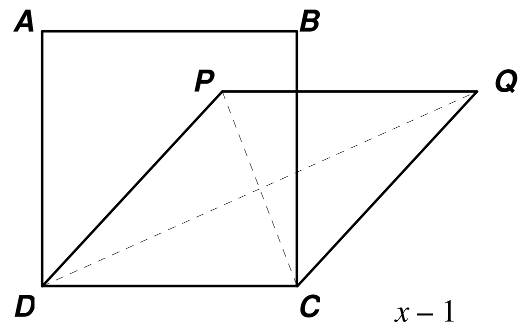
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008 SOLUTION KEY

Round 2

A) Let L denote the length of the ladder (the hypotenuse of the right triangle).
Then: $(a, b, c) = (15, 36, L) = 3(5, 12, ?)$
From special right triangles, the '?' must denote 13 and $L = 39$.
Looking for a pattern, a 3 foot ladder would have 2 rungs (__.__.__) and
a 4 foot ladder 3 rungs (__.__.__.__). Clearly, the 39 foot ladder would have **<u>38</u>** rungs.

B) The diagonals of any rhombus are perpendicular and bisect each other. Let $PC = 2a$ and $QD = 2b$.
The diagonals intersect and form 4 right triangles with legs a, b and hypotenuse 50.
Either $(a, b, 50) = 10(3, 4, 5)$ or $2(7, 24, 25)$
→ $(PC, QD) = (60, 80)$ or $(28, 96)$
→ minimum sum = **<u>124</u>**



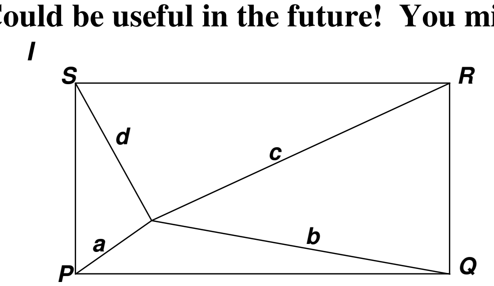
C) The length of the hypotenuse can not be $x - 1$ ($x + 3 > x - 1$ for all values of x).
Case 1: $h = x + 3$
$(x - 1)^2 + (2x - 6)^2 = (x + 3)^2 \rightarrow x^2 - 2x + 1 + 4x^2 - 24x + 36 = x^2 + 6x + 9$
$\rightarrow 4x^2 - 32x + 28 = 4(x^2 - 8x + 7) = 4(x - 1)(x - 7) = 0 \rightarrow$ **<u>7</u>** only
Check: sides are $x - 1 = 6$, $x + 3 = 10$ and $2x - 6 = 8$

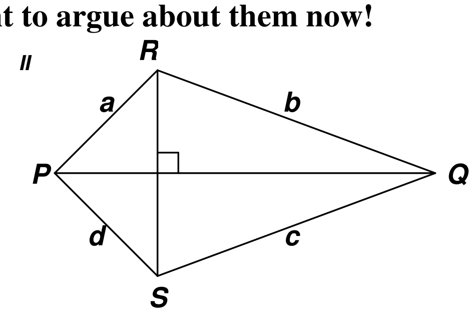
Case 2: $h = 2x - 6$
$(x - 1)^2 + (x + 3)^2 = (2x - 6)^2 \rightarrow x^2 - 2x + 1 + x^2 + 6x + 9 = 4x^2 - 24x + 36$
$\rightarrow 2x^2 - 28x + 26 = 2(x^2 - 14x + 13) = 2(x - 1)(x - 13) = 0 \rightarrow$ **<u>13</u>** only

Alternate solution:
Case 1: $2x - 6 < x + 3$, but $2x - 6 > x - 1 \rightarrow x < 9$ and $x > 5 \rightarrow 5 < x < 9$
By trial and error and using the fact that one leg is 4 less than the hypotenuse,
$x =$ **<u>7</u>** → sides of 6, 8 and 10

Case 2: $2x - 6 > x + 3 \rightarrow x > 9$ ($2x - 6 > x - 1 \rightarrow x > 5$ - a weaker condition)
By trial and error, 12 - 16 - 20 would be a right triangle in which the lengths of the legs differ by 4 and $x =$ **<u>13</u>** produces this triple.

Two nice relations derived from applying the Pythagorean Theorem.
Could be useful in the future! You might want to argue about them now!





I IF $PQRS$ is a rectangle, $a^2 + c^2 = b^2 + d^2$
II IF $\overline{PQ} \perp \overline{RS}$, then $a^2 + c^2 = b^2 + d^2$