MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2014 SOLUTION KEY

Team Round

A) $\begin{vmatrix} 2 & 4 & 8 \\ 0 & 8 & 4 \\ 0 & 1 & 2 \end{vmatrix} x^3 + \begin{vmatrix} 9 & 7 \\ -4 & 5 \end{vmatrix} x^2 - \begin{vmatrix} 7 & 10 \\ -3 & x \end{vmatrix} = 0$ Evaluate $\begin{vmatrix} r_1 & r_3 \\ -r_2 & r_2 + r_3 \end{vmatrix}$.

The lead coefficient is easily evaluated since the left column is all zeros except the first entry. Reducing the determinant of a square $n \text{ x } n$ matrix to a series of determinants of $(n-1)$ x $(n-1)$ matrices is called expansion by minors.

$$\begin{vmatrix} 2 & 4 & 8 \\ 0 & 8 & 4 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} \boxed{2} & \cancel{4} & \cancel{8} \\ \cancel{0} & 8 & 4 \\ \cancel{0} & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 8 & 4 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 4 & 8 \\ 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 4 & 8 \\ 8 & 4 \end{vmatrix} = 2(8 \cdot 2 - 1 \cdot 4) = 24$$

The quadratic coefficient is simply $9 \cdot 5 + 4 \cdot 7 = 73$.
The last determinant is $7 \cdot x + 3 \cdot 10 = 7x + 30$
Thus, the cubic equation is $24x^3 + 73x^2 - 7x - 30 = 0$.
The determinant to be evaluated is $r_1r_2 + r_1r_3 + r_2r_3$.
We could certainly use the clues to factor the cubic expression and then plug in the roots and evaluate the expression. But none of that is necessary!! Notice that when $(x-r_1)(x-r_2)(x-r_3) = 0$ is expanded, you get $x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3 = 0$.
If the lead coefficient were 1, the expression we seek would be just the coefficient of the *x*-term.
So dividing both sides of our cubic by 24, we get the required determinant, namely $\underline{\mathbf{-\frac{7}{24}}}$.

Here's a check: $24x^3 + 73x^2 - 7x - 30 = (x+3)(8x-5)(3x+2) = 0$

$\Rightarrow (r_1, r_2, r_3) = \left(-3, \frac{5}{8}, -\frac{2}{3},\right)$ and

$$r_1r_2 + r_1r_3 + r_2r_3 = \left(-3 \cdot \frac{5}{8}\right) + \left(-3 \cdot -\frac{2}{3}\right) + \left(\frac{5}{8} \cdot -\frac{2}{3}\right) = \frac{-15}{8} + 2 + \frac{-10}{24} = \frac{-45 + 48 - 10}{24} = -\frac{7}{24}$$

B) We know that $1^2 + 2^2 + 3^2 + \ldots n^2 = \frac{n(n+1)(2n+1)}{6}$ and $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

Thus, $Q = \frac{1^2 + 2^2 + 3^2 + \ldots n^2}{1 + 2 + 3 + \ldots + n} = \frac{n(n+1)(2n+1)}{6} \cdot \frac{2}{n(n+1)} = \frac{2n+1}{3}$

Q is an integer for n = 1, 4, 7, … $(3k - 2)$, where k is a positive integer.

$\frac{2(3k-2)+1}{3} \le 2014 \Leftrightarrow k \le \frac{6042+3}{6} = 1007.5$

Thus, Q is an integer for **<u>1007</u>** values of n.