MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2008 SOLUTION KEY

Team Round

A) Given: $\left| i+2i^2+3i^3+4i^4+5i^5+...+(4k+1)i^{4k+1} \right| = 29$.

Grouping in blocks of 4, the first four terms sum to $i - 2 - 3i + 4 = 2 - 2i$
In fact, each successive block of four terms also sums to $2 - 2i$.
The absolute value expression consists of k blocks of 4 terms plus one additional.
The equation simplifies to
$|k(2-2i)+(4k+1)i| = |2k+(2k+1)i| = 29 \rightarrow (2k)^2 + (2k+1)^2 = 29^2$
Noting that $k = 10 \rightarrow 20 - 21 - 29$ which is a Pythagorean Triple avoids the necessity of solving this quadratic equation.

Therefore, the expression to be evaluated is the <u>sum</u> of 11 terms divided by the product of the same 11 terms.

The terms are: $i, i^2, i^4, i^8, i^{16}, \ldots i^{1024}$ The last 9 terms are all 1s.

Thus, the quotient is $\dfrac{i-1+9}{i(-1)(1)^9} = \dfrac{i+8}{-i} = \dfrac{i^2+8i}{-i^2} = \underline{\mathbf{-1+8}\boldsymbol{i}}$

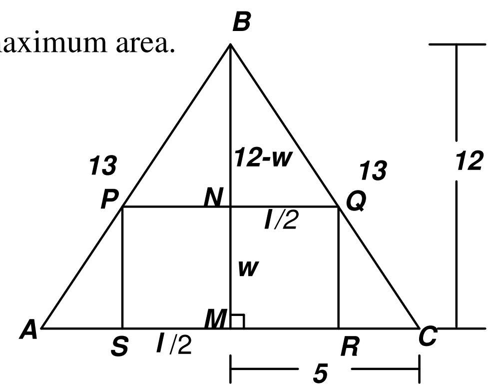
B) Given: $\dfrac{2}{1-\dfrac{1}{1-\dfrac{2}{t}}} \geq t^2 - 4$

Note: $t = 0, 2$ cause division by zero in the expression on the left side.

$$\frac{2}{1-\dfrac{1}{1-\dfrac{2}{t}}} = \frac{2}{1-\dfrac{1}{\dfrac{t-2}{t}}} = \frac{2}{1-\dfrac{t}{t-2}} = \frac{2}{\dfrac{t-2-t}{t-2}} = 2\cdot\frac{t-2}{-2} = 2-t$$

$2 - t \geq t^2 - 4 \rightarrow t^2 + t - 6 = (t+3)(t-2) \leq 0 \rightarrow -3 \leq t \leq 2$
However, with the restriction that $t \neq 0, 2$, we have $\underline{\mathbf{-3 \leq \boldsymbol{t} < 2\ (\boldsymbol{t} \neq 0)}}$ (or equivalent)

C) Let l and w denote the length and width of the rectangle with maximum area.
Since $\Delta BNQ \sim \Delta BMC$,



$\dfrac{12-w}{12} = \dfrac{l/2}{5} \rightarrow l = \dfrac{60-6w}{5} = 12 - \dfrac{6}{5}w$

Area $= lw = \left(12 - \dfrac{6}{5}w\right)w = -\dfrac{6}{5}w^2 + 12w =$

$-\dfrac{6}{5}\left(w^2 - 10w + 25\right) + \dfrac{6}{5}\cdot 25 = -\dfrac{6}{5}(w-5)^2 + 30$

Clearly, an expression of this form always has a value less than or equal to 30 and attains its maximum value of 30 when $w = 5$ (and correspondingly $l = 6$) $\rightarrow$ dimensions: <u>**5 x 6**</u>