MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012 SOLUTION KEY

Team Round

A) Let m denote the slope. Then: $mx - y = 7m - (-1) = 7m + 1$ or $mx + (-1)y - (7m + 1) = 0$

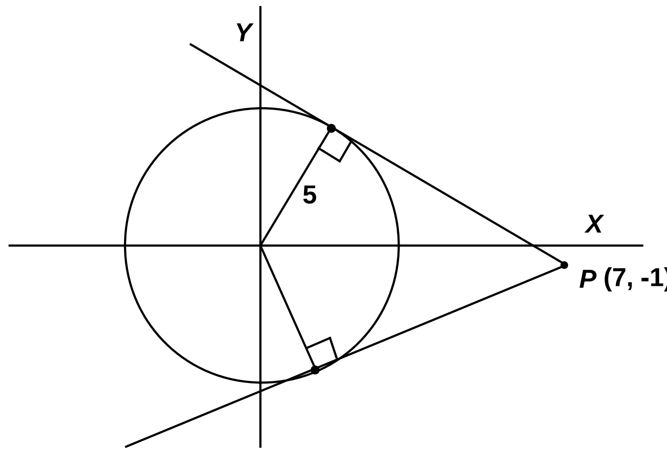
Applying the point to distance formula $\dfrac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}}$ (from $P(h, k)$ to $Ax + By + C = 0$), we have $\dfrac{|m(0) + (-1)0 - (7m+1)|}{\sqrt{m^2+1}} = 5$

$(7m+1)^2 = 25(m^2+1)$

$49m^2 + 14m + 1 = 25m^2 + 25$

$24m^2 + 14m - 24 = 2(3m+4)(4m-3) = 0$

$m = \underline{\boldsymbol{-\frac{4}{3}, \frac{3}{4}}}$



B) Since $\dfrac{x^4 + 3x^2 + k}{x^2 + 3} = x^2 + \dfrac{k}{x^2 + 3}$, we see that $x^2 + 3$ must be a factor of k, if the original quotient is to be an integer.

Plan: List the values of $x^2 + 3$ for consecutive positive integer values of x until we find the smallest value in the list that is divisible by exactly four values preceding it.
$x = 1, 2, 3, \ldots \Rightarrow$ divisors: *4*, *7*, *12*, 19, *28*, 39, 52, 67, **84**, …
Since $k = 84$ is divisible by each of the underlined values, $k = 84$ is the smallest value for which the given quotient has integral values for exactly five positive integer values of x, namely, 1, 2, 3, 5 and 9.