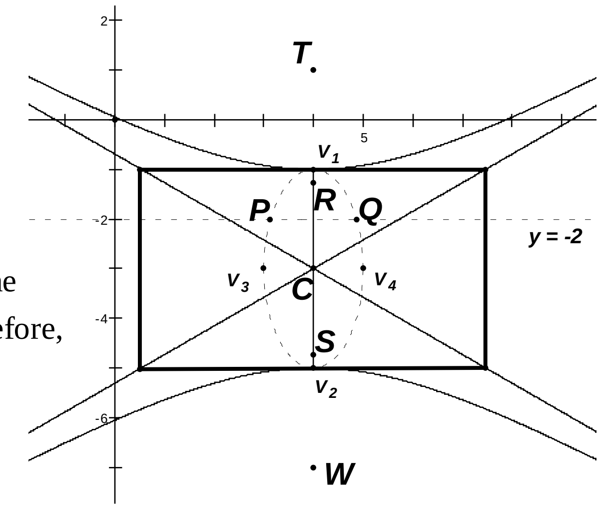
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016 SOLUTION KEY

Team Round

A) $3(y+3)^2-(x-4)^2=12 \Leftrightarrow \dfrac{(y+3)^2}{4}-\dfrac{(x-4)^2}{12}=1$

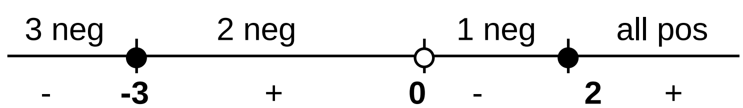


This is a vertical hyperbola with center at $C(4,-3)$,
$a=2, b=2\sqrt{3}$ and $c=4$.
Thus, the vertices of the hyperbola are at
$(4,-3+2)=V_1(4,-1)$ and $(4,-3-2)=V_2(4,-5)$
The center of the ellipse is $(4,-3)$, $a=2, b=1$ (since the major axis is twice as long as the minor axis) and, therefore, its equation is $\dfrac{(y+3)^2}{4}+(x-4)^2=1$.
Substituting, when $y=-2$, we have
$\dfrac{1}{4}+(x-4)^2=1 \Rightarrow (x-4)^2=\dfrac{3}{4} \Rightarrow x=4\pm\dfrac{\sqrt{3}}{2}$ and the length of the segment $\overline{PQ}$ is $\left(4+\dfrac{\sqrt{3}}{2}\right)-\left(4-\dfrac{\sqrt{3}}{2}\right)=\underline{\boldsymbol{\sqrt{3}}}$.

FYI: R and S are the foci of the ellipse, while T and W are the foci of the hyperbola. V_3 and V_4 are the endpoints of the minor axis of the ellipse.
You should be able to specify the coordinates of these points, as well as the equations of the lines containing the diagonals of the rectangle which are referred to as the asymptotes of the hyperbola. Although neither branch of the hyperbola ever intersects these lines, as $x \to \pm\infty$, the distance between the hyperbola and these lines becomes arbitrarily small.

B) $xy(y-2)=x^2-6 \Rightarrow y^2-2y=\dfrac{x^2-6}{x} \Rightarrow (y-1)^2=\dfrac{x^2-6}{x}+1=\dfrac{x^2+x-6}{x}=\dfrac{(x+3)(x-2)}{x}$. Thus,
$y=1\pm\sqrt{\dfrac{(x+3)(x-2)}{x}}$ and since it was given that $y \ge 1$, we have $y=f(x)=1+\sqrt{\dfrac{(x+3)(x-2)}{x}}$.
The critical values are $-3, 0, 2$ and, as each critical value is passed from left to right, there is one less negative factor in the radicand. On the left (as $x \to -\infty$) all three factors are negative; whereas, on the right (as $x \to +\infty$) all three factors are positive. The following chart summarizes this discussion.

Radicand:



Thus, the domain is $\underline{\{x \mid \mathbf{-3 \le x < 0} \text{ or } \mathbf{x \ge 2}\}}$. The symbol "$\vee$" may be used in place of "or".