MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2010 SOLUTION KEY

Team Round - continued

F) Which coefficients are the largest coefficients is not so obvious.
How do we avoid calculating all the coefficients in order to decide?

The combinatorial coefficients are: **1 11 55 165 330 462** 462 330 165 55 55 1

The last six coefficients are identical to the first six since $\binom{11}{i}=\binom{11}{j}$ whenever $i,\ j \geq 0$ and $i+j=11$.

Think of successive combinatorial coefficients in terms of multiplicative factors of

11, 5, 3, 2, 7/5, 1, etc.

Successive coefficients in the expansion of $(3a+2b)^{11}$ introduce one more factor of 2 and one less factor of 3, i.e. a multiplicative factor of 2/3.
Successive coefficients will continue to increase as long as the multiplicative factor is greater than 1.
Combining these multipliers we can determine which coefficient is the largest, without a great deal of tedious arithmetic.

The first coefficient is $\binom{11}{0}3^0 2^{11} = 2^{11}$.

The composite multipliers are:

11(2/3) = 22/3, 5(2/3) = 10/3, 3(2/3) = 2, 2(2/3) = 4/3, (7/5)(2/3) = 14/15

Specifically, the 5th term is 4/3 times the 4th, but the 6th term is only 14/15 times the 5th.
Therefore, the 5th and 6th terms are the two largest coefficients.

$c_5 = \binom{11}{4}(3)^7(2)^4 = 330\cdot(3)^7(2)^4$ and $c_6 = \binom{11}{5}(3)^6(2)^5 = 462\cdot(3)^6(2)^5$

$\rightarrow c_5 - c_6 = 2^4\cdot 3^6(330\cdot 3 - 462\cdot 2) = 2^4\cdot 3^6(990-924) = 2^4\cdot 3^6\cdot 66 = \underline{\mathbf{2^5\cdot 3^7\cdot 11}}$

Notice we did not bother to find the numerical value of either c_5 or c_6 or their difference.