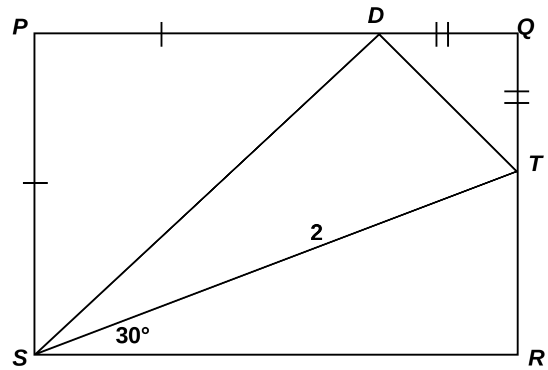
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - NOVEMBER 2011 SOLUTION KEY

Team Round - continued

E) $ST = 2$ and $m\angle TSR = 30° \Rightarrow TR = 1$ and $SR = \sqrt{3}$.
Isosceles triangles PSD and $DQT \Rightarrow m\angle SDT = 90°$
Let $DQ = QT = x \Rightarrow PS = PD = x + 1 \Rightarrow PQ = 2x + 1$



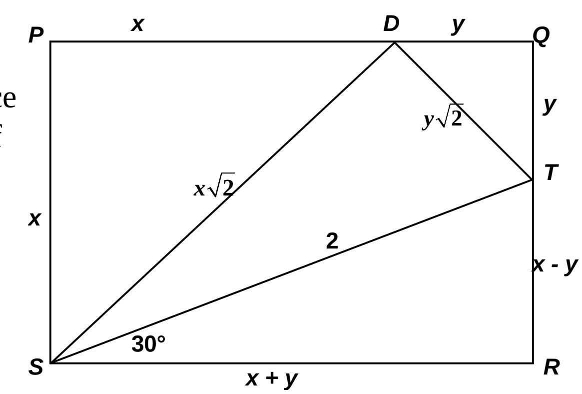
$$\cot(\measuredangle STD) = \frac{DT}{DS} = \frac{QT\sqrt{2}}{PD\sqrt{2}} = \frac{x}{x+1}$$

$PQ = SR \Rightarrow 2x + 1 = \sqrt{3} \Rightarrow x = \dfrac{\sqrt{3}-1}{2}$

$\Rightarrow (x + 1) = \dfrac{\sqrt{3}+1}{2}$

Substituting, $$\cot(\measuredangle STD) = \frac{\left(\frac{\sqrt{3}-1}{2}\right)}{\left(\frac{\sqrt{3}+1}{2}\right)} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\left(\sqrt{3}-1\right)^2}{2} = \underline{2-\sqrt{\mathbf{3}}}$$

Alternate Solution:
Let $PD = x$ and $QD = y$.
The remaining lengths are as indicated in the diagram. Since $ST = 2$ and $m\angle TSR = 30°$, we have the following system of equations: $\begin{cases} x - y = 1 \\ x + y = \sqrt{3} \end{cases}$



$\Rightarrow (x, y) = \left(\dfrac{1+\sqrt{3}}{2}, \dfrac{-1+\sqrt{3}}{2}\right)$ and

$$\cot(\angle STD) = \frac{DT}{DS} = \frac{y\sqrt{2}}{x\sqrt{2}} = \frac{y}{x} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{4-2\sqrt{3}}{3-1} = \underline{\mathbf{2}-\sqrt{\mathbf{3}}}$$

Note: $m\angle STD = 180° - (45° + 60°) = 75°$, but is was not necessary to invoke the expansions for $\sin(A + B)$ and $\cos(A + B)$ to evaluate $\cot(75°)$ as $\dfrac{\cos(30°+45°)}{\sin(30°+45°)}$ to get an exact answer.
This makes the problem solvable by anyone with minimal knowledge of right triangle trig, i.e. SOHCAHTOA, reciprocal relationships and special angles (30°, 60°, 45°).