MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2012 SOLUTION KEY

Round 6

A) Sum greater than 9: (4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6) – 6 possibilities
Sum not 3 and not 4: Of the 36 possible ordered pairs, reject (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), leaving 31 possibilities. Thus, the required ratio is **<u>6 : 31</u>**.

B) In all cases

Seats | 1 | 2 | 3 | 4 | 5 | 6 |

- Pick seats for Alice and Carol
- Seat Alice and Carol in these seats
- Fill the in-between seats
- Fill the other seats (leaving one empty)

Case 1: (1 person in-between) 4·2·3·3! = 144 [A _ C in seats 123 … 456]
Case 2: (2 persons in-between) 3·2·(3·2) ·2! = 72 [A _ _ C in seats 1234… 3456]
Case 3: (3 persons in-between) 2·2·(3·2·1) ·1 = 24 [A _ _ _ C in seats 12345 … 23456]
Total: **<u>240</u>**

C) For $\left(x^3+\dfrac{1}{x^2}\right)^{15}$, the ($k$ + 1)st term is $\dbinom{15}{k}\left(x^3\right)^{15-k}\left(x^{-2}\right)^k=\dbinom{15}{k}x^{45-5k}$ and this will be the constant term when x = 9, i.e. the constant term is $\dbinom{15}{9}=\dfrac{15!}{9!\cdot 6!}$.

Similarly, for $\left(x^4+\dfrac{1}{x^3}\right)^{n}$, the ($k$ + 1)st term is $\dbinom{n}{k}\left(x^4\right)^{n-k}\left(x^{-3}\right)^k=\dbinom{n}{k}x^{4n-7k}$ and the constant term requires 4n – 7k = 0, so $k=\dfrac{4n}{7}$. The required ratio is $\dfrac{\dbinom{15}{9}}{\dbinom{n}{k}}=\dfrac{15!\cdot k!\cdot(n-k)!}{9!\cdot 6!\cdot n!}=\dfrac{5}{3}$

Since 7 is prime, n must be divisible by 7.

n = 7 and k = 4 clearly will not produce a fraction which reduces to $\dfrac{5}{3}$.

Try n = **<u>14</u>**. Then: k = 8 and $\dfrac{15!\cdot k!\cdot(n-k)!}{9!\cdot 6!\cdot n!}=\dfrac{15!\cdot 8!\cdot 6!}{9!\cdot 6!\cdot 14!}=\dfrac{15}{9}=\dfrac{5}{3}$, the required ratio.