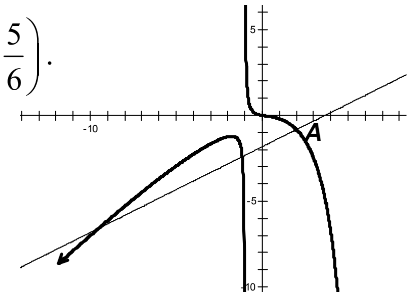
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2010 SOLUTION KEY

Team Round

A) The graph of $f(x)=\dfrac{x^3+x}{x^2-5x-6}$ and $g(x)=\dfrac{3x-11}{6}$ intersect at $\left(2,-\dfrac{5}{6}\right)$.



Compute the coordinates (x, y) of the point of intersection furthest from the origin.

$\dfrac{x^3+x}{x^2-5x-6}=\dfrac{3x-11}{6}$ ➔ $6\left(x^3+x\right)=\left(x^2-5x-6\right)(3x-11)$

➔ $6x^3+6x=3x^3-26x^2+37x+66$ ➔ $6x^3+26x^2-31x-66=0$

We know $x = 2$ is a solution and by synthetic substitution we have

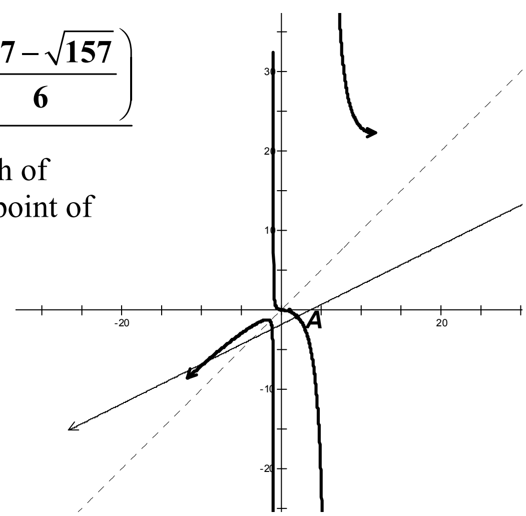
$$6x^3+26x^2-31x-66=(x-2)\left(3x^2+32x+33\right)=0$$

Applying the quadratic formula, $x=\dfrac{-32\pm\sqrt{32^2-12(33)}}{6}=\dfrac{-32\pm\sqrt{4(256-99)}}{6}=\dfrac{-16\pm\sqrt{157}}{3}$

The abscissa (i.e. the x-coordinate) of the point furthest from the origin is $\dfrac{-16-\sqrt{157}}{3}$.

Substituting in the linear function, we easily determine that the ordinate (i.e. the y-coordinate) is

$$\frac{3\left(\dfrac{-16-\sqrt{157}}{3}\right)-11}{6}=\frac{-27-\sqrt{157}}{6} \rightarrow \boxed{\left(\frac{-16-\sqrt{157}}{3},\frac{-27-\sqrt{157}}{6}\right)}$$



Actually, if the window were expanded, there is a third branch of $y = f(x)$ for $x > 6$. How do we know that there is not another point of intersection which is even further from the origin?
The dotted line represents $y = x$.

$f(x)=\dfrac{x^3+x}{x^2-5x-6}$ may be rewritten as $\dfrac{x+\frac{1}{x}}{1-\frac{5}{x}-\frac{6}{x^2}}$.

As x gets larger $(\rightarrow+\infty)$, the fractions in the numerator and denominator approach 0 and $f(x)$ is approximated by

$y=\dfrac{x+0}{1-0-0}=x$. In other words, this branch is asymptotic to $y = x$.

When is $\dfrac{x^3+x}{x^2-2x-6}>x$? $\dfrac{x^3+x-x\left(x^2-2x-6\right)}{x^2-2x-6}=\dfrac{2x^2+7x}{x^2-2x-6}=\dfrac{x(2x+7)}{(x-1)^2-7}>0$

The 4 critical points (-3.5, $1-\sqrt{7}$, 0 and $1+\sqrt{7}\approx 3.6$) divide the number line into 5 regions and the quotient is positive to the extreme left, extreme right and in the middle. Thus, for $x > 6$, $f(x) > x$ and $y = f(x)$ approaches $y = x$ (from above) and, therefore, will never cross $y = g(x)$.]