MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2014 SOLUTION KEY

Team Round

A) Since $f(x)$ is odd, $f(-x) = -f(x)$ and $f(x) = Ax^5 + Bx^3 + Cx$.

$f(1)=0 \Rightarrow$ (1) $A+B+C=0$ (4) $C=-(A+B)$

$f(2)=42 \Rightarrow$ (2) $32A+8B+2C=42 \Rightarrow$ (5) $16A+4B+C=21$

$f\left(\frac{1}{2}\right)=\frac{3}{16} \Rightarrow$ (3) $\frac{A}{32}+\frac{B}{8}+\frac{C}{2}=\frac{3}{16}$ (6) $A+4B+16C=6$

Substituting for C in (5), $15A+3B=21 \Rightarrow B=7-5A$.

Substituting for B in (4), $C=-(A+7-5B)=4A-7$

Substituting for B and C in (6), $A+28-20A+64A-112=6 \Rightarrow 45A=118-28=90 \Rightarrow A=2$

Thus, $(A,B,C)=(2,-3,1) \Rightarrow f(x)=2x^5-3x^3+x$

$f(x)=2x^5-3x^3+x=x(2x^4-3x^2+1)=x(2x^2-1)(x^2-1)=0 \Rightarrow x=\underline{\mathbf{0, \pm 1, \pm\frac{\sqrt{2}}{2}}}$.

B) Suppose the 5 weights were d = 54, a, b, c, e = 62.
There would be 10 possible pairings, namely ab, ac, ad, ae, bc, bd, be, cd, ce and de.
Note each individual weight occurs exactly 4 times. The total weight of the 10 pairs is 1156
Thus, the total weight is $4(a+b+c+d+e) = 1156 \Rightarrow a+b+c+d+e = 289$.
$\Rightarrow a+b+c+116 = 289 \Rightarrow a+b+c = 173$ and $a \geq 55$ and $c \leq 61$
$a = 55 \Rightarrow b+c = 118 \Rightarrow (b, c) =$ ~~(56, 62)~~, (57, 61), (58, 60)
$a = 56 \Rightarrow b+c = 117 \Rightarrow (b, c) =$ (57, 60), (58, 59)
$a = 57 \Rightarrow b+c = 116 \Rightarrow (b, c) =$ ~~(58, 58)~~
$a > 57$ produces no additional ordered pairs.
Therefore, there are 4 <u>possible</u> ordered triples (a, b, c). However, 6 of the pair-sums are odd and this eliminates all but (*54*, **55**, **57, 61**, *62*). Thus, there is only <u>**1**</u> ordered triple (a, b, c).
Alternate solution: (Lexington HS)
Since 110 is the sum of the two smallest numbers, $a=56$. Since 121 is the sum of the two largest numbers, $c=59$. Thus, $4(54+56+b+59+62)=1156 \Rightarrow b=58$ and there can be only <u>**one**</u> quintuple.

C) $y=Sin^{-1}(6x^2-5x) \Rightarrow \sin(y)=6x^2-5x \Rightarrow -1 \leq 6x^2-5x \leq 1$

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$\Leftrightarrow 6x^2-5x \geq -1$ and $6x^2-5x \leq 1 \Leftrightarrow 6x^2-5x+1 \geq 0$ and $6x^2-5x-1 \leq 0$

$\Leftrightarrow (3x-1)(2x-1) \geq 0$ and $(6x+1)(x-1) \leq 0 \Leftrightarrow \left(x \leq \frac{1}{3} \text{ or } x \geq \frac{1}{2}\right)$ and $\left(-\frac{1}{6} \leq x \leq 1\right)$

Taking the intersection of these two rays and the overlapping segment, we have

$\underline{\mathbf{-\frac{1}{6} \leq x \leq \frac{1}{3} \text{ or } \frac{1}{2} \leq x \leq 1}}$ or using interval notation, $\underline{\left[-\frac{1}{6},\frac{1}{3}\right],\left[\frac{1}{2},1\right]}$. In each case, the word "or" , a comma or the union operator "U" may be used as the connector.