MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017 SOLUTION KEY

Round 3

A) $1+\tan^2 x \Leftrightarrow 1+\dfrac{\sin^2 x}{\cos^2 x}=\dfrac{\cos^2 x+\sin^2 x}{\cos^2}=\dfrac{1}{\cos^2 x}=\underline{\dfrac{\mathbf{1}}{\mathbf{1-\sin^2 x}}}$

B) $\sin 2\theta=\tan\theta \;\Leftrightarrow\; 2\sin\theta\cos\theta-\dfrac{\sin\theta}{\cos\theta}=0 \Leftrightarrow$

$$\frac{2\sin\theta\cos^2\theta-\sin\theta}{\cos\theta}=\frac{\sin\theta\left(2\cos^2\theta-1\right)}{\cos\theta}=\tan\theta\left(2\cos^2\theta-1\right)=0$$

$\Rightarrow \tan\theta=0 \Rightarrow \theta=\underline{\mathbf{0,\ \pi}}$

$\Rightarrow \cos\theta=\pm\dfrac{\sqrt{2}}{2} \Rightarrow \theta=\underline{\dfrac{\boldsymbol{\pi}}{\mathbf{4}},\dfrac{\mathbf{3}\boldsymbol{\pi}}{\mathbf{4}},\dfrac{\mathbf{5}\boldsymbol{\pi}}{\mathbf{4}},\dfrac{\mathbf{7}\boldsymbol{\pi}}{\mathbf{4}}}$

C) $\text{Arccos}(x)+2\text{Arcsin}(-1)=-\dfrac{\pi}{6}$

$2\text{Arcsin}(-1)=2\left(-\dfrac{\pi}{2}\right)=-\pi$ (Since, by definition, $\text{Arcsin}\,A=x \Leftrightarrow \sin x=A$ and $-\dfrac{\pi}{2}\le x\le\dfrac{\pi}{2}$)

Solution #1: (Appealing only to the definition of inverse functions)

Substituting, $\text{Arccos}(x)\ +\ 2\text{Arcsin}(-1)\ =-\pi/6 \Leftrightarrow \boxed{\text{Arccos}(x)=\dfrac{5\pi}{6}}$.

Since $\text{Arccos}\,A=x \Leftrightarrow \cos x=A$ and $0\le A\le\pi$,

$\text{Arccos}(x)=\dfrac{5\pi}{6} \Leftrightarrow x=\cos\left(\dfrac{5\pi}{6}\right) \Leftrightarrow x=\underline{-\dfrac{\sqrt{\mathbf{3}}}{\mathbf{2}}}$.

Solution #2: (Taking the sine of both sides)
Let θ = Arccos(x). By definition, this is equivalent to $x=\cos\theta$ and $0\le\theta\le\pi$.

Thus, $\sin\left(\text{Arccos}(x)+2\text{Arcsin}(-1)\right)=\sin\left(-\dfrac{\pi}{6}\right) \Leftrightarrow \sin(\theta-\pi)=-\dfrac{1}{2}$.

Since the sine is an <u>odd</u> function, $\sin(\theta-\pi)=-\sin(\pi-\theta)=-\sin(\theta)$

$\Rightarrow\ \sin\theta=\dfrac{1}{2}\ \Rightarrow\ \theta=\dfrac{\pi}{6},\dfrac{5\pi}{6}\ \Rightarrow\ x=\cos\theta=\pm\dfrac{\sqrt{3}}{2}$ (but the positive value is extraneous).

Why is that?

$\text{Arc}\cos\left(-\dfrac{\sqrt{3}}{2}\right)=\dfrac{5\pi}{6}$ and $\dfrac{5\pi}{6}-\pi=-\dfrac{\pi}{6}$ [or $150^\circ-180^\circ=-30^\circ$]

$\text{Arc}\cos\left(\dfrac{\sqrt{3}}{2}\right)=\dfrac{\pi}{6}$ and $\dfrac{\pi}{6}-\pi\ne-\dfrac{\pi}{6}$ [or $30^\circ-180^\circ\ne-30^\circ$]

Thus, $x=\underline{-\dfrac{\sqrt{\mathbf{3}}}{\mathbf{2}}}$ only.