MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2011 SOLUTION KEY

Team Round

A) If the dimensions of the solid are a , b and c, then $\begin{cases} (1)\ ab = 180 \\ (2)\ bc = 240 \\ (3)\ ac = 144 \end{cases}$.

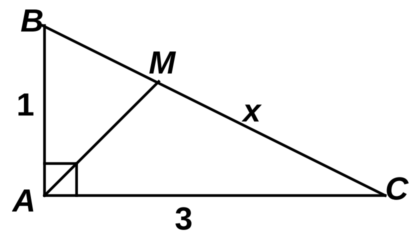
Divide (1) by (2), multiply the left hand side by c/c and substitute for ac using (3):

$$\frac{a}{c} = \frac{180}{240} \Rightarrow \frac{ac}{c^2} = \frac{3}{4} \Rightarrow \frac{144}{c^2} = \frac{3}{4} \Rightarrow c^2 = 4(48) \Rightarrow c = 8\sqrt{3} \Rightarrow a = 6\sqrt{3} \text{ and } b = 10\sqrt{3}$$

Find the edge as above and use the relationship $d^2 = L^2 + W^2 + H^2$

$$\Rightarrow d^2 = \left(6\sqrt{3}\right)^2 + \left(8\sqrt{3}\right)^2 + \left(10\sqrt{3}\right)^2 = 3(36+64+100) = 600 \Rightarrow d = \underline{\mathbf{10\sqrt{6}}}.$$

B) Let $MC = x$. Applying the Pythagorean Theorem to ΔABC, $AC = 3 \Rightarrow AM = 1$.



$\cos(\measuredangle ACB) = \dfrac{3}{\sqrt{10}}$. Now use the Law of Cosines on ΔAMC

$$1^2 = 3^2 + x^2 - 2\cdot 3\cdot x\cdot \frac{3}{\sqrt{10}} \Rightarrow x^2 - \frac{18}{\sqrt{10}}x + 8 = 0 \Rightarrow$$

$$\sqrt{10}x^2 - 18x + 8\sqrt{10} = 0$$

$$\Rightarrow x = \frac{18 \pm \sqrt{18^2 - 32(10)}}{2\sqrt{10}} = \frac{18 \pm 2}{2\sqrt{10}} = \sqrt{10} \text{ (rejected) or } \frac{8}{\sqrt{10}} = \underline{\mathbf{\frac{4\sqrt{10}}{5}}}.$$

Alternative Solution #1 (Norm Swanson)

$AC = 3$ and $\dfrac{AM}{AC} = \dfrac{1}{3} \Rightarrow AM = 1$, so ΔABM is isosceles. $m\angle MAC = 90 - (180 - 2B) = 90 - 2B$

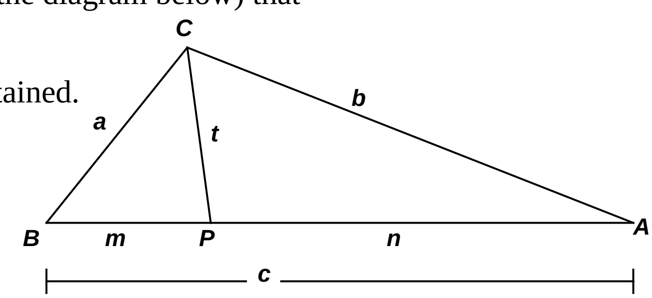
$$\cos\left(2B - 90\right) = \cos\left(90 - 2B\right) = \sin 2B = 2\sin B\cos B = 2\left(\frac{3}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}\right) = \frac{3}{5}$$

Using the Law of Cosines on ΔMAC,

$$MC^2 = 1^2 + 3^2 - 2\cdot 1\cdot 3\cdot \frac{3}{5} = 10 - \frac{18}{5} = \frac{32}{5} = \frac{16\cdot 10}{25} \Rightarrow MC = \underline{\mathbf{\frac{4\sqrt{10}}{5}}}$$

Alternative solution #2 applies Stewart's Theorem to ΔABC.
Stewart Theorem states that if a segment is drawn from the vertex of <u>any</u> triangle to <u>any</u> point on the opposite side (with lengths as indicated in the diagram below) that

$$\boxed{a^2n + b^2m = t^2c + cmn}.$$



It is left to you to check that the same result is obtained.
The proof requires some basic trig and some heavy algebraic lifting, but is not out of reach. You might want to try deriving it on your own or peeking at the end of this solution key.

Hint: Use Law of Cosines on ΔBPC and ΔCPA.