MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2009 SOLUTION KEY

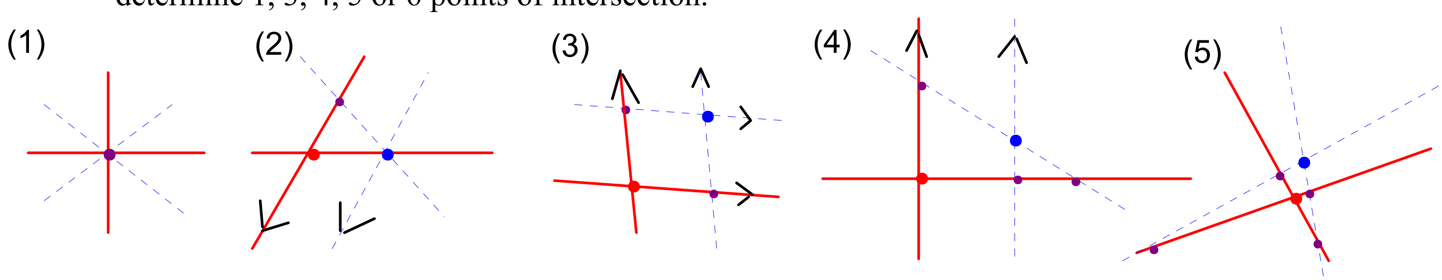
Team Round

A) Since the side opposite the 120° must be the longest side, the side opposite 120° can't be $x - 1$.
Using the law of cosines:
Case 1: $(x + 3)^2 = (x - 1)^2 + (2x - 3)^2 - 2(x - 1)(2x - 3)(-1/2)$
$\rightarrow x^2 + 6x + 9 = 5x^2 - 14x + 10 + 2x^2 - 5x + 3$
$\rightarrow 6x^2 - 25x + 4 = (6x - 1)(x - 4) = 0 \rightarrow 4$ only
Per $= (4 - 1) + (2\cdot 4 - 3) + (4 - 3) = 3 + 5 + 7 = \underline{\mathbf{15}}$

Case 2: $(2x - 3)^2 = (x - 1)^2 + (x + 3)^2 - 2(x - 1)(x + 3)(-1/2)$
$\rightarrow 4x^2 - 12x + 9 = 2x^2 + 4x + 10 + x^2 + 2x - 3$
$\rightarrow x^2 - 18x + 2 = 0 \rightarrow (x - 9)^2 = -2 + 81 = 79 \rightarrow x = 9 + \sqrt{79}$
Per $= (9+\sqrt{79}-1) + (9+\sqrt{79}+3) + \left(2(9+\sqrt{79})-3\right)$
$= (8 + 12 + 15) + (1 + 1 + 2)\sqrt{79} = \underline{\mathbf{35+4\sqrt{79}}}$ (but this is a larger perimeter)

B) The numbers in the n^{th} row are: $2n + 7$, m and $(m + 1)$
$n = 11 \rightarrow 29^2 + m^2 = (m + 1)^2 \rightarrow 841 = 2m + 1 \rightarrow m = 420$
We must evaluate $29^2 + 420^2 + 421^2$, but we know from the Pythagorean Theorem that the sum of the first two terms equals the third.
Thus, the required sum is $2(421)^2 = 2(177241) = \underline{\mathbf{354482}}$

C) Each graph consists of two intersecting lines. In general, two pair of intersecting lines can determine 1, 3, 4, 5 or 6 points of intersection.



$(x - y + 2)(3x + y - 4) = 0 \rightarrow$ lines with slopes of +1 and −3 and intersecting at $P\left(\frac{1}{2},\frac{5}{2}\right)$

$(x + y - 2)(2x - 5y + 7) = 0 \rightarrow$ lines with slopes of −1 and 2/5 and intersecting at $Q\left(\frac{3}{7},\frac{11}{7}\right)$

Thus, since none of the given lines are parallel and at least 2 points were determined, the maximum number of points were determined, i.e. 6 - diagram (5) above. However, each graph consists of a pair of intersecting lines and the intersection of the graphs consists of only points determined by a red line and a blue line, $6 - 2 = \underline{\mathbf{4}}$.
An aside:
Call the lines A, B, C and D. Two lines determine at most one point of intersection.
Thus, four lines determine at most ${}_4C_2 = 6$ possible points of intersection. Clearly, four parallel lines would produce no points of intersection. The diagrams above indicate the five other possible cases. Try as you may, exactly two points of intersection is impossible.