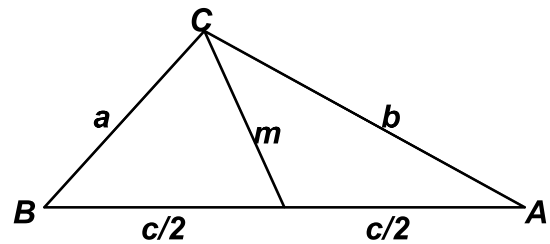
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2009 SOLUTION KEY

Team Round

F) Using Stewart's Theorem, $a^2\frac{c}{2}+b^2\frac{c}{2}=m^2c+c\cdot\frac{c}{2}\cdot\frac{c}{2}$



$c \neq 0 \rightarrow \frac{a^2+b^2}{2}=m^2+\frac{c^2}{4} \rightarrow 2(a^2+b^2)-c^2=4m^2$

$\rightarrow m=\frac{1}{2}\sqrt{2(a^2+b^2)-c^2}$

<u>Case 1</u>: $c = 8$ and $a + b = 16$

$m=\frac{1}{2}\sqrt{2(a^2+(16-a)^2)-8^2} = \frac{1}{2}\sqrt{4a^2-64a+448} = \sqrt{a^2-16a+112} = \sqrt{(a-8)^2+48}$

Since $c = 8$, $a + b = 16$, $a < b$, the triangle inequality implies we must only try $a = 5\ldots8$.

$5 \rightarrow \sqrt{57}$, $6 \rightarrow \sqrt{52}$, $8 \rightarrow \sqrt{48}$, $a = 7 \rightarrow m = \underline{\mathbf{7}}$ (and $b = 9$)

(Check: $7^2\frac{8}{2}+9^2\frac{8}{2}=7^2 8+8\cdot\frac{8}{2}\cdot\frac{8}{2}$ = 520)

<u>Case 2</u>: $c = 10$ and $a + b = 14$

$m=\frac{1}{2}\sqrt{2(a^2+(14-a)^2)-10^2} = \frac{1}{2}\sqrt{4a^2-56a+292} = \sqrt{a^2-14a+73} = \sqrt{(a-7)^2+24}$

We must try $a = 1 \ldots 7$.

$a = 1 \rightarrow \sqrt{60}$, $3 \rightarrow \sqrt{40}$, $4 \rightarrow \sqrt{33}$, $5 \rightarrow \sqrt{28}$, $7 \rightarrow \sqrt{24}$

$a = 2$, $m = 7$, $b = 12$, $c = 10$ (rejected – Triangle Inequality fails)

$a = 6$, $m = \underline{\mathbf{5}}$, $b = 8$, $c = 10$

(Check: $6^2\frac{10}{2}+8^2\frac{10}{2}=5^2 10+10\cdot\frac{10}{2}\cdot\frac{10}{2}$ = 500)

The following code snippet found two other triangles with a smaller perimeter.

```
FOR a = 1 TO 9
  FOR b = a TO 10
    FOR c = 1 TO 10

      IF ItsaTriangle(a, b, c) THEN
        m = 2 * (a ^ 2 + b ^ 2) - c ^ 2
        IF m > 0 THEN m = .5 * SQR(m)

        IF m = INT(m) THEN
          IF ItsaTriangle(a, m, c / 2) AND ItsaTriangle(m, b, c / 2) THEN
              PRINT a; b; c; "M = "; m

FUNCTION ItsaTriangle% (p, q, r)
  ItsaTriangle% = ((p + q > r) AND (p + r > q) AND (q + r > p))
```

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→       5       5       6       M = 4
        5       5       8       M = 3
        6       8      10       M = 5
        7       9       8       M = 7
```