MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014 SOLUTION KEY

Round 3

A) The smallest positive angle is 90°. $x^2 - x = 90 \Leftrightarrow x^2 - x - 90 = (x-10)(x+9) = 0$.
Thus, $x = \underline{\mathbf{10, -9}}$.

B) $\sec(2x)\csc(2x) = -4 \Leftrightarrow \dfrac{1}{2\cos 2x \sin 2x} = -2 \Rightarrow \dfrac{1}{\sin(4x)} = -2 \Rightarrow \sin(4x) = -\dfrac{1}{2}$

$\Rightarrow 4x = \dfrac{7\pi}{6} + 2n\pi$ or $\dfrac{11\pi}{6} + 2n\pi$

$\Rightarrow x = \dfrac{(12n+7)\pi}{24}$ or $\dfrac{(12n+11)\pi}{24} \Rightarrow x = \underline{\boldsymbol{\dfrac{7\pi}{24}}}, \underline{\boldsymbol{\dfrac{11\pi}{24}}}$

C) Factoring the difference of cubes,

$3(\sin x - \cos x) + 4(\cos^3 x - \sin^3 x) = 0 \Rightarrow$

$-3(\cos x - \sin x) + 4(\cos x - \sin x)(\cos^2 x + \cos x \sin x + \sin^2 x) = 0$

$\Rightarrow (\cos x - \sin x)(-3 + 4(1 + \cos x \sin x)) = 0$

$\Rightarrow (\cos x - \sin x)(1 + 4\cos x \sin x) = 0$

$\Rightarrow (\cos x - \sin x)(1 + 2\sin 2x) = 0$

$\Rightarrow \tan x = 1$ or $\sin 2x = -\dfrac{1}{2} \Rightarrow x = \underline{\boldsymbol{\dfrac{\pi}{4}}}$ $x = 2x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6} \Rightarrow x = \underline{\boldsymbol{\dfrac{7\pi}{12}, \dfrac{11\pi}{12}}}$

If you know your identities really well, the solution is much shorter.
Multiply out and regroup as follows:

$$(4\cos^3 x - 3\cos x) + (3\sin x - 4\sin^3 x) = 0$$

These are expansions for $\cos(3x)$ and $\sin(3x)$ respectively.

$\Rightarrow \cos(3x) = -\sin(3x) \Rightarrow \tan(3x) = -1 \Rightarrow 3x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}, \dfrac{11\pi}{4}, \ldots \Rightarrow x = \underline{\boldsymbol{\dfrac{\pi}{4}, \dfrac{7\pi}{12}, \dfrac{11\pi}{12}}}$.