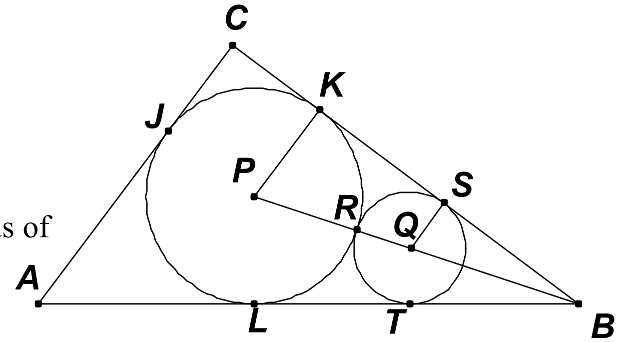
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2011 SOLUTION KEY

Team Round



E) - continued
Solution #2 (Tuan Lee)

After showing that $CK = 2$, $BK = 6$ and the radius of the larger circle (PK) is 2, apply the Pythagorean Theorem to ΔPKB, getting $PB = 2\sqrt{10}$

➔ $BR = 2\left(\sqrt{10}-1\right)$

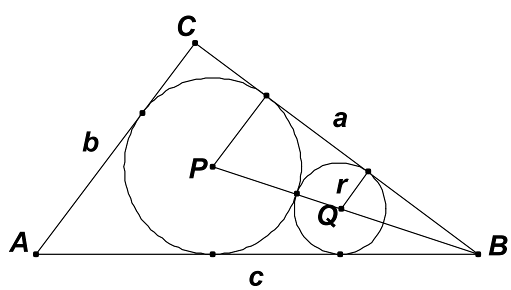
$QR = QS$ ➔ $BR = BQ + QS = 2\left(\sqrt{10}-1\right)$ (Eqtn #1)

Now $\Delta BSQ \sim \Delta BKP$ ➔ $\frac{BQ}{QS} = \frac{BP}{PK} = \frac{2\sqrt{10}}{2} = \sqrt{10}$ ➔ $BQ = \sqrt{10}\,QS$ (Eqtn #2).

Substituting for BQ in eqtn #1, $QS\,(\sqrt{10}+1) = 2\left(\sqrt{10}-1\right)$ ➔ $QS = \frac{2}{9}\left(11-2\sqrt{10}\right)$

Using the same pair of similar triangles, $\frac{QS}{PK} = \frac{BS}{BK}$ ➔ $\frac{\frac{2}{9}\left(11-2\sqrt{10}\right)}{2} = \frac{BS}{6}$ ➔ $BS = \underline{\mathbf{\frac{2}{3}\left(11-2\sqrt{10}\right)}}$.

Conjecture: (Norm Swanson)



For any right triangle with hypotenuse c and legs a and b (a, b and c integers) and two circles externally tangent to each other and internally tangent to the three sides of the right triangle, as shown in the diagram above, the radius of the <u>larger</u> circle is $\frac{ab}{a+b+c}$ or equivalently $\frac{a+b-c}{2}$ and the radius of the <u>smaller</u> circle is $\frac{(a+b-c)\left((a+c)^2+2b^2-2b\sqrt{(a+c)^2+b^2}\right)}{2(a+c)^2}$.

Will you accept the challenge of proving (or disproving) these conjectures?

Insight gives us conjectures.
Proof gives us theorems (generalizations).