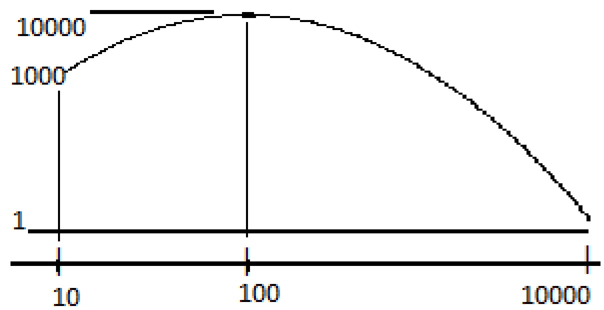
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013 SOLUTION KEY

Team Round

D) $f(x) = x^{4-\log_{10} x}$ By inspection, we are tempted to let x = 10, getting M = $f(10) = 10^{4-1} = 1000$, let x = 10000, getting N = $f(10000) = 4^{4-4} = 1$ and assume these are the maximum and minimum values respectively. However, this is not the case.

Taking the logarithm (base 10) of both sides, we have
$$\log_{10} f(x) = (4 - \log_{10} x)\log_{10} x$$
$$= -(\log_{10} x)^2 + 4\log_{10} x$$

This is a quadratic expression in $\log_{10} x$. For clarity, let $A = \log_{10} x$. Completing the square, we have $-A^2 + 4A = -\left(A^2 - 4A \underline{+4}\right)\underline{+4} = -(A-2)^2 + 4$ Think parabola opening down with maximum value at the vertex (2, 4). Thus, the function attains a maximum when $A = \log_{10} x = 2$ or x = 100.



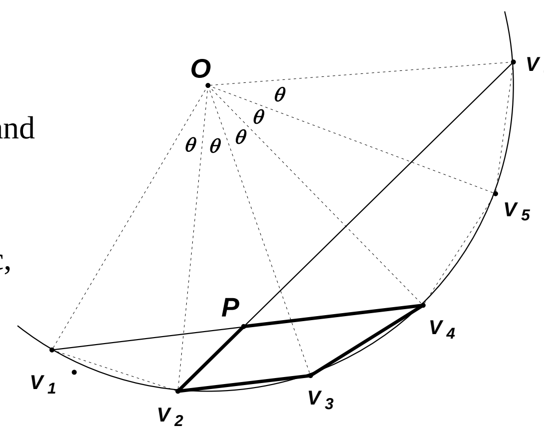
$f(100) = 100^{4-2} = 10000$ and $\frac{M}{N} = \underline{\mathbf{\frac{1}{10000}}}$.

E) Joint variation implies as the product.
Let A_1 and A_2 denote the areas of the transformed quadrilaterals.
The new area A_1 is given by $k\left(9n^2 D\right)S$, while $A_2 = kD(27n-20)S$

$$A = A' \Leftrightarrow \cancel{k}(9n^2\cancel{D})\cancel{S} = \cancel{k}\cancel{D}(27n-20)\cancel{S} \Rightarrow$$

$$9n^2 - 27n + 20 = 0 \Leftrightarrow (3n-4)(3n-5) = 0 \Rightarrow n = \underline{\mathbf{\frac{4}{3}, \frac{5}{3}}}$$

F) If there are 17 diagonals from each vertex, then n – 3 = 17 and the polygon has 20 sides.
Here's a sketch of the pertinent portion of the 20-gon.



Each side of the 20-gon subtends (cuts off) $\frac{360}{20} = 18°$ of arc, i.e. $\frac{1}{20}$ of the circumscribed circle. Therefore, each central angle θ measures 18°. We require $m\angle V_2PV_4$.

Consider quadrilateral $PV_2V_3V_4$. At V_2, the inscribed angle $V_6V_2V_3$ measures $\frac{1}{2}(3 \cdot 18) = 27$.

At V_3, $m\angle V_2V_3V_4 = \frac{1}{2}(18 \cdot 18) = 162$. At V_4, $m\angle V_1V_4V_3 = \frac{1}{2}(2 \cdot 18) = 18$.

Therefore, $m\angle P = (360 - 27 - 162 - 18) = 360 - 207 = \underline{\mathbf{153}}°$.
Alternate solution: (Angle formed by two chords in a circle)
Let x denote an arc cut off by two successive vertices of the 20-gon. $x = 360/20 = 18$
The pair of vertical angles at P in which we are interested cut off (subtend) arcs of $2x$ and $15x$.

$$m\angle P = \frac{1}{2}(2 \cdot 18 + 15 \cdot 18) = \frac{1}{2} \cdot 17 \cdot 18 = 17 \cdot 9 = \underline{\mathbf{153}}.$$