MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2011 SOLUTION KEY

Team Round

F) Suppose the first term of the GP is a and the common multiplier is r.

$$S_3 = \frac{a(1-r^3)}{1-r} = 1792 \text{ and } S_{11} = \frac{a(1-r^{11})}{1-r} = 2047$$

Dividing, $\dfrac{S_{11}}{S_3} = \dfrac{\left(1-r^{11}\right)}{\left(1-r^3\right)} = \dfrac{2047}{1792}$

If the sum of the terms in the infinite geometric progression converges to a finite sum, $|r| < 1$.
Noting that 2047 is 1 less than a power of 2 and that all the terms are rational numbers,

I try $r = \dfrac{1}{2}$. $\left[\left(\dfrac{1-\dfrac{1}{2048}}{1-\dfrac{1}{8}}\right)\dfrac{2048}{2048} = \dfrac{2048-1}{2048-156} = \dfrac{2047}{1792}\right]$ Bingo!

Substituting, $\dfrac{a\left(1-\left(\dfrac{1}{2}\right)^3\right)}{1-\dfrac{1}{2}} = 1792 \rightarrow \dfrac{7}{4}a = 1792 \rightarrow a = 4(256) = 1024$.

The sum of the infinite G.P. is $\dfrac{a}{1-r} = \dfrac{1024}{1-\dfrac{1}{2}} = 2048$.

Now, for the arithmetic progression, $t_{56} = a + 55d = 2048$

$\rightarrow d = \dfrac{2048-a}{55} = 37 - \boxed{\dfrac{13-a}{55}}$

For the boxed expression to be an integer, $a = 13 + 55k$, for integer values of k.
$a < 50 \rightarrow a = 13, d = 37 \rightarrow t_{55} = 13 + 54(37) =$ **<u>2011</u>**.

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