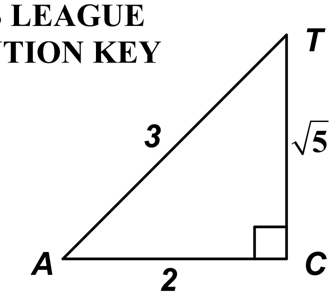
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2009 SOLUTION KEY



Round 1

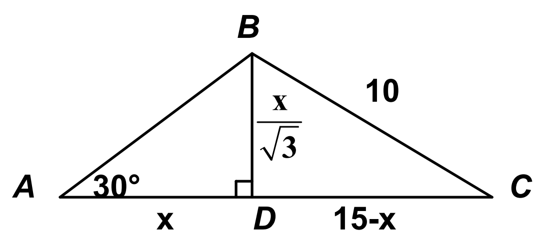
A) $\frac{AC}{180} = \cos(\angle A)$ ➔ $AC = 180\left(\frac{2}{3}\right) = \underline{\mathbf{120}}$

B) $\frac{\sin 30}{4} = \frac{\sin B}{n}$

➔ $\sin B = \frac{n}{8}$ ➔ $0 < \frac{n}{8} < 1$ ➔ $n = 1, 2, \ldots, 7$. This condition is <u>necessary</u> to guarantee the existence of a non-right ΔABC, but not <u>sufficient</u> to guarantee that the triangle is acute.
$m\angle C = 30°$ ➔ $m\angle A + m\angle B = 150°$.
Both A and B must be acute; hence $m\angle B < 90°$ ➔ $150 - m\angle A < 90$ ➔ $m\angle A > 60$
Applying the same reasoning to $\angle A$, we have $60 < m\angle A, m\angle B < 90$.
Since the sine is a strictly increasing function over this interval, we have
$\sin 60° < \sin B < \sin 90°$ ➔ $\frac{\sqrt{3}}{2} < \frac{n}{8} < 1$ ➔ $4\sqrt{3} < n < 8$. $4\sqrt{3} \approx 4(1.7) \approx 6.8$ and only $n = 7$ satisfies this requirement. Therefore, there is only <u>**one**</u> value of n for which ΔABC is acute.

C) Using the Law of Sines, $\frac{\sin 30°}{10} = \frac{\sin B}{15}$ ➔ $\sin B = 15/20 = ¾$ (and B is obtuse)

Method 1:
Using the Pythagorean Theorem on ΔBDC,



$\frac{x^2}{3} + (15 - x)^2 = 100$ ➔ $x^2 + 675 - 90x + 3x^2 = 300$

➔ $4x^2 - x + 375 = 0$ ➔ $x = \frac{90 \pm \sqrt{8100 - 16(375)}}{8}$

$= \frac{90 \pm 10\sqrt{21}}{8}$ and since $x < 15$, $x = \frac{5(9 - \sqrt{21})}{4}$

and $\sin C = \frac{x}{10\sqrt{3}} = \frac{5(9-\sqrt{21})}{4 \cdot 10\sqrt{3}} = \frac{9 - \sqrt{21}}{8\sqrt{3}} = \frac{9\sqrt{3} - 3\sqrt{7}}{8 \cdot 3} = \underline{\mathbf{\frac{3\sqrt{3} - \sqrt{7}}{8}}}$

Method 2: far easier w/this approach, but students in this round are not expected
to know the expansion of $\sin(A - B)$

$\sin B = ¾$ and $\angle B$ obtuse ➔ $\cos B = -\frac{\sqrt{7}}{4}$

$A + B + C = 180$ ➔ $C = 150 - B$

Thus, $\sin C = \sin(150 - B) = \sin 150 \cos B - \sin B \cos 150 = \frac{1}{2} \cdot \frac{-\sqrt{7}}{4} - \frac{3}{4} \cdot \frac{-\sqrt{3}}{2} = \underline{\mathbf{\frac{3\sqrt{3} - \sqrt{7}}{8}}}$