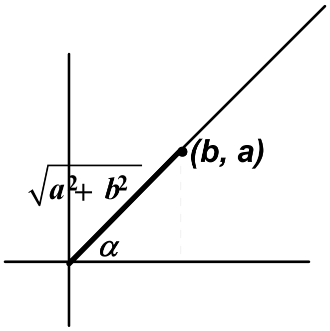
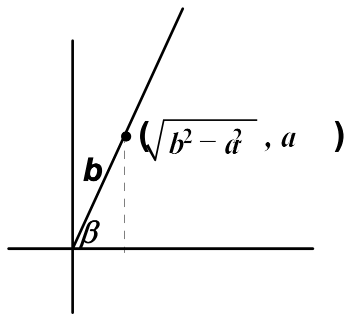
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2011 SOLUTION KEY

Team Round
C) - continued

Alternate solution #1: Let $\alpha = Arc\tan\left(\dfrac{a}{b}\right)$ and $\beta = Arc\sin\left(\dfrac{a}{b}\right)$





Taking the cosine of both sides, $\cos\left(Arc\tan\left(\dfrac{a}{b}\right)+Arc\sin\left(\dfrac{a}{b}\right)\right)=\cos 90°$

➔ $\cos\left(Arc\tan\left(\dfrac{a}{b}\right)\right)\cos\left(Arc\sin\left(\dfrac{a}{b}\right)\right)-\sin\left(Arc\tan\left(\dfrac{a}{b}\right)\right)\sin\left(Arc\sin\left(\dfrac{a}{b}\right)\right)=0$

➔ $\dfrac{b}{c}\cdot\dfrac{\sqrt{b^2-a^2}}{b}-\dfrac{a}{c}\cdot\dfrac{a}{b}=0$, where c replaces $\sqrt{a^2+b^2}$

➔ $b\sqrt{b^2-a^2}=a^2$

Squaring both sides, $b^2\left(b^2-a^2\right)=a^4$ ➔ $b^4-a^2b^2-a^4=0$ and then proceed as above.

Alternate solution #2 (Norm Swanson):

$\cos\left(Arc\tan\left(\dfrac{a}{b}\right)+\arcsin\left(\dfrac{a}{b}\right)\right)=\boxed{\left(\dfrac{b}{c}\right)\left(\dfrac{\sqrt{b^2-a^2}}{b}\right)-\left(\dfrac{a}{c}\right)\left(\dfrac{a}{b}\right)=0}^{***}$, where $c=\sqrt{a^2+b^2}$

Multiplying through by $c \neq 0$, eliminates c and we have $\sqrt{b^2-a^2}=\dfrac{a^2}{b}$.

Dividing by a ($\sqrt{a^2}$ on the left side), we have $\sqrt{\dfrac{b^2}{a^2}-1}=\dfrac{a}{b}$ ➔ $\dfrac{b^2}{a^2}-1=\dfrac{a^2}{b^2}$

or letting $x=\dfrac{b^2}{a^2}$, $x-1=\dfrac{1}{x}$ and the result follows.

Even easier: Let $b = 1$. Then *** immediately simplifies to $\left(\dfrac{1}{c}\right)\sqrt{1-a^2}=\dfrac{a^2}{c}$

$c \neq 0$ ➔ $\sqrt{1-a^2}=a^2$ ➔ $a^4+a^2-1=0$ ➔ $a^2=\dfrac{-1+\sqrt{5}}{2}$ (since $a > 0$).

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