MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2009 SOLUTION KEY

Team Round

E) $CE = ED = DF = 6$, $BE = 6\sqrt{5}$, $EF = 6\sqrt{2}$

Altitude $BG = \sqrt{BE^2 - EG^2} = \sqrt{180-18} = \sqrt{162} = 9\sqrt{2}$

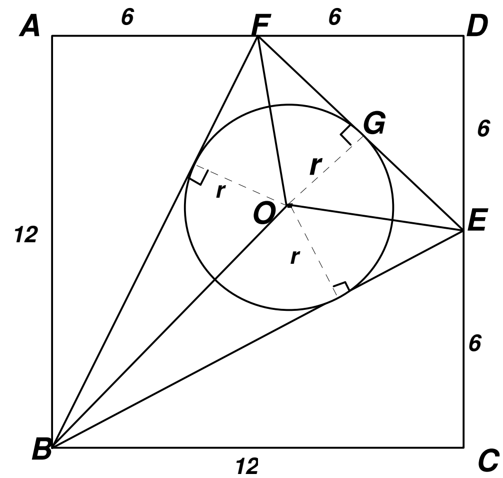
Area(ΔBEF) = $\frac{1}{2}\cdot 9\sqrt{2}\cdot 6\sqrt{2} = 54$

But Area(ΔBEF) = $\frac{1}{2}\cdot 6\sqrt{5}\cdot r + \frac{1}{2}\cdot 6\sqrt{5}\cdot r + \frac{1}{2}\cdot 6\sqrt{2}\cdot r$

$= 3r(2\sqrt{5}+\sqrt{2})$

Equating and solving for r,

$r = \frac{18}{(2\sqrt{5}+\sqrt{2})} = \frac{18(2\sqrt{5}-\sqrt{2})}{20-2} = \underline{\mathbf{2\sqrt{5}-\sqrt{2}}}$



F) Cesaro sum of $(a_1, a_2, a_3, \ldots, a_t)$ is 180 → $S_1 + S_2 + S_3 + \ldots + S_t = 180t$

The Cesaro sum of $(1, a_1, a_2, a_3, \ldots, a_t)$ is $\frac{1+(1+S_1)+(1+S_2)+\ldots+(1+S_t)}{1+t}$

$= \frac{(1+t)+S_1+S_2+\ldots+S_t}{1+t} = \frac{(1+t)+180t}{1+t} = 1+\frac{180t}{1+t}$

For integers $t > 1$, $1 + t$ is never a factor of t. Thus, if $\frac{180t}{1+t}$ is to be an integer, $(1 + t)$ must be a factor of 180. Since $180 = 2^2\cdot 3^2\cdot 5^1$, 180 has $(2 + 1)(2 + 1)(1 + 1) = 18$ factors.
The only factors that must be excluded are 1 and 2 and, therefore, there are 16 values of t for which the required Cesaro sum will be an integer.