Team Round - continued

B) Notice that $\left(1+\sqrt{35}\right)^2=36+2\sqrt{35}$. Dividing both sides by 2, we have $\left(\dfrac{1+\sqrt{35}}{\sqrt{2}}\right)^2=18+\sqrt{35}$.

Taking the square root, $\sqrt{18+\sqrt{35}}=\dfrac{1+\sqrt{35}}{\sqrt{2}}\Rightarrow$ **<u>(1, 35, 2)</u>**. (Clearly, 1+35+2 = 38 is a minimum.)

<u>An alternative solution</u>, finding an infinite set of solutions, might proceed as follows:

Squaring both sides, $\sqrt{18+\sqrt{35}}=\dfrac{x+\sqrt{y}}{\sqrt{z}}$ becomes $\begin{cases}(1)\quad \dfrac{x^2+y}{z}=18\\ (2)\quad \dfrac{2x\sqrt{y}}{z}=\sqrt{35}\Rightarrow 4x^2y=35z^2\end{cases}$

If $z > y > x$ and each variable represents a positive integer, then for some positive integers a and b, $z=y+a$ and $y=x+b$.

Substituting in (1), $\left(y-b\right)^2+y=18(y+a)\Rightarrow (y-b)^2=17y+18a$.

Substituting in (2), $4\left(y-b\right)^2y=35(y+a)^2 \Rightarrow 4(17y+18a)y=35\left(y+a\right)^2$

$\Rightarrow 68y^2+72ay=35y^2+70ay+35a^2 \Rightarrow 33y^2+2ay-35y^2=(33y+35a)(y-a)=0$

$\Rightarrow a=y\Rightarrow z=2y$.

Eqtn (1) above now becomes $x^2+y=36y\Rightarrow x^2=35y\Rightarrow x^2=35(x+b)\Rightarrow x^2-35x-35b=0$

Using the quadratic formula, $x=\dfrac{35\pm\sqrt{35^2+35(4)b}}{2}=\dfrac{35\pm\sqrt{35(35+4b)}}{2}$.

The radicand must be a perfect square.

To force a perfect square radicand, $(35+4b)$ must be 35 times a perfect square.

To that end, let $b=35n$. Then: $35+4b=35(1+4n)$. For $n=0, 2, 6, 12, 20, \ldots$ the radicand satisfies the requirement. Thus, $b=35n$ and $x=\dfrac{35\left(1\pm\sqrt{4n+1}\right)}{2}$.

Note: If we use the "–" sign for any value of n, $x\le 0$, violating x's status as a positive integer.

$$n=\begin{cases}0\\2\\6\\12\\20\\\ldots\end{cases}\Rightarrow x=\frac{35}{2}\cdot\begin{cases}1+1\\1+3\\1+5\\1+7\\1+9\\\ldots\end{cases}\Rightarrow x=35\cdot\begin{cases}1\\2\\3\\4\\5\\\ldots\end{cases}$$

The <u>smallest</u> sum is generated by the triple for $n=0$ (or $b=35$), namely $(x, y, z)=(35, 35, 70)$.

However, $\dfrac{x+\sqrt{y}}{\sqrt{z}}=\dfrac{35+\sqrt{35}}{\sqrt{70}}=\dfrac{35/\sqrt{35}+\sqrt{35}/\sqrt{35}}{\sqrt{70}/\sqrt{35}}=\dfrac{1+\sqrt{35}}{\sqrt{2}}\Rightarrow(x,y,z)=\underline{\mathbf{(1,35,2)}}$.