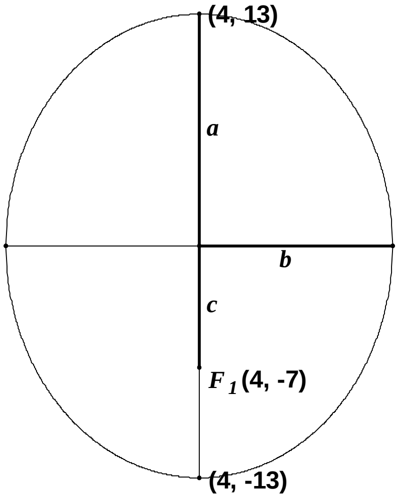
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015 SOLUTION KEY

Round 1



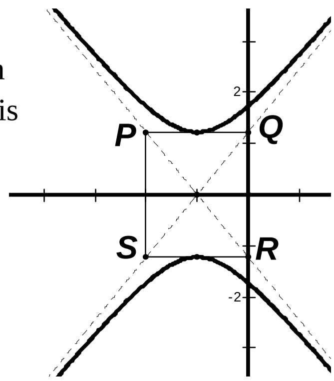
A) Given: ellipse w/focus $F(4, -7)$ and vertices $(4, \pm 13)$.
The ellipse must be vertical w/center at (4, 0), a = 13, c = 7
For an ellipse, $a^2 = b^2 + c^2 \Rightarrow b^2 = 120$.
For the ellipse, the a-value is always larger than the b-value.
Since the ellipse is vertical, the y-term will have the larger denominator, producing an equation $\dfrac{(x-4)^2}{120} + \dfrac{y^2}{169} = 1$.
Thus, $\left(h,k,a^2,b^2\right) = \underline{\left(\mathbf{4,0,120,169}\right)}$.

B) Given: a circle passing through $P(-5, 2)$, $Q(-3, 4)$ and $R(1, 2)$
The equation of any circle can be expressed in the form $x^2 + y^2 + Cx + Dy + E = 0$.
Substituting, we have
1) $-5C + 2D + E = -29$
2) $-3C + 4D + E = -25$
3) $C + 2D + E = -5$

Solving this system of simultaneous equations, $C = 4$, $D = -2$, and $E = -5$
Thus, equation is $x^2 + y^2 + 4x - 2y - 5 = 0$ or, completing the square, $\underline{\mathbf{(x + 2)^2 + (y - 1)^2 = 10}}$.

An alternate solution takes advantage of the fact that chord $\overline{PR}$ is horizontal.
The perpendicular bisector of any chord in a circle passes through the center of the circle.
Talk about this approach with your teammates and/or coach. Some additional hints are at the end of the solution key.

C) $9x^2 - 6y^2 + 18x + 18 = 0 \Leftrightarrow 2y^2 - 3x^2 - 6x - 6 = 0 \Leftrightarrow 2y^2 - 3\left(x+1\right)^2 = 3$ or $\boxed{\dfrac{y^2}{\frac{3}{2}} - \dfrac{\left(x+1\right)^2}{1} = 1}$



Since $y = \pm m(x-h) + k \Leftrightarrow \left(y-k\right) = \pm m(x-h)$, we note that the equations of the asymptotes $\overleftrightarrow{PR}$ and $\overleftrightarrow{SQ}$ are in point-slope form. From the equation of the hyperbola (boxed above), we see that the hyperbola is vertical, point $\left(h,k\right)$ is the center of the hyperbola and the slopes of the asymptotes are $\pm\dfrac{a}{b}$. The center is at $(-1, 0)$ and

$a = \sqrt{\dfrac{3}{2}} = \dfrac{\sqrt{6}}{2}, b = 1 \Rightarrow m = \dfrac{\sqrt{6}}{2}$. Thus, $\left(m,h,k\right) = \underline{\left(\mathbf{\dfrac{\sqrt{6}}{2}, -1, 0}\right)}$.