MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2010 SOLUTION KEY

Round 1

A) To determine f^{-1}, let $y = f(x)$, switch x and y and solve the resulting equation for y.

$x = \dfrac{3}{y+2}$ ➔ $xy + 2x = 3$ ➔ $y = \dfrac{3-2x}{x}$ ➔ $f^{-1}(x) = \dfrac{3-2x}{x}$

Equating, $\dfrac{3}{x+2} = \dfrac{3-2x}{x}$ ➔ $3x = (3-2x)(x+2) = 3x + 6 - 2x^2 - 4x$

➔ $2x^2 + 4x - 6 = 2\left(x^2 + 2x - 3\right) = 2(x+3)(x-1) = 0$ ➔ $x = -3, 1$ ➔ **<u>(–3, –3), (1, 1)</u>**

Alternate (better) solution:
$y = f(x)$ and $y = f^{-1}(x)$ always intersect along $y = x$. Thus, we may find the points of intersection without explicitly finding the inverse function! We must solve $\dfrac{3}{x+2} = x$.

Cross multiplying, $x(x+2) - 3 = x^2 + 2x - 3 = (x+3)(x-1) = 0$ ➔ **<u>(–3, –3), (1, 1)</u>**

B) Can we avoid having to find the numerical value of the individual roots and evaluating the given tedious expression? YES. The answer is actually determined by just two of the coefficients!!
Since the lead coefficient is 2, the polynomial has factors of 2 and $(x - A)$, $(x - B)$, $(x - C)$, $(x - D)$.
Thus, $2x^4 - 7x^3 - 2x^2 + 13x + 6 = 0 = 2(x - A)(x - B)(x - C)(x - D)$.
Multiplying out these factors, there will be 16 terms, but many of them can be combined.
Convince yourself that:
Only 1 term contains x^4, 4 terms contain x^3, 6 terms contain x^2, 4 terms contain x and 1 term is a constant.
Specifically, the expansion is:

$$2\left(x^4 - (A+B+C+D)x^3 + (\underbrace{AB + AC + \ldots + CD}_{6\ pairs})x^2 - (\boldsymbol{ABC} + \boldsymbol{ABD} + \boldsymbol{ACD} + \boldsymbol{BCD})x + ABCD\right) = 0$$

Thus, $-2\left(ABC + ABD + ACD + BCD\right) = 13$ ➔ **<u>–13/2</u>**

Note: The required quantity was just <u>*the opposite of the x-coefficient divided by the lead coefficient*</u>.
The actual factorization is $(2x + 1)(x + 1)(x - 2)(x - 3)$ ➔ roots: –1/2, –1, 2 and 3.

$$\left(-\frac{1}{2}\cdot -1\cdot 2\right) + \left(-\frac{1}{2}\cdot -1\cdot 3\right) + \left(-\frac{1}{2}\cdot 2\cdot 3\right) + \left(-1\cdot 2\cdot 3\right) = 1 + \frac{3}{2} - 3 - 6 = \frac{5-18}{2} = \underline{\boldsymbol{-\frac{13}{2}}}$$

C) $f(-a) = -f(a)$ ➔ f is an odd function ➔ $B = D = 0$
4 is a zero ➔ $f(4) = 64A + 4C = 0$ or $C = -16A$ (Condition #1)
$f(-3) + 2f(3) + f(5) = 3$ ➔ $f(3) + f(5) = 3$ ➔ $(27A + 3C) + (125A + 5C) = 3$

➔ $152A + 8C = 3$ (Condition #2) Solving the system of equations, we have $(A, C) = \left(\dfrac{1}{8}, -2\right)$ ➔ $\underline{\left(\boldsymbol{\dfrac{1}{8}, 0, -2, 0}\right)}$.

Alternate solution:

$$f \text{ is an odd function} \Rightarrow \begin{cases} (1)\ \text{Since 4 is a zero of the function,}\quad 4 \text{ must be as well.} \\ (2)\ D \text{ must be zero} \\ (3)\ 0 \text{ must also be a zero of the function} \end{cases}$$

Thus, $f(x) = ax(x+4)(x-4) = a\left(x^3 - 16x\right)$ and $\begin{cases} f(-3) + 2f(3) + f(5) = 3 \\ f(-3) = -f(3) \end{cases}$ ➔ $f(5) = 3 - f(3)$

Substituting, $(125 - 80)a = 3 - (27 - 48)a$ ➔ $24a = 3$ ➔ $a = 1/8$ and the result follows.