MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2010 SOLUTION KEY

Round 2

A) The first number has 13 digits and the second number has 11 digits.
The product of an a-digit integer and a b-digit integer will have $a + b$ or $(a + b - 1)$ digits.
[Ex: (2-digit)(3-digit) - min (10)(100) = 1000 (4 digits)
- max $(99)(999) = (100 - 1)(1000 - 1) = 10^5 - 10^3 - 10^2 + 1$
100,000 – 1,000 – 100 + 1 = 98,901 (5 digits)]
Thus, the product P has either 23 or 24 digits. Since (348…)(285…) = 99180…, the product will not have the extra digit. The number of digits in P is **<u>23</u>**.
Using scientific notation, we can trap AB as follows:
$AB > 3.4(10^{12}) \text{ x } 2.8(10^{10}) = 9.52(10^{22}) \rightarrow$ 23 digits (lower bound)
$AB < 3.49(10^{12}) \text{ x } 2.86(10^{10}) = 9.9148(10^{22}) \rightarrow$ 23 digits (upper bound)
The actual numerical value is approx. $9.949626125 \text{ x } 10^{22}$.

B) $360 = 2^3 \cdot 3^2 \cdot 5 \rightarrow$ # divisors = (3 + 1)(2 + 1)(1 + 1) = 24
Clearly, 23, a prime with divisors of 1 and 23 satisfies this requirement.
All integers smaller than 11 or bigger than 23 will fail to satisfy this requirement.
We are left to test integers in the range [11, 22].
$14 \rightarrow 1 + 2 + 7 + 14 = 24$
$15 \rightarrow 1 + 3 + 5 + 15 = 24$
There are no others. It is left to you to verify this.

In all of these cases, the sum of the divisors can be computed by simply adding them all up. However, where there are several divisors, using the shortcut is preferable. Here it is.

If $N = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots$, where the p's denote the primes in the prime factorization of N,

then the sum of the divisors is given by $\dfrac{p_1^{e_1+1}-1}{p_1-1} \cdot \dfrac{p_2^{e_2+1}-1}{p_2-1} \cdot \ldots$

That is, for <u>each</u> prime factor,
- increase the power by 1
- raise the prime to this new power
- subtract 1 from the above result and from the prime itself
- divide the smaller difference into the larger difference

Then take the product of each of these quotients.

For example, to determine the sum of the divisors of 360 would require adding up 24 integers, a tedious task! The shortcut starts with the prime factorization given above: $360 = 2^3 \cdot 3^2 \cdot 5$

Then we compute $\left(\dfrac{2^4-1}{2-1}\right)\left(\dfrac{3^3-1}{3-1}\right)\left(\dfrac{5^2-1}{5-1}\right) = 15(13)(6) = 15(78) = 780 + 390 = 1170$

Check this result out by brute force or by writing a short computer program.