MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2014 SOLUTION KEY

C) Let $\alpha = Tan^{-1}(.4)$ and $\beta = Tan^{-1}\left(\frac{1}{A}\right)$. Taking tan of both sides, $\tan(2\alpha+\beta)=1$. Expanding,

$$\frac{\tan 2\alpha + \tan\beta}{1-\tan 2\alpha \cdot \tan\beta} = \frac{\tan 2\alpha + \frac{1}{A}}{1-\tan 2\alpha \cdot \frac{1}{A}} = 1 \Rightarrow \tan 2\alpha + \frac{1}{A} = 1 - \tan 2\alpha \cdot \frac{1}{A} \Rightarrow \tan 2\alpha\left(1+\frac{1}{A}\right) = 1-\frac{1}{A}$$

$\Rightarrow \tan 2\alpha = \dfrac{1-\frac{1}{A}}{1+\frac{1}{A}} = \dfrac{A-1}{A+1}$. We know that $\tan\alpha = 0.4 = \frac{2}{5} \Rightarrow \tan 2\alpha = \dfrac{2\left(\frac{2}{5}\right)}{1-\left(\frac{2}{5}\right)^2} = \dfrac{20}{25-4} = \dfrac{20}{21}$

Therefore, $\dfrac{A-1}{A+1} = \dfrac{20}{21} \Rightarrow 21A-21 = 20A+20 \Rightarrow A = \underline{41}$.

FYI: $Tan^{-1}(.4) \approx 21.80°$ and $Tan^{-1}\left(\frac{1}{41}\right) \approx 1.40°$ and $2(21.80)+1.40 = 45° = \frac{\pi}{4}^{rad}$

Alternately, start with $\tan\alpha = 0.4 = \frac{2}{5}$ and get $\tan 2\alpha = \frac{20}{21}$ as above. Rearrange the terms in the equation and take the tangent of both sides $\left(\text{Recall}: \tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}\right)$:

$$2Tan^{-1}(.4) + Tan^{-1}\left(\frac{1}{A}\right) = \frac{\pi}{4} \Leftrightarrow Tan^{-1}\left(\frac{1}{A}\right) = \frac{\pi}{4} - 2Tan^{-1}(.4) \Rightarrow \frac{1}{A} = \frac{1-\frac{20}{21}}{1+1\cdot\frac{20}{21}} = \frac{1}{41} \Rightarrow A = \underline{\mathbf{41}}.$$

D)

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

A/P	Q	R	S
B			
C			
D			

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

The entries along the main diagonal (upper left to lower right) are equal.
Unless what we add to row 1, we also subtract from column 1, the entry in the upper left corner of the resulting grid will change. Therefore, $A = P$.
Similarly, $B = Q$, $C = R$ and $D = S$.
Since 2 in the original grid is replaced by 5, what we add to row 1 must be 3 more than what we subtract from column 2. Thus, $A = Q + 3$.
Similarly, by examining the entries in the top row, $A = R + 6$, $A = S + 9$
To minimize the sum, let $S = 0 \Rightarrow D = 0$, $A = P = 9$, $R = C = 3$, $Q = B = 6$.
The minimum sum is $A+B+C+D+P+Q+R+S$ = $\underline{\mathbf{36}}$.