MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016 SOLUTION KEY

Round 1

A) Solution #1:

Since $i^4 = 1$, $\left(i^4\right)^{\text{any integer power}}$ equals 1.

Therefore, $i^A \cdot i^{2A} \cdot i^{4A} \cdot i^{8A} \cdot i^{16A} = i^A \cdot i^{2A} \cdot \cancel{i^{4A}} \cdot \cancel{i^{8A}} \cdot \cancel{i^{16A}} = i^{3A} = i^1 = i^5 = i^9$ and we find the minimum value of A by equating exponents. $3A = 9 \Rightarrow A = \underline{\mathbf{3}}$.

Solution #2:

$i^A \cdot i^{2A} \cdot i^{4A} \cdot i^{8A} \cdot i^{16A} = i^{31A}$

$A = 1 \Rightarrow i^{31} = \left(i^4\right)^7 i^3 = 1^7 \left(-i\right) = -i$

$A = 2 \Rightarrow i^{62} = \left(i^4\right)^{15} i^2 = -1$

$A = 3 \Rightarrow i^{93} = \left(i^4\right)^{23} i = i \Rightarrow A_{\min} = \underline{\mathbf{3}}$.

B) $z = 0.1 + 6i \Rightarrow \overline{z} = 0.1 - 6i$. If $a = \dfrac{1}{z + \overline{z}}$ and $bi = z - \overline{z}$,

$a = \dfrac{1}{.2} = 5$, $b = 12 \Rightarrow \left|5 + 12i\right| = \sqrt{5^2 + 12^2} = \sqrt{169} = \underline{\mathbf{13}}$ (or recall 5-12-13 Pythagorean Triple)

C) $\left(3+4i\right)^{\frac{1}{2}} = C + Di \Rightarrow (C + Di)^2 = 3 + 4i \Rightarrow \begin{cases} C^2 - D^2 = 3 \\ 2CD = 4 \end{cases} \Rightarrow (C, D) = (2, 1)$ or $\left(-2, -1\right)$.

$\left(3+4i\right)^{\frac{3}{2}} = \left(\left(3+4i\right)^{\frac{1}{2}}\right)^3 = (2+i)^3 = (2+i)^2(2+i) = (3+4i)(2+i) = 2 + 11i$

Thus, $\left(A, B\right) = \underline{\mathbf{(2, 11)}}$.

$\left(-2, -1\right)$ is rejected, since $(-2-i)^3 = \left(-1\right)^3 (2+i)^3 = -2 - 11i$ and it was required that A and B be positive integers.