MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2009 SOLUTION KEY

Team Round

A) $\left|\sqrt{z}\cdot\sqrt[3]{z^2}\cdot\sqrt[6]{z^5}\right| = \left|z^{\frac{1}{2}}\cdot z^{\frac{2}{3}}\cdot z^{\frac{5}{6}}\right| = \left|z^{\frac{1}{2}+\frac{2}{3}+\frac{5}{6}}\right| = \left|z^{\frac{6+8+10}{12}}\right| = \left|z^2\right|$

$z^2 = 1-2\sqrt{3}i-3 = -2-2\sqrt{3}i$ ➔ $\left|z^2\right| = \sqrt{4+12} = \underline{\mathbf{4}}$

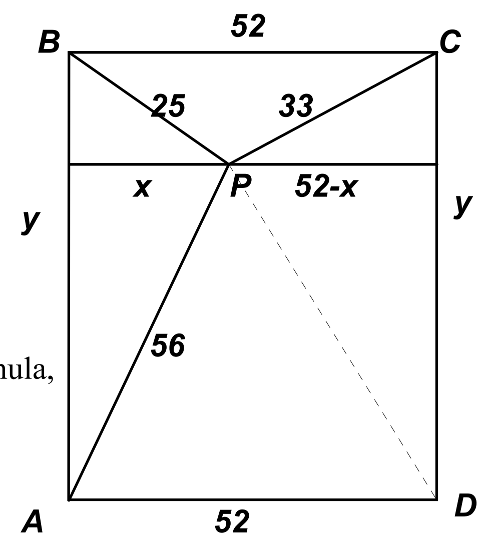
B) Let $\frac{1}{x}$ denote the rate at which the brick mason works, i.e. the fraction of the job he does in one hour. Then $\frac{4}{x}+\frac{(4+3)}{x+6}=1$ ➔ $4(x + 6) + 7x = x^2 + 6x$ ➔ $x^2 - 5x - 24 = (x - 8)(x + 3) = 0$
➔ $x = 8$ Thus, the mason and apprentice take 8 hours and 14 hours respectively to complete the job. Assume a minimum of A apprentices are needed

$\frac{1}{8}(1)+A\left(\frac{1}{14}\right)(1)\geq 1$ ➔ $\frac{A}{14}\geq\frac{7}{8}$ ➔ $A\geq\frac{98}{8}=12.25$ ➔ $A_{\min} = \underline{\mathbf{13}}$

[12 apprentices and 1 brick mason take T hours to finish

$\frac{1}{8}T+12\left(\frac{1}{14}T\right)=1$ ➔ $\frac{T}{8}+\frac{6T}{7}=1$ ➔ $7T + 48T = 56$

➔ $T = 56/55 > 1$]



C) $25^2 + PD^2 = 33^2 + 56^2$ ➔ $PD^2 = 3600$
➔ $PD = 60$ (Refer to note on Contest 1 Round 2.) Using Heron's formula,

Area(ΔPBC) = $\sqrt{55(30)(22)(3)} = \sqrt{3^2\cdot 11^2\cdot 10^2} = 330$

Area(ΔPAD) = $\sqrt{84(28)(32)(24)} = \sqrt{2^{12}\cdot 3^2\cdot 7^2} = 1344$

Let $y = AB = CD$.

Area(rectangle) = $52y = 1674+\frac{1}{2}xy+\frac{1}{2}y(52-x) = 1674 + 26y$ ➔ $26y = 1674$ ➔ $y = AB = \underline{\mathbf{\frac{837}{13}}}$

D) Hoping to take advantage of a binomial of the form $\left(a^x+c\right)$, where c is a constant and thinking of Pascal's triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$a^{4x}-4a^{3x}+a^{2x}+6a^x = a^{4x}-4a^{3x}+(6-5)a^{2x}+(10-4)a^x+(1-5+4)$

Regrouping, we have $\left(a^{4x}-4a^{3x}+6a^{2x}-4a^x+1\right)-5\left(a^{2x}+2a^x+1\right)+4$

$= \left(a^x-1\right)^4-5\left(a^x-1\right)^2+4 = \left(\left(a^x-1\right)^2-1\right)\left(\left(a^x-1\right)^2-4\right)$

$= \left(a^x-1+1\right)\left(a^x-1-1\right)\left(a^x-1+2\right)\left(a^x-1-2\right) = \underline{\boldsymbol{a^x\left(a^x-2\right)\left(a^x+1\right)\left(a^x-3\right)}}$