**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 3 - DECEMBER 2012 SOLUTION KEY**

## Round 2

A) The unit digits of both the powers of 2 and the powers of 3 repeat in cycles of 4.

$2^1 = \underline{2}, 2^2 = \underline{4}, 2^3 = \underline{8}, 2^4 = 1\underline{6}, 2^5 = 3\underline{2}$ For any positive integer $k$, $2^{4k}$ has a unit digit of 6.

$3^1 = \underline{3}, 3^2 = \underline{9}, 3^3 = 2\underline{7}, 3^4 = 8\underline{1}$ For any positive integer $k$, $3^{4k}$ has a unit digit of 1.

$2^{71} = \left(2^4\right)^{17} \cdot 2^3 = \left(...6\right)^{17} \cdot 8$.

Since the powers of a number ending in 6 always end in 6, $2^{71}$ ends in 8.

$3^{44} = \left(3^4\right)^{11} = \left(...1\right)^{11}$.

Since the powers of a number ending in 1 always end in 1, $3^{44}$ ends in 1.

$\left|(8-1)^3\right| = 7^3 = \underline{\mathbf{343}}$

B) $314_{(b)} = 3b^2 + 1b + 4$ and $132_{(b)} = 1b^2 + 3b + 2$

Therefore, $3b^2 + b + 4 = 2\left(b^2 + 3b + 2\right) \Leftrightarrow b^2 - 5b = b(b-5) = 0$ and $b$ = **<u>5</u>**.

or, alternately, with digits of 1,2,3 and 4, the base must be at least 5. By trial and error, the first trial works.

$314_{(5)} = 3(25) + 5 + 4 = 84$
$132_{(5)} = 25 + 3(5) + 2 = 42$ and 84 = 2(42), so $b$ = **<u>5</u>**.

C) Consider the first 12 natural numbers. 1 <u>2</u> <u>3</u> 4 5 6 7 8 <u>9</u> <u>10</u> 11 12

Exactly four of them are divisible by 2 or 3, but not 4 or 6, namely the underlined values.
Since 12 is the least common multiple of 2, 3, 4 and 6, it follows that
($N$ + 12) satisfies the required conditions if and only if $N$ does. Thus, in the second block of 12 natural numbers, 14, 15, 21 and 22 satisfy the divisibility requirements.
In each block of 12 natural numbers, the four numbers satisfying the required conditions will always be the second, third, ninth and tenth numbers.
Since 117 = 29 · 4 + 1, the first 29 blocks will contain 116 numbers satisfying the divisibility requirements and the 117$^{th}$ natural number will be in the 30$^{th}$ block.
The first block ends in 12 = 12(1), the second in 24 = 12(2). The $N^{th}$ block ends in 12$N$.
Thus, the last number in the 29$^{th}$ block is 29·12 = 348. The required number is the second number in the next block, namely **<u>350</u>**.