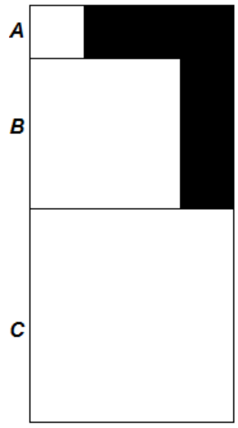
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2015 SOLUTION KEY



Team Round - continued

D) The area of the entire region is
$A^2+B^2+C^2+910=C(A+B+C)=AC+BC+C^2$

Cancelling and transposing terms, $910=\left(AC+BC\right)-\left(A^2+B^2\right)$

Factoring and substituting for A^2+B^2, $C(A+B)-C^2=910$

$\Rightarrow C\left((A+B)-C\right)=C\left(A-(C-B)\right)=(B+2)(A-2)=910$

Examining a table of Pythagorean Triples, where the difference between the lengths of the hypotenuse and the longer leg is 2:
A is increasing by 2, the *B*-gap increases by 2 each time, *C* = *B* + 2
Every other row is a ***primitive*** Pythagorean Triple.

	A	B	C	(B+2)(A-2)	****
1	6	8	10	10·4	40
2	8	15	17	17·6	102
3	10	24	26	26·8	208
4	12	35	37	37·10	370
5	14	48	50	50·12	600
6	16	63	65	65·14	**910**
7	18	80	82	17·6	
8	20	99	101		

Thus, $A+B+C$ = 16 + 63 + 65 = **144**. Note (as a check) that the area of the rectangle as the sum of the areas of 4 regions $16^2+63^2+65^2+910=256+3969+4225+910=9360$ and as a length times width computation $65\left(16+63+65\right)=65\left(144\right)=9360$ are equal.

Solution #2 (Norm Swanson – Hamilton Wenham - retired)
Alternately, without resorting to a table, we could think of $\left(B+2\right)\left(A-2\right)=910=2\cdot5\cdot7\cdot13$ as

$\dfrac{\left(B+2\right)\left(A-2\right)}{10}=7\cdot13$. Since $B>A$, we try $\dfrac{B+2}{5}=13$ and $\dfrac{A-2}{2}=7$.

This gives us $B=65-2=63$ and $A=14+2=16$ and $\left(A,B,C\right)=\left(16,63,65\right)$ works!
Think about the second step.
A keen number sense inspired this insight, along with the recognition that 7 and 13 are primes.
[In honesty, we could have considered $91=13\cdot7$ or $91\cdot1$ and $10=5\cdot2$ or $10\cdot1$.]
None of the other possibilities produces a *C*-value which satisfies the P.T. Check it out.

The formula for the areas listed in the rightmost column of the chart above is $2\left(n+1\right)\left(n^2+4n+5\right)$

which can be written as $2\left(n+1\right)\left[\left(n+2\right)+i\right]\left[\left(n+2\right)-i\right]$. $n=6\Rightarrow 2\cdot7\cdot\left(8+i\right)\left(8-i\right)=14\cdot65=910$

If the *B/C* relation were $C=B+1$ (instead of $C=B+2$), the first few triples and areas would be $\left(3,4,5,10\right)$, $\left(5,12,13,52\right)$, $\left(7,24,25,150\right)$,… . Can you determine a formula for the area?