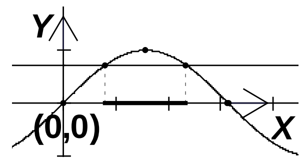
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012 SOLUTION KEY

Round 3

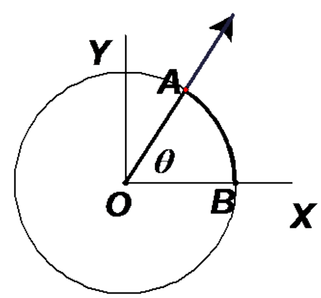
A) $3x = 330° + 360n$ or $210° + 360n \Rightarrow x = 110° + 120n$ or $70° + 120n$
$n = -1 \Rightarrow x =$ **$-10°, -50°$**

B) $3(\cot x + \csc x) = 3\left(\dfrac{\cos x + 1}{\sin x}\right) = 2\sin x \quad (x \neq 0° + 180n)$
$\Rightarrow 3\cos x + 3 = 2\sin^2 x = 2 - 2\cos^2 x$
$\Rightarrow\ 2\cos^2 x + 3\cos x + 1 = (2\cos x + 1)(\cos x + 1) = 0$
$\Rightarrow\ \cos x = -\dfrac{1}{2}, -1 \Rightarrow$ **120°, 240°**, (180° is extraneous.)

C) $(\cos^2 x - \sin^2 x) - 1 + \sin^2 x < -0.5 \Rightarrow \cos^2 x - 1 < -0.5 \Rightarrow \sin^2 x > 0.5$
Since $\sin x$ is always positive over the specified interval,
we have $\sin x > \dfrac{\sqrt{2}}{2}$ and we know that $\sin 45° = \dfrac{\sqrt{2}}{2}$.
Appealing to the graph of $y = \sin x$, we have **$45° < x < 135°$**.



[In the graph above, x is measured in radians, so the first point of intersection occurs between 0 and 1 $\left(45° = \dfrac{\pi}{4}^{rad} \approx 0.785 \text{ radians}\right)$, the second point of intersection occurs between 2 and 3 $\left(135° = \dfrac{3\pi}{4}^{rad} \approx 2.356 \text{ radians}\right)$ and the function intersects the x-axis at $\left(180° = \pi^{rad} \approx 3.142 \text{ radians}\right)$.]



For those unfamiliar with radian measure, you may want to discuss the following with your coach or a teammate.
$m\angle AOB = \theta = 1$ radian if and only if the radius of the circle ($\overline{OA}$) has the same length as the intercepted (minor) arc ($\overset{\frown}{AB}$). Imagine a series of concentric circles, with center at O. In all cases, the intercepted arc has the same length as the radius and the central angle measures 1 radian! So to keep things simple, consider the unit circle (radius 1).
The circumference of the unit circle is 2π.
Thus, 2π copies of central angle AOB complete 1 revolution or 360°.
Dividing by 2, we have the equivalence $\boxed{180° = \pi^{\text{ rad}}}$ or $\boxed{1^{\text{rad}} = \dfrac{180}{\pi}°}\left(\approx 57.296° \approx 57°17'44.8''\right)$.
Radian measure of angles allows us to use the same scale on both axes in the graph of $y = \sin x$ above!