MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012 SOLUTION KEY

Round 5

A) $.2A+.6B=B \Leftrightarrow 2A+6B=10B \Leftrightarrow A=2B$

$.3B+.1A=.3\left(\frac{A}{2}\right)+.1A=\left(\frac{3}{20}+\frac{1}{10}\right)A=\frac{1}{4}A \Rightarrow k=\underline{\mathbf{25}}$.

Alternately, pick a convenient value of *A*, say 100.
Applying the first condition, $20+.6B=B \Rightarrow 20=.4B \Leftrightarrow B=50$.
Remember *k*% is equivalent to *k* parts out of 100, i.e. $\frac{k}{100}$.

Applying the second condition, $10+15=\frac{k}{100}\cdot 100 \Rightarrow k=\underline{\mathbf{25}}$.

B) The variation rule is $F=\frac{km_1m_2}{d^2}$.

$F_1=0.004=\frac{4}{1000}=\frac{1}{250}$ Substituting, $\frac{1}{250}=k\cdot\frac{8}{144}=\frac{k}{18} \Rightarrow k=\frac{9}{125}$ (or 0.072)

For the second scenario, $F_2=\frac{\frac{9}{125}\cdot 3\cdot 6}{64}=\frac{81}{4000}$ (or 0.02025)

Thus, $(k,F_2)=\underline{\left(\mathbf{\frac{9}{125},\frac{81}{4000}}\right)}$ or $\underline{(\mathbf{0.072, 0.02025})}$. Of course, the lead zeros are optional.

C) $\frac{(N+1)^3-N^3}{(A+1)^2-A^2}=\frac{3N^2+3N+1}{2A+1}=\frac{3N(N+1)+1}{2A+1}$

Note the numerator, as 3 times the product of two consecutive integers, plus 1 must be odd.
Therefore, the denominator must also be odd.
$A = 1 \Rightarrow$ denom = 3$\Rightarrow$ numerator = 39 (impossible $3N^2+3N-38=0$ has no integer solutions)
$A = 3 \Rightarrow$ denom = 7$\Rightarrow$ numerator = 91
($3N^2+3N-90=3\left(N^2+N-30\right)=3(N+6)(N-5)=0$)

Thus, for N = 5 and A = 3, the larger cube is 216 and the smaller square is 9 $\Rightarrow$ $\underline{\mathbf{225}}$.
As *A* increases, so does *N*. Therefore we have the smallest possible sum.

Alternate solution:
The differences *Q* between consecutive squares 1, 4, 9, 16, 25, … are 3, 5, 7, 9,… .
The differences *P* between consecutive cubes 1, 8, 27, 64, 125, … are 7, 19, 37, 61, …
Notice: In this second sequence, the differences between consecutive terms are 12, 18, 24, …, an amount that is increasing by 6. Thus, the next term is 61 + 30 = 91 = 13(7). Since we are looking for a term in the *P* sequence which is 13 times a term in the *Q* sequence, we have the first occurrence and $6^3 + 3^2 = 216 + 9 =$ $\underline{\mathbf{225}}$.