MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2017 SOLUTION KEY

Team Round - continued

D) Given: $S = x^2 + 4xy + 9y^2 + 4x + \underline{\underline{18y}} + 2017$

Recall: $(A+B+C)^2 = A^2 + B^2 + C^2 + 2AB + 2AC + 2BC$

Since there is an *xy*-term, we assume that the expression can be written in a form involving the square of a trinomial in *x*; in fact, the trinomial would have to be $(x + 2y + 2)$ to insure the correct coefficients for x^2 and xy. Proceeding with this assumption,

$(x+2y+2)^2 = \underline{x^2} + 4y^2 + 4 + \underline{4xy} + \underline{4x} + 8y$ and the underlined terms are accounted for.

The mismatched terms are $4y^2 + 8y + 4$. We need an additional $5y^2$, so we formulate an additional term of $5(y+c)^2$, where c is a constant to be determined.

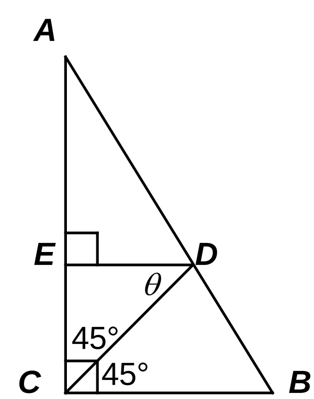
Since $5(y+c)^2 = 5y^2 + 10cy + 5c^2$, the *y*-coefficient will be matched if $10c + 8 = 18 \Rightarrow c = 1$.

Now $(x+2y+2)^2 + 5(y+1)^2$ matches S except for the constant and adding $\underline{2008}$ produces a perfect match.

Since $(x+2y+2)^2 + 5(y+1)^2 \geq 0$, for all real numbers x and y,

$(x+2y+2)^2 + 5(y+1)^2 + 2008$ has a *minimum* value of <u>**2008**</u> when $(x, y) =$ <u>**(0, –1)**</u>.

E) $\overline{DE} \parallel \overline{BC} \Rightarrow \theta = 45° \Rightarrow DE = EC$



As corresponding sides, $\Delta ADE \sim \Delta ABC \Rightarrow \dfrac{AE}{AC} = \dfrac{DE}{BC}$.

Switching the means in the proportion, $\dfrac{AE}{DE} = \dfrac{AC}{BC}$.

Adding "1" to both sides of the proportion,

$$\frac{AE}{DE} + \boxed{\frac{DE}{DE}} = \frac{AC}{BC} + \boxed{\frac{BC}{BC}} \Rightarrow \frac{AE+DE}{DE} = \frac{AC+BC}{BC}$$

Since $DE = EC$, we have $\dfrac{AC}{DE} = \dfrac{AC+BC}{BC}$

Cross multiplying, $DE(AC+BC) = AC \cdot BC$.

Now the magic!

Divide both sides by $AC \cdot BC \cdot DE$ and we have $\dfrac{AC+BC}{AC \cdot BC} = \dfrac{1}{BC} + \dfrac{1}{AC} = \dfrac{1}{DE}$.

Therefore, we just take the reciprocal of the given dimension! $\dfrac{2}{\sqrt{5}+1} \cdot \dfrac{\sqrt{5}-1}{\sqrt{5}-1} = \underline{\mathbf{\dfrac{\sqrt{5}-1}{2}}}$.