MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2006 BRIEF SOLUTIONS

Round One:

A. $f(2y^2 + 1) = 2(2y^2 + 1)^2 + 1 = 2(4y^4 + 4y^2 + 1) + 1 = 8y^4 + 8y^2 + 3$.

B. $G(F(x)) = 2(1 - 3x)^2 = 18x^2 - 12x + 2$. $F(G(x)) = 1 - 6x^2$. Substitute to get $24x^2 - 11x + 1 = x$, so $(8x - 1)(3x - 1) = 0$.

C. $h^{-1}(x) = ½\,x - ½$; $g^{-1}(x) = -½\,x + 1\,½$ so $h^{-1}(x) + g^{-1}(x) = 1$. $h(x) + g(x) = 4$. $h(x)\cdot g(x) = -4x^2 + 4x + 3$. If $1 = (-4x^2 + 4x + 3)/4$ then $4x^2 - 4x + 1 = (2x - 1)^2 = 0$.

Round Two:

A. $abcabc = abc\,(1001) = abc(11)(7)(13)$. $13 + 11 = 24$.

B. 15©21 = 16 + 18 + 20 = 54; 28©33 = 30 + 32 = 62 ;
54©62 = 55 + 56 + 57 + 58 + 60 = 286

C. $2xy - y = 4x + 1$ so $y = \dfrac{4x+1}{2x-1} = \dfrac{2(2x-1)+3}{(2x-1)}$ so $2x - 1$ is a factor of 3 (±1 or ±3).
Thus, x must be 0, ±1 or 2. This yields the ordered pairs (1, 5), (0, -1), (2, 3) and (-1, 1). The first of these is furthest from the origin.

Round Three:

A. Right triangle has opposite side $\sqrt{x}$, adjacent 1, hypotenuse $\sqrt{1+x}$

B. Common denominator gets
$$\frac{\sin^2\theta + (1+\cos\theta)^2}{2\sin\theta(1+\cos\theta)} = \frac{\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta}{2\sin\theta(1+\cos\theta)} = \frac{2(1+\cos\theta)}{2\sin\theta(1+\cos\theta)} = \frac{1}{\sin\theta}.$$

C. $\sin(2\cdot 2\theta) = 2\sin(2\theta)\cos(2\theta) = 2(2\sin\theta\cos\theta)(1-2\sin^2\theta) = 4\sin\theta\cos\theta(1+-2\sin^2\theta)$,
so $A = 4$, $B = 1$, $C = -2$.

Round Four:

A. The absolute minimum number of coins would be 16 dollars and 2 quarters, but since there must be at least 3 quarters, we have 15 dollars and 6 quarters → $K = 21$.
The teller's mistake credited my account 3(75) = 225 extra cents → $X = 1875$.

B. In one minute 12(60)+x(60)=1320 so sister jogs at 10 ft/sec. In same direction I gain 2 ft/sec or 120 ft/minute. 1320/120 = 11.

C. Original mix was $n/100$. $\dfrac{n+6}{100} = \dfrac{(n/100)10 + .30(20)}{30}$ Solving, $n = 21$.

Round Five:

A. Rt. triangle with radius as hypotenuse has legs of 20 and ½ (96), so hypotenuse is 52 [4x(5 - 12 - 13) triangle] . 20 + 52 = 72

B. $(AE)(BE)=(DE)(CE)$, so $(5x - 3)(x + 1) = (3x-1)(2x)$. Solve quadratic to get $x = 1$ or 3. Both give all positive lengths so $AE = 2$ or 12.

C. Rt ΔPOA gives $OA = 7$. If $DE = x$, $x(2x) = (7\sqrt{6})^2 = 294$, so $x = DE = 7\sqrt{3}$
Thus, $m\angle DOE = 120°$ and sector is 1/3 of the circle.