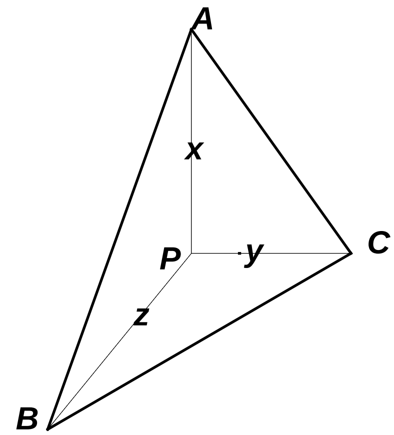
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013 SOLUTION KEY



Team Round

A) $x^2+y^2=224+65=289 \Rightarrow AC=17$

$x^2+z^2=224+560=784 \Rightarrow AB=28$

$y^2+z^2=65+560=625 \Rightarrow BC=25$

Using Heron's formula, the area of ΔABC can be computed.

The semi-perimeter s is $\dfrac{28+25+17}{2}=35$. Thus, the area is given by

$\sqrt{35(35-17)(35-25)(35-28)}=\sqrt{35\cdot 18\cdot 10\cdot 7}=\sqrt{5^2\cdot 6^2\cdot 7^2}=210$.

The volume of any pyramid (and this tetrahedron is a pyramid with a triangular base) is given by $\dfrac{1}{3}Bh$, where B denotes the base and h the height.

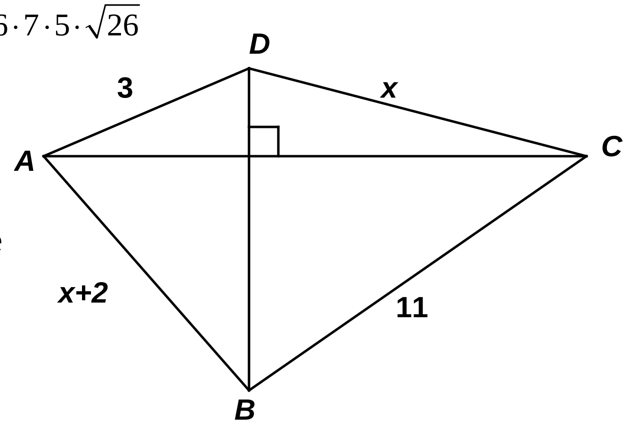
Using PBC as the base, the volume is $\dfrac{1}{3}\cdot x\cdot\left(\dfrac{1}{2}yz\right)=\dfrac{xyz}{6}$

$\sqrt{224}=\sqrt{16\cdot 14}=4\sqrt{14}$, $\sqrt{560}=\sqrt{16\cdot 35}=4\sqrt{35}$

Using ABC as the base, we have

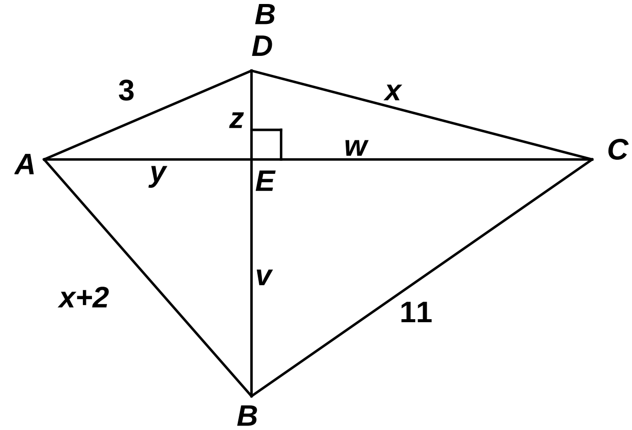
$\dfrac{1}{3}\cdot h\cdot 210=\dfrac{xyz}{6}\Rightarrow 420h=xyz=4\sqrt{14}\cdot\sqrt{65}\cdot 4\sqrt{35}=16\cdot 7\cdot 5\cdot\sqrt{26}$

$\Rightarrow h=\dfrac{560}{420}\sqrt{26}=\dfrac{4}{3}\sqrt{26}$ $\Rightarrow$ (A, B, C) = **(4, 3, 26)**



B) As noted in a previous contest, since the diagonals of $ABCD$ are perpendicular, the sum of the squares of the lengths of the opposite sides are equal.
See the proof of this fact in the notes included with the solution key for 2009 Round 2 questions.

$3^2 + 11^2 = x^2 + (x + 2)^2$
$\Rightarrow 130 = 2x^2 + 4x + 4$
$\Rightarrow x^2 + 2x - 63 = (x + 9)(x - 7) = 0 \Rightarrow x = 7$
Thus, the perimeter of $ABCD$ is 30.



Alternate Solution:

In ΔADE, (1) $y^2+z^2=9$.

In ΔDCE, (2) $z^2+w^2=x^2$

In ΔBCE, (3) $v^2+w^2=121$

In ΔBAE, (4) $v^2+y^2=x^2+4x+4$

To eliminate z^2, subtract (1) – (2): $y^2-w^2=9-x^2$

Add to (3): $y^2+v^2=130-x^2$

Substitute in (4): $130-x^2=x^2+4x+4$

$\Rightarrow 2x^2+4x-126=0\Rightarrow x^2+2x-63=(x+9)(x-7)=0 \Rightarrow x=7 \Rightarrow$ Per = 3 + 11 + 7 + 9 = **30**.