MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2015 SOLUTION KEY

Team Round - continued

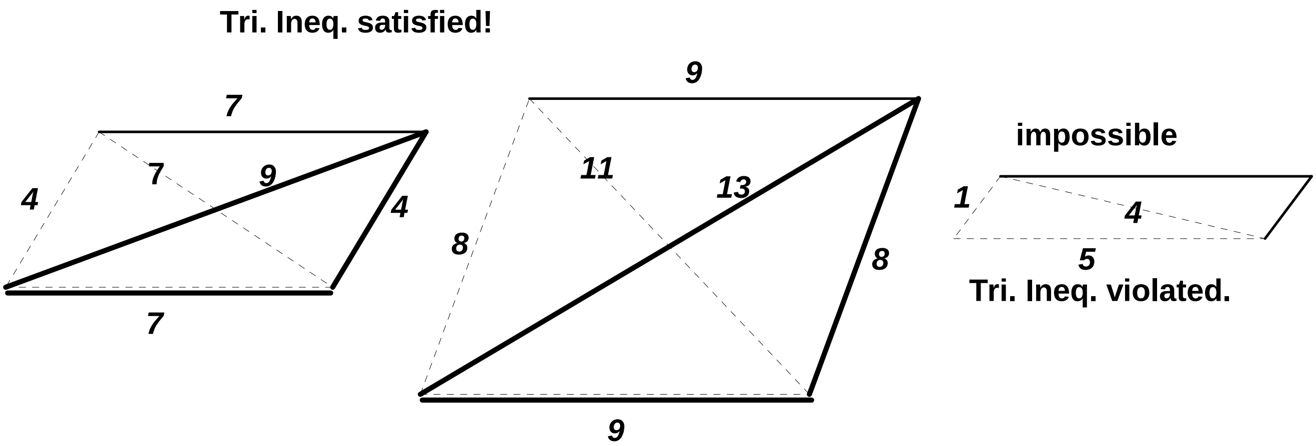
E) In any parallelogram, the sum of the squares of the lengths of the sides equals the sum of the squares of the lengths of its diagonals.

$$2\left(x^2+(x+c)^2\right)=(x+3)^2+(x+5)^2 \Rightarrow 2(x+c)^2=6x+9+10x+25=16x+34$$

$$(x+c)^2=8x+17 \Rightarrow x^2+(2c-8)x+\left(c^2-17\right)=0$$

$$\Rightarrow x=\frac{(8-2c)\pm\sqrt{(2c-8)^2-4(1)\left(c^2-17\right)}}{2}=\frac{(8-2c)\pm\sqrt{4(33-8c)}}{2} = (4-c)\pm\sqrt{(33-8c)}$$

Since x must be an integer, $(33 - 8c)$ must be a perfect square.
The only possibilities are 1, 2, 3 and 4.
$c = 1 \Rightarrow (33 - 8c) = 25$ and $x = 3 \pm 5 = 8$ and other side 9, diagonals 11 and 13 (ok)
$c = 3 \Rightarrow (33 - 8c) = 9$ and $x = 1 \pm 3 = 4$ and other side 7, diagonals 7 and 9 (ok)
$c = 4 \Rightarrow (33 - 8c) = 1$ and $x = 0 \pm 1 = 1$ and other side 5, diagonals 4 and 6 (rejected)
Therefore, there are two possible perimeters, 2(8 + 9) = **<u>34</u>** and 2(4 + 7) = **<u>22</u>**.
See diagram below.



Note:
$$2\left(4^2+7^2\right)=7^2+9^2=130$$
$$2\left(8^2+9^2\right)=11^2+13^2=290$$