MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2007 SOLUTION KEY

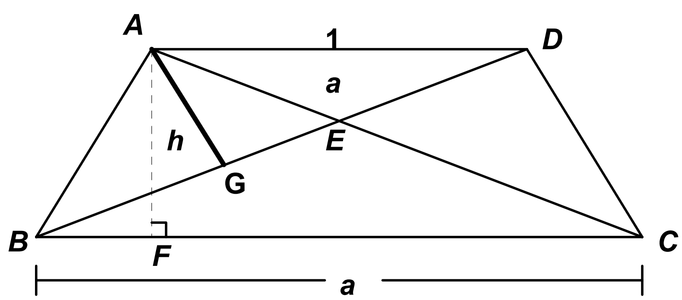
Team Round - continued

D) $x = 2+\cfrac{1}{a+\cfrac{1}{\mathbf{2}+\cfrac{\mathbf{1}}{\mathbf{a}+\ldots}}}$ → $x = 2+\cfrac{1}{a+\cfrac{1}{x}}$ → $x - 2 = \dfrac{x}{ax+1}$

→ $ax^2 - 2ax - 2 = 0$ → $x = \dfrac{2a \pm \sqrt{4a^2+8a}}{2a}$ → $x = 1 \pm \dfrac{\sqrt{a^2+2a}}{a}$

To be rational, the radicand $a^2 + 2a$ must be a perfect square.
Thus, $a^2 + 2a = k^2$ for some integer k.
Then $a^2 + 2a + 1 = (a + 1)^2 = k^2 + 1$ or $(a + 1)^2 - k^2 = 1$
The only integer perfect squares that differ by 1 are 0 and 1.
$a = -1$ → $k^2 = -1$ which contradicts the condition that k is an integer.
$a = 0$ causes division by 0
$a = -2$ → $k = 0$ → $x = 1$
Therefore, the only value of a for which the given continued fraction is rational is **-2**.

E) $A_{trap} = \dfrac{h}{2}(1+a)$ Since $\Delta ADE \sim \Delta CBE$,



$BE : DE = a : 1$ and

$\dfrac{a}{\text{Area}(\Delta CBE)} = \left(\dfrac{1}{a}\right)^2$ → Area(ΔCBE) = a^3

Since ΔADE and ΔABE share a common altitude from point A, namely $\overline{AG}$, their areas are in the same ratio as their bases.

Thus, $\dfrac{\text{Area}(\Delta ABE)}{a} = \dfrac{a}{1}$ → Area(ΔABE) = Area(ΔDCE) = a^2

and $\dfrac{h}{2}(1+a) = a + 2a^2 + a^3 = a(a + 1)^2$ → $\underline{\boldsymbol{h = 2a(a+1)}}$

Alternate: Draw altitude $\overline{HI}$ through E(H on $\overline{AD}$ and I on $\overline{BC}$), and let $HE = x$ and $EI = h - x$

Then $\dfrac{x}{h-x} = \dfrac{1}{a}$ → $x = \dfrac{h}{a+1}$ and the area of $\Delta AED = a = \dfrac{1}{2}\cdot 1\cdot \dfrac{h}{a+1}$ → $\underline{\boldsymbol{h = 2a(a+1)}}$

F) Assume the grandfather (GF) was x years old on his grandson's first birthday.
Next birthday: GF = $(x + 1)$ must be divisible by 2 → x is odd.
3rd birthday: GF = $(x + 2)$ must be divisible by 3.
(The only possibilities are: $x = 3n$, $3n + 1$ or $3n + 2$) Only $x = 3n + 1$ works→ GF = $3n + 3$
4th birthday: GF = $3n + 4$ which must be divisible by 4 and, in turn, this implies n must be divisible by 4. n must be of the form $4k$ → GF = $12k + 4$
5th birthday: GF = $12k + 5$ which must be divisible by 5 → $k = 5j$ → GF = $60j + 5$
6th birthday $60j + 6$. Only j =1 produces a possible age for the grandfather that is less than 100, namely **66**.