MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2014 SOLUTION KEY

Round 1

A) $(3+4i)^2+a+bi=9-16+24i+a+bi=(a-7)+(b+24)i=3+ki$
Equating the real and imaginary parts, $a=10$ and $k=b+24 \Leftrightarrow b=k-24$.
Therefore, $k-24=-10 \Rightarrow k=\underline{\mathbf{14}}$.

B) Squaring both sides of $\sqrt{12-5i}=x+yi$, $\begin{cases} x^2-y^2=12 \\ 2xy=-5 \\ x^2+y^2=13 \end{cases}$.

Where did this third equation come from? We could have solved the second equation for *y* in terms of *x* and substituted in the first, but adding the first and third will be much easier. Consider $|12-5i|$ - the absolute value of the radicand. As on the real number line, the absolute value of a complex number is its distance from the origin $O(0,0)$. Let $x+yi$ be represented by the point $P(x,y)$ in the complex plane and we have $|x+yi|=OP=\sqrt{x^2+y^2}$, regardless of the quadrant in which *P* is located.
Extracting, $\sqrt{(12)^2+(-5)^2}=\sqrt{144+25}=\sqrt{169}=13$ (or recall the Pythagorean Triple 5-12-13)
Thus, adding the first and third equations and dividing by 2, $x^2=\dfrac{12+13}{2}=\dfrac{25}{2}$.
Subtracting the same equations, $2y^2=1 \Rightarrow y^4=\dfrac{1}{4}$. Thus, $\dfrac{x^2}{y^4}=\underline{\mathbf{50}}$. Note that:
$\sqrt{12-5i}$ denotes either $\dfrac{5\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}i$ or $-\dfrac{5\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i$. In both cases, $2xy=-5$ and $x^2-y^2=12$.

C) Simply! Simplify! Simplify!
Recall: $(A\pm B)^3=A^3\pm 3A^2B+3AB^2\pm B^3$
Thus, in the expansion, the first and third terms cancel
Therefore, $(A+Bi)^3-(A-Bi)^3=6A^2Bi-2B^3i=2B(3A^2-B^2)i$.

$$\frac{\left(3\sqrt{3}-3i\right)^3-\left(3\sqrt{3}+3i\right)^3}{108}=\frac{\cancel{3^3}\left(\left(\sqrt{3}-i\right)^3-\left(\sqrt{3}+i\right)^3\right)}{\cancel{3^3}\cdot 4}$$

Since $A=\sqrt{3}$ and $B=-1$, we have $x+yi=\dfrac{2\cdot(-1)\cdot\left(3\cdot 3-(-1)^2\right)i}{4}=-4i \Rightarrow (x,y)=(0,-4)$.
Thus, $x^3+y^3=(-4)^3=\underline{\mathbf{-64}}$.