Team Round - continued

D) $\begin{cases} (1)\ a^2+b^2+ab=12 \\ (2)\ b^2+c^2+bc=13 \\ (3)\ a^2+c^2+ac=19 \end{cases}$ Subtracting $(2)-(1) \Rightarrow 1 = c^2-a^2+b(c-a)=(c-a)(c+a+b)$

Similarly, $(3)-(2) \Rightarrow\ 6=(a-b)(a+b+c)$

By transitivity, $\dfrac{1}{c-a}=\dfrac{6}{a-b} \Rightarrow \boxed{b=7a-6c}$ (4).

Substituting in (1),

$a^2+(7a-6c)^2+a(7a-6c)=12 \Leftrightarrow a^2+\left(49a^2-84ac+36c^2\right)+7a^2-6ac=12$

$\Leftrightarrow\ 57a^2-90ac+36c^2=12\ \Leftrightarrow\ \boxed{19a^2-30ac+12c^2=4}$

Substituting in (2),

$(7a-6c)^2+c^2+(7a-6c)c=13 \Leftrightarrow 49a^2-84a+36c^2+c^2+7ac-6c^2$

$\Leftrightarrow\ \boxed{49a^2-77ac+31c^2=13}$

Factoring these trinomials would be fruitless, unless they were equal to zero!
Multiplying the first equation by 13 and the second by 4, we get our wish.

$13\left(19a^2-30ac+12c^2=4\right)+4\left(49a^2-77ac+31c^2\right)=51a^2-82ac+32c^2=0$

$\Leftrightarrow\ (3a-2c)(17a-16c)=0 \Rightarrow c=\dfrac{3}{2}a, \dfrac{17}{16}a$

Substituting in (4), $b=7a-6\left(\dfrac{3}{2}a\right)=-2a$

Substituting in (1), $a^2+(-2a)^2+a(-2a)=3a^2=12$ and $a>0 \Rightarrow a=2 \Rightarrow \underline{\mathbf{(2,-4,3)}}$.

Alternately, subtracting (2) from (1) and factoring, we have $(a-c)(a+b+c)=-1$. Using (4), $(a-c)(8a-5c)=-1$. For *integer* solutions, one factor would be 1 and the other would be -1.

$\begin{cases} a-c=-1 \\ 8a-5c=1 \end{cases} \Rightarrow (a,b,c)=\underline{\mathbf{(2,-4,3)}}$, but $\begin{cases} a-c=1 \\ 8a-5c=-1 \end{cases} \Rightarrow (a,b,c)=(-2,4,-3)$, rejected since $a<0$.

FYI:
The other substitution for *c* produces *irrational* solutions.

$b=7a-6\left(\dfrac{17}{16}a\right)=\dfrac{5}{8}a \Rightarrow\ a^2+\left(\dfrac{5}{8}a\right)^2+a\left(\dfrac{5}{8}a\right)=\dfrac{129}{64}a^2=12 \Rightarrow\ a^2=\dfrac{4(64)}{43}$ and $a>0 \Rightarrow a=\dfrac{16}{\sqrt{43}}$

$\Rightarrow\ (a,b,c)=\left(\dfrac{16}{\sqrt{43}},\dfrac{10}{\sqrt{43}},\dfrac{17}{\sqrt{43}}\right)$.