MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2007 SOLUTION KEY

Round 2

A) Taking the square root of 1741 → 41.7^+
The product 40(41) = 1640 is obviously smaller then 1741, since both factors are smaller than the square root of 1741. Likewise 42(43) = 1806 is obviously larger then 1741, since both factors are larger than the square root of 1741. Thus, the product closest to 1741 is produced by the pair of integers that sandwich the square root, 41(42) = 1722. $a > b$ → (a, b) = **<u>(42, 41)</u>**.

B) n = 3, 5 and 7 produces 7, 31 and 127 respectively, all of which are primes.
$n = 9$ → 511 = 7(73)
7 + 73 = **<u>80</u>**

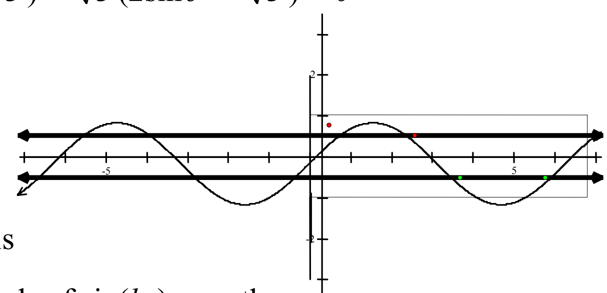
C) Let $A = \frac{x-3}{x-7}$. Then $\left(6\left(\frac{x-3}{x-7}\right)-4\right)^2 - 5\left(2-3\left(\frac{x-3}{x-7}\right)\right) = 21$ simplifies to
$(2(3A-2))^2 + 5(3A-2) - 21 = 0$
Letting $B = 3A - 2$, we have $4B^2 + 5B - 21 = (4B - 7)(B + 3) = 0$ or substituting back
$(4(3A-2) - 7)(3A - 2 + 3) = (12A - 15)(3A + 1) = 0$ → A = 5/4 or -1/3
Finally, substituting for A,
$\frac{x-3}{x-7} = \frac{5}{4}$ → $4x - 12 = 5x - 35$ → x = **<u>23</u>**

$\frac{x-3}{x-7} = \frac{-1}{3}$ → $3x - 9 = -x + 7$ → $4x = 16$ → x = **<u>4</u>**

Round 3

A) $3\cos(x) + 3 = 2(1 - \cos^2(x))$ → $2\cos^2(x) + 3\cos(x) + 1 = (2\cos x + 1)(\cos x + 1) = 0$
→ $\cos x = -1/2$ → x = **<u>120°, 240°</u>**
→ $\cos x = -1$ → x = **<u>180°</u>**

B) $2\sin\theta\tan\theta + \sqrt{3}\tan\theta - 2\sqrt{3}\sin\theta + 3 = \tan\theta(2\sin\theta + \sqrt{3}) - \sqrt{3}(2\sin\theta + \sqrt{3}) = 0$
→ $(\tan\theta - \sqrt{3})(2\sin\theta + \sqrt{3}) = 0$
→ $\tan\theta = \sqrt{3}$ → θ = **<u>60°, 240°</u>**
→ $\sin\theta = -\frac{\sqrt{3}}{2}$ → θ = 240°, **<u>300°</u>**



C) Let k = 2007. We need $\sin(kx) = \pm\frac{1}{2}$. The sine function is periodic with period 2π. This implies there will be k periods of $\sin(kx)$ over the given interval $[0, 2\pi)$ or $k/8$ periods over $[0, \pi/4)$. Both $y = +\frac{1}{2}$ and $y = -\frac{1}{2}$ cross one cycle of $\sin(kx)$ twice. Thus, there are $[k/2]$ = [2007/2] = **<u>1003</u>** points of intersection.
[2007/8 = 250.875 cycles → 4(250) + 3, the last three points of intersection occurring at the one quarter, halfway and three quarters points in the 251st cycle]