MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2016 SOLUTION KEY

Round 2

A) Simply build a table of n!-values.

n	$n!$	Digitsum	n	$n!$	Digitsum
2	2	2	7	5040	9
3	6	6	8	40320	9
4	24	6	**9**	362880	**27**
5	120	3	10		
6	720	9			

Thus, $(P,Q)=\underline{(\mathbf{9,27})}$

B) Looking for a pattern: $7^1=4\cdot1+\boxed{3},\ 7^2=49=4\cdot12+\boxed{1},\ 7^3=343=4\cdot85+\boxed{3},$
$7^4=2401=4\cdot600+\boxed{1},\ 7^5=16807=4\cdot4201+\boxed{3},\ldots$

This *suggests* that the remainders alternate between 3 and 1 and that the required remainder is **3**, since the exponent 355 is odd.

This can be summarized as $7^{\text{odd}}\equiv3\ (\text{mod }4)$ and $7^{\text{even}}\equiv1\ (\text{mod }4)$, where (mod n) denotes the remainder upon division by n and $\equiv$ is read "is congruent to".

Does this alternating pattern really continue? Removing any doubt .....

Consider that $7^n=(4+3)^n$. Each term in the expansion will contain a factor of 4, except the last term 3^n, so we must examine powers of 3 to determine the remainder.

$3^n=(2+1)^n$ and the last terms in the expansion will be $2n+1$. If n is even, then this is 1 more than a multiple of 4 $\left(n=2k\ (\text{i.e., even})\Rightarrow 2n+1=2(2k)+1=4k+\boxed{1}\right)$; if n is odd, this is 3 more than a multiple of 4 $\left(n=2k+1\ (\text{i.e., odd})\Rightarrow 2n+1=2(2k+1)+1=4k+\boxed{3}\right)$.

Thus, the alternating pattern really does continue!

C) If N and N' denote the two-digit numbers and a and b denote the digits, then

$\begin{cases}N=10a+b=5k+1\\ N'=10b+a=5j+3\end{cases}$. Adding, $N+N'=11(a+b)=5(j+k)+4$.

If $j+k$ is even, then $5(j+k)$ is a multiple of 10 and a + b = 4 or 14.

If $j+k$ is odd, a + b = 9. [19 is rejected, since the maximum digit sum is 9 + 9 = 18.]

$a+b=4\Rightarrow N=\cancel{13},\ \cancel{22},\ 31$

$a+b=9\Rightarrow N=\cancel{18},\ \cancel{27},\ 36,\ \cancel{45},\ \cancel{54},\ \cancel{63},\ \cancel{72},\ 81$

$a+b=14\Rightarrow N=\cancel{77},\ 86,\ \cancel{95}$

The sum of all the numbers satisfying the specified conditions is 234 which leaves a remainder of **0** when divided by 9.