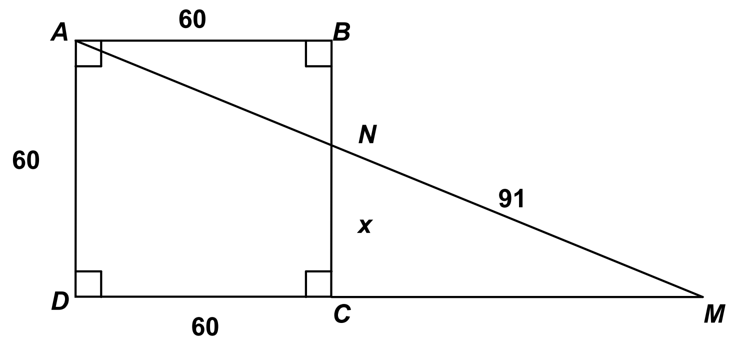
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2010 SOLUTION KEY

Team Round - continued

D) Bottle #1 contains 4 parts and bottle #2 contains $(A + B)$ parts.
To insure equal volumes – multiply the bottle #1 ratio by $(A + B)$ → $3(A + B) : (A + B)$
and the bottle #2 ratio by 4 → $4A : 4B$.

The alcohol : water ratio, after mixing, is $(7A + 3B) : (A + 5B) = 27 : 13$
→ $91A + 39B = 27A + 135B$ → $64A = 96B$ → $2A = 3B$.
The only relatively prime pair of values satisfying this equality is $(A, B) =$ **(3, 2)**.

E) Since $\Delta NBA \sim \Delta NCM$, we require $\dfrac{NC}{CM} = \dfrac{NB}{BA}$



Solution #1: (Banking on a hunch)
A triangle with sides 5 – 12 – 13 is a right triangle and since $91 = 7(13)$, we might guess that $NC = x = 7(5) = 35$,
$CM = 7(12) = 84$, $\dfrac{NC}{CM} = \dfrac{5}{12}$

$NB = 60 - 35 = 25$ and $\dfrac{NB}{BA} = \dfrac{25}{60} = \dfrac{5}{12}$, so our hunch was correct, $x =$ **35**.

In the absence of a hunch, here are two analytical solutions.
Solution #2: (Thanks to Andrew Geng – Westford Academy/M.I.T.)
Let $y = 60 - x$. By applying the Pythagorean Theorem and similarity relations:

$$\frac{60^2 + y^2}{91^2} = \frac{y^2}{x^2} \rightarrow \left(60^2 + y^2\right)x^2 = 91^2 y^2 \quad (***)$$

Rewriting $60^2 + y^2$ as $(60 - y)^2 + 120y$ and noting that $60 - y = x$, we have $60^2 + y^2 = x^2 + 120y$.
This substitution will make it possible to reduce equation (***) to a quadratic.

$$\left(x^2 + 120y\right)x^2 = 91^2 y^2 \rightarrow x^4 + 120x^2 y - 91^2 y^2 = 0$$

The quadratic formula can be used to solve for x^2 (in terms of y) or alternately, the left side can be factored by noticing that 91^2 is the product of 7^2 and 13^2, which differ by 120.

$$x^2 = \frac{1}{2}\left(-120y \pm \sqrt{120^2 y^2 + 4 \cdot 91^2 y^2}\right) = -60y \pm \sqrt{60^2 y^2 + 91^2 y^2}$$

$$= -60y \pm \sqrt{11881 y^2} = -60y \pm 109y = -49y, -169y$$

Since both x and y must be positive, $-169y$ can be discarded. Applying the substitution $y = 60 - x$ and using the quadratic formula (or factoring again) finishes the problem:

$$x^2 = 49(60 - x) \rightarrow x^2 + 49x - 49 \cdot 60 = 0$$

$$x = \frac{1}{2}\left(-49 \pm \sqrt{49^2 + 4 \cdot 49 \cdot 60}\right) = \frac{1}{2}\left(-49 \pm 7\sqrt{49 + 240}\right) = \frac{7}{2}\left(-7 \pm \sqrt{289}\right)$$

$$= \frac{7}{2}(-7 \pm 17) = \mathbf{35}, \cancel{-84}$$