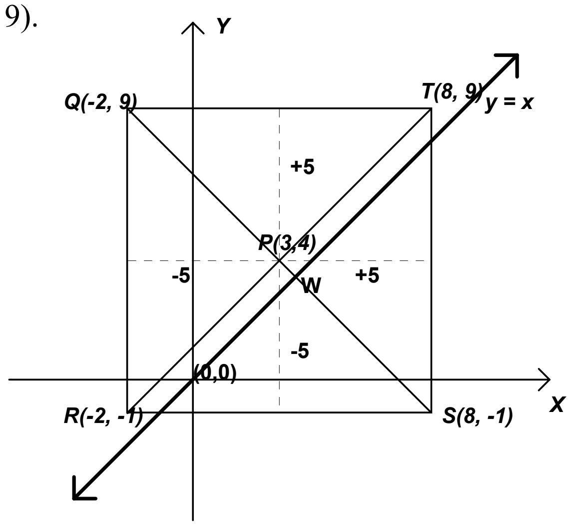
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2011 SOLUTION KEY

Team Round

A) The 4 vertices are $Q(-2, 9)$, $R(-2, -1)$, $S(8, -1)$ and $T(8, 9)$.
Note: The distance from the point $P(h, k)$ to the line L: $Ax + By + C = 0$ is given by the formula $\dfrac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}}$



The equation $y = x$ may be rewritten $x - y + 0 = 0$
Thus, from point Q, the calculations are:

$$\frac{|1(-2)+(-1)(9)+0|}{\sqrt{(1)^2+(-1)^2}} = \frac{|-2-9|}{\sqrt{2}} = \frac{11\sqrt{2}}{2}$$

$R \rightarrow \frac{1}{2}\sqrt{2}$

$S \rightarrow \frac{9}{2}\sqrt{2}$

$T \rightarrow \frac{1}{2}\sqrt{2}$

Thus, the required sum is $\frac{11}{2}\sqrt{2} + 2\left(\frac{\sqrt{2}}{2}\right) + \frac{9}{2}\sqrt{2} = \underline{\mathbf{11\sqrt{2}}}$.

Alternate solution (Norm Swanson):
The sum of the distances from $(-2, 9)$ and $(8, -1)$ to the line $y = x$ is the length of a diagonal of the square, i.e. $10\sqrt{2}$. The sum of the distances of the other vertices from $y = x$ is twice the distance from $x - y = 0$ to $x - y = -1$ which is $2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} \rightarrow \underline{\mathbf{11\sqrt{2}}}$.

B) $x^8 + x^4 + 1 = (x^8 + 2x^4 + 1) - x^4 = (x^4 + 1)^2 - (x^2)^2 = (x^4 - x^2 + 1)(x^4 + x^2 + 1)$
$= (x^4 - x^2 + 1)\left((x^4 + 2x^2 + 1) - x^2\right) = (x^4 - x^2 + 1)\left((x^2 + 1)^2 - x^2\right)$
$= (x^4 - x^2 + 1)(x^2 + 1 - x)(x^2 + 1 + x) = \underline{\mathbf{(x^4 - x^2 + 1)(x^2 - x + 1)(x^2 + x + 1)}}$ or equivalent.

C) $\text{hav}(2A) = \frac{1}{2}(1 - \cos 2A) = \frac{1}{2}\left(1 - (2\cos^2 A - 1)\right) = \frac{1}{2}(2 - 2\cos^2 A) = 1 - \cos^2 A = \sin^2 A$

Thus, $\text{hav}(2A) + \frac{\text{covers}(A)}{2} = 1 \rightarrow \sin^2 A + \frac{1 - \sin A}{2} = 1 \rightarrow 2\sin^2 A - \sin A - 1 = 0$

$\rightarrow (\sin A - 1)(2\sin A + 1) = 0 \rightarrow \sin A = 1, -\frac{1}{2} \rightarrow A = \underline{\mathbf{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}}}$