MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2015 SOLUTION KEY

Round 1 (Additional comments on 1B at the end of the solution key)

A) $\begin{vmatrix} 3 & -5k \\ 4 & k+100 \end{vmatrix} = 3(k+100) - 4(-5k) = 23k + 300 < 0 \Leftrightarrow k < \dfrac{-300}{23} = -13\dfrac{1}{23} \Rightarrow \underline{\mathbf{-14}}$

B) Given: (1) $14x + 3y - 7z = 8$
(2) $-8x + 5y + 4z = c$ (3) $\Rightarrow z = 2 + 2x - 3y$
(3) $-2x + 3y + z = 2$

Substituting in (1), $14x + 3y - 14 - 14x + 21y = 8 \Rightarrow 24y = 22 \Rightarrow \begin{cases} y = \dfrac{11}{12} \\ z = 2x - \dfrac{3}{4} \end{cases}$

Substituting in (2), $-8x + \dfrac{55}{12} + 8x - 3 = c \Rightarrow c = \underline{\dfrac{\mathbf{19}}{\mathbf{12}}}$, otherwise, there is no solution.

<u>FYI</u>: Provided $c = \dfrac{19}{12}$, the solution set is $\left\{\left(x, \dfrac{11}{12}, 2x - \dfrac{3}{4}\right)\right\}$. The equations become $\begin{cases} \dfrac{11}{4} + \dfrac{21}{4} = 8 \\ \dfrac{55}{12} - 3 = c \\ \dfrac{11}{4} - \dfrac{3}{4} = 2 \end{cases}$

as the x-terms drop out. The 3 equations are satisfied if and only if $c = \dfrac{19}{12}$.

Note that if the z-coefficients are multiplied by –2, they equal the x-coefficients.

Therefore, the 3 equations are dependent and $c = \dfrac{19}{12}$ makes the system consistent.

Alternately, since the columns are dependent, the determinant of the matrix of coefficients is zero, and, for there to be an infinite number of solutions, the determinant of the matrix of coefficients, where any column is replaced by the constants, must also be zero.

For example, $\begin{vmatrix} 8 & 3 & -7 \\ c & 5 & 4 \\ 2 & 3 & 1 \end{vmatrix} = (40 + 24 - 21c) - (-70 + 96 + 3c) = 0 \Rightarrow 38 - 24c = 0 \Rightarrow c = \dfrac{19}{12}$.

Replacing the z-coefficients gives the same c-value. If the y-coefficients are replaced, the determinant of the matrix is zero, for <u>all</u> values of c; so again 0/0 $\Rightarrow$ an infinite number of solutions.

C) $\begin{matrix} 10 & -7 & 1 & 10 & -7 \\ -6 & 11 & 1 & -6 & 11 \\ 3 & k & 1 & 3 & k \end{matrix} \Rightarrow A = \dfrac{1}{2}\left|\det\begin{bmatrix} 10 & -7 & 1 \\ -6 & 11 & 1 \\ 3 & k & 1 \end{bmatrix}\right|$

$= \dfrac{1}{2}\left|\left(10\cdot 11\cdot 1 + (-7)\cdot 1\cdot 3 + 1\cdot(-6)\cdot k\right) - \left(3\cdot 11\cdot 1 + k\cdot 1\cdot 10 + 1\cdot(-6)\cdot(-7)\right)\right|$

$= \dfrac{1}{2}\left|110 - 21 - 6k - 33 - 10k - 42\right| = \dfrac{1}{2}\left|14 - 16k\right| = \left|7 - 8k\right|$.

$\left|7 - 8k\right|$ evaluates to a perfect square for k = **<u>1</u>**, **<u>2</u>**, **<u>4</u>**, and **<u>7</u>**.