MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012 SOLUTION KEY

Team Round – continued

C) $\sin 4x - \cos 2x = 4\sin x\cos x - 1 \Leftrightarrow 2\sin 2x\cos 2x - \cos 2x = 2\sin 2x - 1$

$\Leftrightarrow \cos 2x\left(2\sin 2x - 1\right) = 2\sin 2x - 1$

$\Leftrightarrow \left(\cos 2x - 1\right)\left(2\sin 2x - 1\right) = 0$

$\Leftrightarrow \cos 2x = 1 \text{ or } \sin 2x = 1/2$

$\Leftrightarrow 2x = 0° + n\cdot 360° \text{ or } 2x = \begin{cases} 30° + n\cdot 360° \\ 150° + n\cdot 360° \end{cases}$

Dividing by 2 and letting n = 0, 1 $\Rightarrow x$ = <u>**0°, 180°, 15°, 195°, 75°, 255°**</u>.
(Answers in any order.)

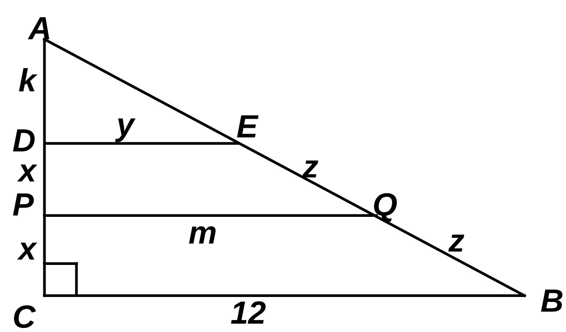
D) $\begin{cases} (1)\ \ k - 4 = 2A \\ (2)\ \ 6k - 2 = A^2 + B^2 \end{cases}$ Substituting $2A + 4$ for k in (2), we have $12A + 22 = A^2 + B^2$.

Completing the square, $\left(A^2 - 12A \underline{+36}\right) = \left(A - 6\right)^2 = 22 - B^2 \underline{+36} = 58 - B^2$

For positive integer values of B ($1 \leq B \leq 7$), $58 - B^2$ must evaluate to a perfect square.
When $B = 3$, we have $(A - 6)^2 = 49 \Rightarrow A = 6 \pm 7 \Rightarrow A = 13$ (–1 is rejected).
When $B = 7$, we have $(A - 6)^2 = 9 \Rightarrow A = 6 \pm 3 \Rightarrow A = 3, 9$
For all other values of B, $58 - B^2$ is not a perfect square.
Thus, there are three solutions <u>**(30, 13, 3)**</u>, <u>**(10, 3, 7)**</u> and <u>**(22, 9, 7)**</u>.
(The ordered triples may be listed in any order.)

E) Since $\overline{PQ}$ is a median of trapezoid $BCDE$, $x > 0$ and, consequently $k < 5$.

Since $AC = 5$, $x = \dfrac{5-k}{2}$



$\Delta ADE \sim \Delta ACB \Rightarrow \dfrac{k}{5} = \dfrac{y}{12} = \dfrac{AE}{13} \Rightarrow \begin{cases} y = \dfrac{12k}{5} \\ AE = \dfrac{13k}{5} \end{cases}$

Therefore, $m = \dfrac{y+12}{2} = \dfrac{\frac{12k}{5}+12}{2} = \dfrac{12(k+5)}{10}$ and

$z = \dfrac{13 - \frac{13k}{5}}{2} = \dfrac{13(5-k)}{10}$. The perimeter of $PQBC = x + m + z + 12 \leq 25 \Rightarrow$

$\dfrac{5-k}{2} + \dfrac{12(k+5)}{10} + \dfrac{13(5-k)}{10} \leq 13 \Rightarrow 25 - 5k + 12k + 60 + 65 - 13k \leq 130 \Rightarrow 20 \leq 6k \Rightarrow k \geq \dfrac{10}{3}$

Combining the restrictions, $\underline{\boldsymbol{\dfrac{10}{3} \leq k < 5}}$.