MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013 SOLUTION KEY

Round 2

A) For what numbers is the cube of the number equal to the number? $\Rightarrow x - 3 = -1, 0, 1 \Rightarrow$ **<u>2, 3, 4</u>**

or cube both sides and factor, $(x-3) = (x-3)^3 \Rightarrow (x-3)\left((x-3)^2 - 1\right) = 0$

$\Rightarrow (x-3)\left(x^2 - 6x + 8\right) = (x-3)(x-2)(x-4) = 0 \Rightarrow x =$ **<u>2, 3, 4</u>**

B) Rather than expanding the sum, let's take out the common factor of $(x + 2)$.

$3\left(x^2 - 4\right) + x^2(x+2) = (x+2)\left(3(x-2) + x^2\right) = (x+2)\left(x^2 + 3x - 6\right) = 0$,

implying $x =$ **<u>–2</u>** and factoring the quadratic trinomial, $x = \dfrac{-3 \pm \sqrt{9+24}}{2} = \underline{\boldsymbol{\dfrac{-3 \pm \sqrt{33}}{2}}}$.

C) Suppose $(x + 2y + 2)(x - 2y + 3)$ factors as $(x + Ay + B)(x - Ay + C)$

The coefficients of the y-term must be the same, but opposite in sign, since there is no xy-term.

Multiplying out, we have $\begin{cases} A^2 = 4 \\ B + C = 5 \\ AC - AB = A(B - C) = 2 \\ BC = 6 \end{cases}$, implying $A = 2$ or $A = -2$.

$A = 2 \Rightarrow (B, C) = (2, 3), \quad A = -2 \Rightarrow (B, C) = (3, 2)$

In either case, we have $\underline{\boldsymbol{(x + 2y + 2)(x - 2y + 3)}}$**.**

An alternate solution uses completing the square. Show that

$x^2 - 4y^2 + 5x + 2y + 6 = \left(x + \dfrac{5}{2}\right)^2 - \left(2y - \dfrac{1}{2}\right)^2$ and the same result follows.