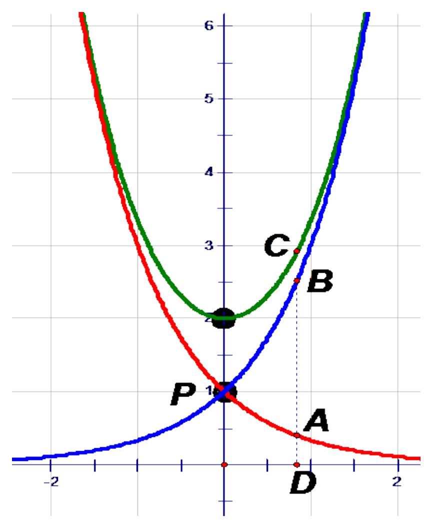
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2012 SOLUTION KEY

Team Round
A) - continued
The strategy will be to:

Use elementary row operations on the matrix of coefficients $\begin{bmatrix} 1 & 7 & 5 \\ 2 & 9 & 4 \\ 6 & A & 3 \end{bmatrix}$ and convert it to a triangular matrix where all the entries below the main diagonal are zero.

This square matrix will take on the form $\begin{bmatrix} 1 & 7 & 5 \\ 0 & a & b \\ 0 & 0 & c \end{bmatrix}$. In fact, we will tack on the constants from the right side of the equation before we start the triangularization process and get a matrix of coefficients for an **equivalent** system of equations, namely $\begin{bmatrix} 1 & 7 & 5 & n_1 \\ 0 & a & b & n_2 \\ 0 & 0 & c & n_3 \end{bmatrix} \Rightarrow \begin{cases} x + 7y + 5z = n_1 \\ ay + bz = n_2 \\ cz = n_3 \end{cases}$ which is easily solved by **backtracking**. Substitute $z = \dfrac{n_3}{c}$ into the 2nd equation to get y. Substitute both of these values into the 1st equation to get x. This is a systematic process which is relatively easy to adapt to a computer algorithm and let the computer do the number crunching.
Details are in the addendum at the end of the solution key.



B) $\left(a^x + a^{-x}\right)^2 = \left(a^{2x} + a^{-2x}\right) + 2 = K + 2 \Rightarrow a^x + a^{-x} = +\sqrt{K+2}$

Examining the graph of $y = a^x + a^{-x}$ (for $a > 0$), we see that $-\sqrt{K+2}$ is extraneous. Any vertical line selects corresponding points on the graphs of $y = a^x$ and $y = a^{-x}$. Adding these two values always produces a value greater than or equal to 1 + 1 = 2 (at P, the point of intersection).
For example, on the vertical dotted line, if $BC = AD$, then point C would represent the point on $y = a^x + a^{-x}$ for the selected x - value. Applying the same arguments to $y = a^{2x} + a^{-2x}$, we see that $K \geq 2$.

$a^{4x} + a^{-4x} = \left(a^{2x} + a^{-2x}\right)^2 - 2 = K^2 - 2$

Thus, $J = \sqrt{K+2} + \left(K^2 - 2\right)$. Consequently, ($k$ + 2) must be a perfect square.
Examining K = 2, 7, 14, 23, …, we get $(2^2 + 0)$, $(7^2 + 1)$, $(14^2 + 2)$, $(23^2 + 3)$… .
Continuing $34^2 + 4 < 2012$, but $47^2 + 5 = (50 - 3)^2 + 5 = 2500 - 300 + 9 + 5 = 2214 > 2012$.
Thus, (K, J) = **(47, 2214)**.