MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2015 SOLUTION KEY

Round 1

A) $f(x)=\frac{2}{3}x-6,\ g(x)=-\frac{3}{2}x+6 \Rightarrow h(x)=\left(\frac{2}{3}x-6\right)\left(-\frac{3}{2}x+6\right)$

$h(x)=0 \Rightarrow \left(\frac{2}{3}x-6\right)=0$ or $\left(-\frac{3}{2}x+6\right)=0 \Rightarrow x=\underline{\mathbf{9,\ 4}}$

B) Interchanging *x* and *y* and resolving for *y*, we have

$y=f^{-1}(x)=\frac{1-2x}{3} \Leftrightarrow x=\frac{1-2y}{3} \Leftrightarrow 3x+2y=1 \Leftrightarrow y=f(x)=\frac{1-3x}{2}$ Now $8\le f(x)\le 20$

$\Leftrightarrow 8\le \frac{1-3x}{2}\le 20 \Leftrightarrow 16\le 1-3x\le 40 \Leftrightarrow 15\le -3x\le 39 \Leftrightarrow -5\ge x\ge -13$

$\Rightarrow (a,b)=\underline{(\mathbf{-13,-5})}$ The order was important, since it was required that $a\le b$!

C) If the zeros of $y=f(x)\ =3x^2+2x-4$ are *u* and *v*, then $\begin{cases}(1)\ \ u+v=-\frac{2}{3}\\ (2)\quad uv=-\frac{4}{3}\end{cases}$.

The sum of the zeros of $y=g(x)$ is $(2u+3v)+(3u+2v)=5(u+v)=5\cdot -\frac{2}{3}=-\frac{10}{3}$.

The product of the zeros of $y=g(x)$ is $(2u+3v)(3u+2v)=6u^2+13uv+6v^2=6\left(u^2+v^2\right)+13uv$.

Squaring (1), we have $(u+v)^2=\left(u^2+v^2\right)+2uv=\frac{4}{9} \Rightarrow \left(u^2+v^2\right)=\frac{4}{9}-2uv$.

Substituting, $(2u+3v)(3u+2v)=6\left(\frac{4}{9}-2uv\right)+13uv=\frac{8}{3}+uv=\frac{4}{3}$.

A quadratic equation with roots $2u+3v$ and $3u+2v$ is $x^2+\frac{10}{3}x+\frac{4}{3}=0$.

Therefore, $y=g(x)=3x^2+10x+4 \Rightarrow (b,c)=\underline{(\mathbf{10,4})}$.