MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2013 SOLUTION KEY

Round 2

A) Expanding $(A + B)^3$, we know $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$ or we expand in stages:
$(A+B)^3 = (A+B)(A+B)^2 = (A+B)\left(A^2 + 2AB + B^2\right)$ and the same result follows.
$A^3 + 3A^2B + 3AB^2 + B^3 = \left(A^3 + B^3\right) + 3AB(A+B) = 258$
Subtracting, using the second condition, we have $A^3 + B^3 = 258 - 3\cdot 5 =$ **<u>243</u>**.

B) To insure that the radicand is itself a positive integer, we require that
$82 - x \geq 2x + 1 \Rightarrow 3x \leq 81 \Rightarrow x \leq 27$
Thus, the largest possible x-value is 27 and $\sqrt[3]{\dfrac{82-27}{2\cdot 27+1}} = \sqrt[3]{\dfrac{55}{55}} = 1$, so $x =$ **<u>27</u>** works.

<u>FYI</u>: For $x = 1$, 2 and 5, $\dfrac{82-x}{2x+1}$ evaluates to 27, 16 and 7.
For $x = 7$, 16 and 27, $\dfrac{82-x}{2x+1}$ evaluates to 5, 2 and 1. Did you expect this to happen?

C) Squaring both sides, $(4x+1) + 2\sqrt{4x+1}\sqrt{10-x} + (10-x) = 49$.
Combining like terms and isolating the radicals, $2\sqrt{4x+1}\sqrt{10-x} = 38 - 3x$
Squaring,
$4(4x+1)(10-x) = \left(38-3x\right)^2 \Leftrightarrow -16x^2 + 156x + 40 = 9x^2 - 284x + 1444$
$\Leftrightarrow 25x^2 - 384x + 1404 = 0$

1404
2 702
2 351
3 117
3 39
3 13

This would be a pain to factor over the integers (if indeed it factored at all), except for the hint that there was an integer solution.
One of the factors must be of the form $(x - c)$, where c is an integer constant.
Thus, we start with $\left(25x - \square\right)\left(x - \square\right)$ and consider possible factorizations of $1404 = 2^2 \cdot 3^3 \cdot 13$ until we find the one which gives the proper coefficient of the middle term.
$(25x - 234)(x - 6)$ gives the correct middle term $(150 + 234 = 384)$, implying that the fractional solution is $\underline{\boldsymbol{\dfrac{234}{25}}}$. Note: Both solutions check: $\sqrt{25} + \sqrt{4} = 7$ and, converting $\dfrac{234}{25}$ to the equivalent decimal (9.36), $\sqrt{38.44} + \sqrt{0.64} = 6.2 + 0.8 = 7$

Alternate (**Better!**) Solution (Norm Swanson – Hamilton Wenham)
Guessing the integer solution 6 is a matter of a little trial and error. As above squaring both sides twice, we get $25x^2 - 384x + (\text{We don't care!}) = 0$ and, by inspection, the sum of the roots is $\dfrac{384}{25}$. Since the integer root is 6, the fractional root must be $\dfrac{384-150}{25} = \underline{\boldsymbol{\dfrac{234}{25}}}$.