MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2014 SOLUTION KEY

F) $P_n(x) = xP_{n-1}(x) - P_{n-2}(x)$, where $P_0(x) = 1$ and $P_1(x) = 1 \Rightarrow$

$P_2(x) = xP_1(x) - P_0(x) = x(1) - 1 = x - 1$
$P_3(x) = xP_2(x) - P_1(x) = x(x-1) - 1 = x^2 - x - 1$
$P_4(x) = xP_3(x) - P_2(x) = x(x^2 - x - 1) - (x-1) = x^3 - x^2 - 2x + 1$ etc.

$$\sum_{n=0}^{n=6} P_n(3) = P_0(3) + P_1(3) + P_2(3) + P_3(3) + P_4(3) + P_5(3) + P_6(3)$$

Rather than trying to evaluate explicit formulas for each of the polynomials P_i, let's use the recursive definition.

$P_2(3) = 3(1) - 1 = 2$
$P_3(3) = 3(2) - 1 = 5$
$P_4(3) = 3(5) - 2 = 13$
$P_5(3) = 3(13) - 5 = 34$
$P_6(3) = 3(34) - 13 = 89$

Thus, $\sum_{n=0}^{n=6} P_n(3) = 1 + 1 + 2 + 5 + 13 + 34 + 89 = \underline{\mathbf{145}}$.

Notice that from 2 on it appears that we are getting every other Fibonacci number.
Recall that the Fibonacci sequence was defined by
$a_n = a_{n-1} + a_{n-2}$, where $a_0 = a_1 = 1$.
Specifically, the sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …
What other interesting facts came be derived from this innocent sequence of polynomials?
Time will tell.
Send your ideas to olson.re@gmail.com.