MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2016 SOLUTION KEY

Team Round

D) $2x+3=A(x-2)^2+B(x-2)+C=A\left(x^2-4x+4\right)+Bx-2B+C$

$$\begin{cases}(1):\ x=0\Rightarrow 4A-2B+C=3\\(2):\ x=1\Rightarrow 3A-B+C=5\\(3):\ x=2\Rightarrow 4A+C=7\end{cases}\Rightarrow\begin{cases}(1)-(2):\ A-B=-2\\(3)-(2):\ A+B=2\end{cases}\Rightarrow A=0,B=2,C=7$$

Thus, $A^2+B^2+C^2=\underline{\mathbf{53}}$.

E) Since x^2+1 is positive for all values of *x*, the inequality simplifies to

$2\left|x^2-1\right|\le\left(x^2+1\right)+9x+7=x^2+9x+8=(x+1)(x+8)$

Since the left side always produces a nonnegative value, the right side of the inequality must also be nonnegative and this is the case for $\boxed{x\le-8\text{ or }x\ge-1}$.

Values outside this range are extraneous.

To remove the absolute value, we must consider two separate cases.

Case 1: $x\le-1$ or $x\ge1\Rightarrow\left|x^2-1\right|=x^2-1$

$2\left(x^2-1\right)\le x^2+9x+8\Leftrightarrow x^2-9x-10\le0\Leftrightarrow(x-10)(x+1)\le0\Rightarrow-1\le x\le10$

Some of these values fall outside the domain of definition of this equivalent equation.

The acceptable values are $x=-1$ or $1\le x\le10$. [$-1<x<1$ are extraneous *for this case*.]

Case 2: $-1<x<1\Rightarrow\left|x^2-1\right|=1-x^2$

None of these values can be rejected out of hand, since all of these values are a subset of $\boxed{x\le-8\text{ or }x\ge-1}$. The equivalent inequality is

$2\left(1-x^2\right)\le x^2+9x+8\Leftrightarrow 3x^2+9x+6\ge0\Leftrightarrow 3\left(x^2+3x+2\right)\ge0\Leftrightarrow 3(x+2)(x+1)\ge0$

$\Rightarrow x\le-2$ or $x\ge-1$. Over the domain of definition for this equivalent equation, we pick up solutions $-1<x<1$. Combining the two cases, the required condition is $\underline{\mathbf{-1\le x\le10}}$.

Since, for any *positive constant k*, $|x|\le k\Leftrightarrow(-k\le x\le k)\Leftrightarrow(x\ge-k)$ and $(x\le k)$, you might want to complete the following solution with your teammates and/or coach:

$2\left|x^2-1\right|-\left|x^2+1\right|\le9x+7\ \Leftrightarrow\ 2\left|x^2-1\right|-\left(x^2+1\right)\le9x+7\ \Leftrightarrow\ 2\left|x^2-1\right|\le x^2+9x+8$

$\Leftrightarrow\ -x^2-9x-8\le2\left(x^2-1\right)\le x^2+9x+8\ \Leftrightarrow\ \left(3x^2+9x+6\ge0\right)$ **<u>and</u>** $\left(x^2-9x-10\le0\right)$

F) The Fibonacci numbers in blocks of 5 are:

<u>1,2,3,5,8</u> <u>13,21 ^ 34,55 ^ 89</u> ^ <u>144 ^ 233 ^ 377 ^ 610, ^ 987</u> ^^ 1597,… .

^ denotes the location of a perfect square.

The perfect cubes are 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, … .

We require that two cubes fit between consecutive Fibonacci numbers.

This first happens for the interval $(987,1597)$ which contains both 1000 and 1331 $\Rightarrow$

$(j,k)=\underline{(\mathbf{10,15})}$.