MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013 SOLUTION KEY

Round 4

A) Since $2^4 = 16$ and $4^2 = 16$, we have (x, y) = (2,4) or (4, 2).

Therefore, $\dfrac{\log_x y + \log_y x}{x - y} = \dfrac{\log_2 4 + \log_4 2}{\pm 2} = \dfrac{2 + \frac{1}{2}}{\pm 2} = \underline{\mathbf{\pm 1.25}}$.

How do you argue that there are no other ordered pairs of positive integers???

B) $\log_b x - \log_b\left(x^{1/3}\right) + \log_b\left(x^{1/5}\right) = 4 \Rightarrow \log_b\left(\dfrac{xx^{1/5}}{x^{1/3}}\right) = 4 \Rightarrow \log_b\left(\dfrac{x^{18/15}}{x^{5/15}}\right) = \log_b\left(x^{13/15}\right) = 4$

$\Rightarrow b^4 = x^{13/15}$ Raising each side to the 15/13th power, $x = \underline{\boldsymbol{b^{\frac{60}{13}}}}$

C) If $f(x) = 2^x - 2^{-x} = k$, then

$2^x - \dfrac{1}{2^x} = k \Leftrightarrow \dfrac{2^{2x} - 1}{2^x} = k \Leftrightarrow 2^{2x} - k \cdot 2^x - 1 = 0 \Leftrightarrow \left(2^x\right)^2 - k \cdot 2^x - 1 = 0$

If $N = 2^x$, then we have $N^2 - kN - 1 = 0$ or $N = \dfrac{k \pm \sqrt{k^2 + 4}}{2} \Rightarrow$

For $x = A$ and $k = 8$, we have $N = 2^A = \dfrac{8 \pm \sqrt{8^2 + 4}}{2} = 4 + \sqrt{17}$.

For $x = B$ and $k = 4$, we have $N = 2^B = \dfrac{4 \pm \sqrt{4^2 + 4}}{2} = 2 + \sqrt{5}$

Thus, $2^A - 2^B = 4 + \sqrt{17} - \left(2 + \sqrt{5}\right) = \underline{\mathbf{2 + \sqrt{17} - \sqrt{5}}}$.