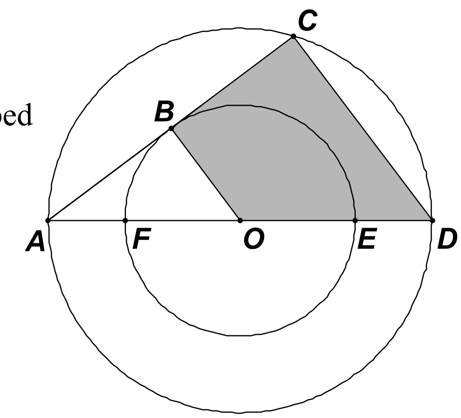
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2007 SOLUTION KEY

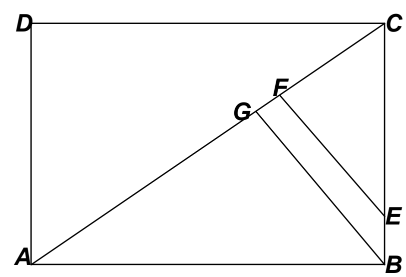
Round 5

A) There are 6 possible line segments, 4 sides and 2 diagonals.
The diagonals have length 37. 2(12 + 35 + 37) = **168**

B) Let the radius $OF = r$. Using the secant-tangent theorem, $AB^2 = AF(AD) \rightarrow 8^2 = 4(4 + 2r) \rightarrow r = 6$. Since $\angle ACD$ is inscribed in a semi-circle, it must be a right angle. Thus, $\Delta ABO \sim \Delta ACD$. Since the corresponding sides are in a 2 : 1 ratio, $CD = 12$. Quadrilateral $BCDO$ is a trapezoid and its area is $\frac{1}{2}\cdot 8\cdot(6+12) = \underline{\mathbf{72}}$



C) $AC = 200$, $CE = 120 – 20 = 100$, Δs ABC, ABG and CFE are all 3-4-5 Δs.
Thus, $CF = 60$, $FE = 80$, $BG = 96$, $AG = 128$
Area $BEFG$ = Area ABC – Area ABG – Area CFE =
½(160(120) - ½(118)(96) - ½(60)(80)
= 9600 – 6144 – 2400 = **1056**



Round 6

A) $P(\text{same color}) = \frac{5\cdot4\cdot3+6\cdot5\cdot4+4\cdot3\cdot2}{15\cdot14\cdot13} = \frac{60+120+24}{15\cdot14\cdot13} = \frac{204}{15\cdot14\cdot13} = \underline{\mathbf{\frac{34}{455}}}$

B) The general term is $\binom{16}{k}(x^{2/3})^{16-k}\cdot\left(\frac{1}{2x^{3/2}}\right)^k = \binom{16}{k}2^{-k}x^{\frac{32-2k}{3}-\frac{3k}{2}} = \binom{16}{k}2^{-k}x^{\frac{64-13k}{6}}$

Thus, $\frac{64-13k}{6} = 2 \rightarrow k = 4$ and the coefficient $k = \binom{16}{4}2^{-4} = \frac{16\cdot15\cdot14\cdot13}{1\cdot2\cdot3\cdot4\cdot2^4} = \frac{5\cdot7\cdot13}{4} = \underline{\mathbf{\frac{455}{4}}}$

C) There are 5 choices for the leftmost digit, namely 1, 4, 6, 8 and 9. There are 4 choices for the rightmost digit, namely 2, 3, 5 and 7.
Thus, there are 5(8!) arrangements that begin with a non-prime and 4(8!) arrangements that end in a prime. But we must subtract the arrangements which satisfy both conditions since these have been counted twice.
5(8!) + 4(8!) – **5(4)(7!)** → 7!(40 +32 – 20) = 7!(52). This is out of the total of 9! possible arrangements. So we have $\frac{52(7!)}{9!} = \frac{4(13)(7!)}{9(8)(7!)} = \underline{\mathbf{\frac{13}{18}}}$