MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2007 SOLUTION KEY

Team Round – continued

D) Factoring out the common factor of a^{2x} and noticing that there are 6 terms with alternating signs and the first and last terms are perfect fifth powers, we check to see if the expansion is the fifth power of a binomial.

$$a^{2x}\left(\left(2a^x\right)^5 - 240a^{4x} + 720a^{3x} - 1080a^{2x} + 810a^x - 3^5\right)$$

$$a^{2x}\left(\left(2a^x\right)^5 - (5)(2a^x)^4(3) + (10)(2a^x)^3(9) - (10)(2a^x)^2(27) + (5)(2a^x)(81) - 3^5\right)$$

$$= \underline{\boldsymbol{a^{2x}\left(2a^x - 3\right)^5}}$$

E) $\sin(4x) = \sin(2(2x)) = 2\sin(2x)\cos(2x) = 2[2\sin(x)\cos(x)][1 - 2\sin^2(x)]$
$= 4\sin(x)\cos(x) - 8\sin^3(x)\cos(x) \rightarrow a = 4$ and $b = -8 \rightarrow a + 3b =$ **<u>-20</u>**

or let $x = \pi/4 \rightarrow 0 = 2a + b$ Then let $x = \pi/6 \rightarrow 4a + b = 8 \rightarrow (a, b) = (4, -8) \rightarrow a + 3b =$ **<u>-20</u>**

Note: $(X + Y)^5 = X^5 + \binom{5}{1}X^4Y^1 + \binom{5}{2}X^3Y^2 + \binom{5}{3}X^2Y^3 + \binom{5}{4}X^1Y^4 + Y^5$

where $\binom{n}{r}$ denotes a combination of n items taken r at a time and is evaluated by $\dfrac{n!}{r!\cdot(n-r)!}$.

F) Since the interior and exterior angles in a regular polygon with n sides are given by $\dfrac{180(n-2)}{n}$ and $\dfrac{360}{n}$ respectively, the ratio of the interior angle to the exterior angle is $(n - 2) : 2$.

Let P and Q have n and m sides respectively.
$\dfrac{n-2}{2} = \dfrac{a}{b} \rightarrow bn - 2b = 2a \rightarrow n = \dfrac{2(a+b)}{b} = \dfrac{30}{b}$ and b must be a factor of 30 (and $b \leq 15$)
Thus, $(a, b) =$ (14, 1) corresponding to $n = 30$ or (13, 2) corresponding to $n = 15$.
All other ordered pairs (12, 3), (10, 5), (9, 6) and (10, 5) correspond to <u>un</u>reduced ratios.
Possible interior angles of P are: 168° (for a 30-gon) or 156° (for a 24-gon)

$\dfrac{2}{m-2} = \dfrac{c}{d} \rightarrow cm - 2c = 2d \rightarrow m = \dfrac{2(c+d)}{c} = \dfrac{24}{c}$ and c must be a factor of 24 (and $c \leq 12$)
Thus, $(c, d) = (1,11)$ corresponding to $m = 24$.
All other ordered pairs (2, 10), (3, 9), (4, 8), (56, 6) and (8, 4) correspond to <u>un</u>reduced ratios.
The only possible exterior angle for Q: 15°
Therefore, possible ratios are: **<u>56 : 5</u>** and **<u>52 : 5</u>**