MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2016 SOLUTION KEY

Team Round

E) Knowing the expansions for sin(A ± B) and cos(A ± B), we see that

$$\begin{cases} (1)\ \cos(A-B)+\cos(A+B)=2\cos A\cos B \\ (2)\ \cos(A-B)-\cos(A+B)=2\sin A\sin B \\ (3)\ \sin(A+B)+\sin(A-B)=2\sin A\cos B \\ (4)\ \sin(A+B)-\sin(A-B)=2\cos A\sin B \end{cases}$$

$\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ =$

Regrouping for implementation of rule #2: $\frac{1}{4}\left(2\sin 10^\circ \sin 70^\circ\right)\left(2\sin 30^\circ \sin 50^\circ\right)$

Applying rule #2: $\frac{1}{4}\left(\cos 60^\circ - \cos 80^\circ\right)\left(\cos 20^\circ - \cos 80^\circ\right)$

FOILing: $\frac{1}{4}\left(\cos 60^\circ \cos 20^\circ - \cos 60^\circ \cos 80^\circ - \cos 80^\circ \cos 20^\circ + \cos 80^\circ \cos 80^\circ\right)$

Regrouping: $\frac{1}{4}\cdot\frac{1}{2}\left(2\cos 60^\circ \cos 20^\circ - 2\cos 60^\circ \cos 80^\circ - 2\cos 80^\circ \cos 20^\circ + 2\cos 80^\circ \cos 80^\circ\right)$

Applying rule #1 (to each of the 4 products):

$$\frac{1}{8}\left(\left(\cos 40^\circ + \cos 80^\circ\right) - \left(\cos(-20^\circ) + \cos 140^\circ\right) - \left(\cos 60^\circ + \cos 100^\circ\right) + \left(\cos 0^\circ + \cos 160^\circ\right)\right)$$

$$\frac{1}{8}\left(\left(\cos 40^\circ + \cos 80^\circ\right) - \left(\cos 20^\circ - \cos 40^\circ\right) - \left(\cos 60^\circ - \cos 80^\circ\right) + \left(1 - \cos 20^\circ\right)\right)$$

$$\frac{1}{8}\left(2\cos 40^\circ + 2\cos 80^\circ - 2\cos 20^\circ - \frac{1}{2} + 1\right)$$

$$\frac{1}{8}\left(2(\cos 40^\circ + \cos 80^\circ) - 2\cos 20^\circ - \frac{1}{2} + 1\right)$$

Now applying rule #1 in reverse, $A - B = 40$ and $A + B = 80 \Rightarrow (A, B) = (60, 20)$.

$$\frac{1}{8}\left(2(2\cos 60^\circ \cos 20^\circ - \cos 20^\circ) - \frac{1}{2} + 1\right) = \frac{1}{8}\left(2(\cos 20^\circ - \cos 20^\circ) - \frac{1}{2} + 1\right) = \frac{1}{8}\left(\frac{1}{2}\right) = \underline{\mathbf{\frac{1}{16}}}$$