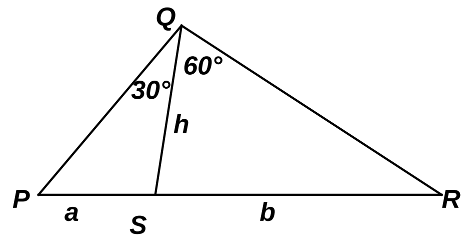
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2011 SOLUTION KEY

Team Round



A) Using the Law of Sines on ΔPQS and ΔRQS,

$$\frac{\sin 30}{a}=\frac{\sin P}{h} \text{ and } \frac{\sin 60}{b}=\frac{\sin R}{h}.$$

Squaring each equation and adding,

$$\frac{\sin^2 30}{a^2}+\frac{\sin^2 60}{b^2}=\frac{\sin^2 P}{h^2}+\frac{\sin^2 R}{h^2}=\frac{\sin^2 P+\cos^2 P}{h^2}=\frac{1}{h^2}$$

(since P and R are complementary)

$$\frac{1}{4a^2}+\frac{3}{4b^2}=\frac{b^2+3a^2}{4a^2b^2}=\frac{1}{h^2}\Rightarrow (L, M, N) = \underline{\mathbf{(4, 3, 1)}}.$$

B) In group A, the first number in each triple is $2k + 1$.
We want the 3rd term in the 13th row, i.e. we need the triple $(27, x, x + 1)$

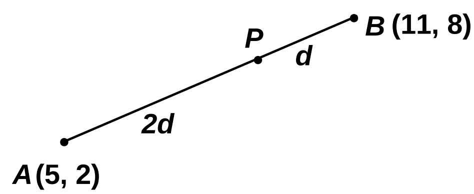
$27^2+x^2=(x+1)^2 \Rightarrow 729 = 2x + 1 \Rightarrow x = 364 \Rightarrow$ 3rd term = 365 = 5(73)

In group B, the first number in each triple is $4(k + 1)$.
We want the 2nd term in the 36th row, i.e. we need the triple $(148, x, x + 2)$
Avoid as much computation as possible.

$148^2+x^2=(x+2)^2 \Rightarrow 148^2=4x+4 \Rightarrow 4x=148^2-2^2=(148+2)(148-2)=150(146)$

$\Rightarrow x = 75(73) =$ 2nd term

The required ratio is $\dfrac{5(73)}{75(73)} = \underline{\dfrac{\mathbf{1}}{\mathbf{15}}}$.



C) $\sqrt{(x-5)^2+(y-2)^2}=2\sqrt{(x-11)^2+(y-8)^2}$

$\Leftrightarrow (x-5)^2+(y-2)^2=4\left((x-11)^2+(y-8)^2\right)$

$\Leftrightarrow x^2-10x+25+y^2-4y+4=4\left(x^2-22x+121+y^2-16y+64\right)=4x^2-88x+484+4y^2-64y+256$

$\Leftrightarrow 3x^2-78x+3y^2-60y=-711$

$\Leftrightarrow 3\left(x^2-26x\underline{+169}\right)+3\left(y^2-20y\underline{+100}\right)=-711\underline{+507+300}=96$

$\Leftrightarrow (x-13)^2+(y-10)^2=32$ (A circle with center at (13, 10) and radius $4\sqrt{2}$.)

We need to examine perfect squares which sum to 32.
The list of candidates is 1, 4, 9, 16, 25. Only 16 + 16 = 32!

$(x-13)^2=16\Rightarrow x-13=\pm 4\Rightarrow x=17,9$

$(y-10)^2=16\Rightarrow y-10=\pm 4\Rightarrow y=14,6$

Thus, the possible points are **<u>(17, 14), (17, 6), (9, 14), (9, 6)</u>**.