MASSACHUSETTS MATHEMATICS
CONTEST 6 - MARCH 2013 SOLUTION KEY

Team Round - continued

F) Consider the trinomial expansion $\left(x^2+x-1\right)^6$ to be a binomial expansion $\left(x^2+\left(x-1\right)\right)^6$.

The k^{th} term in the expansion is $\binom{6}{k}\left(x^2\right)^{6-k}(x-1)^k = \binom{6}{k}x^{12-2k}\left(x-1\right)^k$

If $k \geq 4$, then the largest exponent of x never exceeds 8.
If $k \leq 1$, then the smallest exponent of x always exceeds 9.
Therefore, we restrict our attention to k = 2 and 3.

$k = 2 \Rightarrow \binom{6}{2}x^8\left(x-1\right)^2 = 15(x^8)(x^2-2x+1) \;\Rightarrow\; -30x^9$
$k = 3 \;\Rightarrow\; 20x^9$
Combining terms, we have <u>**–10**</u>x^9.

<u>An alternative solution</u> applies the <u>Multinomial Theorem</u>. For the expansion of $\left(x^2+x-1\right)^6$,

the only terms that will contain x^9 are: $\left(x^2\right)^3\left(x\right)^3\left(-1\right)^0$ or $\left(x^2\right)^4\left(x\right)^1\left(-1\right)^1$

The corresponding multinomial coefficients are $\dfrac{6!}{3!3!0!}=\dfrac{5!}{3!}=20$ and $-\dfrac{6!}{4!1!1!}=-\dfrac{6\cdot 5}{1}=-30$

and the required coefficient is again <u>**–10**</u>.

<u>Here's the idea behind the multinomial expansion</u> of $\left(x_1+x_2+...+x_m\right)^n$, illustrated for $\left(A+B+C\right)^9$.

Clearly, the first term in the expansion would be A^9 and the last term, C^9.

Will there be terms like A^3B^6, $A^2B^3C^4$ and B^3C^7 ? There are 9 factors of $\left(A+B+C\right)$ to be multiplied and from each trinomial factor, a single monomial multiplier must be selected each time. Thus, the sum of the exponents must be 9 and the last term is <u>not</u> possible. If A is selected as the monomial multiplier 3 times, B 6 times (and C not at all), we will get an A^3B^6 term.

How many are there in the expansion of $\left(A+B+C\right)^9$? 9 objects in groups of 3 identical and 6 identical $\Rightarrow \dfrac{9!}{3!6!}=84$, which would the coefficient of A^3B^6.

Any *AA BBB CCCC* arrangement will produce $A^2B^3C^4$ and there are $\dfrac{9!}{2!3!4!}=1260$ such arrangements. Thus, the numerator of the multinomial coefficient will always be n! The sum of the exponents in any term of the expansion must always be n.

The denominator will always be the product of the factorials of the individual exponents.

In general, the multinomial coefficient of the term $x_1^{\,a_1}x_2^{\,a_2}\cdot\ldots\cdot x_m^{\,a_m}$, where $a_1+a_2+...+a_m=n$

in the expansion of $\left(x_1+x_2+...+x_m\right)^n$ is $\binom{n}{a_1\;,a_2\,,...,a_m}=\dfrac{n!}{a_1!\cdot a_2!\cdot\,...\,\cdot a_m}$.