MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008 SOLUTION KEY

Team Round - continued

F) There are 31 fractions in this list of reduced fractions with denominators ≤ 10.

Denominator	2	3	4	5	6	7	8	9	10
Count	1	2	2	4	2	6	4	6	4

It seems reasonable that there would always be just as many fractions less than ½ as there would be greater than ½ . Thus, ½ is the 16^{th} fraction in this increasing list, implying we want the next largest fraction. The possible suspects: 2/3, 3/4, 3/5, 4/7, 5/8, 5/9 and 7/10.

By comparing the decimal equivalents ($0.\overline{6}, 0.75, 0.6, 0.\overline{571428}, 0.625, 0.\overline{5}, 0.7$) or invoking the fact that $\frac{a}{b} > \frac{c}{d} \Leftrightarrow ad > bc$ for $a, b, c, d > 0$, we have the seventeenth fraction in the list, namely **5/9**.

Alternate solution (especially useful for longer lists)
For example, find the 495^{th} fraction in list of reduced fractions w/ denominators ≤ 50.
There are 773 fractions in this list!

Start with the "seed" list: $\frac{0}{1}, \frac{1}{1}$ and apply this rule: Between successive elements $\frac{a}{b}, \frac{c}{d}$ insert $\frac{a+c}{b+d}$
as long as $b + d$ does not exceed the denominator of the previous list PLUS 1.
In each list, the fractions will automatically be in increasing order!
A programmable solution is now easily within reach.
These sequences are referred to as sequences of **Farey Fractions**.

List 2: $\frac{0}{1}, \mathbf{\frac{1}{2}}, \frac{1}{1}$

List 4: $\frac{0}{1}, \mathbf{\frac{1}{4}}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \mathbf{\frac{3}{4}}, \frac{1}{1}$ (2/5 and 3/5 were not added)

List 3: $\frac{0}{1}, \mathbf{\frac{1}{3}}, \frac{1}{2}, \mathbf{\frac{2}{3}}, \frac{1}{1}$

List 5: $\frac{0}{1}, \mathbf{\frac{1}{5}}, \frac{1}{4}, \frac{1}{3}, \mathbf{\frac{2}{5}}, \frac{1}{2}, \mathbf{\frac{3}{5}}, \frac{2}{3}, \frac{3}{4}, \mathbf{\frac{4}{5}}, \frac{1}{1}$ (2/7 and 5/7 were not added)

List 6: $\frac{0}{1}, \mathbf{\frac{1}{6}}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \mathbf{\frac{5}{6}}, \frac{1}{1}$

Lis 7: $\frac{0}{1}, \mathbf{\frac{1}{7}}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \mathbf{\frac{2}{7}}, \frac{1}{3}, \frac{2}{5}, \mathbf{\frac{3}{7}}, \frac{1}{2}, \mathbf{\frac{4}{7}}, \frac{3}{5}, \frac{2}{3}, \mathbf{\frac{5}{7}}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \mathbf{\frac{6}{7}}, \frac{1}{1}$

List 8: $\frac{0}{1}, \mathbf{\frac{1}{8}}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \mathbf{\frac{3}{8}}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \mathbf{\frac{5}{8}}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \mathbf{\frac{7}{8}}, \frac{1}{1}$

List 9: $\frac{0}{1}, \mathbf{\frac{1}{9}}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \mathbf{\frac{2}{9}}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \mathbf{\frac{4}{9}}, \frac{1}{2}, \mathbf{\frac{5}{9}}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \mathbf{\frac{7}{9}}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \mathbf{\frac{8}{9}}, \frac{1}{1}$

List 10:

$$\frac{0}{1}, \mathbf{\frac{1}{10}}, \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{2}{7}, \mathbf{\frac{3}{10}}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{1}{2}, \underset{\uparrow}{\frac{5}{9}}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \mathbf{\frac{7}{10}}, \frac{5}{7}, \frac{3}{4}, \frac{7}{9}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \mathbf{\frac{9}{10}}, \frac{1}{1}$$