MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2007 SOLUTION KEY

Team Round

A) $x^8 - 81 = (x^4 - 9)(x^4 + 9) = (x^2 - 3)(x^2 + 3)(x^2 - 3i)(x^2 + 3i) = 0$
→ roots: $\pm\sqrt{3}$, $\pm\sqrt{3}i$ $\pm\sqrt{3i}$ and $\pm\sqrt{-3i}$

The last two pairs must be further simplified.

Let $\sqrt{3i} = a + bi$. Squaring, $(a^2 - b^2) + 2abi = 0 + 3i$.

Thus, $\begin{cases} a^2 - b^2 = 0 \\ 2ab = 3 \end{cases}$ → $b = \dfrac{3}{2a}$ and $a^2 - \dfrac{9}{4a^2} = 0$

Substituting, $4a^4 - 9 = (2a^2 + 3)(2a^2 - 3) = 0$ → $a^2 = 3/2$ → $a = \dfrac{\sqrt{6}}{2}$ and $b = \dfrac{3}{\sqrt{6}} = \dfrac{\sqrt{6}}{2}$

and $\sqrt{3i} = \pm\left(\dfrac{\sqrt{6}}{2} + \dfrac{\sqrt{6}}{2}i\right)$. You should verify that $\sqrt{-3i} = \pm\left(\dfrac{\sqrt{6}}{2} - \dfrac{\sqrt{6}}{2}i\right)$.

Therefore, $(A, B) = \underline{\left(\boldsymbol{\sqrt{3}, \dfrac{\sqrt{6}}{2}}\right)}$.

B) Subtracting the second equation from the first, $4a - 4c = 20$ → $a = c + 5$
Substituting in the first equation, $(c + 5) + 3b + 5c = 50$ → $6c + 3b = 45$ → $b = 15 - 2c$

Since all variables must be positive integers, $c \geq 1$ → $a > 6$ and $b = 13, 11, 9, \ldots, 1$
b prime → $b = 13, 11, 7, 5$ or 3

Substituting in the two equations, $\begin{cases} b = 13 \quad 11 \quad 7 \quad 5 \quad 3 \\ a + 5c = 11 \quad 17 \quad 29 \quad 35 \quad 41 \\ 5a + c = 31 \quad 37 \quad 49 \quad 55 \quad 61 \end{cases}$

Adding $6(a + c) = 42, 54, 78, 90$ or 102 → $a + c = 7, 9, 13, 15$ or 17
Only 9 and 15 are composite $a = c + 5$, 9 → $(a, c) = (7, 2)$ and 15 → $(a, c) = (10, 5)$
→ **<u>(7, 11, 2)</u>** and **<u>(10, 5, 5)</u>**