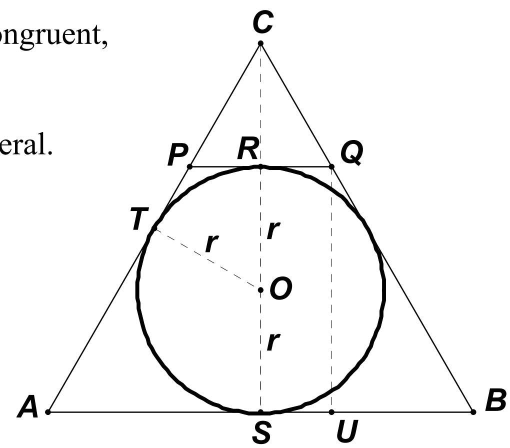
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2008 SOLUTION KEY

Team Round – continued

E) Let $CS = h$
Then the facts that $\Delta PQC \sim \Delta ABC$ and $PQ : AB = 1 : 3$
$\rightarrow CR = h/3$, $RS = 2h/3$ and, therefore $r = h/3$

Let $PR = x$. Then $AS = 3x$
Since tangents to a circle from a common external point are congruent,
$PT = x$ and $AT = 3x$
To maintain the 1 : 3 ratio for $CP : CA$, $CP = 2x$
Thus, since all sides of ΔABC have length $6x$, ΔABC is equilateral.



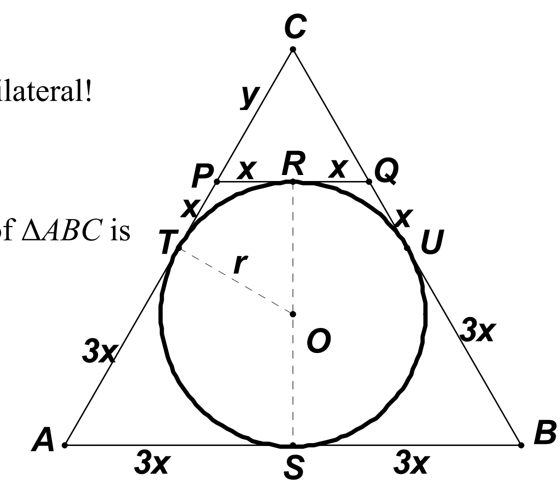
ΔCPR = 30-60-90 triangle and $CR = h/3 \rightarrow PR = \dfrac{h\sqrt{3}}{9}$

and the bases of the trapezoid are $\dfrac{2h\sqrt{3}}{9}$ and $\dfrac{2h\sqrt{3}}{3}$

$$\frac{\pi\left(\frac{h}{3}\right)^2}{\frac{1}{2}\cdot\frac{2}{3}h\left(\frac{2h\sqrt{3}}{9}+\frac{2h\sqrt{3}}{3}\right)} = \frac{\frac{\pi}{9}h^2}{\frac{1}{3}\left(\frac{8\sqrt{3}}{9}\right)h^2} = \frac{\pi}{9}\cdot\frac{27}{8\sqrt{3}} = \frac{3\pi}{8\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \underline{\boldsymbol{\frac{\pi\sqrt{3}}{8}}}$$

Easier Alternate Method:
Convince yourself that ΔABC is not only isosceles, it's equilateral!
Here's why!
$\overline{PQ} \parallel \overline{AB} \rightarrow \Delta PQC \sim \Delta ABC$

Therefore, $\dfrac{y}{y+4x} = \dfrac{2x}{6x} = \dfrac{1}{3} \rightarrow y = 2x$ and since each side of ΔABC is $6x$, it is equilateral.



The required ratio is $\dfrac{\pi r^2}{\frac{1}{2}(2r)(2x+6x)} = \dfrac{\pi r}{8x}$

Draw $\overline{OP}$. ΔOPT is a 30-60-90 right triangle
Thus, $r = x\sqrt{3}$.

Substituting and canceling, $\dfrac{\pi r}{8x} = \underline{\boldsymbol{\frac{\pi\sqrt{3}}{8}}}$