MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2011 SOLUTION KEY

Team Round

A) $A\cdot\det\left(\begin{bmatrix} x+1 & 1 \\ 1 & x+2 \end{bmatrix}\right)+B\cdot\det\left(\begin{bmatrix} x-1 & 1 \\ 1 & x-2 \end{bmatrix}\right)=0 \rightarrow A\left(x^2+3x+1\right)+B\left(x^2-3x+1\right)=0$

$\rightarrow \left(A+B\right)x^2+3\left(A-B\right)x+(A+B)=0 \rightarrow x=\dfrac{-3\left(A-B\right)\pm\sqrt{9\left(A-B\right)^2-4\left(A+B\right)^2}}{2(A+B)}$.

If there is only one real root then, $9\left(A-B\right)^2-4\left(A+B\right)^2=0$ and $x=\dfrac{-3(A-B)}{2(A+B)}$

The radicand as a difference of perfect squares is
$\left(3(A-B)+2(A+B)\right)\left(3(A-B)-2(A+B)\right)=(5A-B)(A-5B)=0$
$A>B>0 \rightarrow A=5B$.
A and B relatively prime $\rightarrow$ $(A, B) = (5, 1)$ $\rightarrow x=\dfrac{-3(5-1)}{2(5+1)}=$ **<u>–1</u>** $\rightarrow$ (A, B, R) = **<u>(5, 1, –1)</u>**.

B) $\sqrt{1+4n}$ generates integers for $n = 0, 2, 6, 12, \ldots$
Note these values of n are of the form $k(k-1)$ for $k = 1, 2, 3, \ldots$

$\sqrt{1+4n}=\sqrt{1+4k(k-1)}=\sqrt{4k^2-4k+1}=\sqrt{(2k-1)^2}=2k-1$ for $k = 1, 2, 3, \ldots$.
$10 < 2k-1 < 100 \rightarrow 6 \le k \le 50$
$k = 6 \rightarrow n = 30 \rightarrow \sqrt{121}=11$ (or $2\cdot6-1$)
$k = 7 \rightarrow n = 42 \rightarrow \sqrt{169}=13$ (or $2\cdot7-1$)
…
$k = 50 \rightarrow n = 2450 \rightarrow \sqrt{9801}=99$ (or $2\cdot50-1$)
Thus, S contains the 45 odd integers between 11 and 99 inclusive.
The condition "less than 20" is satisfied by only 11,13,15,17 and 19.
Thus, the required probability is $\dfrac{45-5}{45}=\dfrac{\mathbf{8}}{\mathbf{9}}$.

Alternate solution [Michael Zanger-Tishler (BB & N)]
The square of any integer is congruent to either 0 or 1 mod 4, specifically, the squares of even integers are congruent to 0 mod 4 and the squares of odd integers are congruent to 1 mod 4. [Note: Being "congruent to 1 mod 4" is a fancy way of saying "leaves a remainder of 1 when divided by 4".]
So the question then becomes "what percent of two digit odd numbers are greater than 20?"
which is $\dfrac{\{21,23,\ldots,99\}}{\{11,13,\ldots,99\}} \rightarrow \dfrac{40}{45}=\dfrac{\mathbf{8}}{\mathbf{9}}$.

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