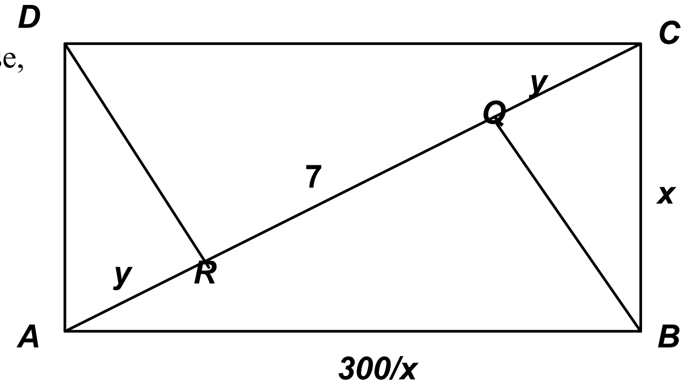
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006 SOLUTION KEY

Team Round

A) In right ΔABC, $x^2 + (300/x)^2 = (2y + 7)^2$ and since $\overline{BQ}$ is an altitude to the hypotenuse, $x^2 = y(2y + 7)$.



Substituting in the 1st equation for x^2,

$$y(2y + 7) + \frac{300^2}{y(2y + 7)} = (2y + 7)^2$$

$y^2(2y + 7)^2 + 300^2 = y(2y + 7)^3$
$= 4y^4 + 56y^3 + 245y^2 + 343y - 90000$
$= (y - 9)(4y^3 + 92y^2 + 1073y + 10000)$

Clearly $y = 9$ is a solution and since the coefficients of the cubic factor are all positive, there are no additional positive roots. Substituting in the 1st equation, $x^2 = 9(25) \rightarrow x = 15$. Thus, the only possible perimeter of rectangle $ABCD$ is $2(15 + 20) =$ **<u>70</u>**.

Alternative:

Let $BQ = z$. Then $\begin{cases} z^2 = y(y+7) \\ z(2y+7) = 300 \end{cases}$. Solving for z in the 2nd equation and substituting in the 1st, $y(y + 7)(2y + 7)^2 = 300^2 = 3^2\cdot 2^4\cdot 5^4$. By inspection, if $y = 9$, $(y + 7) = 16 = 2^4$ and $(2y + 7)^2 = 5^4$. Thus, $y = 9$ is a positive solution and, for $y > 9$, the left hand side $> 300^2$ and, for $0 < y < 9$, the left hand side is $< 300^2$. $y = 9 \rightarrow z = 12$, $x = 15$ and finally, $P =$ **<u>70</u>**.

B) We must count the total number of factors of 2 and of 3 in the product of the 732 consecutive integers denoted by 732!, since these are the only prime factors of 12.
In 732!, every 2nd integer is a multiple of 2, every 4th a multiple of 4, every 8th a multiple of 8, etc. Some multiples of 2 need to be counted once (e.g. 2, 6, 10, …), some twice (e.g. 4, 12, 20, …), some three times (e.g. 8, 24, 40, …) etc.
The multiples of 2: 2, 4, 6, 8, 10, 12, 14, 16, …., 732 [= 2(366)] - a total of 366 numbers
The multiples of 4: 4, 8, 12, 16, …, 732 [= 4(183)] - a total of 183 numbers
The multiples of 8: 8, 16, 16, …, 728 [= 8(91)] – a total of 91 numbers
The number of factors of 2 is equal to the following sum:

$$366 + 183 + 91 + 45 + 22 + 11 + 5 + 2 + 1 + 0 = 726$$

Start by dividing 732 by 2 and record the quotient. Continue dividing by 2 and recording the quotient (disregarding any remainder), until a quotient of zero is obtained. In the above sum,
5 → exactly 5 multiples of 128 (2^7) are less than 732, namely 128, 256, 384, 512 and 640.
2 → exactly 2 multiples of 256 (2^8) are less than 732, namely 256 and 512.
1 → only 1 multiple of 512 (2^9) is less than 732.
0 → no multiples of 1024 (2^{10}) are less than 732.

The powers of 3 can be counted similarly as

$$244 + 81 + 27 + 9 + 3 + 1 + 0 \rightarrow 365$$

Thus, $732! = 2^{726}\cdot 3^{365}\cdot$(a bunch of other primes raised to various powers)
Since $12 = 2^2 3^1$, twice as many 2s are needed as 3s to form factors of 12.
$2^{726}3^{365}\,[\ldots] = 2^{2(363)}3^{363}3^2\,[\ldots] = (2^2 3)^{363}\cdot 9\cdot [\ldots] = 12^{363}\cdot 9\cdot [\ldots] \rightarrow$ **<u>363</u>** factors of 12.