MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017 SOLUTION KEY

Team Round – continued

C) $P = 16cis\frac{\pi}{3}\cdot cis\frac{\pi}{6}\cdot cis\frac{\pi}{12}\cdot cis\frac{\pi}{24}\cdot ... = 16cis\left(\pi\left(\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+...\right)\right)$

Since the argument is an infinite geometric sequence with a multiplier of $\frac{1}{2}$, it converges to $\frac{1/3}{1-1/2}=\frac{2}{3}$ and $P = 16cis\left(\frac{2\pi}{3}+2n\pi\right)$. Thus, $\sqrt{P}=\sqrt{16}cis\left(\frac{1}{2}\cdot\left(\frac{2\pi}{3}+2n\pi\right)\right) = 4cis\left(\frac{\pi}{3}+n\pi\right)$

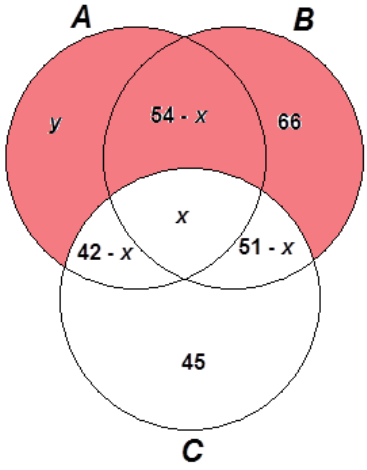
$n = 0 \Rightarrow 4\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right) = 4\left(\frac{1}{2}\right)+4\left(\frac{\sqrt{3}}{2}\right)i = 2+(2\sqrt{3})i$

$n = 1 \Rightarrow 4\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right) = 4\left(-\frac{1}{2}\right)+4\left(-\frac{\sqrt{3}}{2}\right)i = -2+(-2\sqrt{3})i$

$\Rightarrow (A, B) = \underline{\left(\mathbf{2,\ 2\sqrt{3}}\right)}\ ,\ \underline{\left(\mathbf{-2,\ -2\sqrt{3}}\right)}$.

D) Given:

- 54 people voted for *A* and *B*
- 66 people voted for *B* only
- 186 people voted for *A* or *B*, but not *C*
- 42 voted for *A* and *C*
- 51 voted for *B* and *C*
- 45 voted for *C* only



Let the Venn Diagram at the right summarize the voting.
The shaded region represents *A* or *B*, but not *C*.
Let *x* denote the people who voted for all three candidates.
Let *y* denote those people who voted for *A* only.
Since all 300 people voted for at least one candidate, the 7 disjoint regions account for all the votes.

$y+(54-x)+66=186 \Rightarrow y = 66+x$

Adding the *x*-expressions for all 7 regions, $324-x=300 \Rightarrow x=24$.
Since $P \Leftrightarrow Q$ is equivalent to $(P$ and $Q)$ or $(\text{not}\,P$ and $\text{not}\,Q)$,
the bi-conditional "voted for *B* if and only if NOT *C*" is equivalent to
"*B* and NOT *C*" or "*C* and NOT *B*".
Thus, the required regions in the Venn diagram are shaded (in green), and, we have $(30+66)+(18+45)=\underline{\mathbf{159}}$.