MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013 SOLUTION KEY

Team Round

B) Definitions: $x \heartsuit y = \dfrac{x+y}{2}$ (arithmetic average) and $x \diamondsuit y = \dfrac{2xy}{x+y}$ (harmonic average)

Method #1:

Let $(a, b, c) = (1, 0, 1)$. (Testing conditions 2 and 5)

$1 \heartsuit (0 \diamondsuit 1) = 1 \heartsuit 0 = 1/2$ and $(1 \heartsuit 0) \diamondsuit (1 \heartsuit 1) = (1/2) \diamondsuit 1 = 2/3$
Since the results are unequal, in general, conditions 2) and 5) fail.

Let $(a, b, c) = (1, 1, 0)$. (Testing conditions 3 and 4)

$1 \heartsuit (1 \diamondsuit 0) = 1 \heartsuit 0 = 1/2$ and $(1 \heartsuit 1) \diamondsuit (1 \heartsuit 0) = 1 \diamondsuit (1/2) = 2/3$
Since the results are unequal, in general, conditions 3) and 4) fail.

All my attempts to eliminate 1 and/or 6 have failed. I can *assume* **1 and 6** always results in equality.
But what if I missed ordered triples which would have eliminated one or both conditions?

Method #2: (brute force substitution)

Compute formulas for $a \heartsuit (b \diamondsuit c)$ and $(a \heartsuit b) \diamondsuit (a \heartsuit c)$ and then substitute for each of the conditions and find formulas for each expression.

$$a \heartsuit (b \diamondsuit c) = a \heartsuit \frac{2bc}{b+c} = \frac{a+\frac{2bc}{b+c}}{2} = \frac{ab+ac+2bc}{2(b+c)} \quad (\#1)$$

$$(a \heartsuit b) \diamondsuit (a \heartsuit c) = \frac{2\left(\frac{a+b}{2}\right)\left(\frac{a+c}{2}\right)}{\left(\frac{a+b}{2}\right)+\left(\frac{a+c}{2}\right)} = \frac{(a+b)(a+c)}{2a+b+c} \quad (\#2)$$

Both sides are defined provided , provided $b \neq -c$ and $a \neq -\dfrac{b+c}{2}$. Verdict

			Verdict
1) $a=0$	$0 \heartsuit (b \diamondsuit c) = \dfrac{2bc}{2(b+c)} = \dfrac{bc}{b+c}$	$(0 \heartsuit b) \diamondsuit (0 \heartsuit c) = \dfrac{bc}{b+c}$	ok
2) $b=0$	$a \heartsuit (0 \diamondsuit c) = \dfrac{ac}{2c} = \dfrac{a}{2}$	$(a \heartsuit 0) \diamondsuit (a \heartsuit c) = \dfrac{a(a+c)}{2a+c}$	fails
3) $c=0$	$a \heartsuit (b \diamondsuit 0) = \dfrac{ab}{2b} = \dfrac{a}{2}$ $(b \neq 0)$	$(a \heartsuit b) \diamondsuit (a \heartsuit 0) = \dfrac{(a+b)a}{2a+b}$	fails
4) $a=b$	$b \heartsuit (b \diamondsuit c) = \dfrac{b^2+3bc}{2(b+c)}$	$(b \heartsuit b) \diamondsuit (b \heartsuit c) = \dfrac{2b(b+c)}{3b+c}$	fails
5) $a=c$	$c \heartsuit (b \diamondsuit c) = \dfrac{c^2+3bc}{2(b+c)}$	$(c \heartsuit b) \diamondsuit (c \heartsuit c) = \dfrac{2c(c+b)}{b+3c}$	fails
6) $b=c$	$a \heartsuit (c \diamondsuit c) = \dfrac{2ac+2c^2}{4c}$ $= \dfrac{2c(a+c)}{4c} = \dfrac{a+c}{2}$ $(c \neq 0)$	$(a \heartsuit c) \diamondsuit (a \heartsuit c) = \dfrac{(a+c)(a+c)}{2a+c+c}$ $= \dfrac{(a+c)^2}{2(a+c)} = \dfrac{a+c}{2}$ $(a+c \neq 0)$	ok