MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2015 SOLUTION KEY

Team Round

A) $(1+i)^2 = 2i \Rightarrow (1+i)^{11} = (2i)^5(1+i) = 32\cancel{i^4}i(1+i) = -32+32i$

Thus, we require that $(1-i)^n = -(-32+32i)+(-16+16i) = 16-16i$.

$(1-i)^2 = -2i \Rightarrow (1-i)^8 = (-2i)^4 = 16i^4 = 16$

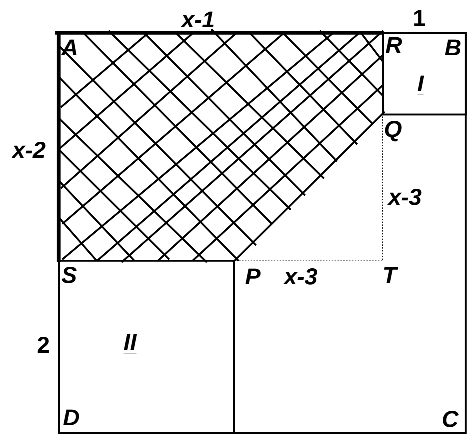
Therefore, $(1-i)^9 = 16(1-i) = 16-16i \Rightarrow \ n = \underline{\mathbf{9}}$.

Alternate Solution:

Using polar (or cis form), $(1-i)^n = 16-16i \Leftrightarrow \left(\sqrt{2}, -45°\right)^n = \left(16\sqrt{2}, -45°\right)$

$\Rightarrow\ 2^{n/2} = 2^{4.5} \Rightarrow n = \underline{\mathbf{9}}$.

B) Suppose $L = abcxyz$ is a lucky lottery ticket.
Consider the companion lottery ticket M with the number $(9-a)(9-b)(9-c)xyz$.
Since $a+b+c = x+y+z$, we have
$(9-a)+(9-b)+(9-c)+x+y+z = 27-(a+b+c)+(x+y+z) = 27$
Thus, each companion lottery ticket's digits total 27.
For each ticket L, there is exactly one ticket M and vice versa.
Because of this one-to-one correspondence, we see there are as many lucky lottery tickets as there are these companion lottery tickets. Amazingly, $N(A) = N(B)$ and (3) is true.
If a lottery ticket is lucky then $a+b+c = x+y+z$ and, regardless of whether these sums are even or odd, the sum of the 6 digits will be even; hence, never equal to one of the companion lottery tickets.
Since (4) is true, (5) must be false. Thus, (**<u>3) and (4)</u>** are true.



C) Let x be the side of square $ABCD$.
D, P, Q and B are collinear, so
$BD = 2\sqrt{2} + PQ + \sqrt{2} = x\sqrt{2}$
$\Rightarrow PQ = (x-3)\sqrt{2}$
Area(I) + Area(II) = Area(Square on $\overline{PQ}$) $\Rightarrow$

$PQ^2 = 1^2 + 2^2 = 5$ and we have $\sqrt{5} = (x-3)\sqrt{2} \Rightarrow x = 3 + \dfrac{\sqrt{10}}{2}$.

Thus, area of the shaded region is equal to the area of rectangle $ARTS$ minus the area of triangle PQT.

$$\left(1+\frac{\sqrt{10}}{2}\right)\left(2+\frac{\sqrt{10}}{2}\right) - \frac{1}{2}\cdot\left(\frac{\sqrt{10}}{2}\right)^2 = 2 + \frac{3}{2}\sqrt{10} + \frac{5}{2} - \frac{5}{4} = \underline{\mathbf{\frac{13+6\sqrt{10}}{4}}}$$