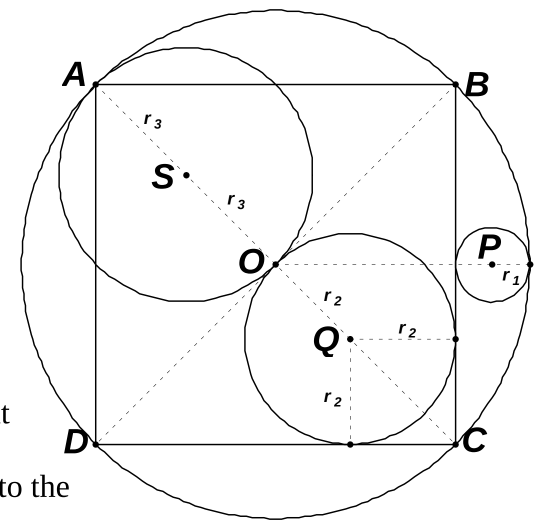
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012 SOLUTION KEY

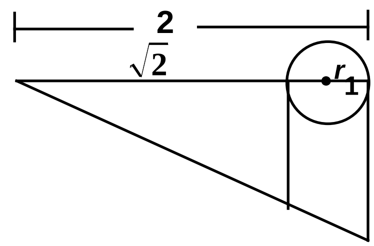
Team Round - continued

C) Let $R = OB = 2$ and $AS = r_3 = 1$.
Then $r_1 + r_2 + r_3 = kR \Leftrightarrow r_1 + r_2 + 1 = 2k$.



Since $m\angle OBC = 45°$ and the side of square $ABCD$ is $2\sqrt{2}$,

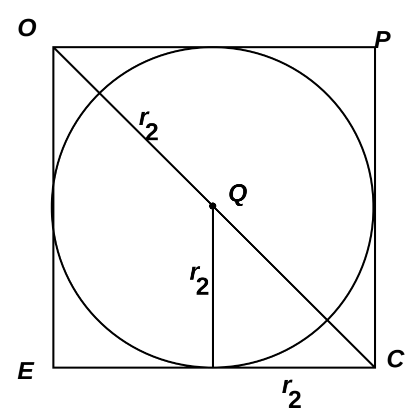
$r_1 = \dfrac{2-\sqrt{2}}{2} = 1 - \dfrac{\sqrt{2}}{2}$



Finding r_2 is the hardest.
Recall that the incenter of a triangle (as the intersection point of the angle bisectors) is equidistant from the three vertices.
The radius of the inscribed circle (center at Q) is equivalent to the area of the triangle (ΔBCD) divided by its semi-perimeter.

Since the area of the square is 8, we have $r_2 = \dfrac{4}{\frac{\left(4\sqrt{2}+4\right)}{2}} = \dfrac{2}{\sqrt{2}+1} \cdot \dfrac{\sqrt{2}-1}{\sqrt{2}-1} = 2\left(\sqrt{2}-1\right)$

Thus, $2k = \left(1 - \dfrac{\sqrt{2}}{2}\right) + 2\left(\sqrt{2}-1\right) + 1 \Rightarrow k = \underline{\mathbf{\dfrac{3\sqrt{2}}{4}}}$.



If you were unfamiliar with the relationship of the radius of the inscribed circle in a triangle and the area/perimeter of the triangle, consider the cutout diagram of the lower right corner of the overall diagram.
$OC = 2 \Rightarrow QC = r_2\sqrt{2} \Rightarrow r_2 + r_2\sqrt{2} = 2$

Solving, $r_2 = \dfrac{2}{\sqrt{2}+1} \cdot \dfrac{\sqrt{2}-1}{\sqrt{2}-1} = 2\left(\sqrt{2}-1\right)$ and the result follows.