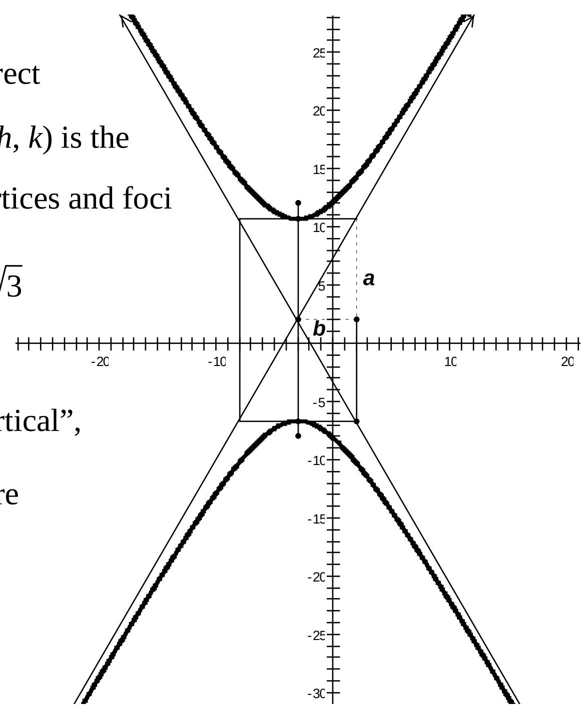
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013 SOLUTION KEY

Round 1

A) The semi-major axis (a) is 5; the semi-minor axis (b) is 3.
The area of the lightly shaded region is $\pi \cdot 3 \cdot 5 - \pi \cdot 3^2 = 6\pi$.
The area of the darkly shaded region is $\pi \cdot 5^2 - \pi \cdot 3 \cdot 5 = 10\pi$.
Thus, the required ratio is 5 : 3.
Convince yourself this ratio is always a : b for any ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b$.

B) Adding the equations, we get $5x^2 + 5y^2 - 20x + 10y - 100 = 0 \Leftrightarrow x^2 + y^2 - 4x + 2y - 20 = 0$.
Completing the square,
$\left(x^2 - 4x \underline{+4}\right) + \left(y^2 + 2y \underline{+1}\right) = 20 \underline{+4+1} = 25 \Leftrightarrow (x-2)^2 + (y+1)^2 = 5^2$
The center is **<u>(2, –1)</u>** and the radius is **<u>5</u>**.
<u>FYI</u>: One of the 4 points is (6, 2) and the other three can be found without too much additional effort, using symmetry w.r.t. the center.



C) Since the transverse axis is parallel to the y-axis, the correct format of the equation is $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$, where ($h$, k) is the center, a and c are the distance from the center to the vertices and foci respectively. For a hyperbola, $c^2 = a^2 + b^2$.
Rewriting the equation of the asymptote $\sqrt{3}x + y = 2 - 3\sqrt{3}$ in point slope form, we have $y - 2 = -\sqrt{3}(x+3)$ and the center must be at (–3, 2).
Given foci at (–3, –8) and knowing the hyperbola is “vertical”, we have c = 10.
The slopes of the asymptotes of a “vertical” hyperbola are always $\pm\frac{a}{b}$. Therefore, $\frac{a}{b} = \frac{\sqrt{3}}{1}$ or $a^2 = 3b^2$.
Substituting, $c^2 = 4b^2 \Rightarrow b^2 = 25$.
Thus, $\left(a^2, b^2, h, k\right) = \underline{\mathbf{(75,\ 25,\ -3,\ 2)}}$.