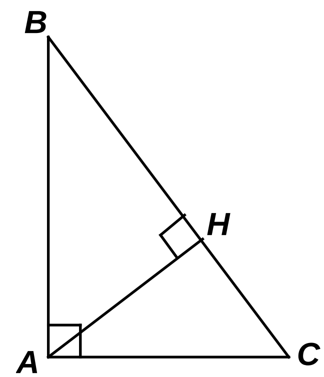
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016 SOLUTION KEY

Team Round



B) 1) $BC^2 = 9^2 + \left(2\sqrt{10}\right)^2 = 121 \Rightarrow BC = 11$

2) In ΔABC, $AB^2 + AC^2 = BC^2$ and $BC = HB + HC = 11$.

3) In ΔAHC, $AH^2 + HC^2 = AC^2$.

4) In ΔAHB, $AH^2 + HB^2 = AB^2$.

Subtracting equation 2) from equation 3),

$HB^2 - HC^2 = (HB + HC)(HB - HC) = AB^2 - AC^2$

$\Rightarrow 11(HB - HC) = AB^2 - AC^2 = 81 - 40 = 41$

Thus, $HB - HC = \underline{\boldsymbol{\frac{41}{11}}}$. Sure beats solving for HB and HC separately and subtracting!

Alternative solution (Brute Force, ***but*** putting off the inevitable as long as possible):

$BC = 11$ and area(ΔABC) = $\frac{1}{2} \cdot 9 \cdot 2\sqrt{10} = \frac{1}{2} \cdot 11 \cdot AH \Rightarrow AH = \frac{18\sqrt{10}}{11}$ (Let $k = AH^2 = \frac{18^2 \cdot 10}{11^2}$.)

From similar triangles, $AH^2 = (HB)(HC) \Rightarrow k = x(11 - x) \Rightarrow x^2 - 11x + k = 0$.

Applying the quadratic formula, $x = \frac{11 \pm \sqrt{11^2 - 4k}}{2}$.

Notice the difference we need is simply $\sqrt{11^2 - 4k}$

Substituting, $11^2 - 4k = 11^2 - \frac{18^2 \cdot 40}{11^2} = \frac{11^4 - 40 \cdot 18^2}{11^2} = \frac{14641 - 12960}{11^2} = \frac{1681}{11^2} = \frac{41^2}{11^2} \Rightarrow \underline{\boldsymbol{\frac{41}{11}}}$.