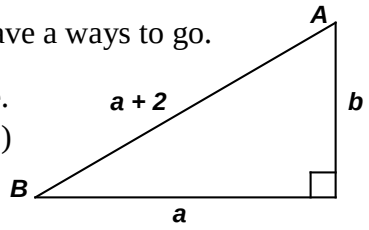
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012 SOLUTION KEY

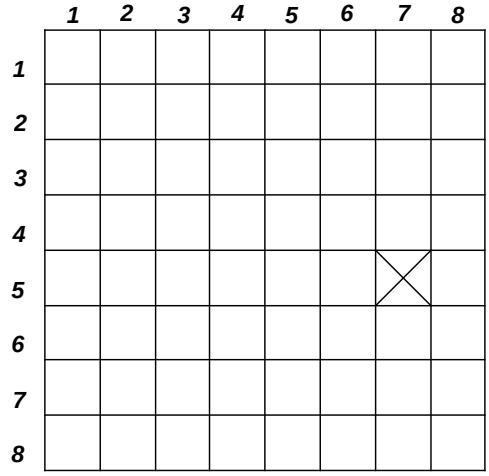
Team Round

b	*a*	*c*
6	8	10
8	15	17
10	24	26
12	35	37
14	48	50

A) Since $\cos A = \dfrac{b}{a+2}$, we could simply examine Pythagorean Triples, where the long leg and hypotenuse differ by 2. However, looking for a pattern could take a while. Let's take a different tact. Applying the Pythagorean Theorem,
$a^2 + b^2 = (a+2)^2 \Rightarrow b^2 = 4(a+1) \Rightarrow b = 2\sqrt{a+1}$
Since *b* must be an integer, *a* + 1 must be a perfect square
So, we only need consider *a*-values like 3, 8, 15, 24, 35, …. and the *b*-values are easy to compute, as are the hypotenuses (simply add 2 to *a*).



$a + 1 = 6^2 = 36 \Rightarrow\ a = 35 \Rightarrow$ 35-12-37 $\Rightarrow \cos A = \dfrac{12}{37} > \dfrac{12}{120} = 0.1$, so we have a ways to go.
$16^2 = 256 \Rightarrow a = 255 \Rightarrow b = 32$, $a + 2 = 257 \Rightarrow \cos A > 0.1$, so we continue.
$18^2 = 324 \Rightarrow a = 323 \Rightarrow b = 36$, $a + 2 = 325 \Rightarrow \cos A > 0.1$ (getting close!)
$19^2 = 361 \Rightarrow a = 360$ $b = 38$, $a + 2 = 362 \Rightarrow \cos A > 0.1$ (closer).
$20^2 = 400 \Rightarrow a = 399$ $b = 40$, $a + 2 = 401 \Rightarrow \cos A < 0.1$ (Bingo!).
Thus, the minimum perimeter is **<u>840</u>**.



B) Counting the squares of all possible sizes:
1 x 1: 64
A 2 x 2 square can have its upper left cell in any column, except column 8, and in any row, except row 8. Thus, there are 7 x 7 = 49 squares of side 2.
3 x 3: 36 4 x 4: 25 5 x 5: 16
6 x 6: 9 7 x 7: 4 8 x 8: 1
Counting those that do <u>not</u> contain the "X":
1 x 1 squares $\Rightarrow$ 63
Of the 49 2 x 2 squares only <u>4</u> contain the "X" $\Rightarrow$ 45
The "X" would have to be in the UL, UR, LL or LR cell of the 2 x 2 square and all these squares fit on the grid.
Of the 36 3 x 3 squares only 6 contain the "X" $\Rightarrow$ 30
The "X" could only be in the center or rightmost columns of a 3 x 3 square.
Of the 25 4 x 4 squares only 8 contain the "X" $\Rightarrow$ 17
The "X" could only be in the rightmost two columns of a 4 x 4 square.
Of the 16 5 x 5 squares only 8 contain the "X" $\Rightarrow$ 8
Of the 9 6 x 6 squares 6 contain the "X" $\Rightarrow$ 3
All the 7 x 7 and 8 x 8 squares contain the "X".
The total "X"-less squares is then **<u>166</u>**.