MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2011 SOLUTION KEY

Team Round

A) Let $y = f(x) = \dfrac{3x+1}{2(x-1)}$. Interchanging variables: $x = \dfrac{3y+1}{2(y-1)}$

Solving for y: $2xy - 2x = 3y + 1 \Rightarrow 2xy - 3y = y(2x - 3) = 2x + 1 \Rightarrow y = f^{-1}(x) = \dfrac{2x+1}{2x-3}$

Let $y = g(t) = \dfrac{1}{3t-2}$ Interchanging variables: $t = \dfrac{1}{3y-2}$

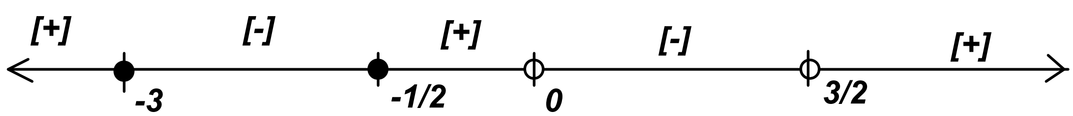
Solving for y: $3ty - 2t = 1 \Rightarrow 3ty = 2t + 1 \Rightarrow y = g^{-1}(t) = \dfrac{2t+1}{3t}$

Thus, we require that $\dfrac{2m+1}{2m-3} \le \dfrac{2m+1}{3m} \Rightarrow \dfrac{2m+1}{2m-3} - \dfrac{2m+1}{3m} \le 0 \Rightarrow (2m+1)\left(\dfrac{1}{2m-3} - \dfrac{1}{3m}\right) \le 0$

$\Rightarrow (2m+1)\left(\dfrac{3m-(2m-3)}{(2m-3)(3m)}\right) \le 0 \Rightarrow \dfrac{(2m+1)(m+3)}{(2m-3)(3m)} \le 0 \quad (m \ne 0, 3/2)$

The critical values are: $-3, -\dfrac{1}{2}, 0, \dfrac{3}{2}$

At the extreme left on the number line all four factors are negative, producing a positive quotient and as we move to the right, the sign of the quotient alternates as we pass each critical point. This is summarized in the following diagram:



Thus, the inequality is satisfied if and only if $\underline{\boldsymbol{-3 \le m \le -\frac{1}{2}} \text{ or } \boldsymbol{0 < m < \frac{3}{2}}}$.

B) Nine two-digit integers can be formed, but only 5 of them are even, namely 18, 36, 54, 72 and 90. Examining the factorization of each of these

$$18 = 2^1 \cdot 3^2, 36 = 2^2 \cdot 3^2, 54 = 2^1 \cdot 3^3, 72 = 2^3 \cdot 3^2, \ 90 = 2^1 \cdot 3^2 \cdot 5^1,$$

we can determine the number of factors by adding 1 to each exponent and then taking the product of all these sums.

18: 2(3) = 6 36: 3(3) = 9 54: 2(4) = 8 **72**: 4(3) = 12 **90**: 2(3)(2) = 12 $\Rightarrow$ **162**.

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