MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006 SOLUTION KEY

Team Round

A) Rotating around the vertical axis, a single cone(call it #1) with height AD is produced by ΔADC. Rotating around the horizontal axis, two cones sharing a common base and heights BD and DC (call them #2 and #3 respectively) are produced by Δs ABD and ADC.

Let $DC = AD = x$. $DB = 10 - x = \dfrac{x}{\sqrt{3}}$ → $x = \dfrac{10\sqrt{3}}{\sqrt{3}+1} = 15 - 5\sqrt{3} = 5(3 - \sqrt{3})$

Using V(cone) = $\frac{1}{3}\pi r^2 h$, V(cone #1) = $\frac{1}{3}\pi (DC)^2(AD) = \frac{1}{3}\pi x^2 \cdot x = \frac{\pi x^3}{3}$,

V(cone #2) = $\frac{1}{3}\pi (AD)^2(BD) = \frac{1}{3}\pi x^2(\frac{x}{\sqrt{3}}) = \frac{\sqrt{3}}{9}\pi x^3$

V(cone #3) = $\frac{1}{3}\pi (AD)^2(DC) = \frac{1}{3}\pi x^2 \cdot x = \frac{\pi x^3}{3}$

Thus, the difference is $\frac{\sqrt{3}}{9}\pi x^3 = \frac{125\pi\sqrt{3}}{9}(3-\sqrt{3})^3 = \frac{125\pi\sqrt{3}}{9}(54-30\sqrt{3}) = \underline{\mathbf{250\pi(3\sqrt{3}-5)}}$

B) $1^2 + 2^2 = 5$, $5 + 3^2 = 14$, $14 + 4^2 = 30$, $30 + 5^2 = 55$
The difference between successive terms is the square of the next integer.
→ 91, 140, 204, 285, 385, 506, 650, 819, 1015, 1240, 1496, 1785, 2109, 2470, 2870, 3311, 3795, 4324, and finally, $4324 + 24^2 = 4900 = 70^2$ → **<u>70</u>**

C) Let (a, b, c) denote the number of 2-, 3- and 4-legged Venutians respectively.
Then $2a + 3b + 4c = 68$, $a + b + c = 26$, $c \geq 3b$ and $a, b, c \geq 1$. The first two equations give us $b + 2c = 16$ or $c = \dfrac{16-b}{2}$. Substituting in the inequality, we have $\dfrac{16-b}{2} \geq 3b$, hence $16 \geq 7b$.
Thus, $b = 1$ or 2 and because $2c = 16 - b$, it follows that b must be even, so $b = 2$.
This implies $c = 7$ and, therefore, $a = 26 - (2 + 7) = \underline{\mathbf{17}}$.