MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2011 SOLUTION KEY

Round 1

A) The 2nd condition requires $y = x$ or $y = -x$.
Substituting in the first equation , $y = x$ ➔ $-6x = 96$ ➔ **(–16, –16)**
$y = -x$ ➔ $16x = 96$ ➔ **(6, –6)**

B) $\det\left(\begin{bmatrix} x & 2 \\ x+1 & 10-x \end{bmatrix}\right) = 10x - x^2 - 2x - 2 = -x^2 + 8x - 2$

Completing the square, $-\left(x^2 - 8x + 16\right) + 16 - 2 = -(x-4)^2 + 14$ which defines a downward opening parabola with vertex at **(4, 14)**.

C) Since the point is in the xy-plane, $z = 0$.
Thus, in the xy-plane, we want the point on L: $3x + 2y = 6$ closest to (0, 0).
This point lies on the line through the origin perpendicular to L ➔ $y = \frac{2}{3}x$

Substituting, $3x + 2\left(\frac{2}{3}x\right) = 6$ ➔ $13x = 18$ ➔ $(x, y, z) = \left(\mathbf{\frac{18}{13}, \frac{12}{13}, 0}\right)$.

Round 2

A) Just need to evaluate $x^2 - x^3$ for the three given values and pick the largest difference.
Substitution is easier if the expression is factored as $x^2(1-x)$.

$\frac{1}{3}$ ➔ $\frac{1}{9}\left(1 - \frac{1}{3}\right) = \frac{2}{27}$ $\quad$ $\frac{1}{2}$ ➔ $\frac{1}{4}\left(1 - \frac{1}{2}\right) = \frac{1}{8}$ $\quad$ $\frac{2}{3}$ ➔ $\frac{4}{9}\left(1 - \frac{2}{3}\right) = \frac{4}{27}$

'Cross Multiplying', $\frac{4}{27} > \frac{1}{8}$ because $4 \cdot 8 > 27 \cdot 1$. Thus, $D = \mathbf{\frac{4}{27}}$.

B) 2^{2^3} is evaluated from right to left. $2^{2^3} = 2^8 = 256$ (not $4^3 = 64$).

$\begin{cases} (A+B)^3 = -8 \\ (A-B)^2 = 256 \end{cases}$ ➔ $\begin{cases} A + B = -2 \\ A - B = \pm 16 \end{cases}$

Adding, $2A = -2 \pm 16$ ➔ $A = -1 \pm 8$ ➔ 7, –9
Therefore, $(A, B) =$ **(7, –9), (–9, 7)**

C) If the given radical can be simplified, then the radicand must be expressible as a perfect square.

$37 - 20\sqrt{3} = \left(a + b\sqrt{3}\right)^2 = a^2 + 3b^2 + 2ab\sqrt{3}$ ➔ $a^2 + 3b^2 = 37$ and $ab = -10$

a and b have opposite signs. The ordered pairs (5, –2) and (–5, 2) satisfy both equations.
$a + b\sqrt{3}$ must represent a positive number, so $-5 + 2\sqrt{3}$ is rejected.
$(a, b) = (5, -2)$ ➔ quadrant = 4, distance = $\sqrt{29}$ ➔ $\left(\mathbf{4, \sqrt{29}}\right)$.