MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2016 SOLUTION KEY

Team Round - continued

F) The series is 2 + 5 + 9 + 14 + 20 + 27 + 35 + …
The sequence of partial sums is 2, 7, 16, 30, 50, 77, …
The sequence of first differences is: 5, 9, 14, 20, 27, …..
The sequence of second differences is: 4, 5, 6, 7, …..
The sequence of third differences is: 1, 1, 1, 1, …..
Thus, the general term for the sequence of partial sums is a 3rd degree polynomial of the form $An^3 + Bn^2 + Cn + D$.
[A system of *difference equations* looks intimidating, but, although tedious, it is actually very easy to solve, in stark contract to a system of *differential equations*. But that's a story for another day.]

Letting n = 1, 2, 3 and 4

Subtracting....

$$\begin{cases} s_1 = A+B+C+D=2 \\ s_2 = 8A+4B+2C+D=7 \\ s_3 = 27A+9B+3C+D=16 \\ s_4 = 64A+16B+4C+D=30 \end{cases} \Rightarrow \begin{cases} 37A+7B+C=14 \\ 19A+5B+C=9 \\ 7A+3B+C=5 \end{cases} \Rightarrow \begin{cases} 18A+2B=5 \\ 12A+2B=4 \end{cases} \Rightarrow A = \frac{1}{6}$$

Substituting back, (B, C, D) = $\left(1, \frac{5}{6}, 0\right)$. Therefore,

$s_n = \frac{1}{6}n^3 + n^2 + \frac{5}{6}n = \frac{n^2+6n+5}{6} = \frac{n(n+1)(n+5)}{6}$. We require $\frac{n(n+1)(n+5)}{6} = 2016$ or as close as possible. Since the product of the three terms in the numerator is approximately n^3 and $20^3 < 12{,}096 < 30^3$, we start with $n = 20$.

$\frac{20(21)(25)}{6} = 10(7)(25) = 1750$, $\frac{21(22)(26)}{6} = 7(11)(26) = 2002$, $\frac{22(23)(27)}{6} = 11(23)(9) = 2277$

Thus, the closest value is <u>**2002**</u>.

Alternately, let $S(n) = 2+7+16+...$ and $A(n) = 1+2+3+...+n = n(n+1)/2$.

n	$S(n)$	$A(n)$	$S(n)/A(n)$
1	2	1	2/1 = 6/3
2	7	3	7/3
3	16	6	16/6 = 8/3
4	30	10	30/10 = 9/3
5	50	15	50/15 = 10/3

From this table, we see that $\frac{S(n)}{A(n)} = \frac{n+5}{3}$, so $S(n) = A(n)\cdot\frac{(n+5)}{3} = \frac{n(n+1)(n+5)}{6}$ and have the same result. [Thanks to Norm Swanson - A very slick solution indeed!]