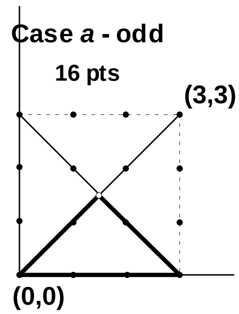
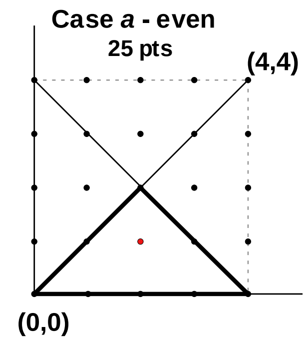
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2014 SOLUTION KEY

D) If a is odd, the diagonals do not intersect at a lattice point; for a even, they do.
Continuing to draw pics would quickly become tedious. Let's generalize.





Case even (a = 4):
There are $(a + 1)^2$ lattice points, but we must exclude:

points on the boundary: $4(a + 1) - 4 = 4a$
(4 · points on each edge – 4 corners which have been counted twice)

points on the diagonals (not already counted): $2\left(\left(a+1\right)-2\right)-1=2a-3$

(minus center point which is on both diagonals and has been counted twice)

Simplifying, $\left(a+1\right)^2-4a-(2a-3)=a^2-4a+4=(a-2)^2$

Since each of the 4 regions has the same number of lattice points, we have $\boxed{\dfrac{(a-2)^2}{4}}$ (a even)

Case odd (a = 3):
There are $(a + 1)^2$ lattice points, but we must exclude:
points on the boundary: $4a$
points on the diagonals: $2(a + 1) - 4 = 2(a - 1)$

Simplifying, $\left(a+1\right)^2-4a-2(a-1)=a^2-4a+3=(a-1)(a-3) \Rightarrow \boxed{\dfrac{(a-1)(a-3)}{4}}$ (a odd)

n odd$\Rightarrow$ use $(n + 1)$ for a in the even formula:

$$\frac{\left((n+1)-2\right)^2}{4}-\frac{(n-1)(n-3)}{4}=5 \Leftrightarrow \left(n^2-2n+1\right)-\left(n^2-4n+3\right)=20 \Rightarrow 2n=22 \Rightarrow n=\underline{\mathbf{11}}$$

n even $\Rightarrow$ use $(n + 1)$ for a in the odd formula:

$$\frac{\left((n+1)-1\right)\left((n+1)-3\right)}{4}-\frac{(n-2)^2}{4}=5 \Leftrightarrow \left(n^2-2n\right)-\left(n^2-4n+4\right)=20 \Rightarrow 2n=24 \Rightarrow n=\underline{\mathbf{12}}$$

Ask you teammates/coach about Pic's Theorem.