MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2012 SOLUTION KEY

Team Round

A) $\begin{cases} x + 7y + 5z = 12 \\ 2x + 9y + 4z = 20 \\ 6x + Ay + 3z = 19 \end{cases}$

Solution #1: (Pretty much brute force – Tedious but nothing difficult)

- Use first two equations to get expressions for *x* and *z* in terms of *y*.
- Substitute into third equation and get an expression for *y* in terms of *A*
- Find smallest positive value for *A* which makes this expression an integer
- Substitute back to get *x*, *y* and *z*.

Multiply equation 1 by –2 and add to second equation: –5y – 6z = –4 or $z = \dfrac{4-5y}{6}$

Using equation 1, $x = 12 - 7y - 5z \Rightarrow x = 12 - 7y - 5\left(\dfrac{4-5y}{6}\right) = \dfrac{72 - 42y - 20 + 25y}{6} = \dfrac{52 - 17y}{6}$

In equation 3, $6\left(\dfrac{52-17y}{6}\right) + Ay + 3\left(\dfrac{4-5y}{6}\right) = 19 \Leftrightarrow 104 - 34y + 2Ay + 4 - 5y = 38$

$\Leftrightarrow 70 + (2A - 39)y = 0 \Leftrightarrow y = \dfrac{70}{39 - 2A}$

Since *A* = 2 is the smallest positive value of *A* that produces an integer value of *y*,

$y = \dfrac{70}{35} = 2$, $x = \dfrac{52-34}{6} = 3$ and $z = \dfrac{4-10}{6} = -1 \Rightarrow$ **(3, 2, –1)**.

Solution #2: Triangularization – The Big Idea

Elementary row operations (EROs) on a square matrix are the equivalent of using linear combinations on the system of coefficients of (linear) equations like the system we have in this problem. Just as linear combinations produce equivalent systems that are "easier" to solve, EROs focus on the coefficients of the equations in the system and ignore the variables. The variables are understood – from left to right the columns hold the *x*, *y*, *z*-coefficients (unless columns are interchanged)

There are three types of EROs:

- interchanging any two rows (or columns)
- replacing a row (column) by that row (column) times and nonzero constant.
- replacing a row with a linear combination of itself and any other row, e.g. for any two rows *A* and *B* and nonzero constants *m* and *n*, row *B* may be replaced by *mA* + *nB*. Again this property also holds for columns.