MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2008 SOLUTION KEY

Round 3

A)

$$\begin{array}{r|rrrl} & 1 & k & -13 & 6 \\ & & 2 & 2k+4 & 4k-18 \\ \hline 2 & 1 & k+2 & 2k-9 & 4k-12 = k \to k = \underline{4} \end{array}$$

B) $p(x) = (x-1)Q(x) + 1$ and $p(x) = (x+1)Q_2(x) + 7$, where Q_2 is also unknown to us.
Substituting $x = -1$ in the second equation, $p(-1) = 7$. Even though Q_2 is unknown to us, it is being multiplied by zero!
Substituting, $x = -1$ in the first equation → $p(-1) = -2Q(-1) + 1$ → $7 = -2Q(-1) + 1$ → $Q(-1) = \underline{\mathbf{-3}}$

C) Method 1 (brute force):
Using synthetic substitution, find the roots of the original cubic equation

$$\begin{array}{r|rrrr} & 2 & -5 & -4 & 3 \\ \hline -1 & 2 & -7 & 3 & 0 \\ 3 & 2 & -1 & 0 & \\ 1/2 & 2 & 0 & & \end{array}$$

Thus, the original roots are -1, 3 and ½ and the corresponding roots of the new equation are -2, -2/3 and 1 producing factors of $(x + 2)$, $(3x + 2)$ and $(x - 1)$.
Expanding the product, the new equation is $(x + 2)(3x^2 - x - 2) = \underline{\mathbf{3x^3 + 5x^2 - 4x - 4 = 0}}$

Method II
The equation with reciprocal roots is $2(1/x)^3 - 5(1/x)^2 - 4(1/x) + 3 = 0$ and, multiplying through by x^3 to clear fractions, we have $3x^3 - 4x^2 - 5x + 2 = 0$. Notice that the coefficients have been reversed. Instead of replacing x by $(x - 1)$ and expanding, we'll again use synthetic substitution.

$$\begin{array}{r|rrrr} & 3 & -4 & -5 & 2 \\ \hline 1 & 3 & -1 & -6 & \boxed{-4} \\ 1 & 3 & 2 & \boxed{-4} & \\ 1 & 3 & \boxed{5} & & \\ 1 & \boxed{3} & & & \end{array}$$

→ $\underline{\mathbf{3x^3 + 5x^2 - 4x - 4 = 0}}$