MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009 SOLUTION KEY

Team Round – continued

B) Multiply out and re-arrange terms: $G^4+T^4-8G^2-8T^2+16-4G^3T-4GT^3+6G^2T^2+16GT$

→ $\left(G^4-4G^3T+6G^2T^2-4GT^3+T^4\right)-8\left(G^2-2GT+T^2\right)+16$

= $\left(G-T\right)^4-8\left(G-T\right)^2+16$

= $\left((G-T)^2-4\right)^2$ = $\underline{\boldsymbol{(G-T-2)^2(G-T+2)^2}}$

C) Squaring both sides, $1+a\sin x=\cos^2 x$ → $a\sin x=\cos^2 x-1=-\sin^2 x$

→ $\sin x(\sin x+a)=0$

→ $\sin x = 0$ → $x = 0°, 180°$ (180° checks, but 0° is extraneous)

or $\sin x = -a$ and $x = 150°$ → $a = -1/2$

Check: $\sin x = 1/2$ → $x = 30°, 150°$

30: $\sqrt{1+\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}=\frac{\sqrt{3}}{2}\neq-\frac{\sqrt{3}}{2}$

150: $\sqrt{1+\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}=\frac{\sqrt{3}}{2}=-\left(-\frac{\sqrt{3}}{2}\right)$

Thus, a must be $-\frac{1}{2}$ and the only additional solution is **<u>180°</u>**.

D) If you tried finding the actual coordinates of points A and B, the computation quickly became painful. How can this be avoided?

Suppose $x^2 = mx + b$ and that r and s are the roots of $x^2 - mx - b = 0$. Then:

The coordinates of A, B and the midpoint M would be (r, r^2), (s, s^2) and $\left(\frac{r+s}{2},\frac{r^2+s^2}{2}\right)$.

From the root coefficient relationship, $r + s = m$ and $rs = -b$.

Squaring and substituting, $m^2=r^2+2rs+s^2$ = r^2-2b+s^2 → $r^2+s^2=m^2+2b$

Thus, the midpoint M has coordinates $\left(\frac{m}{2},\frac{m^2+2b}{2}\right)$ and $m = 7$, $b = 13$ → $\left(\boldsymbol{\frac{7}{2},\frac{75}{2}}\right)$