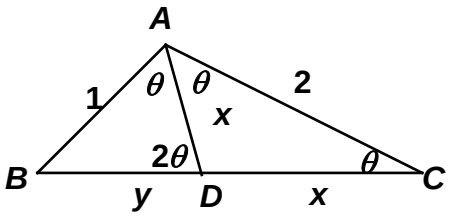
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016 SOLUTION KEY

Team Round - continued



E) Let $DC = x$, $BD = y$ and $m\measuredangle DAC = m\measuredangle DCA = \theta$.
As an exterior angle of ΔADC, $m\measuredangle BDA = 2\theta$
$m\measuredangle DAC = m\measuredangle DCA \Rightarrow DA = DC$.
$\Delta BAC \sim \Delta BDA \Rightarrow m\measuredangle BCA = m\measuredangle BAD = \theta$. Thus, $\overrightarrow{AD}$ is an angle bisector and $\frac{y}{1} = \frac{x}{2} \Rightarrow x = 2y$. $\Delta BAC \sim \Delta BDA \Rightarrow \frac{BC}{BA} = \frac{AC}{DA} \Leftrightarrow \frac{x+y}{1} = \frac{2}{x}$.

Cross multiplying, $x^2 + xy = 2 \Leftrightarrow 4y^2 + 2y^2 = 2 \Leftrightarrow y^2 = \frac{1}{3} \Rightarrow y = \frac{\sqrt{3}}{3} \Rightarrow AD = x = \underline{\mathbf{\frac{2\sqrt{3}}{3}}}$.

F) JJ's current mile time is 270 seconds; big brother's is 243. After one year, JJ's time will be $270(1-k\%) = 270\left(\frac{100-k}{100}\right)$. After 2 years, his time will be

$\left(270\left(\frac{100-k}{100}\right)\right)\left(\frac{100-k}{100}\right) = 270\left(\frac{100-k}{100}\right)^2$. We require that

$270\left(\frac{100-k}{100}\right)^2 < 243 \Rightarrow (100-k)^2 < \frac{243}{270}\cdot 10^4 = 9\cdot 10^3 \Rightarrow (100-k) < \sqrt{9\cdot 10^3} = 30\sqrt{10}$

$k > 100 - 30\sqrt{10} \approx 100 - 94.86 \approx 5.14$. Therefore, k must be at least **6**.

FYI: A 6% improvement over 2 years would drop little brother's time to 238.57 seconds and he would break the 4:00 mile (240 seconds), whereas a 5% improvement would result in a mile-time of 243.675 seconds and he would fall just short of beating big brother.

Some real world perspective:
The first American to run a sub 4-minute mile was Don Bowdon in 1957 (3:58.7). Jim Ryan was the first American high school runner to break the 4-minute mile (3:55.3 in 1965 - Witchita, KS). 5 others have joined him since - Tim Danielson (1966), Marty Liquori (1967), Alan Webb (2001), Lukas Verzbicas (2011), and Matthew Maton (2015). Only Webb has had a faster time (3:53.43). The current world record time for the mile of 3:43.13 (set in 1999 in Rome) belongs to Morocco's Hicham El Guerrouj who bested Kenyan Noah Ngeny's time of 3:43.40 **in the same race** with quarter-mile splits of 55.6, 56.0, 56.3 and 55.2 . Two runners bettered the then world record of Algerian Noureddine Morceli (3:44.39) by almost a full second! As of this date (2016), these three times are still the fastest mile times ever recorded! You can watch the 1999 record setting race on Youtube. The link is https://www.youtube.com/watch%3Fv%3DJi0yK7fV5Rk.

Roger Bannister
First to break 4-minute barrier
(May 6th, 1954 3:59.4)

Hicham El Guerrouj bests Noah Ngeny by 0.27 seconds in a world-record time of 3:43.13
July 7th, 1999