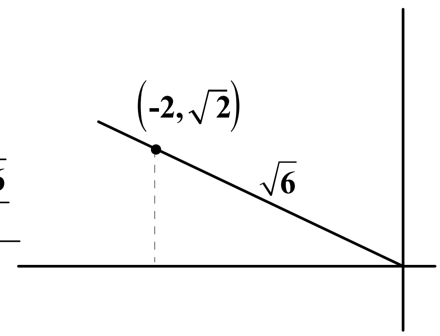
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2010 SOLUTION KEY

Round 3

A) $x^0 (x \neq 0)$ and $(1)^x$ both produce 0. By inspection, $x = \pi/2$ solves the equation.
But is it the smallest? Taking the log of both sides, we have $x\log(\sin x) = \log(1) = 0$
Since $x > 0$, $\log(\sin x) = 0$ ➔ $\sin x = 1$ ➔ $x = \pi/2 + 2n\pi$ and $\boldsymbol{\frac{\pi}{2}}$ is the smallest solution.

B) $\dfrac{2}{\tan(2x)} - \tan x = \dfrac{2(1-\tan^2 x)}{2\tan x} - \tan x = 0$ ➔ $1 - \tan^2 x - \tan^2 x = 0$



➔ $\tan^2 x = \dfrac{1}{2}$ ➔ $\tan x = -\dfrac{\sqrt{2}}{2}$ (x lies in quadrant 2) ➔ $\cos(x) = \boldsymbol{-\dfrac{\sqrt{6}}{3}}$

C) Potential extraneous answers: $(\cos\theta = -1)\ \theta \neq 180 + 360n$
$\sin\theta = -\sqrt{3}\,(1 + \cos\theta)$ ➔ $\sin^2\theta = 3(1 + 2\cos\theta + \cos^2\theta)$
$1 - \cos^2\theta = 3 + 6\cos\theta + 3\cos^2\theta$ ➔ $4\cos^2\theta + 6\cos\theta + 2 = 0$
➔ $2\cos^2\theta + 3\cos\theta + 1 = (2\cos\theta + 1)(\cos\theta + 1) = 0$
➔ $\cos\theta = -1/2$ ➔ $\theta = 120°, 240°$ or $\cos\theta = -1$ ➔ $\theta = 180°$ (extraneous)

Checking:

$\theta = 120°$: $\dfrac{\sqrt{3}/2}{\sqrt{3}+\sqrt{3}(-1/2)} = \dfrac{1/2}{1-1/2} = 1$ (extraneous)

$\theta = 240°$: $\dfrac{-\sqrt{3}/2}{\sqrt{3}+\sqrt{3}(-1/2)} = \dfrac{-1/2}{1-1/2} = -1$ (ok)

Alternate solution: Using the identity $\dfrac{\sin x}{1+\cos x} = \tan\left(\dfrac{x}{2}\right)$,

$\dfrac{\sin\theta}{\sqrt{3}+\sqrt{3}\cos\theta} = \dfrac{\sin\theta}{\sqrt{3}(1+\cos\theta)} = \dfrac{\tan(\theta/2)}{\sqrt{3}}$ ➔ $\tan\left(\dfrac{\theta}{2}\right) = -\sqrt{3}$ ➔ $\dfrac{\theta}{2} = \begin{cases}120° \\ 300°\end{cases} + 360n$ ➔ $\theta = 240°$ only

Round 4

A) $2a = 3 + 1$ ➔ $a = 2$
Equal root ➔ discriminant = 0 ➔ $3^2 - 4(1)(2b) = 0$ ➔ $b = 9/8$
$2c = 10$ ➔ $c = 5$
Thus, $abc = 10\left(\dfrac{9}{8}\right) = \boldsymbol{\dfrac{45}{4}}$ or **11.25**