MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2016 SOLUTION KEY

Team Round - continued

B) Adding N to its reversal frequently produces an integer with identical digits (call them twins). So we pay attention to the cycles associated with 11, 22, 33, etc.
The number in* [*brackets*] *represents the cycle of the corresponding value of N, i.e. the number of reversals and additions required to reach 99. Note: In the following chains, the cycle numbers were filled in after the chain reached 99.
10 [9] $\Rightarrow$ **11** [8] $\Rightarrow$ **22** [7] $\Rightarrow$ **44** [6] $\Rightarrow$ **88** [5] $\Rightarrow$ 76[4] $\Rightarrow$ 43[3] $\Rightarrow$ **77**[2] $\Rightarrow$ 54[1] $\Rightarrow$ 99
12[14] $\Rightarrow$ **33**[13] $\Rightarrow$ **66**[12] $\Rightarrow$ 32[11] $\Rightarrow$ **55**[10] $\Rightarrow$ 10 continues in above list
13[7] $\Rightarrow$ 44 …. 14[11] $\Rightarrow$ 55 …. 15[13] $\Rightarrow$ 66 … 16[3] $\Rightarrow$ **77** $\Rightarrow$ 54 $\Rightarrow$ 99
This accounts for the cycles associated with all possible twins.

So far 33 has the longest cycle length of any twin, namely 13 and, consequently, 12 has the longest cycle length (14) of any number in the teens.

Larger values of N which eventually produce 33 will have even longer cycles.
Numbers in the 20s, 30s, 40s, etc. produce twins, until they overflow.
We need to concentrate on numbers that produce overflows and eventually 33.
29 $\Rightarrow$ 21 $\Rightarrow$ 33, 38 $\Rightarrow$ 21 $\Rightarrow$ 33, 47 $\Rightarrow$ 21 $\Rightarrow$ 33, 56 $\Rightarrow$ 21 $\Rightarrow$ 33, 65 $\Rightarrow$ 21 $\Rightarrow$ 33 (all cycles of 15).
However, 69 $\Rightarrow$ 65 $\Rightarrow$ 21 $\Rightarrow$ 33 (a cycle of 16).
78 and 87 also have cycle lengths of 16.
But, **<u>89</u>** $\Rightarrow$ 87 $\Rightarrow$ 65 $\Rightarrow$ 21 $\Rightarrow$ 33 and **<u>98</u>** $\Rightarrow$ 87 $\Rightarrow$ 65 $\Rightarrow$ 21 $\Rightarrow$ 33 (both cycles of 17).

C) $y = Sin^{-1}(x) \Rightarrow x = \sin y \;\; \therefore Sin^{-1}(x) = \dfrac{3}{5} \Rightarrow x = \sin\left(\dfrac{3}{5}\right)$.

$$\Rightarrow \frac{x}{y} = \frac{x}{Sin^{-1}(x)} = \frac{\sin\left(\frac{3}{5}\right)}{\frac{3}{5}}.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \Leftrightarrow \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots.$$

Plugging in $\dfrac{3}{5}$ (radians) for x, we have $\dfrac{x}{y} = \dfrac{\sin(.6)}{.6} \approx 1 - \dfrac{(.6)^2}{3!} + \dfrac{(.6)^4}{5!} \approx 1 - 0.06 + 0.001 \approx \underline{\mathbf{0.941}}$.

D) $\begin{cases} R \cdot 110 + (R+k)110 = 1320 \\ R \cdot 88 + kR \cdot 88 = 1320 \end{cases} \Rightarrow \begin{cases} 2R + k = 12 \\ R + kR = 15 \end{cases}$

Substituting for k, $R + (12 - 2R)R = 15 \Rightarrow 2R^2 - 13R + 15 = (2R - 3)(R - 5) = 0 \Rightarrow R = 1.5, 5$
$R = 1.5 \Rightarrow k = 9 \Rightarrow$ faster runner runs at $k + R = 10.5$ and $kR = 13.5$ feet/sec.
$R = 5 \Rightarrow k = 2 \Rightarrow$ faster runner runs at $k + R = 7$ and $kR = 10$ feet/sec
Thus, (f, s) = **<u>(13.5, 7)</u>**.