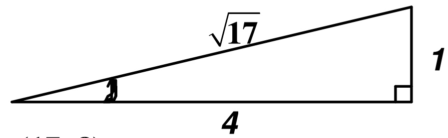
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2009 SOLUTION KEY

Round 3



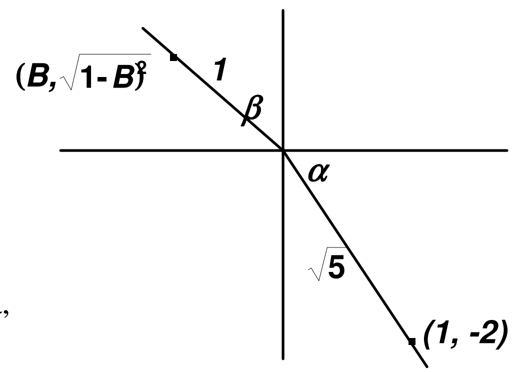
A) $\csc(2Arc\cot 4) = \csc\left(2Arc\tan\frac{1}{4}\right) =$

$$\frac{1}{2\sin\left(Arc\tan\frac{1}{4}\right)\cdot\cos\left(Arc\tan\frac{1}{4}\right)} = \frac{1}{2\left(\frac{1}{\sqrt{17}}\right)\left(\frac{4}{\sqrt{17}}\right)} = \frac{17}{8}$$ → **<u>(17, 8)</u>**

(Even though there are no $\boxed{\csc}$ or $\boxed{Arc\cot}$ buttons on a scientific calculator, with a little thought, the calculator gives 2.125 = 17/8 and the same result follows)

B) $$\frac{\sin 36°\sin 78° + \cos 36°\sin 12°}{\cos 72°\sin 66° + \sin 72°\sin 24°} = \frac{\sin 36°\cos 12° + \cos 36°\sin 12°}{\cos 72°\cos 24° + \sin 72°\sin 24°}$$

Since the numerator and denominator are expansions of sin(A + B) and cos(A – B) respectively , we have $\frac{\sin(36°+12°)}{\cos(72°-24°)} = \frac{\sin 48°}{\cos 48°} = \tan 48°$ Over the specified range, we have two answers, namely x = **<u>48 , 228</u>** (or **<u>48° , 228°</u>**).



C) Let $\alpha = Arc\sin\left(-\frac{2}{\sqrt{5}}\right)$ and $\beta = Arc\cos(B)$, where $B < 0$.

Because of principle values, α must be in quadrant 4 and β must be in quadrant 2, as diagrammed to the right.

If x = 135, $\tan\left(Arc\sin\left(-\frac{2}{\sqrt{5}}\right) - Arc\cos B\right) = \cot(180° + x)$ simplifies to $\tan(\alpha - \beta) = -1$. Applying the expansion formula,

$\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = -1$. Substituting, $$\frac{-2 - \left(\frac{\sqrt{1-B^2}}{B}\right)}{1 + (-2)\left(\frac{\sqrt{1-B^2}}{B}\right)} = \frac{1}{-1}$$

Cross multiplying, $1 - 2\left(\frac{\sqrt{1-B^2}}{B}\right) = 2 + \left(\frac{\sqrt{1-B^2}}{B}\right)$ → $B - 2\sqrt{1-B^2} = 2B + \sqrt{1-B^2}$

→ $3\sqrt{1-B^2} = -B$ (Note: B is negative, so we don't have a contradiction, $-B$ is <u>positive</u>!)

→ $9\left(1-B^2\right) = B^2$ → $10B^2 = 9$ → $B = \pm\frac{3}{\sqrt{10}}$

Since B < 0, we have, in rationalized form, $B = \frac{-3\sqrt{10}}{10}$ → (P, Q) = **<u>(–3, 10)</u>**

Note: (3, –10) is unacceptable since it would correspond to $\frac{3\sqrt{-10}}{-10}$.