MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016 SOLUTION KEY

Team Round - continued

B) $n^3-3n^2+3n=k^3+3k^2+3k-17 \Leftrightarrow \left(n^3-3n^2+3n-1\right)=\left(k^3+3k^2+3k+1\right)-19$

$\Leftrightarrow (n-1)^3-(k+1)^3=-19$

As the difference of perfect squares, the left hand side of this equation factors as

$(n-k-2)\left[(n-1)^2+(n-1)(k+1)+(k+1)^2\right]=-19$

Since n and k are integers and 19 is prime, we have two possibilities,

(1) $\begin{cases} n-k-2=-19 \\ (n-1)^2+(k+1)^2+(n-1)(k+1)=1 \end{cases}$ or (2) $\begin{cases} n-k-2=-1 \\ (n-1)^2+(k+1)^2+(n-1)(k+1)=19 \end{cases}$

Notice we are tacitly assuming that the second equation must be equal to a positive number. The reasoning will be given at the end.

(1) $\Rightarrow n=-17+k$

Substituting, we have $(k-18)^2+(k+1)^2+(k-18)(k+1)=1 \Leftrightarrow 3k^2-51k+18(17)+1=0$

which has <u>no</u> rational solutions, since the discriminant is negative (-1083).

(2) $\Rightarrow n=k+1$

Substituting, we have

$k^2+(k+1)^2+k(k+1)=19 \Leftrightarrow 3k^2+3k-18=3\left(k^2+k-6\right)=3(k-2)(k+3)=0 \Rightarrow k=2,\,-3$

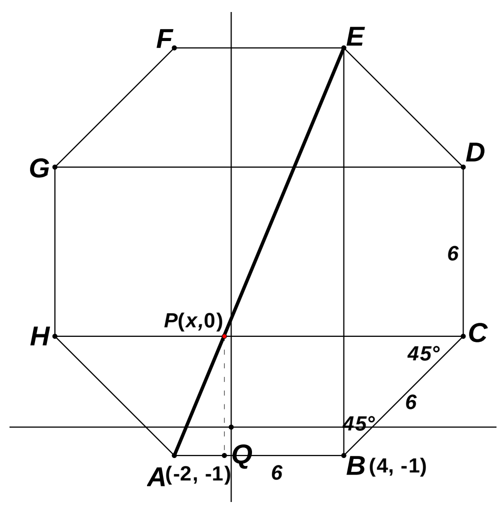
$\Rightarrow (n, k)=$ <u>**(3, 2) or (–2, –3)**</u>.

Clearly, an expression of the form A^2+AB+B^2 is positive if A and B are either both positive or both negative, but what about when they have opposite signs?

Completing the square, $A^2+AB+B^2=A^2+2AB+B^2-AB=(A+B)^2-AB$.

If A and B have opposite signs, then we are subtracting a negative and again the expression is positive.

C) Solution #1: $\Delta AQP \sim \Delta ABE$



$$\frac{AQ}{AB}=\frac{PQ}{EB} \Leftrightarrow \frac{x+2}{6}=\frac{1}{6+2\left(3\sqrt{2}\right)}$$

$$\Rightarrow x+2=\frac{6}{6+6\sqrt{2}}=\frac{1}{1+\sqrt{2}}\cdot\frac{1-\sqrt{2}}{1-\sqrt{2}}=\sqrt{2}-1$$

$$\Rightarrow x=\sqrt{2}-3$$

Thus, the x-intercept of $\overline{AE}$ is at $\underline{\left(\mathbf{-3+\sqrt{2},0}\right)}$.

Solution #2:

The equation of $\overleftrightarrow{AE}$ is

$(y+1)=\dfrac{6+6\sqrt{2}}{6}(x+2)=\left(\sqrt{2}+1\right)(x+2)$. Substituting $y=0$ into $(y+1)=\left(\sqrt{2}+1\right)(x+2)$ gives the same result.