MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2017 SOLUTION KEY

Team Round - continued

F) Since $4000 = 4(1000) = 2^2 \cdot 10^3 = 2^5 \cdot 5^3$, factors which are not multiples of 5 are 1, 2, 4, 8, 16, and 32. Factors which are multiples of 5 are products of 5, 25, and 125 with any of these 6 numbers. Thus, there are 6·3 = 18 possible factors which are multiples of 5.
Clearly, no prime numbers other than 2 and 5 divide evenly into 4000. Thus, a factor of 4000 must be of the form $2^a \cdot 5^b \Rightarrow a = \mathbf{0},1,2,3,4,5$ and $b = \mathbf{0},1,2,3$, resulting in $6 \cdot 4 = 24$ total factors and $k = \dfrac{18}{24} = 75\%$.
In general, to find the number of factors of any integer:

- determine the prime factorization of the integer
- add 1 to each of the exponents
- multiply

A and *B* must be of the form $5^3 \cdot n$, where *n*, like 2^5, has 6 factors (none of which are divisible by 5).
[$\dfrac{1}{2}$ of the numbers of the form $5^1 \cdot n$, where *n* is not divisible by 5, will be divisible by 5.
$\dfrac{2}{3}$ of the numbers of the form $5^2 \cdot n$, where *n* is not divisible by 5, will be divisible by 5.
$\dfrac{3}{4}$ of the numbers of the form $5^3 \cdot n$, where *n* is not divisible by 5, will be divisible by 5 - Bingo!]
Consider *n*-values less than 32 with 6 factors (none of which are multiples of 5):
$\cancel{31}(2), \cancel{29}(2), \mathbf{28} = 2^2 \cdot 7\ (6)$ - Bingo!
Consider *n*-values greater than 32 with 6 factors (none of which are multiples of 5):
$\cancel{33}(4), \cancel{34}(4), \cancel{36}(9), \cancel{37}(2), \cancel{38}(4), \cancel{39}(4), \cancel{41}(2), \cancel{42}(8), \cancel{43}(2), \mathbf{44} = 2^2 11\ (6)$ - Bingo!
Thus, the minimum value of *B* – *A* is 125(44 –28) = 125(16) = **<u>2000</u>**.

Note: 75% of the factors of $A = 31 \cdot 5^3$ are multiples of 5 (6 out of 8, i.e., all factors except 1 and 31), but *A* does not have the same number of factors as 4000, namely 24.
Likewise, 75% of the factors of $B = 33 \cdot 5^3 = 3 \cdot 11 \cdot 5^3$ are multiples of 5 (12 out of 16, i.e. all factors except 1, 3, 11, and 33), but *B* fails to have the same total number of factors as 4000.