MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2016 SOLUTION KEY

Round 3

A) $2\sin^2 2x+\sin 2x-1=0 \Leftrightarrow (2\sin 2x-1)(\sin 2x+1)=0 \Rightarrow \sin 2x=\frac{1}{2},-1$

$$2x=\begin{cases}30^\circ\\150^\circ\\270^\circ\end{cases}+n(360^\circ)\Rightarrow x=\begin{cases}15^\circ\\75^\circ\\135^\circ\end{cases}+n(180^\circ).\ n=0\Rightarrow x=15,75,135\Rightarrow (k,T)=\underline{\mathbf{(3,225)}}.$$

B) Using the conversion equations $\begin{cases}x=r\cos\theta\\y=r\sin\theta\\x^2+y^2=r^2\end{cases}$, $r=3\cos\theta \Rightarrow \sqrt{x^2+y^2}=\dfrac{3x}{\sqrt{x^2+y^2}}$

Cross multiplying, $x^2+y^2=3x \Rightarrow x^2-3x+9x^2=0 \Rightarrow 10x^2-3x=x(10x-3)=0$

$\Rightarrow x=0,\frac{3}{10} \Rightarrow (x,y)=\underline{\mathbf{(0,0),\left(\frac{3}{10},\frac{9}{10}\right)}}$.

C) Using reduction formulas and the fact that cosine is an even function,

$\cos 140^\circ-\sin 230^\circ+\cos 100^\circ=2\cos(-300^\circ)(\cos x^\circ)$

$\Leftrightarrow\ -\cos 40^\circ+\sin 50^\circ-\cos 80^\circ=2\cos(300^\circ)(\cos x^\circ)=2\cos 60^\circ\cos x^\circ=\cos x^\circ$

$\Leftrightarrow\ \cos x^\circ=-(\cos 80^\circ+\cos 40^\circ)+\sin 50^\circ$

Using the Sum and Difference Formulas, this simplifies to

$$\cos x^\circ=-\left(2\cos\frac{80^\circ+40^\circ}{2}\cdot\cos\frac{80^\circ-40^\circ}{2}\right)+\sin 50^\circ=-2\cos 60^\circ\cos 20^\circ+\sin 50^\circ=\sin 50-\cos 20^\circ$$

$\Leftrightarrow$

$$\cos 40^\circ-\cos 20^\circ\Leftrightarrow -2\sin\frac{40^\circ+20^\circ}{2}\sin\frac{40^\circ-20^\circ}{2}=-2\sin 30^\circ\sin 10^\circ=-\sin 10^\circ=-\cos 80^\circ.$$

Since $-\cos 80^\circ$ denotes a negative number, we require the related members of the 80° family in quadrants II and III, where the cosine is negative, namely $180^\circ\pm 80^\circ=\underline{\mathbf{100^\circ,\ 260^\circ}}$.