MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2008 SOLUTION KEY

Round 2

A) The powers of 7 and 3 are of cyclical order 4, i.e. $3^4 \equiv 1$ mod 4 and $7^4 \equiv 1$ mod 4
In other words, the rightmost digit of each product is 1.
Thus, 3 (and 7) raised to a power that is a multiple of 4 has a rightmost digit of 1.

$7^{218} \cdot 3^{507} = (7^{216} \cdot 3^{504})7^2 3^3 = (\_1)(\_1)(49)((27) = (\_3) \rightarrow$ units digit = **<u>3</u>**

B) $x^2 - y^2 = (x + y)(x - y) = 31$.
Since the only possible factorization of a prime number is itself times 1,
we have $x + y = 31$ and $x - y = 1 \rightarrow x = 16$ and $y = 15$.
Thus, $N = 16(15) = 2^4 \cdot 3 \cdot 5$. Since a factor of this product is only divisible by the prime factors 2, 3 and 5, the factor has the form $2^a 3^b 5^c$. Clearly, the largest possible value of a is 4 and the smallest is 0 (not 1). For 3 and 5, $(b_{max}, b_{min}) = (c_{max}, c_{min}) = (1, 0)$.
Thus, the total number of positive factors is (5)(2)(2) = **<u>20</u>**.

Note: Once you have the <u>prime</u> factorization of an integer, the number of positive integer factors depends only on the exponents. In this case, it was $(a + 1)(b + 1)(c + 1)$.

C) To be divisible by 15, an integer must be divisible by both 3 and 5.
Thus, the rightmost digit must be 5 and the sum of all the digits used must be divisible by 3.
The only two-digit possibility is 75.
The 3-digit possibilities can only be formed using {3, 5, 7} → 375 or 735
There are no 4-digit possibilities since the sum of the 4 digits is 17. → sum = **<u>1185</u>**