MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2008 SOLUTION KEY

Round 4

A) Integer coefficients → roots must occur in conjugate pairs.
Thus, the two roots are $2 \pm i\sqrt{5}$ → sum = 4 and product = 9 → $\underline{\mathbf{x^2 - 4x + 9 = 0}}$

B) Let L denote the larger of the positive numbers.

$$\begin{cases} L^2 + W^2 = 81 \\ 2L = 9 + W \end{cases}$$

$$\rightarrow L^2 + (2L - 9)^2 = 81 \rightarrow 5L^2 - 36L = L(5L - 36) = 0 \rightarrow L = \frac{36}{5} \text{ and } W = \frac{27}{5} \rightarrow |L - W| = \underline{\mathbf{\frac{9}{5}}}$$

C) Assume the roots of the original quadratic are r_1 and r_2 and the corresponding roots of the new equation are s_1 and s_2. Then $s_1 = 2r_1 + 3$ and $s_2 = 2r_2 + 3$

According to the root/coefficient relationship for quadratics, $p = -(r_1 + r_2)$ and $q = r_1r_2$.
Also $A = -(s_1 + s_2) = -(2(r_1 + r_2) + 6) = \underline{\mathbf{2p - 6}}$ or $\underline{\mathbf{2(p - 3)}}$
$B = s_1s_2 = (2r_1 + 3)(2r_2 + 3) = 4r_2r_2 + 6(r_1 + r_2) + 9 = \underline{\mathbf{9 - 6p + 4q}}$

Continuing, $\dfrac{2p-6}{9-6p+4q} = \dfrac{-2}{3}$ → $6p - 18 = 18 - 12p + 8q$ → $6p = 8q$ → $\dfrac{p}{q} = \underline{\mathbf{4 : 3}}$

Note: If $A = 6$ and $B = 9$, then the first equation, $x^2 + 6x + 9 = 0$ has a double root of -3.
Since $2(-3) + 3 = -3$, the second equation would be identical.
In the above solution, $A = 6 - 2p = 6 \rightarrow p = 0$ and $B = 9 - 6p + 4q = 9 \rightarrow q = 0$
In this situation the ratio of $p : q$ would be indeterminant. Thus, it was necessary to require that the equations be different.