MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017 SOLUTION KEY

Round 6

A) Solution #1:
There will be a total of 8 terms, and Pascal's triangle is the fastest ways to evaluate all the coefficients. We require the 7th row (which has all odd numbers) $\Rightarrow$ **0** (or none).

0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

Solution #2: (Annalisa Peterson - Mt. Alvernia)
To determine the number of *odd* numbers in row *k* of Pascal's triangle:

- Count the number of 1s in the binary representation of row number *k*.
- Raise 2 to this power.

$7 = 111_{(2)} \Rightarrow 3$ - 1s $\quad 2^3 = 8 \Rightarrow 8$ coefficients are odd

Since row *k* always contains $(k+1)$ terms, **none** of the coefficients are even.

B) The interval contains $999-10+1=990$ integers, 495 even and 495 odd.
Multiples of 3 range from $12=3\cdot 4$ to $999=3\cdot 333$, 330 values.
Multiples of 6 range from $12=6\cdot 2$ to $996=6\cdot 166$, 165 values.
Thus, $P(\div 6\,|\div 2 \text{ or } 3)=\dfrac{165}{495+330-165}=\dfrac{1}{3+2-1}=\mathbf{\dfrac{1}{4}}$.

C) The expansion is $k^4+4k^2+6+\dfrac{4}{k^2}+\dfrac{1}{k^4}$

Solution #1 (Direct Approach)

$4k^2+6+\dfrac{4}{k^2}=23 \Rightarrow 4k^4-17k^2+4=0 \Leftrightarrow \left(4k^2-1\right)\left(k^2-4\right)=0 \;\; k=\mathbf{\pm\dfrac{1}{2},\ \pm 2}$.

Solution #2: (Symmetry)

$4k^2+6+\dfrac{4}{k^2}=23 \Leftrightarrow 4k^2-17+\dfrac{4}{k^2}=0 \Leftrightarrow \left(4k^2-1\right)\left(1-\dfrac{4}{k^2}\right)=0$ and the same result follows.

Solution #3 (Even Function: $f(-x)=f(x)$ and symmetry)

Consider the function $f(k)=4k^2-17+\dfrac{4}{k^2}$ which is an even function.

We require that $f(k)=0$, i.e. we are looking for the zeros of this function.

Suspecting that $\dfrac{4}{k^2}$ might be an integer, we try factors of 4. $k=2$ works.

Since the given function is even, $k=\mathbf{\pm 2}$. By the symmetry of the trinomial, $k=\mathbf{\pm\dfrac{1}{2}}$.