MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2009 SOLUTION KEY

Team Round

D) $\frac{1}{A}+\frac{1}{B}=k\left(\frac{1}{A+B}\right) \rightarrow \frac{A+B}{AB}=\frac{k}{A+B} \rightarrow (A+B)^2=kAB \rightarrow A^2+(2-k)AB+B^2=0$

Dividing through by B^2 ($\neq 0$), we have the quadratic $\left(\frac{A}{B}\right)^2+(2-k)\left(\frac{A}{B}\right)+1=0$

Appling the quadratic formula, $\frac{A}{B}=\frac{(k-2)\pm\sqrt{(2-k)^2-4}}{2}=\frac{(k-2)\pm\sqrt{k(k-4)}}{2}$

Clearly, for $k = 0$ and $k = 4$ the radical expression drops out, leaving $R = \frac{A}{B} = -1$ and $+1$ respectively and (k, R) = **<u>(4, 1)</u>** and **<u>(0, –1)</u>**. These are the only possibilities.

To prove this, the reasoning might go something like this:
For all other integer values of k, $k(k-4)$ is <u>not</u> a perfect square and $\sqrt{k(k-4)}$ is <u>irrational</u>.
Suppose $k(k-4) = m^2$, for some integer m. Then $k^2 - m^2 = 4k \rightarrow (k + m)(k - m) = 4k$ (****)
The factors on the left side are the sum and difference of the same two numbers.
Regardless of the values of k and m, the sum and difference will both be even or both be odd.
Since the right hand side of the equation must be even, the sum and difference must both be even. Let $k + m = 2p$ and $k - m = 2q$, for some integers p and q, i.e. the sum and difference are both even.

Solving for k and m in terms of p and q, $\begin{cases} k = p+q \\ m = p-q \end{cases}$

Substituting in (****), $(2p)(2q) = 4(p + q) \rightarrow pq = p + q$
This is satisfied by $(p, q) = (0, 0)$ and $(2, 2)$. Are there any others?

$pq = p + q \rightarrow q = 1+\frac{q}{p}$

If $p = 1$, then $q = 1 + q$ (a contradiction.)

If $p = 3$, then $\frac{2}{3}q=1 \rightarrow q=\frac{3}{2}$ (a non-integer)

If $p = 4, 5, 6, \ldots, r$ then $q = \frac{4}{3},\frac{5}{4},\frac{6}{5},\ldots,\frac{r}{r-1}$ (all non-integers)

If $p=-1$, $2q=1 \rightarrow q=\frac{1}{2}$ (a non-integer)

If $p = -2, -3, -4, \ldots, -r$, then $q = \frac{2}{3},\frac{3}{4},\frac{4}{5},\ldots,\frac{r}{r+1}$ (all non-integers)

Thus, (0, 0) and (2, 2) are the only two possible ordered pairs and $k = p + q = 0, 4$, producing the ordered pairs (k, R) listed above.