MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2013 SOLUTION KEY

Round 1

A) The powers of i repeat after a cycle of 4 $\left(i,\ i^2=-1,\ i^3=-i,\ i^4=1,\ i^5=i,\ ...\right)$.

$4i^{199}-5i^{365}+25i^{68}-(4i)^3=4i^{4(49)+3}-5i^{4(91)+1}+25i^{4(17)}-(4i)^3=-4i-5i+25+64i=25+55i$

Thus, $(x,y)=\underline{\mathbf{(25,55)}}$.

B) $\left(1-i\right)^3=(1-i)^2(1-i)=\left(-2i\right)\left(1-i\right)=-2i-2$

$\left(2+i\right)^2+k(1+2i)+c=3+4i+k+2ki+c=(3+k+c)+(2k+4)i$

Thus, $\begin{cases}3+k+c=-2\\2k+4=-2\end{cases}\Rightarrow(k,c)=(-3,-2)\Rightarrow\dfrac{-3-2}{-3+2}=\underline{\mathbf{5}}$.

C) $\left(4-4i\right)^{100}\cdot\left(8+8i\right)^{60}=4^{100}(1-i)^{100}8^{60}(1+i)^{60}=2^{200}2^{180}\left((1-i)^2\right)^{50}\left((1+i)^2\right)^{30}=2^{380}(-2i)^{50}(2i)^{30}$

$2^{380}2^{50}(-1)2^{30}(-1)=2^{460}$ and $460=2^2\cdot5\cdot23$

To minimize k, we must maximize A with the restriction that $A<1000$.

This occurs when $A=2^5$ and $k=2^2\cdot23=92\Rightarrow(A,k)=$ $\underline{\mathbf{(32,\ 92)}}$.

(Other factorizations of 460 produce a smaller A-value or an A-value exceeding 1000, e.g. $2^{2\cdot5}=2^{10}=1024>1000$.)