MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2011 SOLUTION KEY

Round 4

A) $N = 2^5 \cdot 3^3$
Any divisor (other than 1) will have factors of 2 or 3, but no other prime.
Thus, the exponents of 2 may be any integer from 0 to 5 inclusive – 6 possibilities.
The exponents of 3 may be any integer from 0 to 3 inclusive – 4 possibilities.
Choosing both exponents to be 0 gives us 1.
Thus, there are 4(6) = **24** possible positive divisors.
In general, determine the unique prime factorization of N, add 1 to each exponent and multiply.

B) $4x^4 + 1 - 5x^2 = (4x^2 - 1)(x^2 - 1) = (2x + 1)(2x - 1)(x + 1)(x - 1)$.
The sum of the factors is $\mathbf{6x}$.

C) $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 + 3xy(x + y)$
$\Rightarrow 6^3 = 58.5 + 3xy(6)$
$\Rightarrow 216 - 58.5 = 18xy \Rightarrow xy = \dfrac{157.5}{18} = \dfrac{315}{36} = \dfrac{\mathbf{35}}{\mathbf{4}} = \mathbf{8.75}$
The alternate solution below uses this assertion (fact): ***if a + b + c = 0, then $a^3 + b^3 + c^3 = 3abc$***
In other words, if the sum of 3 numbers is 0, then the sum of the cubes of the 3 numbers will always be 3 times the product of the 3 numbers.

$x + y = 6 \Rightarrow x + y - 6 = 0 \Rightarrow x^3 + y^3 + (-6)^3 = 3xy(-6) \Rightarrow 58.5 - 216 = -18xy$
which is the same result as above.

Proof of the assertion
$(x - a)(x - b)(x - c) = 0$ has solutions $x = a, b, c$
Expanding, and regrouping, $x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc = 0$

Substituting for x: $\begin{cases} a^3 - (a+b+c)a^2 + (ab+ac+bc)a - abc = 0 \\ b^3 - (a+b+c)b^2 + (ab+ac+bc)b - abc = 0 \\ c^3 - (a+b+c)c^2 + (ab+ac+bc)c - abc = 0 \end{cases}$

Adding these three equations,
$a^3 + b^3 + c^3 - (a+b+c)(a^2+b^2+c^2) + (ab+ac+bc)(a+b+c) - 3abc = 0$
Finally, if $(a + b + c) = 0$, $a^3 + b^3 + c^3 - 3abc = 0 \Rightarrow \boxed{a^3 + b^3 + c^3 = 3abc}$.