MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2014 SOLUTION KEY

Team Round

A) Note that $N = \dfrac{1}{(1-i)^k} = \left(\dfrac{1}{1-i}\right)^k = \left(\dfrac{1+i}{2}\right)^k = 2^{-k}\cdot(1+i)^k$

Since 2^{-k} is real for all integer values of k in the specified range, N is real whenever $(1+i)^k$ is real.

For k = 1 to 4, $(1+i)^k$ = 1 + i, 2i, –2 + 2i, –4.

For k = 5 to 8, the values obtained are obtained by multiplying the above values by –4.
There is one real value of N in every block of four consecutive k-values.
If k is a multiple of 4, N is real. N is real for k = 12, 16, …., 96, or 4(3, 4, …, 24).
Thus, we have a total of **<u>22</u>** different values of k.

B) 7 days before (or after) a given date will fall on the same day of the week.
Since $365 = 7\cdot 52 + 1$, a non-leap year consists of 52 weeks and 1 day.
Therefore, from year to year, a given date advances 1 day of the week (DOW), unless 2/29 falls between the two dates. Moving back in time, starting in 2014, consider the following sequence:

14	13	12	11	10	9	8	7	6	5	4
	-1	-1	-2	-1	-1	-1	-2	-1	-1	-1
WED	TUE	MON	SAT	FRI	THU	WED	MON	SUN	SAT	FRI

Since 2/29 falls between 3/2011 and 3/2012 and between 3/2007 and 3/2008, the day of the week changes by 2 days. Over a 4 year period, DOW changes by 5 days, unless the 4 year period spans a century year which is not divisible by 400 (like the year 1900). Therefore, 112 years ago (28 4-year periods), in 1902, the DOW has cycled through $28\cdot 5 = 140$ days. Since 140 is divisible by 7, in 1902, his birthday fell on the same DOW, namely Wednesday. Since none of the remaining 4 intervening years were leap years, the DOW changes only by 4 days. In 1898, his birthday fell on a **<u>SAT</u>**. You are invited to apply Zeller's formula to his actual birthdate 3/5/1898 to confirm this result.

$$z = \left[\frac{13m-1}{5}\right] + \left[\frac{y}{4}\right] + \left[\frac{c}{4}\right] + d + y - 2c, \text{ where}$$

d denotes the day (1..31)
m denotes the month according to the following funky rule:
1 = March 2 = April … 10 = December and January and February are assigned 11 and 12 respectively **for the previous year**
c denotes the "century" in which the date falls (***YY****YY*)
y denotes the year (*YY****YY***), i.e. 0 ... 99
Now, divide z by 7.

The integer <u>remainder</u> determines the day of the week.
(0, Sunday), (1, Monday), (2, Tuesday)… (6, Saturday)