MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013 SOLUTION KEY

Team Round

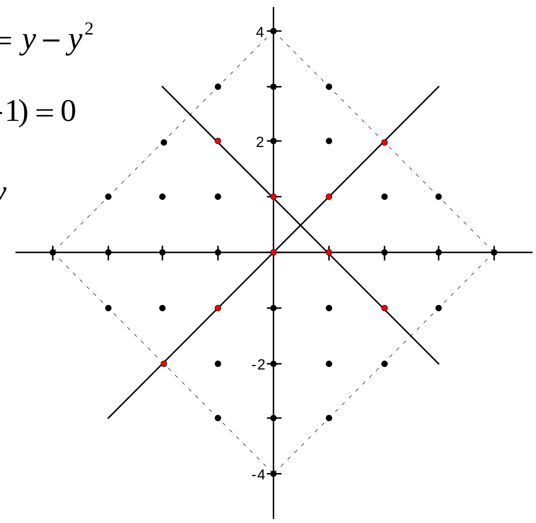
E) $x \blacklozenge y = \dfrac{x+1}{2-y}$ and $S = \{(x, y): |x| + |y| \le 4$, where x and y are integers$\}$

$\dfrac{x+1}{2-y} = \dfrac{y+1}{2-x} \Rightarrow 2x - x^2 + 2 - x = 2y - y^2 + 2 - y \Rightarrow x - x^2 = y - y^2$

$\Rightarrow x^2 - y^2 = x - y \Rightarrow x^2 - y^2 - (x-y) = 0 \Rightarrow (x-y)(x+y-1) = 0$

$\Rightarrow x = y$ (5 solutions) or $x + y = 1$ (4 solutions)

But 3 of these solutions are extraneous, since neither x nor y can be 2. Thus, there are **6** solutions.



F) $\left[6, (2+a)(3+b)\right]$, where $1 < a \le 10$, $0 < b$ and $ab = 1$.

The closed interval is $\left[6, 6 + 2b + 3a + ab\right] = \left[6, 7 + 3a + 2b\right]$.

Since the coefficient of a is larger than the coefficient of b, if we maximize a (i.e. take $a = 10$) and, correspondingly, take $b = 1/10$, the length of the interval is maximized, namely $6 \le x \le 37.2$.

This interval contains 4 integer perfect squares: 9, 16, 25 and 36 and $M = 4$.

To minimize the interval, we want a as small as possible, but there is no minimum positive value for a.

However, since $ab = 1$ and $a > 1$, $0 < b < 1$. $7 + 3a + 2b > 10$.

Thus, the minimum interval contains 9 and $m = 1$. Therefore, $(m, M) =$ **(1, 4)**.