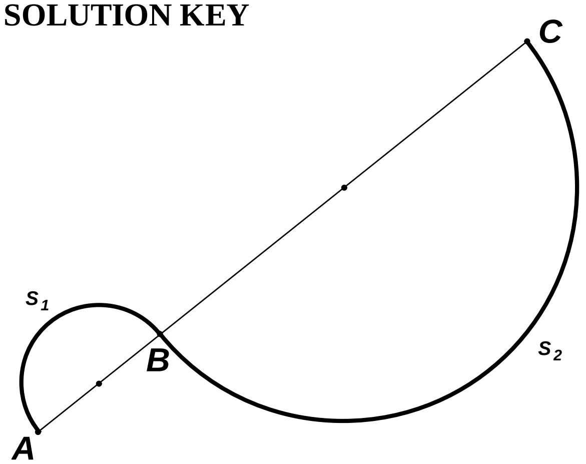
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2015 SOLUTION KEY



Round 3

A) Given: $A(2,4), C(14,20)$ and $AB:BC = 1:3$.

$AC^2 = 12^2 + 16^2 = 400 \Rightarrow AC = 20$

$\Rightarrow (AB, BC) = (5, 15)$

$\Rightarrow (r_1, r_2) = \left(\frac{5}{2}, \frac{15}{2}\right)$

$\Rightarrow S_1 + S_2 = \pi\left(\frac{5}{2} + \frac{15}{2}\right) = \underline{\boldsymbol{10\pi}}$

Note: The numerical value of the ratio $AB:BC$ is irrelevant. B could be any point between A and C. If $AB = a$ and $BC = b$, then $m\left(\widehat{AB}\right) + m\left(\widehat{BC}\right) = \frac{a\pi}{2} + \frac{b\pi}{2} = \frac{\pi}{2}(a+b) = \frac{\pi}{2}(20) = \underline{\boldsymbol{10\pi}}$.
In fact, if $a = 0$, then A and B are the same point, the required distance is a semi-circle on $\overline{AC}$ and again we have $\underline{\boldsymbol{10\pi}}$.

B) $A(-2,-9)$ and $B(8,-5)$

Since the slope of $\overline{AB}$ is $\frac{-5-(-9)}{8-(-2)} = \frac{4}{10} = \frac{2}{5}$, the slope of the perpendicular bisector is $\frac{-5}{2}$.

Since the slope of $ax + 2y = k$ is $\frac{-a}{2}$, we have $a = 5$.

The midpoint of $\overline{AB}$ is $\left(\frac{-2+8}{2}, \frac{-9+(-5)}{2}\right) = (3,-7)$

Substituting in $5x + 2y = k \Rightarrow k = 5(3) + 2(-7) = 1 \Rightarrow (a,k) = \underline{(\mathbf{5,1})}$.

C) $x^2 + y^2 - 10x - 4y - 140 = 0 \Leftrightarrow (x-5)^2 + (y-2)^2 = 169$

$\Rightarrow$ Center: $(h,k) = (5,2)$, Radius: $r = CQ = 13$

We require integers Δx and Δy for which $(\Delta x)^2 + (\Delta y)^2$ has a value as close as possible to 169.
Examining the integer perfect squares 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, we have $1 + 169 = 170$ and $49 + 121 = 170$. Clearly, no other smaller integer value is greater than 169. Therefore, $(\Delta x, \Delta y) = (13,1)$, $(11,7)$ and

$(a,b) = (18,3)$ or $(16,9)$ $\Rightarrow (h+a)^2 + (k+b)^2 = \begin{cases} 23^2 + 5^2 = \underline{\mathbf{554}} \\ 21^2 + 11^2 = \underline{\mathbf{562}} \end{cases}$.