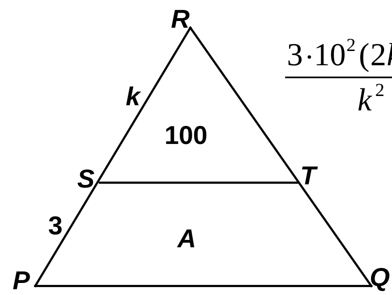
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2014 SOLUTION KEY

Team Round

E) Let A denote the area of trapezoid $STQP$.

$\Delta RST \sim \Delta RPQ \Rightarrow \dfrac{area(\Delta RST)}{area(\Delta RPQ)} = \dfrac{k^2}{(k+3)^2} = \dfrac{100}{A+100}$



$\dfrac{3\cdot 10^2(2k+3)}{k^2} = \dfrac{2\cdot 2\cdot 3\cdot}{}$

Cross multiplying, $k^2A = 100(6k+9) \Rightarrow A =$

Case 1: $\dfrac{2k+3}{k}$ and $\dfrac{2^2\cdot 3\cdot 5^2}{k}$ are integers.

Since $\dfrac{2k+3}{k} = 2 + \dfrac{3}{k}$, we must consider only $k=1$ and $k=3$

k = **<u>1</u>** $\Rightarrow$ $A = 2^2\cdot 3\cdot 5^3 = 1500$

k = **<u>3</u>** $\Rightarrow$ $A = \dfrac{2^2\cdot 3\cdot 5^2\cdot 9}{3^2} = 300$

k is only a factor of $2k+3$ for k = 1 or 3.

Case 2: $\left(2^2\cdot 3\cdot 5^2\right)/k^2$ is an integer

k = **<u>2</u>** $\Rightarrow A = 3(5^2)(7) = 525$

k = **<u>5</u>** $\Rightarrow A = 2^2(5)(13) = 156$

Case 3: Combinations

k = **6** $\Rightarrow A = 2^2(5) = 125$

k = **<u>10</u>** $\Rightarrow A = 3(23) = 69$

k = **<u>15</u>** $\Rightarrow A = 4(11) = 44$

k = **<u>30</u>** $\Rightarrow A = 21$

Larger values of k will produce smaller values of A and since we require that $A+100$ be a perfect square, the next perfect square less than 21+100 = 121 = 11^2 is 10^2 which forces A = 0. Thus, there are no additional solutions to be found!

F) The fraction is undefined for x = 2 and x = A and exactly one other <u>integer</u> value of x.

Simplifying, $\dfrac{10(x+4)}{\dfrac{5}{x-2}-\dfrac{2}{x-A}} = \dfrac{10(x+4)}{\dfrac{5(x-A)-2(x-2)}{(x-2)(x-A)}} = \dfrac{10(x+4)(x-2)(x-A)}{3x-5A+4}$.

Thus, the third troublesome value occurs when $3x-5A+4=0 \Rightarrow x = \dfrac{5A-4}{3}$

To minimize the sum, we want the smallest possible value of A. A = 2 is rejected, since this produces x = 2 which we already have. $(5A-4)$ must be a multiple of 3 and consecutive multiples of 3 differ by 3, so we increase the value of A by 3. $A = 5 \Rightarrow x = 7$.
Therefore, the minimum sum is 2 + 5 + 7 = **<u>14</u>**.

Alternative (brute force): Fraction is undefined for x = 2, x = A and $x = \dfrac{5A-4}{3}$

$A=1 \Rightarrow x = 1, 2, 1/3$ $\quad A=2 \Rightarrow x=2$ only $\quad A=3 \Rightarrow x = 2, 3, 11/3$ $\quad A=4 \Rightarrow x = 2, 4, 16/3$

$A=5 \Rightarrow x = 2, 5, 7 \Rightarrow \min = \underline{\mathbf{14}}$ In general, if $A=3k+2$, $\dfrac{5A-4}{3}$ will be an integer and the x-sum will be $2+(3k+2)+(5k+2) = 8k+6$ for integer $k \ge 1$. $k=1$ produces the minimum.