MASSACHUSETTS MATHEMATICS LEAGUE

JANUARY 2006 BRIEF SOLUTIONS

Round One:

A. $x^2 - 2x + 1 + y^2 + 2y + 1 + 4 = 40$ so $(x-1)^2 + (y+1)^2 = 36$ and r = 6.

B. Slopes are $-a/6$ and $2a/3$ so $-2a^2/18 = -1$.
Thus, $a = 3$ Substitute and solve the system.

C. Intersection at $y = -3$ on line gives vertex as (2, –3) so parabola is $x - 2 = a(y+3)^2$
Other intersection of (0,1) gives $a = -1/8$. Substitute this and y = 0.

Round Two:

A. $(2x - a)(x + b)$ maximizes g when b is maximum, a minimum so b=15, a=1.

B. Factoring gives $6(2x-1)(x-3)$ and $4(2x-1)(x+3)$ common is $2(2x-1)$

C. $\frac{1}{2}x^3 = x(x-24)(x-10)$ so $0 = \frac{1}{2}x^3 - 34x^2 + 240 = \frac{1}{2}x(x^2 - 68x + 480) =$
$\frac{1}{2}x(x-8)(x-60)$. Only $x = 60$ gives a large enough cube.

Round Three:

A. $2\sin(x)\cos(x)=\cos(x)$ so $\cos(x)=0$, $x = \pi/2, 3\pi/2$, or $\sin(x)=1/2$, $x=\pi/6, 5\pi/6$

B. $\tan(180-x) = -\tan(x)$ and $-\tan(x)/\sin(x) = -1/\cos(x)$ so $\cos(x) = -0.5$

C. $\tan^2(\frac{x}{6}) + 1 + \sqrt{3}\tan(\frac{x}{6}) = \sqrt{3} + \tan(\frac{x}{6}) + 1$ becomes

$$\tan^2(\frac{x}{6}) + \sqrt{3}\tan(\frac{x}{6}) - \tan(\frac{x}{6}) - \sqrt{3} = 0 = (\tan(\frac{x}{6}) - 1)(\tan(\frac{x}{6}) + \sqrt{3}) \text{ so}$$

$x/6 = \pi/4 + n\pi$ thus $x = 3\pi/2 + 6n\pi$ or $x/6 = 2\pi/3 + n\pi$ thus $x = 4\pi + 6n\pi$ and the first five positive solutions are 1.5π, 4π, 7.5π, 10π, and 13.5π.

Round Four:

A. $x^2 + x^2 + 2x + 1 = x^2 + 4x + 4$ so $x^2 - 2x - 3 = 0 = (x-3)(x+1)$

B. Second equation gives $y = 7$ or $y = -5$. $2z^2 + 7z + 3 = (2z+1)(z+3)$ so $z = -0.5$ or $z = -3$. $2z^2 - 5z + 3 = (2z-3)(z-1)$ so $z = 1.5$ or $z = 1$.

C. For n increases of \$5, price is $100 + 5n$ while sales is $102 - 3n$. Zeroes are at $n = -20$ and $n = 34$, so vertex is at their average, $n = 7$.

Round Five:

A. The larger triangle is scaled by 10 so its area is scaled by 100. The smaller triangle has area 0.5(3)(4) = 6.

B. Δ#1 is 3-4-5 or 5-12-13. Max difference comes from 5-12-13 whose area and perimeter are both 30. Δ#2 is 5/20, 12/30, 13/30 and area is 1/30.

C. $MN = 15$ (midline) $\Delta MNX \sim \Delta EFX$ ratio 3:2 so MX is 3/5 of ME. AE twice altitude of equil Δ w/side 10 = $10\sqrt{3}$ and ME is hypotenuse of $\Delta AME = \sqrt{300 + 25} = 5\sqrt{13}$,
so $MX = 3\sqrt{13}$

Round Six:

A. $a^2 + 2ab + b^2 = 12 + a^2 - 2ab + b^2$ so $4ab = 12$ and $ab = 3$.

B. By symmetry, $b = c = d$ so $c = 5d - 3$ becomes $c = 5c - 3$ and $c = 3/4$.

C. $14p^2 - 41pq + 15q^2 = (7p - 3q)(2p - 5q)$ so $7p = 3q$, $p/q = 3/7$ or $2p = 5q$, $p/q = 5/2$. Sum is 41/14.