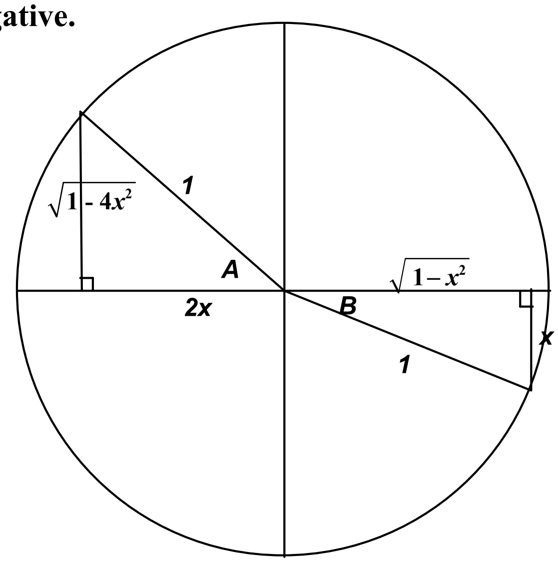
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2008 SOLUTION KEY

Team Round – continued

C) Let $A = \text{Cos}^{-1}(2x)$ and $B = \text{Sin}^{-1}(x)$.
It was not necessary to specify that *x* must be negative.
[For $x = 0$, $A - B = \pi/2 - 0 = \pi/2 < 5\pi/6$
Positive values of x must be $\leq \frac{1}{2}$.
As x increases from 0 to ½,
$\text{Cos}^{-1}(2x)$ decreases from $\pi/2$ to 0 and
$\text{Sin}^{-1}(x)$ increases from 0 to $\pi/6$.
Thus, there are no positive values for which the difference can be $5\pi/6$.]
For $x < 0$, A is in quadrant 2 ($\pi/2 < A < \pi$) and B is in quadrant 4 ($-\pi/2 < B < 0$) as indicated in the diagram at the right
Taking the sin of both sides,
$\text{Sin}(A - B) = \sin A \cos B - \sin B \cos A = 1/2$



→ $\sqrt{1-4x^2} \cdot \sqrt{1-x^2} - (2x)(x) = 1/2$
→ $\left((1-4x^2)\cdot(1-x^2)\right) = (1/2 + 2x^2)^2$
→ $1 - 5x^2 + 4x^4 = 1/4 + 2x^2 + 4x^4$
→ $7x^2 = 3/4$ → $x^2 = \dfrac{3\cdot 7}{4\cdot 7\cdot 7}$ → $x = \underline{\mathbf{-\dfrac{\sqrt{21}}{14}}}$ (the positive root is rejected)

D) $(18+c)+\dfrac{c}{2} = (18-c)H$ → $36 + 2c + c = 2(18 - c)H$ → $H = \dfrac{36+3c}{36-2c} = 1+\dfrac{5c}{36-2c}$
Therefore, the positive integer possibilities for c are 1 … 17.
$c = 1 \ldots 17$ → 8, 1 + 40/20 = **<u>(8, 3)</u>**; 12, 1 + 60/12 = **<u>(12, 6)</u>**; 16, 1 + 80/4 = **<u>(16, 21)</u>**
Other values of c produce fractional values of H.