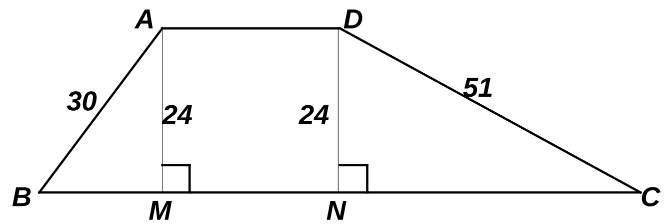
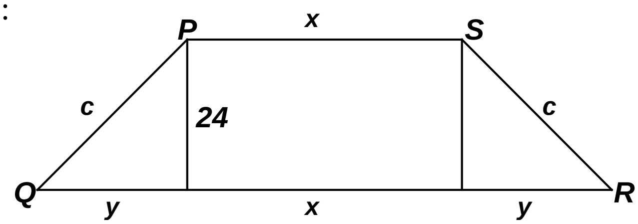
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2014 SOLUTION KEY



C) Drop perpendiculars from A and D to $\overline{BC}$.
$BM = 18$ $(18, 24, 30) = 6(3, 4, 5)$
$NC = 45$ $(24, 45, 51) = 3(8, 15, 17)$
$BC = 87 \Rightarrow MN = AD = (87 - 18 + 45) = 24$
$\Rightarrow \text{Per}(ABCD) = 192$
$\text{Per}(PQRS) = 2x + 2y + 2c = 192 \Rightarrow x + y + c = 96$ and $(y, 24, c)$ is a Pythagorean triple
The possible triples (with a leg of 24) are:



1) $6(3, 4, 5) \Rightarrow (18, 24, 30)$
2) $8(3, 4, 5) \Rightarrow (24, 32, 40)$
3) $2(5, 12, 13) \Rightarrow (10, 24, 26)$
4) $(7, 24, 25)$
5) $3(8, 15, 17) \Rightarrow (24, 45, 51)$
6) $(24, 143, 145)$

Aside:

The system $\begin{cases} a = m^2 - n^2 \\ b = 2mn \\ c = m^2 + n^2 \end{cases}$ may be used to generate any Pythagorean triple. The triple will be primitive whenever the greatest common factor of m and n is 1.
The primitive triples 1) through 6) above were generated by $(m, n) = (2, 1), (3, 2), (4, 3), (4,1)$ and $(12, 1)$.
From the list above, the possible values of $y + c$ are:
48 for $(y, c) = (18, 30)$, 72 for $(y, c) = (32, 40)$, 36 for $(y, c) = (10, 26)$, 32 for $(y, c) = (7, 25)$, ~~96~~ for $(y, c) = (45, 51)$ – rejected ($\Rightarrow x = 0$) ~~288~~ for $(y, c) = (143, 145)$ – also rejected
$x = 96 - (y + c) \Rightarrow$ the allowable values of x are: 48, 24, 60, 64
Thus, the allowable corresponding values of (x, y) are: (48, 18), (24, 32), (60, 10) and (64, 7)

Since the area$(PQRS) = 24(x + y)$, we have $24 \cdot \begin{cases} 66 \\ 56 \\ 70 \\ 71 \end{cases} \Rightarrow$ **<u>1584, 1344, 1680, 1704</u>**