MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017 SOLUTION KEY

Team Round - continued

E) First we find the radius of circle O. Since the radius of any circle inscribed in a triangle is given by the triangle's area divided by its semi-perimeter, we have $r = EO = \dfrac{\frac{1}{2}\cdot 3\cdot 4}{\frac{(3+4+5)}{2}} = \dfrac{6}{6} = 1$.

Concentrate on circle P in the diagram at the left below.

$AE = 3, EO = 1 \Rightarrow AO = \sqrt{10}$

Let $DP = r$ and $AD = x$. Note that $\Delta DAP \sim \Delta EAO$.

$\dfrac{r}{1} = \dfrac{x}{3} = \dfrac{AP}{\sqrt{10}} \Rightarrow x = 3r \Rightarrow AP = r\sqrt{10}$

$AP + PO = AO \Leftrightarrow r\sqrt{10} + r + 1 = \sqrt{10} \Rightarrow r = \dfrac{\sqrt{10}-1}{\sqrt{10}+1}$. Rationalizing, $r_3 = \underline{\mathbf{\dfrac{11-2\sqrt{10}}{9}}}$.

Concentrate on circle Q in the middle diagram below.

$BH = 2, OH = 1 \Rightarrow OB = \sqrt{5}$

Let $QG = r$ and $BG = x$. Note that $\Delta GBQ \sim \Delta HBO \Rightarrow \dfrac{r}{1} = \dfrac{x}{2} = \dfrac{BQ}{\sqrt{5}} \Rightarrow x = 2r \Rightarrow BQ = r\sqrt{5}$

$BQ + QO = OB \Leftrightarrow r\sqrt{5} + r + 1 = \sqrt{5} \Rightarrow r = \dfrac{\sqrt{5}-1}{\sqrt{5}+1}$. Rationalizing, $r_2 = \underline{\mathbf{\dfrac{3-\sqrt{5}}{2}}}$.

Concentrate on circle R in the diagram at the right below.

Let $RK = r$ and $CK = x$. Note that $\Delta KCR \sim \Delta JCO$.

$\dfrac{r}{1} = \dfrac{x}{1} = \dfrac{CR}{\sqrt{2}} \Rightarrow x = r \Rightarrow CR = r\sqrt{2}$. $CR + RO = CO \Leftrightarrow r\sqrt{2} + r + 1 = \sqrt{2} \Rightarrow r = \dfrac{\sqrt{2}-1}{\sqrt{2}+1}$

Rationalizing, $r_1 = 3 - 2\sqrt{2}$

Thus, $(r_1, r_2, r_3) = \underline{\left(\mathbf{3-2\sqrt{3}, \dfrac{3-\sqrt{5}}{2}, \dfrac{11-2\sqrt{10}}{9}}\right)}$.