MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2005 BRIEF SOLUTIONS

Round One:

A. BC/AB = 0.28 =7/25 so one rt Δ has AC = 24 by Pythagoras. Tan(∠B)=AC/BC

B. Law of Sines: $\frac{\sin(\angle D)}{EF} = \frac{\sin(\angle F)}{ED}$ so $\frac{5/9 + 1/3}{x + 2/3} = \frac{5/9}{x}$ thus $8/9\,x = 5/9\,x + 10/27$ so ED = x = 10/9 and EF = 16/9

C. Draw altitude CD. From ∠A AD = 3x, CD = 4x. From ∠B BD = 5y, CD=12y so 12y=9x and AD = 9y. Pythagoras gives BC = 13y and AC = 15y while AB = 14y. Perimeter gives y =10 thus area is 0.5(14y)(12y) = 8400

Round Two:

A. $160 = 2^5\, 5^1$ so there are (5+1)(1+1) = 12 factors.

B. 123_4 in base ten is 27; 567_8 is 375 so product is 10,125 which in base 9 is 14800_9.

C. Increase is (10u + t) – (10t +u) = 9(u – t). % increase is $\frac{9(u-t)}{10t+u} = \frac{108}{100} = \frac{27}{25}$ so 225u – 225t = 270t + 27u so 198u = 495t or 2u=5t thus u=5, t=2.

Round Three:

A. – 8 + 2 = 6; 3 – 5 = – 2.

B. Substitute (9, 16) to get c = 15 thus y = 7/3 x – 5. To get lattice pt x is a multiple of 3 so a = 102, b = 7/3 (102) – 5 = 233 sum is 335

C. Vertices are (0,0) (12, 18) and (36, 0) Consider x= 1 to 12; interior lattice point counts along vertical lines are 1,2, 4, 5, 7, 8, 10,11, 13, 14, 15, 17 sum 108. Counting from x= 35 back to x = 13 gets 0, 1, 2, 2, 3, 4, 5, 5, 6, 7, 8, 8, 9, 10, 11, 11, 12, 13, 14 14, 15 16, 17 sum 193 total is 301. OR Picks Thm: Area = (#Interor pts) + (#Boundary Pts)/2 –1 so 324 = I + (48/2) – 1 so I = 301

Round Four:

A. 4 + (– 3) – ½ + 1 = 3/2 .

B. $\log_5(x^2 - 20x) = \log_5(125)$ so $x^2 - 20x - 125 = 0$ so (x – 25)(x + 5) =0, exclude –5 as not in domain of $\log_5$ function.

C. $\log_x(\frac{4}{x^3}) = \log_2(\frac{1}{x})$ so $\log_x(4) - 3 = -\log_2(x)$ and if $a = \log_x 2$ we solve $2a - 3 = \frac{-1}{a}$ to get a= ½ or 1 so x = 4 or 2.

Round Five:

A. Pow = c (res) $(\text{cur})^2$ = c (4 res) $(\text{curr} / 2)^2$

B. 18a = a + 3b +5c so 17a = 3b + 5c while 84 c= 140 b so c = 5/3 b. Thus 17a= 34/3 b thus a/b = 2/3.

C. The rate is 5/9 egg per hen_day. If we start with 2x hens we have 2x(5) + x(5)=15x hen days yielding 25x/3 eggs. So x=72; start with 144 hens.

Round Six:

A. Exterior angles of 9-gon are each 40 so m∠BPC = 100.

B. Sketch circles of radius 5 and 6 at the endpoints of a segment of length 7 to see possibilities from 1 through 17.

C. Rhombus side 15, half diagonals 9 and 12 full diags 18, 24. Hexagon perimeter 3(24)=72, equil triangle $3(12\sqrt{3}) = 36\sqrt{3}$. Ratio simplifies to $\sqrt{3} : 2$