MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2009 SOLUTION KEY

Round 1

A) $\dfrac{1+2i+3i^2+4i^3}{1-2i+3i^2-4i^3} = \dfrac{1+2i-3-4i}{1-2i-3+4i} = \dfrac{-2-2i}{-2+2i} = \dfrac{1+i}{1-i}\cdot\left(\dfrac{1+i}{1+i}\right) = \dfrac{1+i^2+2i}{1-i^2} = \dfrac{2i}{2} = \underline{\boldsymbol{i}}$

B) $= 3^{40}(1+i)^{40} = 3^{40}(2i)^{20} = 3^{40}2^{20}i^{20} = 3^{40}2^{20}(1) = 9^{20}2^{20} = 18^{20}$ ➔ $r + n =$ **<u>38</u>**

Note: $18^{20} = \left(18^2\right)^{10} = 324^{10}$. Such equivalent expressions produce larger values of $r + n$.

C) If $z = A + Bi$, then $z^2 = -40-9i = A^2+2ABi-B^2 = \left(A^2-B^2\right)+2ABi$. But, $|z| = \sqrt{A^2+B^2}$

and $|z^2| = \sqrt{(-40)^2+(-9)^2} = \sqrt{41^2} = 41$ (9 – 40 – 41 is a Pythagorean Triple.)

Since $|z|^2 = |z^2|$, we have $A^2 + B^2 = 41$.

Equating the real parts, the imaginary parts and the absolute values, we have these three conditions: .

$$\begin{cases} (1)\ A^2-B^2=-40 \\ (2)\ 2AB=-9 \\ (3)\ A^2+B^2=41 \end{cases}$$

(2) ➔ A and B have <u>opposite</u> signs.

(1) + (3) ➔ $2A^2 = 1$, (3) – (1) ➔ $2B^2 = 81$ and $(A, B) = \left(\pm\dfrac{1}{\sqrt{2}}, \mp\dfrac{9}{\sqrt{2}}\right)$ ➔ $\left(\dfrac{A}{B}\right)^2 = \dfrac{\frac{1}{2}}{\frac{81}{2}} = \underline{\boldsymbol{\dfrac{1}{81}}}$

Proof of the fact that for any **<u>complex</u>** number, $|z|^2 = |z^2|$.

Let $z = x + yi$. Then $z^2 = (x+yi)^2 = x^2+2xyi+y^2i^2 = \left(x^2-y^2\right)+(2xy)i$

$$|z|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2$$

$$\left|z^2\right| = \sqrt{\left(x^2-y^2\right)^2+(2xy)^2} = \sqrt{\left(x^4-2x^2y^2+y^4\right)+4x^2y^2} = \sqrt{x^4+2x^2y^2+y^4}$$

$$= \sqrt{\left(x^2+y^2\right)^2} = x^2+y^2$$ By the transitive property, $|z|^2 = \left|z^2\right|$.

Round 2

A) Let the 4 numbers be x, $x + 2$, $x + 4$ and $x + 6$. Then:
$4x + 12 = 213 + x + 6$ ➔ $3x = 207$ ➔ $x = 69$ ➔ **<u>69, 71, 73, 75</u>**

B) By solving $x(3 - x) = -10$ or judicious guess and check, $(x, y) = (5, -2)$ or $(-5, 2)$.

The possible values of $\dfrac{x}{y}$ are –2.5 or –0.4 . The larger value is **<u>–0.4 .</u>**

C) Let R_2 denote the new rate and R_1 the original rate.

Since $R \cdot T = D$, we have $R_2 = \dfrac{150}{w-2}$ and $R_1 = \dfrac{150}{w}$ and $\dfrac{150}{w-2} = \dfrac{150}{w} + x$

Clearing fractions, $150w = 150(w-2)+x(w)(w-2)$ ➔ $150w = 150w-300+xw^2-2xw$

Canceling, we have $300 = xw^2-2xw = x\left(w^2-2w\right)$ ➔ $x = \underline{\boldsymbol{\dfrac{300}{w^2-2w}}}$ or $\underline{\boldsymbol{\dfrac{300}{w(w-2)}}}$.