MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2012 SOLUTION KEY

Team Round - continued

F) An infinite geometric series converges to a sum $\left(\dfrac{a}{1-r}\right)$ if and only if $|r| < 1$.

Thus, $\dfrac{a}{1-r} = a + ar + 1 = a(1+r) + 1 \Leftrightarrow a = a\left(1-r^2\right) + \left(1-r\right) \Leftrightarrow ar^2 + r - 1 = 0 \Leftrightarrow a = \dfrac{1-r}{r^2}$.

This means $2 \le \dfrac{1-r}{r^2} \le 6 \Leftrightarrow 2r^2 \le 1-r$ and $1-r \le 6r^2$ or $\begin{cases} 2r^2 + r - 1 \le 0 \\ 6r^2 + r - 1 \ge 0 \end{cases}$.

We must take the intersection of these two conditions.

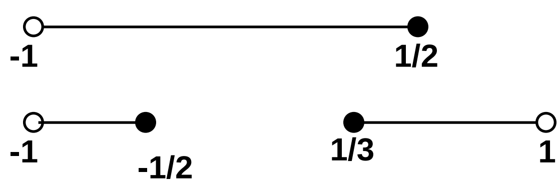
$$\begin{cases} 2r^2 + r - 1 \le 0 \\ 6r^2 + r - 1 \ge 0 \end{cases} \Leftrightarrow \begin{cases} (2r-1)(r+1) \le 0 \\ (2r+1)(3r-1) \ge 0 \end{cases}$$

The first condition requires that $-1 \le x \le \dfrac{1}{2}$, but convergence requires $r \ne -1$; hence,

$-1 < x \le \dfrac{1}{2}$.

The second condition requires $r \le -\dfrac{1}{2}$ or $r \ge \dfrac{1}{3}$, but convergence requires $-1 < r < 1$; hence,

$-1 < r \le -\dfrac{1}{2}$ or $\dfrac{1}{3} \le r < 1$



Carefully taking the intersection, we have $\underline{\boldsymbol{-1 < r \le -\frac{1}{2}} \text{ or } \boldsymbol{\frac{1}{3} \le r \le \frac{1}{2}}}$.