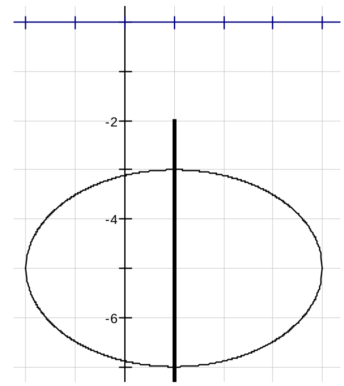
MASSACHUSETTS MATHEMATICS
CONTEST 6 - MARCH 2013 SOLUTION KEY



Team Round - continued

C) $\begin{cases} x = 3\sin(t)+1 \\ y = 2\cos(t)-5 \end{cases} \Rightarrow \left(\dfrac{x-1}{3}\right)^2 = \sin^2(t)$ and $\left(\dfrac{y+5}{2}\right)^2 = \cos^2(t)$

Adding, we have the equation of a **<u>semi-ellipse</u>**, namely

$\dfrac{(x-1)^2}{9} + \dfrac{(y+5)^2}{4} = 1$, where $1 \le x \le 4$

(since $0 \le t \le \pi$ (not 2π)). Thus, the graph is the right half of the ellipse.
The center is at (1, –5) and the major axis is horizontal, a = 3 and b = 2.
The major axis connects (–2, –5) and (4, –5) , and X_M = 4, but X_m = +1.
The minor axis connects (1, –3) and (1, –7) , Y_M = –3 and X_m = –7.
Thus, $X_M Y_M - X_m Y_m$ = –12 + 7 = **<u>–5</u>**.

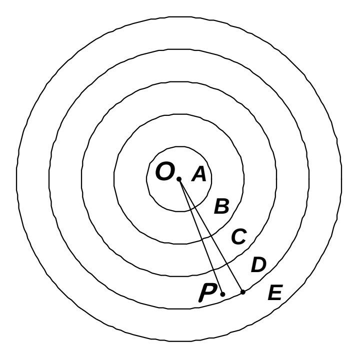
Alternatively, by inspection, the largest and smallest values of x are 1+3(1) = 4 and 1+3(0) = +1.
The largest and smallest values of y are –5±2(1) = –3 and –7. Thus, $X_M Y_M - X_m Y_m$ = **<u>–5</u>**.

D) Assume the club wins N of the remaining k games, i.e. suffers $k - N$ losses. Then:

$\dfrac{72+N}{110+k} > 0.700 = \dfrac{7}{10} \Leftrightarrow 720 + 10N > 770 + 7k \Leftrightarrow N > \dfrac{7k+50}{10}$ and

$k - N \ge 10 \Leftrightarrow N \le k - 10$ By the transitive property, $k - 10 > \dfrac{7k+50}{10} \Leftrightarrow 10k - 100 > 7k + 50 \Leftrightarrow k > 50$

Thus, $k_{\min}$ = 51 and our best record is attained if we lose only 10 of these, i.e. win 41 more games. (g, W) = (51, 72 + 41) = **<u>(51, 113)</u>**.
Check: $113/161 = 0.701^+$ and $112/161 = 0.695^+$



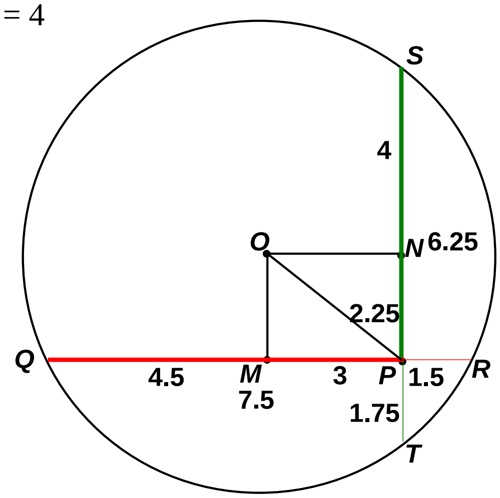
E) On $\overline{ST}$, the 25 : 7 ratio on chord of length 8 $\Rightarrow 32x = 8 \Rightarrow 6\frac{1}{4}, 1\frac{3}{4}$

On $\overline{QR}$, the 5 : 1 ratio on chord of length 9 $\Rightarrow 6x = 9 \Rightarrow 7\frac{1}{2}, 1\frac{1}{2}$

Let O be the center of the circle of radius 5, P be the intersection point of the two chords of lengths 8 and 9.

Since a radius drawn perpendicular to any chord bisects the chord, let M and N be the midpoints. $QM = RM = 4.5$ and $SN = NT = 4$
Applying the Pythagorean Theorem in ΔOMP,

$$2.25^2 + 3^2 = \left(\frac{9}{4}\right)^2 + 9 = \frac{81+144}{16} = \frac{225}{16} \Rightarrow OP = \frac{15}{4} = 3\frac{3}{4}$$



Thus, the dart fell in region D, $\frac{1}{4}$ unit inside the circle of radius 4, implying (k, d) = $\underline{\left(\boldsymbol{D, \frac{1}{4}}\right)}$. **Think we are home free? <u>Read on!</u>**