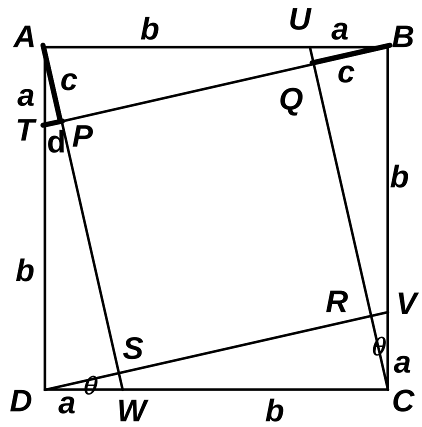
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015 SOLUTION KEY

Team Round - continued



E) Alternate Approach using Trigonometry
(Norm Swanson – Hamilton Wenham – retired)

Since the ratio of the areas of square *ABCD* to square *PQRS* is $\left(\dfrac{DC}{SR}\right)^2$, let's find *DC* and *SR* in terms of $m\angle SDW = m\angle RCV = \theta$.

In ΔDCV , $\cos\theta = \dfrac{DC}{DV} = \dfrac{a+b}{DV}$ and $\sin\theta = \dfrac{VC}{DV} = \dfrac{a}{DV}$.

$\Rightarrow\ DV = (a+b)\sec\theta = a\csc\theta$.

Expanding, $b\sec\theta = a(\csc\theta - \sec\theta) \Rightarrow b = a\left(\dfrac{\csc\theta}{\sec\theta} - 1\right) \Rightarrow \boxed{b = a(\cot\theta - 1)}$.

In ΔDSW , $DS = a\cos\theta$. In ΔRVC , $RV = a\sin\theta$.

Therefore, $\dfrac{DC}{SR} = \dfrac{a+b}{DV - DS - RV} = \dfrac{a + \boxed{a(\cot\theta - 1)}}{a(\csc\theta - \cos\theta - \sin\theta)} = \dfrac{\cot\theta}{\csc\theta - \cos\theta - \sin\theta}$

$$= \frac{\dfrac{\cos\theta}{\sin\theta}}{\dfrac{1}{\sin\theta} - \cos\theta - \sin\theta} = \frac{\cos\theta}{1 - \sin\theta\cos\theta - \sin^2\theta} = \frac{\cos\theta}{\left(\underbrace{1-\sin^2\theta}_{\cos^2\theta}\right) - \sin\theta\cos\theta} = \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}(\cos\theta - \sin\theta)}$$

Squaring this ratio, we have $\left(\dfrac{DC}{SR}\right)^2 = \dfrac{1}{\cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta} = \dfrac{1}{1 - 2\sin\theta\cos\theta} = \dfrac{1}{1 - \sin 2\theta}$

Converting to the required form might follow these lines:

$$\frac{1}{1-\sin 2\theta} = \frac{1}{1 - 2\dfrac{CV}{DV}\cdot\dfrac{CD}{DV}} = \frac{DV^2}{DV^2 - 2CV\cdot CD}$$

Substituting for DV^2 , using the Pythagorean Theorem on ΔDCV , we have

$$\frac{DC^2 + CV^2}{DC^2 + CV^2 - 2CV\cdot DV} = \frac{DC^2 + CV^2}{(DC - CV)^2} = \frac{(a+b)^2 + a^2}{b^2} = 1 + \frac{2(a^2 + ab)}{b^2} \Rightarrow k = \underline{\boldsymbol{a^2 + ab}}\ .$$