MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2010 SOLUTION KEY

Team Round - continued

D) Suppose the original pane had R rows and C columns. Then:

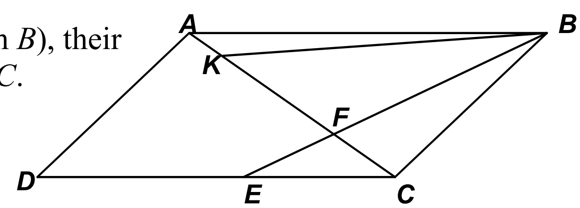
$RC = (R+3)(C-5)+48 \rightarrow 0 = 3C - 5R + 33 \rightarrow C = \dfrac{5R-33}{3} = R - 11 + \dfrac{2R}{3}$

The smallest value of R that returns a positive integer value for C is 9. [$(R, C) = (9, 4)$]
Using slope, we create a table of values until the product RC is a perfect square.

R	9	12	15	18	21	24	27	30	33	36
C	4	9	14	19	24	29	34	39	44	49

Using this lookup table, the last ordered pair gives us $RC = (36)(49) = (6\cdot 7)^2 = 42^2 \rightarrow N =$ **42**
The values satisfying this relationship get quite large very quickly.
The next three values satisfying $RC = N^2$ may be determined with a calculator or spreadsheet. They are: $(324)(529) = 18^2\cdot 23^2 = (414)^2$ $(2025)(3364) = 45^2\cdot 58^2 = (2610)^2$ and $(19881)(33124) = 141^2\cdot 182^2 = (25662)^2$

E) Since ΔBAK, ΔBKF and ΔBFC have a common altitude (from B), their areas are in the same ratio as their bases, namely $AK : KF : FC$.



Let $AK = 2x$, $KF = 7x$ and $FC = y$. Let $DE = 2a$ and $CE = a$.
Since $\Delta ABF \sim \Delta CEF$, their areas are in a 9 : 1 ratio and
$\dfrac{CF}{AF} = \dfrac{CE}{AB} \rightarrow \dfrac{y}{9x} = \dfrac{1a}{3a} \rightarrow y = 3x$.
Let area(ΔBAK) = $2N$, area(ΔBKF) = $7N$ and area(ΔBFC) = $3N$. Then area(ΔCEF) = N and area (ΔADC) = $12N$.

Thus, $\dfrac{area(FADE)}{area(ABCD)} = \dfrac{12N - N}{24N} = \mathbf{\dfrac{11}{24}}$