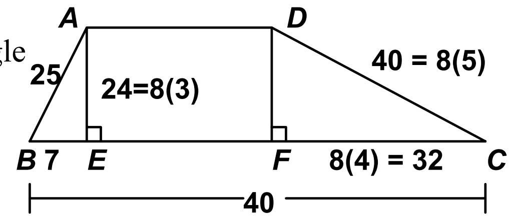
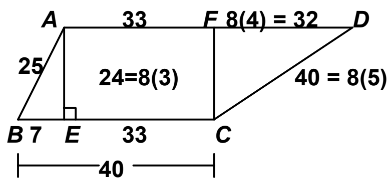
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2006 SOLUTION KEY

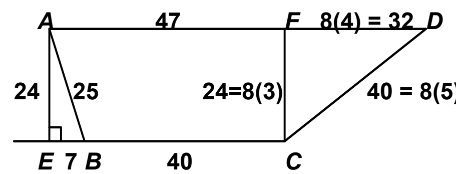
Round 3

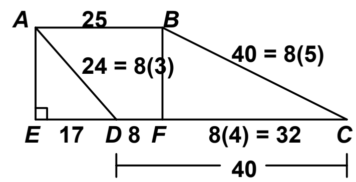


A) Drop perpendiculars from A and D to base $\overline{BC}$, creating a rectangle and two special right triangles as indicated in the diagram.
$EF = 40 - (7 + 32) = 1 \rightarrow AD = 1$. Thus, Area(trapezoid) =
$\frac{1}{2}h(b_1+b_2) = \frac{1}{2}(24)[40+1] = \underline{\mathbf{492}}$.

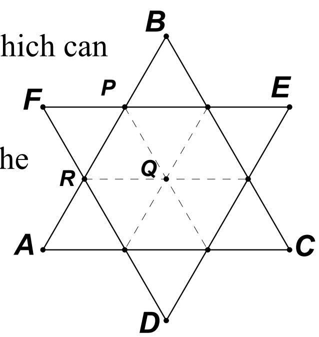
Failing to specify that $AD < BC$ and that $\overline{AD}$ and $\overline{BC}$ are bases, allows additional solutions.
$\frac{1}{2}(24)[40+33+32] = \underline{\mathbf{1260}}$ $\quad\frac{1}{2}(24)[40+47+32] = \underline{\mathbf{1428}}$ $\quad\frac{1}{2}(24)[25+40+17] = \underline{\mathbf{984}}$





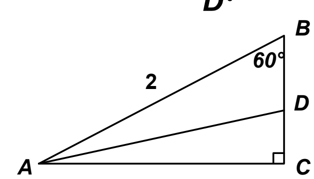


Are there others?



B) Note that the intersection of the two equilateral triangles is a regular hexagon, which can be subdivided into 6 congruent equilateral triangles by drawing the 3 indicated diagonals. It's easy to argue that $FPQR$ is a parallelogram and, therefore, $\Delta FPR \cong \Delta QRP$ and all 12 equilateral triangles are congruent. Thus, the ratio of the area of the entire star to the area of the original ΔABC is 12 : 9 = **<u>4 : 3</u>**

C) Triangles ADC and ADB have the same altitude from point A and, therefore, their areas are in the ratio of bases DC and DB.



By the angle bisector theorem, $\frac{DC}{\sqrt{3}} = \frac{DB}{2} \rightarrow \frac{DC}{DB} = \frac{\sqrt{3}}{2}$

$\text{Area}(\Delta ADC) + \text{Area}(\Delta ADB) = \sqrt{3}x + 2x = 6 \rightarrow x = \frac{6}{2+\sqrt{3}} = 6(2-\sqrt{3})$

and Area(ΔADC) = $\sqrt{3}x$ = $\underline{\mathbf{12\sqrt{3} - 18}}$ or $\underline{\mathbf{6(2\sqrt{3} - 3)}}$

Round 4

A) As the difference of perfect squares, $(a^2 - 3a + 1)^2 - 1 = (a^2 - 3a)(a^2 - 3a + 2)$
$= \underline{\boldsymbol{a(a-3)(a-2)(a-1)}}$ - in any order

B) Let $A = (x + 1)$. Then, grouping in pairs, $A^3 - A^2 - 9A + 9 = A^2(A-1) - 9(A-1)$
$= (A-1)(A^2-9) = (A-1)(A+3)(A-3)$ Substituting for A, we have $\underline{\boldsymbol{x(x+4)(x-2)}}$.

C) Treat the equation as a quadratic equation in x and complete the square.
$x(x+2y) = 1 \rightarrow x^2 + (2y)x + \boldsymbol{y^2} = 1 + \boldsymbol{y^2} \rightarrow (x+y)^2 = 1 + y^2 \rightarrow x + y = \pm\sqrt{1+y^2}$
$x = -y \pm\sqrt{1+y^2}$
Since $\sqrt{1+y^2} > \sqrt{y^2} = |y|$, it follows that $\sqrt{1+y^2} > y$ or $\sqrt{1+y^2} - y > 0$ and the only solution giving $x < 0$ is x = $\underline{\mathbf{-y - \sqrt{1+y^2}}}$