MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2017 SOLUTION KEY

Round 2

A) For $k = 1, 2, 3, ..., 9$, $(10+k, 10-k) = (11,9),(12,8),(13,7),(14,6),...(19,1)$, where each pair of coordinates adds up to 20. Since both 14 and 6 have exactly 4 divisors, $k = 4$ works. It is left to you to verify that all other pairs contain one or more prime numbers, or the numbers have a different number of divisors, so $k = \underline{\mathbf{4}}$ is the only solution.

B) Since we were given $A+B=7$, $A^2B + AB^2 = AB(A+B) = 21 \Rightarrow AB = 3$.
To evaluate $A^2 - AB + B^2$, we notice that, since $(A+B)^2 = A^2 + 2AB + B^2$, we can subtract $3AB$ to get the required expression. Thus, $A^2 - AB + B^2 = (A+B)^2 - 3AB = 7^2 - 3 \cdot 3 = \underline{\mathbf{40}}$.

Solution #2:
$A + B = 7 \Rightarrow (A+B)^3 = A^3 + B^3 + 3(A^2B + AB^2) = 343$.
Thus, $A^3 + B^3 + 3(21) = 343 \Rightarrow A^3 + B^3 = 280$. $A^2 - AB + B^2 = \dfrac{A^3+B^3}{A+B} = \dfrac{280}{7} = \underline{\mathbf{40}}$.

C) $$\frac{2 \cdot \frac{2x+1}{x-1}}{2x+1+\frac{1}{x-1}} = \frac{6}{5} \Rightarrow 10\left(\frac{2x+1}{x-1}\right) = 6(2x+1) + \frac{6}{x-1} \Rightarrow 20x + 10 = 6(2x+1)(x-1) + 6$$
$$12x^2 - 26x - 10 = 2\left(6x^2 - 13x - 5\right) = 2(3x+1)(2x-5) = 0 \Rightarrow x = \underline{-\frac{\mathbf{1}}{\mathbf{3}}, \frac{\mathbf{5}}{\mathbf{2}}}.$$