MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2010 SOLUTION KEY

Round 2 – continued

C) Consider the following sets of 7 consecutive natural numbers

$\{1,2,\underline{3},4,5,\underline{6},\underline{7}\}$, $\{8,\underline{9},10,11,\underline{12},13,\underline{14}\}$, $\{\underline{15},16,17,\underline{18},19,20,\underline{21}\}$, $\{22,23,\underline{24},25,26,\underline{27},\underline{28}\}$, …

Each set contains exactly 4 natural numbers not divisible by either 3 or 7.

Thus, $4n \le 258$ ➔ $n = 64$.

The largest number in the 64th set will be 7(64) = 448 and we have counted 4(64) = 256 natural numbers not divisible by either 3 or 7. Examining 449, 450, 451, … for divisibility by 3, we see that 450 is a multiple of 3. Therefore, 451 is the natural number satisfying our requirements. Since $451 = 11^1 \cdot 41^1$ and 11 and 41 are prime, 451 has a total of 4 factors and only these two are prime. The required sum is **<u>52</u>**.

Round 3

A) The domain of Sin^{-1} and Cos^{-1} are $[-90°, 90°]$ and $[0°, 180°]$ respectively.

Thus, $Cos^{-1}\left(-\frac{1}{2}\right)$ denotes an angle in quadrant 2 whose cosine is $-\frac{1}{2}$, i.e. 120° and

$Sin^{-1}\left(-\frac{1}{2}\right)$ denotes an angle in quadrant 4 whose sine is $-\frac{1}{2}$, i.e. –30°.

Therefore, k = 120 – (–30) = **<u>150</u>**.

B) $\dfrac{1-2\sin^2 x}{\sin x\cos x} = 2\sqrt{3} \leftrightarrow \dfrac{1-2\sin^2 x}{2\sin x\cos x} = \dfrac{\cos 2x}{\sin 2x} = \cot 2x = \sqrt{3}$

➔ $2x = \dfrac{\pi}{6} + n\pi$ ➔ $x = \dfrac{\pi}{12} + \dfrac{n\pi}{2} = \dfrac{(6n+1)\pi}{12}$ ➔ $\underline{\boldsymbol{\dfrac{\pi}{12}, \dfrac{7\pi}{12}}}$

C) $x^{2/3} + y^{2/3} = \left(\cos^3 t\right)^{2/3} + \left(\sin^3 t\right)^{2/3} = \cos^2 t + \sin^2 t = 1$ ➔ $y = \left(1 - x^{2/3}\right)^{3/2}$

$x = \dfrac{64}{125}$ ➔ $y = \left(1 - \left(\left(\dfrac{64}{125}\right)^{\frac{1}{3}}\right)^2\right)^{\frac{3}{2}} = \left(1 - \dfrac{16}{25}\right)^{\frac{3}{2}} = \left(\dfrac{9}{25}\right)^{\frac{3}{2}} = \underline{\pm\boldsymbol{\dfrac{27}{125}}}$ (or **<u>±0.216</u>**)

Round 4

A) $-5W + 8C = 0$, $W + C = 26$ ➔ $5W + 5C = 130$

Adding, we have $13C = 130$ ➔ C = **<u>10</u>**

B) $x(x + 1) + 100 = (x + 4)(x + 1 + 3) = x^2 + 8x + 16$

➔ $x + 100 = 8x + 16$ ➔ $x = 12$ ➔ original # chairs/row = 12 + 1 = **<u>13</u>**