MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010 SOLUTION KEY

Round 3

A) Points A and B lie on the perpendicular bisector of $\overline{PQ}$. The slope of $\overline{PQ}$ is $\dfrac{-6-(-1)}{3-1}=\dfrac{-5}{2}$

Thus, the slope of $\overline{AB}$ must be $\underline{+\dfrac{\mathbf{2}}{\mathbf{5}}}$ (the negative reciprocal).

No need to find the coordinates of *A* and *B*!!! But if you insist:

$A(a, 0)$ and $PA = QA$ ➔ $(a-1)^2+1=(a-3)^2+36$ ➔ $-2a + 2 = -6a + 45$ ➔ $a = \dfrac{43}{4} = 10.75$

$B(0, b)$ and $PB = QB$ ➔ $1 + (b + 1)^2 = 9 + (b + 6)^2$ ➔ $2b + 2 = 12b + 45$ ➔ $b = -4.3$

The slope of $\overline{AB}$ equals $\dfrac{0-b}{a-0}=\dfrac{-b}{a}=\dfrac{4.3}{10.75}=\dfrac{430}{1075}=\dfrac{5(2)(43)}{5(5)(43)}=\underline{\dfrac{\mathbf{2}}{\mathbf{5}}}$

B) $P(0, 0)$ is clearly one of the points of intersection.

$(x-4)^2+(y+2)^2=20 \Leftrightarrow x^2+y^2-8x+4y=0$

$2(x^2+y^2-8x+4y=0)-(2x^2+2y^2-9x-13y=0) \Leftrightarrow 7x+21y=0 \Leftrightarrow x=-3y$

Substituting, $2(9y^2)+2y^2-9(-3y)-13y=0$ ➔ $20y^2-40y=20y(y-2)=0$

➔ $y = 2$, $x = 6$. Thus, $\overline{PQ}$ connects (0, 0) and (6, 2) and $PQ = \sqrt{6^2+2^2}=\sqrt{40}=\underline{\mathbf{2\sqrt{10}}}$.

C) Consider the diagram at the right. Since the circle is tangent to both axes, its center must be at (k, k) and its radius must be k units. In ΔABC, $AB = |8 - k|$ and $BC = |k - 1|$. Applying the Pythagorean theorem,

$(8-k)^2+(k-1)^2=k^2$ ➔ $k^2 - 18k + 65 = 0$

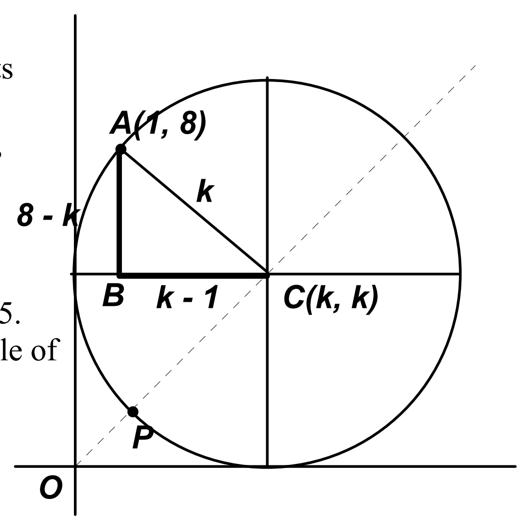
➔ $(k - 5)(k - 13) = 0$ ➔ $k = 5, 13$.

Both k- values are possible. The diagram at the right shows the relative position of P for a circle of radius 5. You are encouraged to re-draw the diagram for a circle of radius 13. The required point is P and

$OP = OC - PC = k\sqrt{2}-k=k(\sqrt{2}-1)$

Thus, $OP = \underline{\mathbf{5(\sqrt{2}-1),\ 13(\sqrt{2}-1)}}$

Both answers are required.



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