MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2015 SOLUTION KEY

Team Round - continued

E) To generate similar problems: $\boxed{\text{Parallelogram w/sides: } x,\ x + c \quad \text{and diagonals: } x + a,\ x + b}$

$$2\left(x^2+(x+c)^2\right)=\left(x+a\right)^2+\left(x+b\right)^2$$

$$\Rightarrow\ (x+c)^2 = x^2+2cx+c^2 = \left(ax+bx\right)+\left(\frac{a^2+b^2}{2}\right) \Rightarrow\ x^2+\left(2c-a-b\right)x+\left(c^2-\frac{a^2+b^2}{2}\right)=0$$

$$\Rightarrow\ x=\frac{a+b-2c\pm\sqrt{\left(2c-a-b\right)^2+4\left(\dfrac{a^2+b^2}{2}-c^2\right)}}{2}$$

$$\Rightarrow\ x=\frac{a+b-2c\pm\sqrt{\left(2c-a-b\right)^2+2\left(a^2+b^2\right)-4c^2}}{2}$$

Simplifying the discriminant,

$$\left(2c-a-b\right)^2+2\left(a^2+b^2\right)-4c^2 = 4c^2+a^2+b^2-4ac-4bc+2ab+2a^2+2b^2-4c^2$$

$$= (a+b)^2+2\left(a^2+b^2\right)-4c(a+b)$$

Try (a, b)

$= (2, 4) \Rightarrow 36 + 2(20) - 24c = 76 - 24c = 4(19 - 6c) \Rightarrow c = 1$ only

$= (3, 6) \Rightarrow 81 + 2(45) - 36c = 171 - 36c = 9(19 - 4c) \Rightarrow$ none

$= (4, 7) \Rightarrow 121 + 2(65) - 44c = 251 - 44c \Rightarrow$ none

$= (3, 5) \Rightarrow 64 + 2(34) - 32c = 132 - 32c = 4(33 - 8c) \Rightarrow c = 1, 3, 4$ Bingo!

$= (3, 7) \Rightarrow 100 + 2(58) - 40c = 216 - 40c = 4(54 - 10c) \Rightarrow c = 5$ only

$= (5, 9) \Rightarrow 196 + 2(106) - 56c = 408 - 56c = 8(51 - 7c) \Rightarrow c = 7$ only

F) There are only two options for the draw from urn #1 – either *RW* or *RR*
(There were only 6 options: *RB* would leave 1 white and 1 red, *BB* would leave 1 white and 1 blue, *BW* would leave 2 reds and 2 blues and *WW* was impossible.)

$$P(RW)=\frac{2}{\binom{6}{2}}=\frac{2}{15},\ P(RR)=\frac{1}{\binom{6}{2}}=\frac{1}{15}$$

#1	#2
W RR BBB	W W W W R R R R B B

Note: $P(WW)=\dfrac{0}{15}, P(RB)=\dfrac{2\cdot 3}{15}=\dfrac{6}{15}, P(BB)=\dfrac{3}{15}, P(BW)=\dfrac{3}{15}$ and the sum of these probabilities is $\dfrac{15}{15}=1$. *RW* from #1 $\Rightarrow$ $5W$ $5R$ $2B$ in #2 *RR* from #1 $\Rightarrow$ $4W$ $6R$ $2B$ in #2

Now

$$P(\text{same color from \#2}) = P(WW \mid RW \text{ from \#1}) + P(RR \mid RW \text{ from \#1}) + P(BB \mid RW \text{ from \#1})$$
$$+ P(WW \mid RR \text{ from \#1}) + P(RR \mid RR \text{ from \#1}) + P(BB \mid RR \text{ from \#1})$$

$$=\frac{2}{15}\left(\frac{\binom{5}{2}+\binom{5}{2}+\binom{2}{2}}{\binom{12}{2}}\right)+\frac{1}{15}\left(\frac{\binom{4}{2}+\binom{6}{2}+\binom{2}{2}}{\binom{12}{2}}\right)=\frac{2}{15}\cdot\frac{21}{66}+\frac{1}{15}\cdot\frac{22}{66}=\frac{64}{15\cdot 66}=\underline{\mathbf{\frac{32}{495}}}$$