MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2013 SOLUTION KEY

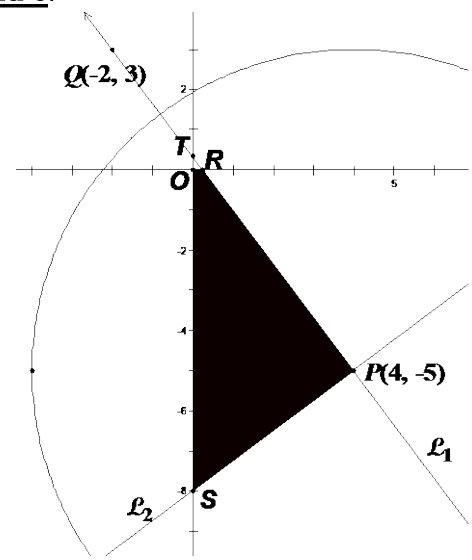
Team Round

B) continued

Method #3: In general, $a \heartsuit (b \diamondsuit c) = (a \heartsuit b) \diamondsuit (a \heartsuit c) \Leftrightarrow \dfrac{ab+ac+2bc}{2(b+c)} = \dfrac{(a+b)(a+c)}{2a+b+c}$

Instead of substituting for 6 special cases, algebraically manipulate the equality.
Cross multiplying, we have equality if and only if
$(ab + ac + 2bc)(2a + b + c) = 2(b + c)(a + b)(a + c)$ or
$\cancel{2a^2b} + \cancel{2a^2c} + 4abc + ab^2 + abc + \underline{2b^2c} + abc + ac^2 + \underline{\mathbf{2bc^2}}$ =
$2(a^2 + ac + ab + bc)(b + c) = \cancel{2a^2b} + 2abc + 2ab^2 + \underline{2b^2c} + \cancel{2a^2c} + 2ac^2 + 2abc + \underline{\mathbf{2bc^2}}$
$\Leftrightarrow 6abc + ab^2 + ac^2 = 4abc + 2ab^2 + 2ac^2$
$\Leftrightarrow 0 = -2abc + ab^2 + ac^2$
$\Leftrightarrow 0 = a(b^2 - 2bc + c^2) = 0$
$\Leftrightarrow 0 = a(b-c)^2 \Leftrightarrow a = 0$ or $b = c$
Thus, the distributive property is satisfied under conditions **<u>1 and 6</u>**.

C) $x^2 + y^2 - 8x + 10y - 23 = 0 \Leftrightarrow (x-4)^2 + (y+5)^2 = 64$



Since the line $\mathcal{L}$ must divide the circle into 2 semi-circles, it must pass through the center of the circle. Thus, $\mathcal{L}_1$ passes through $Q(-2, 3)$ and $P(4, -5)$. Its equation is

$$(y-3) = \frac{-5-3}{4-(-2)}(x+2) \Leftrightarrow y-3 = \frac{-4}{3}(x+2) \Leftrightarrow 4x+3y=1$$

(slope $-\frac{4}{3}$). $\mathcal{L}_2$ has slope $+\frac{3}{4}$, passes through $(4, -5)$ and has equation $3x-4y=32$. The y-intercepts are $\frac{1}{3}$ and -8.

The area of quadrilateral $PROS$ equals the area of ΔPTS minus the area of ΔORT, namely,

$$\frac{1}{2}\cdot\left(\frac{20}{3}\right)\cdot 5 - \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{3} = \frac{50}{3} - \frac{1}{24} = \frac{399}{24} = \underline{\mathbf{\frac{133}{8}}} = \underline{\mathbf{16.625}}.$$

The following procedure works for <u>any convex polygon</u> whose vertices are known.
Start at any vertex and list the vertices in order (clockwise or counterclockwise – your choice).
Repeat the coordinates of the starting vertex. The area is given by ***half the absolute value of the sum of the downward diagonal products minus the sum of the upward diagonal products***.

$$\frac{1}{2}\begin{vmatrix} 0 & 0 \\ 1/4 & 0 \\ 4 & -5 \\ 0 & -8 \\ 0 & 0 \end{vmatrix} \Rightarrow \frac{1}{2}\left|\left(0\cdot 0 + \frac{1}{4}\cdot -5 + 4\cdot -8 + 0\cdot 0\right) - \left(0\cdot -8 + 0\cdot -5 + 4\cdot 0 + \frac{1}{4}\cdot 0\right)\right| = \frac{1}{2}\left|-\frac{5}{4} - 32\right| = \underline{\mathbf{\frac{133}{8}}} = \underline{\mathbf{16.625}}$$