MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2013 SOLUTION KEY

Team Round

A) $\begin{cases} f(-1) = -1 \Rightarrow \dfrac{-A+B+9}{3} = -1 \Rightarrow A - B = 12 \\ f(5) = 27 \;\; \Rightarrow \dfrac{125A+25B-27}{99} = 27 \Rightarrow 125A + 25B = 99(27) + 27 = 27(99+1) = 2700 \\ \qquad\qquad\;\; \Rightarrow 5A + B = 108 \end{cases}$

Thus, $(A, B) = (20, 8)$. By long division, $\dfrac{20x^3 + 8x^2 - 6x + 3}{4x^2 - 1} = 5x + 2 + \dfrac{-x+5}{4x^2-1}$.

As $x \to \infty$, the value of the fractional third term approaches zero, since the degree of the denominator is 1 more than the degree of the numerator implying the denominator grows much faster than the numerator.

Therefore, the functional values become closer and closer to $y = 5x + 2$ and $(m, b) =$ **<u>(5, 2)</u>**.

Is the graph of the function above or below the line as $x \to \infty$? For other values of x?

Think about it and then look at the graph at the end of this solution key.

B) The test: A 4-digit integer <u>*a*</u> <u>*b*</u> <u>*c*</u> <u>*d*</u> is divisible by 11 <u>*if and only if*</u> $(a + c) - (b + d)$ is divisible by 11, that is, equal to either 0 or 11. Sum digits in even positions, sum digits in odd positions and then subtract.

The prime digits are 2, 3, 5 and 7.

Case 1: (all digits the same)

All <u>4</u> possibilities have $(a + c) - (b + d) = 0$ and are divisible by 11.

Case 2: (3 digits same) - $4 \cdot 3 = 12$ digit selections $\Rightarrow$ x 4 arrangements $\Rightarrow$ 48 possible N-values

For <u>any</u> integer of this form, for example, <u>*x*</u> <u>*x*</u> <u>*x*</u> <u>*y*</u>, $(a + c) - (b + d) = |x - y|$.

The minimum and maximum differences are 1 and 5 respectively.

<u>None</u> of these N-values are divisible by 11.

Case 3: (2 digits same, 2 different) - 6 digit selections $\Rightarrow$ x 12 $\Rightarrow$ 72 possible N-values

For 2235, 2237, 2257 min = 2, 4, 2 (sum of largest and smallest minus sum of other two)

max = 4, 6, 8 (sum largest two – sum smallest two) $\Rightarrow$ <u>None</u>

For 3325, 3327, 3357 min = 1, 1, 2 max = 3, 5, 6 $\Rightarrow$ <u>None</u>

For 5523, 5527, 5537 min = 1, 1, **<u>0</u>** max = 5, 5, 4 $\Rightarrow$ <u>4</u> (7535, 3575, 5357, 5753)

For 7723, 7725, 7735 min = 1, 3, 2 max = 9, 7, 6 $\Rightarrow$ <u>None</u>

Case 4: (2 pairs of digits the same) - 6 digit selections $\Rightarrow$ x 6 $\Rightarrow$ 36 possible N-values

Consider one of the 6 selections, e.g., <u>2</u> <u>2</u> <u>3</u> <u>3</u>. The required difference is either $6 - 4 = 2$ or $5 - 5 = 0$. There are 4 arrangements giving a difference of 0, namely <u>2</u> <u>3</u> <u>3</u> <u>2</u>, <u>3</u> <u>2</u> <u>2</u> <u>3</u>, <u>2</u> <u>2</u> <u>3</u> <u>3</u>, <u>3</u> <u>3</u> <u>2</u> <u>2</u>. This is true for each of the 6 selections. Thus, there are <u>24</u> possible N-values.

Case 5: (all digits different) - $4! = 24$ possibilities

Using all distinct digits 2, 3, 5 and 7, the minimum difference $9 - 8 = 1$ and the maximum difference is $12 - 5 = 7$. Therefore, <u>none</u> of these are possible N-values.

Thus, the total is $0 + 4 + 4 + 24 + 0 =$ **<u>32</u>**.