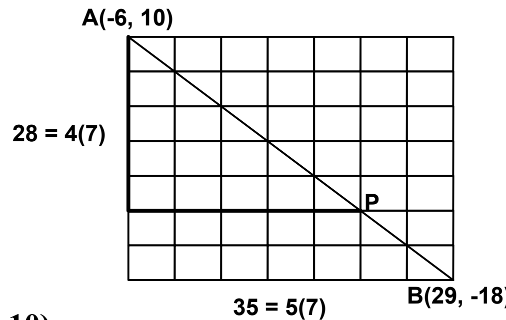
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007 SOLUTION KEY

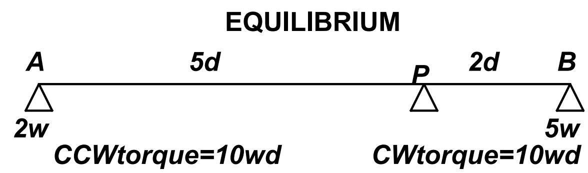
Round 3

A) The coordinates of point P must satisfy both equations.
Thus, both $2\cdot 8 - a^2 = 7$ and $8a - 4a - 12 = 0$ must be true.
$a^2 = 9$ is satisfied by both ± 3, but the second equation is only satisfied by $a = \underline{\mathbf{3}}$.

B) Each block in the vertical direction is 4 units and each block in the horizontal direction is 5 units.
$(-6 + 5 \cdot 5, 10 - 5 \cdot 4) \rightarrow$ **(19, -10)**
Alternate solution:
$5/7 \rightarrow AP : PB = 5 : 2$ Thus, the coordinates of P are determined by 'weighting' the coordinates.
Since P is closer to B than A its 'influence" is greater.
In fact, B will be counted 5 times and A only twice.
$$P\left(\frac{2(-6)+5(29)}{2+5}, \frac{2(10)+5(-18)}{2+5}\right) = \left(\frac{145-12}{7}, \frac{20-90}{7}\right) = \mathbf{(19, -10)}$$



The diagram at the right illustrates this weighting of x- and y-coordinates in terms of a balancing act where the force producing a clockwise turn around point P equals the force producing a counterclockwise turn around point P; hence, the term equilibrium.



C) Since perpendicular lines have negative reciprocal slopes, the perpendicular to $3x + 2y - 13 = 0$ has the form $2x - 3y + c = 0$.
Since this line must also pass through (1, 5), we can find c by substituting for x and y.
$2(1) - 3(5) + c = 0 \rightarrow c = 13$ and the required line is $2x - 3y + 13 = 0$.

Substituting the coordinates of the points that lie on this line, $\begin{cases} P)\ 2a - 3b + 13 = 0 \\ Q)\ 2b - 3a + 13 = 0 \end{cases}$

Subtracting, $5a - 5b = 0 \rightarrow a = b \rightarrow P$ and Q are the same point $\rightarrow PQ = \underline{\mathbf{0}}$

Aside #1:
Since the slope of $\overline{PQ}$, given $P(a, b)$ and $Q(b, a)$ is $\dfrac{b-a}{a-b}$ = -1 and the slope of the given line $\neq$ -1, the only way both P and Q could be on the line is for P and Q to be the same point!

Aside #2:
Suppose both (h, k) and (k, h) lie on a line $Ax + By + C = 0$. Then
$\begin{cases} Ah + Bk + C = 0 \\ Ak + Bh + C = 0 \end{cases}$ Subtracting, $A(h-k) + B(k-h) = 0 \rightarrow A(h-k) = -B(k-h) = B(h-k)$
$\therefore A = B$ or $h = k \rightarrow$ if x- and y- coefficients are unequal, (h, k) and (k, h) must be the same point.