MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014 SOLUTION KEY

Round 2

A) Examine the remainders of consecutive powers of 6 divided by 7.

$6^1 = 6, 6^2 = 36, 6^3 = 216, 6^4 = 1296$

Dividing by 7, the corresponding remainders are 6, 1, 6, 1, … .

Thus, this pattern suggests that 6 raised to an odd power leaves a remainder of 6; an even power leaves 1. Therefore, 6^{153} divided by 7 leaves a remainder of **<u>6</u>**.

Alternate solution/proof (Modular arithmetic):

Note: The notation $a \equiv b \pmod{n}$ reads "*a* is congruent to *b* modulo *n*".

It is equivalent to saying "when *a* is divided by *n*, the remainder is *b*".

Convince yourself that this relation satisfies the transitive property, namely,

if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.

$6 \equiv -1 (\text{mod } 7) \Rightarrow 6^{153} \equiv (-1)^{153} = -1 (\text{mod } 7)$

By transitivity, $6^{153} \equiv \underline{\mathbf{6}} (\text{mod } 7)$.

B) The numbers –5, –4, –3, and –2 work, but are not positive. The LCM of 5, 4, 3, and 2 is 60. Adding 60 to each of these numbers, we have the desired 4 consecutive positive integers.

The smallest is 60 + (-5) = **<u>55</u>**.

55 = 5(11), 56 = 4(14), 57 = 3(19) and 58 = 2(29)

Alternate solution #1 (Algebraic tour de force):

Let the integers be *x*, *x* + 1, *x* + 2 and *x* + 3. Increase the first by 5, the second by 4, the third by 3 and the fourth by 2 to get *x* + 5, *x* + 5, *x* + 5 and *x* + 5.

Thus, *x* + 5 must be divisible by 5, 4, 3, and 2, that is, by 60.

The smallest such positive integer is 60 – 5 = **<u>55</u>**.

Alternate Solution #2 (Brute Force – the first number must be a multiple of 5):

5, <u>6</u>, 7, 8 - fails   10, <u>*11*</u>, 12, 13 - fails   15, 16, <u>*17*</u>, 18 – fails … 35, 36, <u>37</u>, 38 – fails

… 55, 56, 57, 58 – Bingo!

C) A number with exactly 15 factors must be of the form p^{14} or p^4q^2, where *p* and *q* are primes. The smallest two numbers of the first form are $2^{14} = 2^{10} \cdot 2^4 = 16384$ and 3^{14} which is considerably larger. The smallest numbers of the second form are

$2^4 3^2 = 144,\ 2^4 5^2 = 400,\ 3^4 2^2 = 324$ and the winner is **<u>324</u>**.

Alternately, in order to have an odd number of factors, an integer must be a perfect square.

$4 = 2^2 \Rightarrow 3$ factors $9 = 3^2 \Rightarrow 3$ factors $16 = 2^4 \Rightarrow 5$ factors $25 = 5^2 \Rightarrow 3$ factors

$36 = 2^2 3^2 \Rightarrow (2+1)(2+1) = 9$ factors

… $144 = 2^4 3^2 \Rightarrow (4+1)(2+1) = 15$ factors (1st)

… $\underline{\mathbf{324}} = 18^2 = 2^2 3^4 \Rightarrow (2+1)(4+1) = 15$ factors (2nd)