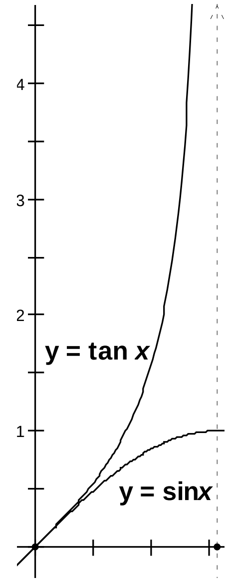
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2013 SOLUTION KEY



Round 3

A) The values of both $\tan x$ and $\sin x$ are 0 for $x = 0$. Over the interval $0 \le x < 90°$, both functions are increasing, but $\tan x$ is increasing faster, as is evidenced by the graphs at the right. Here is the numerical evidence:

At $x = 30°, \frac{\sqrt{3}}{3} - \frac{1}{2} \approx 0.1$; $x = 45°, 1 - \frac{\sqrt{2}}{2} \approx 0.3$; $x = 60°, \sqrt{3} - \frac{\sqrt{3}}{2} \approx 0.9$.

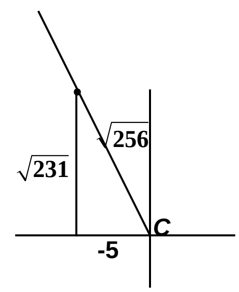
Thus, the smaller the angle, the smaller the difference and the minimum must occur for x = **15**.

Alternately, $\tan x - \sin x = \sin x\left(\frac{1}{\cos x} - 1\right)$. Over the -interval $0 \le x < 90°$, as x increases, $\sin x$ increases, $\cos x$ decreases, $\frac{1}{\cos x}$ increases and, consequently, $\frac{1}{\cos x} - 1$ also increases. Therefore, the product (and equivalently, the given difference) increases over this interval and the minimum occurs for the smallest value of x. For either point of view, number crunching the given values of x was not necessary.

B) Since $B = 180 - (A + C)$, $\cos B = \cos(180 - (A + C)) = -\cos(A + C)$. Expanding, we have

$$\cos B = -\cos A \cos C + \sin A \sin B = -\frac{13}{20} \cdot \frac{37}{40} + \frac{\sqrt{400-169}}{20} \cdot \frac{\sqrt{1600-1369}}{40} = \frac{-481+231}{800} = -\frac{250}{800} = -\frac{5}{16}$$

Since $\cos B < 0$, B must be an obtuse angle. $\boldsymbol{Arc\cos\left(-\frac{5}{16}\right)}$ denotes an obtuse angle (i.e. in quadrant 2). The corresponding angle in quadrant 1 is $Arc\cos\left(\frac{5}{16}\right)$ and the required obtuse angle could also be represented as is $\boldsymbol{180 - Arc\cos\left(\frac{5}{16}\right)}$.



Alternately, using a lesser known identity: (In any ΔABC, $\tan A \cdot \tan B \cdot \tan C = \tan A + \tan B + \tan C$)

$$\frac{\sqrt{231}}{13} \cdot \frac{\sqrt{231}}{37} \cdot \tan C = \frac{\sqrt{231}}{13} + \frac{\sqrt{231}}{37} + \tan C \Leftrightarrow \tan C\left(1 - \frac{231}{13 \cdot 37}\right) = -\frac{50\sqrt{231}}{13 \cdot 37} \Leftrightarrow (481 - 231)\tan C = -50\sqrt{231}$$

Since $\tan C = -\frac{\sqrt{231}}{5} < 0$, C must be obtuse, so $\cos C = -\frac{5}{\sqrt{256}} = -\frac{5}{16}$ and the result follows.

C) $\cos x \cos y - \sin x \sin y = \cos(x + y) = \frac{1}{2} \Rightarrow x + y = \pm 60° + 360n$ (quadrants 1, 4)

$\cos x \cos y + \sin x \sin y = \cos(x - y) = \frac{\sqrt{2}}{2} \Rightarrow x - y = \pm 45 + 360m$ (quadrants 1, 4)

Adding, $2x = \pm 15 + 360k$ or $\pm 105 + 360k$, where $k = n + m$

Thus, $x = \pm 7.5 + 180k$ or $\pm 52.5 + 180k$. The smallest positive value of x is **7.5**.