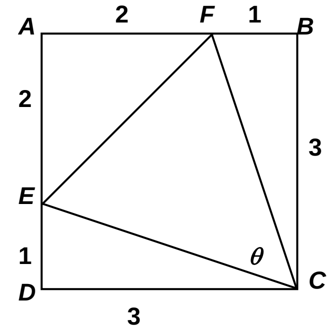
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012 SOLUTION KEY

Round 1



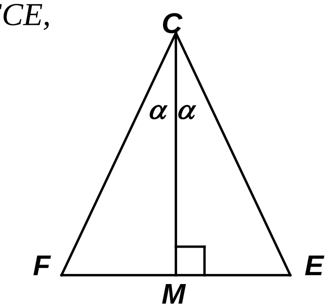
A) The smaller angle is opposite the shorter leg which we find by using the Pythagorean Theorem, $37^2 = 35^2 + x^2 \Rightarrow x^2 = 37^2 - 35^2$.
Recognizing this as the difference of perfect squares, we avoid the need to square these numbers.

$$37^2 - 35^2 = (37+35)(37-35) = 72\cdot 2 = 144 \Rightarrow x = 12$$

Thus, SOHCAHTOA $\Rightarrow \sin\theta = \underline{\mathbf{\dfrac{12}{37}}}$.

B) With no loss of generality, assume the side of square *ABCD* is 3. Using the Pythagorean Theorem, the sides of ΔFCE are easily determined. Using the Law of Cosines on ΔFCE,

$$8 = 10 + 10 - 2\cdot 10\cos\theta \Rightarrow \cos\theta = \frac{12}{20} = \underline{\mathbf{\frac{3}{5}}} .$$



Alternate solution:
Recognizing that ΔEFC is isosceles, draw the perpendicular bisector $\overline{CM}$ of the base $\overline{EF}$, where *M* is the midpoint of $\overline{EF}$. $\overline{CM}$ bisects $\angle FCE$.
Appling the Pythagorean Theorem, $CF = \sqrt{10},\ EF = \sqrt{8} = 2\sqrt{2}$

$$CM = \sqrt{8},\ \sin\alpha = \frac{\sqrt{2}}{\sqrt{10}},\ \cos\alpha = \frac{\sqrt{8}}{\sqrt{10}}\quad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{8}{10} - \frac{2}{10} = \underline{\mathbf{\frac{3}{5}}} .$$

C) Using the Law of Sines $\left(\dfrac{BC}{\sin A} = \dfrac{AC}{\sin B}\right)$, we have $\dfrac{BC}{AC} = \dfrac{\sin A}{\sin B} = \dfrac{2}{\sqrt{7}}$

Squaring both sides and substituting for $\sin^2$, $\dfrac{\sin^2 A}{\sin^2 B} = \dfrac{1-\cos^2 A}{1-\cos^2 B} = \dfrac{4}{7}$

Cross multiplying, $7 - 7\cos^2 A = 4 - 4\cos^2 B \Leftrightarrow 7\cos^2 A - 4\cos^2 B = \underline{\mathbf{3}}$.
Amazingly, for a right triangle the answer is still 3.
If ΔABC is a right triangle, *A* cannot be the right angle. ($\overline{BC}$ would be the hypotenuse and $2 \not> \sqrt{7}$.)

If *B* is the right angle, then $AB = \sqrt{3}$ and $7\left(\dfrac{\sqrt{3}}{\sqrt{7}}\right)^2 - 4(0)^2 = 7\cdot\dfrac{3}{7} - 0 = \underline{\mathbf{3}}$.

If *C* is the right angle, then $AB = \sqrt{11}$ and $7\left(\dfrac{\sqrt{7}}{\sqrt{11}}\right)^2 - 4\left(\dfrac{2}{\sqrt{11}}\right)^2 = 7\cdot\dfrac{7}{11} - 4\cdot\dfrac{4}{11} = \dfrac{49-16}{11} = \underline{\mathbf{3}}$.