MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2010 SOLUTION KEY

Team Round

A) Given : $\begin{cases} \dfrac{1}{z} = \bar{z} \\ a+b=1.4 \\ a>b \end{cases}$ and $z = a + bi$. Substitute $b = \left(\frac{7}{5} - a\right)$ in $\frac{1}{a+bi} = a - bi \Leftrightarrow a^2 + b^2 = 1$

$a^2 + \left(\frac{7}{5} - a\right)^2 = 1 \Rightarrow 2a^2 + \frac{49}{25} - \frac{14a}{5} = 1 \Rightarrow 50a^2 - 70a + 24 = 0$

$\Rightarrow 25a^2 - 35a + 12 = (5a-3)(5a-4) = 0$

$a > b \Rightarrow (a, b) = \underline{\left(\mathbf{\frac{4}{5}, \frac{3}{5}}\right)}$.

Alternate solution (Norm Swanson)

$z \cdot \bar{z} = 1 \Rightarrow z$ lies on the unit circle with center at (0, 0). So, let $a = \cos(t)$ and $b = \sin(t)$ and consequently, $a^2 + b^2 = 1$.

Squaring the second equation, $a^2 + 2ab + b^2 = \left(\frac{7}{5}\right)^2 \Rightarrow 2ab = \frac{49}{25} - 1 = \frac{24}{25} \Rightarrow ab = \frac{12}{25}$.

We want two numbers whose sum is 7/5 and whose product is 12/25.
Clearly, 3/5 and 4/5 satisfy the requirement.

$a > b \Rightarrow (a, b) = \underline{\left(\mathbf{\frac{4}{5}, \frac{3}{5}}\right)}$.

B) Rearranging terms, $1 - \frac{1}{x^2} = \frac{1}{2x} - \frac{1}{2x^3} \Rightarrow \frac{x^2-1}{x^2} = \frac{x^2-1}{2x^3}$

Since $x \neq 0$, this simplifies to $\frac{x^2-1}{1} = \frac{x^2-1}{2x}$

For $x = \pm 1$, both terms are 0; otherwise, equating the denominators, $2x = 1 \Rightarrow x = 1/2$
Thus, $(x^2 + 1)^2 = 2^2$ or $(5/4)^2 \Rightarrow (x^2 + 1)^2 =$ **<u>4 or 25/16</u>**

Alternate Solution: ($x \neq 0$) Multiplying through by $2x^3$, $1 - 2x - x^2 + 2x^3 = 0$
$\Rightarrow (1 - 2x) - x^2(1 - 2x) = 0 \Rightarrow (1 - 2x)(1 - x^2) = 0 \Rightarrow x = \pm 1$ or ½ $\Rightarrow (x^2 + 1)^2 =$ **<u>4 or 25/16</u>**

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