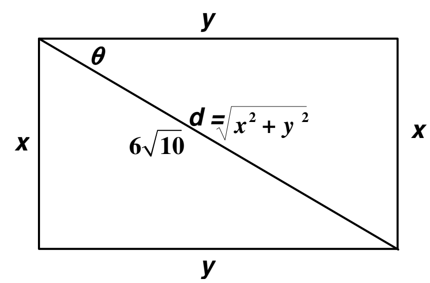
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2009 SOLUTION KEY

Team Round

E) Let x and y denote the lengths of the short and long sides, respectively. Then:

$$\begin{cases} \dfrac{x}{y} = \dfrac{y}{x+10\sqrt{6}} \rightarrow y^2 - x^2 = 10x\sqrt{6} \\ x^2 + y^2 = \left(10\sqrt{6}\right)^2 = 600 \end{cases}$$



Subtracting, $2x^2 = 600 - 10x\sqrt{6}$

→ $2x^2 + 10x\sqrt{6} - 600 = 0$ → $x^2 + 5x\sqrt{6} - 300 = 0$

→ $x = \dfrac{-5\sqrt{6} \pm \sqrt{150+1200}}{2} = \dfrac{-5\sqrt{6} \pm \sqrt{25(54)}}{2} = \dfrac{-5\sqrt{6} \pm 15\sqrt{6}}{2} = 5\sqrt{6}$

→ $y^2 = 600 - 150 = 450 = 25(18)$ → $y = 15\sqrt{2}$ → Area = $75\sqrt{12}$ = $\underline{\mathbf{150\sqrt{3}}}$

<u>More general solutions</u>

#1 Let x and y denote the lengths of the short and long sides, respectively.

The definition gives the equation: $\dfrac{x}{y} = \dfrac{y}{x+\sqrt{x^2+y^2}} = \dfrac{1}{\dfrac{x}{y} + \sqrt{\left(\dfrac{x}{y}\right)^2 + 1}}$ Let $s = \dfrac{x}{y}$. Then:

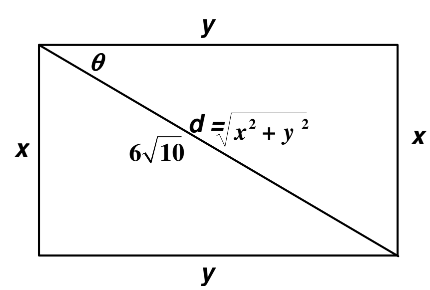
(***) $s = \dfrac{1}{s+\sqrt{s^2+1}}$ → $s^2 + s\sqrt{s^2+1} = 1$ → $s\sqrt{s^2+1} = 1 - s^2$

Squaring both sides, $s^2\left(s^2+1\right) = \left(1-s^2\right)^2$ → $s^4 + s^2 = 1 - 2s^2 + s^4$

→ $3s^2 = 1$ → s = $\dfrac{1}{\sqrt{3}} = \dfrac{x}{y}$ → $y^2 = 3x^2$.

Thus, the sides of the semi-golden rectangle form 30-60-90 right triangles. Since $d = 10\sqrt{6}$, $100(6) = x^2 + 3x^2$ → $x^2 = 150$ → $x = 5\sqrt{6}$ and $y = 15\sqrt{2}$ and, therefore, the area is $\underline{\mathbf{150\sqrt{3}}}$

#2 Let $s = \tan\theta = \dfrac{x}{y}$.



Since $1 + \tan^2\theta = \sec^2\theta$, this trig substitution greatly simplifies equation (***) above.

$\tan\theta = \dfrac{1}{\tan\theta + \sec\theta}$ → $\dfrac{\sin\theta}{\cos\theta}\left(\dfrac{\sin\theta+1}{\cos\theta}\right) = 1$

→ $\sin^2\theta + \sin\theta = \cos^2\theta = 1 - \sin^2\theta$

→ $2\sin^2\theta - \sin\theta - 1 = (2\sin - 1)(\sin\theta + 1) = 0$ → $\sin\theta = \dfrac{1}{2}, -1$ (extraneous) → θ = 30° and the rest of the argument proceeds as above.