MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015 SOLUTION KEY

Round 3

A) $\frac{1}{2}+\sin^2 x=\cos^2 x \Leftrightarrow \cos^2 x-\sin^2 x=\frac{1}{2} \Leftrightarrow \cos 2x=\frac{1}{2}$

Therefore, $2x=\pm\frac{\pi}{3}+2k\pi \Leftrightarrow x=\pm\frac{\pi}{6}+k\pi$

$k=0 \Rightarrow\ x=\pm\frac{\pi}{6} \Rightarrow\ \underline{\boldsymbol{\frac{\pi}{6}}}, -\frac{\pi}{6}+2\pi=\ \underline{\boldsymbol{\frac{11\pi}{6}}}$

$k=1 \Rightarrow\ x=\pm\frac{\pi}{6}+\pi=\ \underline{\boldsymbol{\frac{5\pi}{6}}},\ \underline{\boldsymbol{\frac{7\pi}{6}}}$

Alternate Solution!! (Mike Schockett – Maimonides)
Adding $\sin^2 x$ to both sides,

$\frac{1}{2}+2\sin^2 x=\sin^2 x+\cos^2 x=1 \Rightarrow \sin^2 x=\frac{1-\frac{1}{2}}{2}=\frac{1}{4} \Rightarrow \sin x=\pm\frac{1}{2} \Rightarrow\ \frac{\pi}{6}$ - family.

B) A solution interval of $-\pi<x<0$ implies that the solutions must be in quadrants 3 and 4.

$$8\cos^3 x-4\cos^2 x-2\cos x+1=0$$

Grouping the first two terms and the last two terms, $4\cos^2 x(2\cos x-1)-(2\cos x-1)=0$.

$\Leftrightarrow\ (4\cos^2 x-1)(2\cos x-1)=0$

$\Leftrightarrow\ (2\cos x-1)^2(2\cos x+1)=0 \Rightarrow \cos x=\pm\frac{1}{2} \Rightarrow x=\ \underline{\boldsymbol{-\frac{\pi}{3}, -\frac{2\pi}{3}}}$

Synthetic division could also have been used with the coefficients of $8, -4, -2, 1$,

producing a double root of $+\frac{1}{2}$ and a single root of $-\frac{1}{2}$ and the solution follows as above.

C) A solution interval of $\underline{0<x<\frac{\pi}{2}}$ implies that all solutions must be in quadrant 1.

$\tan\left(2x-\frac{\pi}{4}\right)=\cot\left(3x+\frac{\pi}{6}\right) \Rightarrow\ \tan\left(2x-\frac{\pi}{4}\right)=\tan\left(\frac{\pi}{2}-\left(3x+\frac{\pi}{6}\right)\right)$

Since $\tan A=\tan B\ \Rightarrow\ A=B\pm k\pi$, we have $2x-\frac{\pi}{4}=\frac{\pi}{2}-\left(3x+\frac{\pi}{6}\right)\pm k\pi$

$\Rightarrow\ 5x=\frac{\pi}{2}-\frac{\pi}{6}+\frac{\pi}{4}\pm k\pi=\frac{7\pi}{12}\pm k\pi\ \Rightarrow\ x=\frac{(7\pm 12k)\pi}{60}$ and $k = 0, 1 \Rightarrow x=\ \underline{\boldsymbol{\frac{7\pi}{60}, \frac{19\pi}{60}}}$

Even if we had not been given that there were only 2 solutions, $k=2 \Rightarrow x=\frac{31\pi}{60}>\frac{\pi}{2}$

and we stop searching.