value that $f(x)$ may take on. Compute $\frac{M}{N}$.

E) The area of a certain quadrilateral varies jointly as the distance *D* between two parallel sides and the sum *S* of the lengths of these parallel sides. Two such quadrilaterals are initially congruent and, therefore, have the same area. In the first quadrilateral, *D* is multiplied by a factor of $9n^2$ and *S* is unchanged. In the second quadrilateral, *S* is multiplied by a factor of $27n-20$ and *D* is unchanged. Compute all possible values of *n* for which the areas of these quadrilaterals remain equal.

F) A regular polygon has vertices $V_1, V_2, V_3, \ldots, V_n$. 17 diagonals can be drawn from each vertex. Let *P* be the point of intersection between diagonal $\overline{V_iV_{i+3}}$ and $\overline{V_{i+1}V_{i+5}}$, where $1 \le i \le 10$. Compute $m\angle V_{i+1}PV_{i+3}$.