MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2015 SOLUTION KEY

Team Round

A) Solving the second equation for *y*, $y = \dfrac{x-4}{3x}$.

Substituting in the first equation,

$$x^2 - \frac{x-4}{3} + \frac{x^2-8x+16}{9x^2} = 7 \Rightarrow 9x^4 - 3x^2(x-4) + x^2 - 8x + 16 = 63x^2 \Rightarrow 9x^4 - 3x^3 - 50x^2 - 8x + 16 = 0$$

We know that there is an integer root! By synthetic division $\begin{array}{r|rrrrr} & 9 & -3 & -50 & -8 & 16 \\ \hline -2 & 9 & -21 & -8 & 8 & 0 \end{array}$, we confirm $x = -2$ is a root and the quotient is $9x^3 - 21x^2 - 8x + 8$.

Continuing synthetic division of this quotient, we watch for the remainder to change sign.

$$\begin{array}{r|rrrr} & 9 & -21 & -8 & 8 \\ \hline -1 & 9 & -30 & 22 & -14 \\ 0 & 9 & -21 & -8 & 8 \\ 1 & 9 & -12 & -20 & -12 \\ 2 & 9 & -3 & -14 & -20 \\ 3 & 9 & 6 & 10 & 38 \end{array}$$

Thus, we have roots in the intervals $(-1, 0)$, $(0, 1)$ and $(2, 3)$.

This gives us $b = -1$, $c = 0$, $d = 2$, and the required product $ad = \underline{\mathbf{-4}}$.

B) $4^{2x+a} = 8^{5-bx} \Leftrightarrow 2^{4x+2a} = 2^{15-3bx}$

Equating the exponents, transposing terms and solving for *x*, $4x + 2a = 15 - 3bx \Rightarrow x = \dfrac{15-2a}{3b+4}$

Substituting $a = 2b$, $x = \boxed{\dfrac{-4b+15}{3b+4}} = \dfrac{-4+\dfrac{15}{b}}{3+\dfrac{4}{b}}$. Clearly, $b = -1$ gives division by -1 and *x* will be an integer, namely $\dfrac{-19}{-1} = \underline{\mathbf{19}}$. For even values of *b*, the numerator of the boxed expression is odd, while the denominator is even, and *x* will not be an integer.

Looking at the complex fraction expression for *x*, as $b \to \pm\infty$, we see that $x \to -\dfrac{4}{3}$.

So we must consider "small" odd values of *b*.

Wanting to avoid mindless plug and chug, we try long division, $\begin{array}{r} -4/3 \\ (3b+4)\overline{)-4b+15} \\ \underline{-4b-16/3} \\ 61/3 \end{array}$, getting

$x = -\dfrac{4}{3} + \dfrac{61}{3(3b+4)}$. Since 61 is prime, we have limited choices for a divisor.

If $b = 19$, then $x = -\dfrac{4}{3} + \dfrac{1}{3} = \underline{\mathbf{-1}}$.