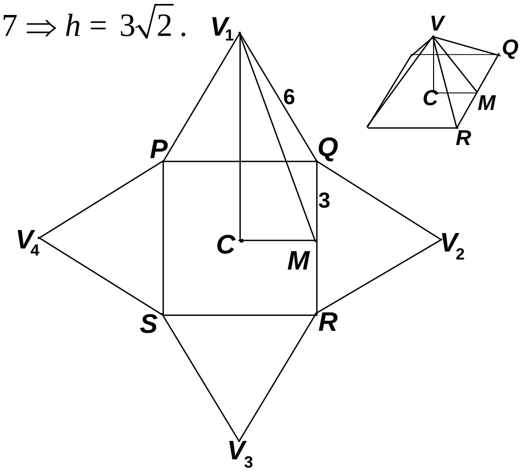
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2011 SOLUTION KEY

Round 1

A) Let s denote the length of a side of the cube. Then the height of the cylinder is also s and the radius of the base is $\frac{s}{2}$. The required ratio is $\dfrac{\pi\left(\frac{s}{2}\right)^2\cdot s}{s^3} = \underline{\boldsymbol{\frac{\pi}{4}}}$.

B) If h denotes the height of the pyramid, the volume of the pyramid is $\frac{1}{3}Bh$ or $12h$.

Let V denote the vertex of the pyramid, C the center of the square and M the midpoint of a side of the square.
Consider right ΔVCM, with hypotenuse VM. $VC = h$, $CM = 3$ and VM is an altitude of the equilateral triangle VQR.
$VM^2 + 3^2 = 6^2 \Rightarrow VM^2 = 27$ and, therefore, $h^2 + 9 = 27 \Rightarrow h = 3\sqrt{2}$.
Thus, the required volume is $12\left(3\sqrt{2}\right) = \underline{\mathbf{36\sqrt{2}}}$.



C) $\begin{cases} d = 4\sqrt{10} \\ l = \sqrt{3}w \\ h = w-2 \\ d^2 = l^2 + w^2 + h^2 \end{cases} \Rightarrow \left(4\sqrt{10}\right)^2 = 160 = w^2 + (w-2)^2 + \left(w\sqrt{3}\right)^2$

$160 = 5w^2 - 4w + 4 \Rightarrow 5w^2 - 4w - 156 = (5w+26)(w-6) = 0$
$\Rightarrow w = 6$
$\Rightarrow (l, w, h) = \left(6\sqrt{3}, 6, 4\right) \Rightarrow$ V = $\underline{\mathbf{144\sqrt{3}}}$.