MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2016 SOLUTION KEY

Round 1

A) $f(2)=11-3\cdot 2=5$

If $f^{-1}(2)=c$, then $f(c)=2$. Therefore, without specifically finding $f^{-1}(x)$, we have $f(c)=11-3c=2 \Rightarrow c=3$ and $\frac{1}{f(2)}-\frac{1}{f^{-1}(2)}=\frac{1}{5}-\frac{1}{3}=\frac{3-5}{5\cdot 3}=\underline{\mathbf{-\frac{2}{15}}}$.

B) 1) Different scales are used on the *x*- and *y*-axes.

2) As *x* increases $(x \to +\infty)$, $y=f(x)$ is unbounded. At point *A*, there is a local maximum only

4) Over the specified interval, $y=f(x)$ is a decreasing function and $f(c)<f(b)<f(a)$.

Thus, only **<u>3</u>** and **<u>5</u>** are true.

C) Given: $r_2=2r_1$

The product of the new zeros is

$$(r_1+1)(r_2+1)=r_1r_2+(r_1+r_2)+1=3r_1r_2 \Rightarrow 2r_1r_2-r_2=r_1+1 \Rightarrow \boxed{r_2=\frac{r_1+1}{2r_1-1}}\ (***)$$

Substituting for r_2 in (***), cross multiplying and transposing terms, we have

$$4r_1^2-3r_1-1=(4r_1+1)(r_1-1)=0 \Rightarrow r_1=-\frac{1}{4},1.$$

Alternately, $r_1r_2=2r_1^2$ and $(r_1+1)(r_2+1)=(r_1+1)(2r_1+1)=3r_1r_2=3(2r_1^2)$

$\Rightarrow 4r_1^2-3r_1-1=0$ and the same result follows.

$$r_1=-\frac{1}{4} \Rightarrow r_2=-\frac{1}{2} \Rightarrow f(x)=(4x+1)(2x+1)=8x^2+6x+1 \Rightarrow a+b+c=15$$

$$r_1=1 \Rightarrow r_2=2 \Rightarrow f(x)=(x-1)(x-2)=x^2-3x+2 \Rightarrow a+b+c=0$$

Thus, (*a*, *b*, *c*) = **<u>(8, 6, 1)</u>**.

FYI - To generate similar problems:
If the new product were *A* times the original (instead of triple) and the roots were in an *N* : 1 ratio (instead of 2 : 1), the equations would be

$$\begin{cases} r_2=Ar_1 \\ r_2=\dfrac{r_1+1}{(N-1)r_1-1} \end{cases} \Rightarrow A(N-1)r_1^2-(A+1)r_1-1=0$$ and adjusting (*N*, *A*) so the discriminant

$(A+1)^2+4A(N-1)=(A-1)^2-4AN$ is a perfect square will generate a quadratic factorable over the integers and, therefore, a function with integer coefficients.

For example, if $A=4$, then the discriminant $9+16N$ must be a perfect square.

$$N=7,10,22,\ldots \Rightarrow 121=11^2,\ 169=13^2,\ 361=19^2,\ldots \Rightarrow 24r_1^2-5r_1-1=(8r_1+1)(3r_1-1)=0,$$

$$36r_1^2-5r_1-1=(9r_1+1)(4r_1-1)=0,\ 84r_1^2-5r_1-1=(12r_1+1)(7r_1-1)=0,\ \ldots .$$