MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016 SOLUTION KEY

Team Round - continued

D) Let A denote $\log 2$.

$$\log_5 250 = \frac{\log 250}{\log 5} = \frac{N}{\log\left(\frac{10}{2}\right)} = \frac{N}{\log 10 - \log 2} = \underline{\boldsymbol{\frac{N}{1-A}}}$$

But $N = \log 250 = \log\left(2 \cdot 5^3\right) = \log 2 + 3\log\left(\frac{10}{2}\right) = 3 - 2\log 2 = 3 - 2A \Rightarrow A = \frac{3-N}{2}$

Substituting, $\log_5 250 = \frac{N}{1-A} = \frac{N}{1-\frac{3-N}{2}} = \frac{N}{\frac{N-1}{2}} = \underline{\boldsymbol{\frac{2N}{N-1}}}$.

E) $\frac{L_g}{W_g} = \frac{L_g + W_g}{L_g} \Rightarrow L_g{}^2 - L_g W_g - W_g{}^2 = 0$. Dividing by $W_g{}^2$, $\left(\frac{L_g}{W_g}\right)^2 - \frac{L_g}{W_g} - 1 = 0$.

Applying the Q.F., $\frac{L_g}{W_g} = \frac{1+\sqrt{5}}{2} = n$.

$$\frac{L_s}{W_s} = \frac{D_s}{L_s} \Rightarrow L_s{}^2 = W_s\sqrt{L_s{}^2 + W_s{}^2} \Rightarrow L_s{}^4 = W_s{}^2\left(L_s{}^2 + W_s{}^2\right)$$

Multiplying out, dividing by $W_s{}^4$ and transposing terms $\Rightarrow \left(\frac{L_s}{W_s}\right)^4 - \left(\frac{L_s}{W_s}\right)^2 - 1 = 0$. Applying the Q.F. again, we have $\left(\frac{L_s}{W_s}\right)^2 = \frac{1+\sqrt{5}}{2} = n$. Therefore, the required ratio is

$$\frac{n}{\sqrt{n}} = \sqrt{n} = \sqrt{\frac{1+\sqrt{5}}{2} \cdot \frac{2}{2}} = \sqrt{\frac{2+2\sqrt{5}}{4}} = \frac{\sqrt{2+2\sqrt{5}}}{2}$$

Multipliers other than $\frac{2}{2}$ were possible. $\frac{8}{8} \Rightarrow \frac{\sqrt{8+8\sqrt{5}}}{4}$, $\frac{18}{18} \Rightarrow \frac{\sqrt{18+18\sqrt{5}}}{6}$, but in all these cases, the product $A \cdot B$ is larger. Since it was given that $A > 1$, $A = 2$ is a minimum and we have $(A, B) = \underline{\boldsymbol{\left(2,\ 2+2\sqrt{5}\right)}}$.