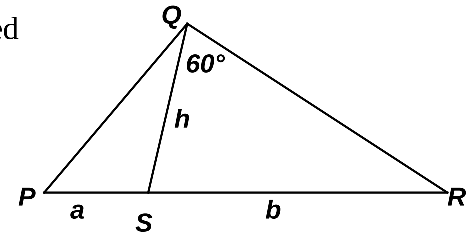
Addendum #1: Team A Question

ΔPQR is a right triangle with right angle at Q.

$PS = a$, $RS = b$ and $\overrightarrow{QS}$ divides $\angle Q$ into 30° and 60° as indicated in the diagram at the right.

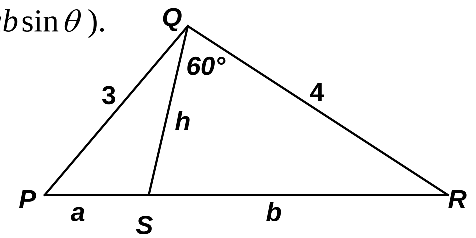
For positive integers L, M and N, $h^2 = \dfrac{La^2b^2}{Ma^2 + Nb^2}$.

Compute the ordered triple (L, M, N).



The original team A question specified a right triangle with $PQ = 3$ and $QR = 4$ and requested an expression for $1/h^2$ exclusively in terms of a and b.

In this case, we could have actually solved for a and b (using $\frac{1}{2}ab\sin\theta$).



$$3h\sin 30° + 4h\sin 60° = 3\cdot 4$$

$$\Rightarrow h = \frac{12}{\frac{3}{2} + 2\sqrt{3}} = \frac{8}{13}\left(4\sqrt{3} - 3\right)\cdot$$

Using the Law of Sines in triangles QPS and QRS, $h = \frac{8}{5}a$ and $h = \frac{6b}{5\sqrt{3}}$.

Equating, $a = \frac{\sqrt{3}}{4}b$, so $a = \frac{5}{13}\left(4\sqrt{3} - 3\right)$ and $b = \frac{20}{13}\left(4 - \sqrt{3}\right)$.

But none of these values needed to come into play when solving for $\frac{1}{h^2}$ in terms of a and b.

Depending on how you proceeded, different-looking 'formulas' would have been possible besides the official solution.

Using Stewart's Theorem, we get

$$3^2 \cdot b + 4^2 \cdot a = h^2 \cdot 5 + 5ab \Rightarrow h^2 = \frac{9b + 16a - 5ab}{5}. \Rightarrow h^2 = \frac{9b + 16a - 5ab}{5} \Rightarrow \frac{1}{h^2} = \frac{5}{9b + 16a - 5ab}.$$

Using the Law of Cosines (twice) we have:

$$\begin{cases} h^2 = 9 + a^2 - 6a\cos P = 9 + a^2 - 6a\left(\frac{3}{5}\right) \\ h^2 = 16 + b^2 - 8b\cos R = 16 + b^2 - 8b\left(\frac{4}{5}\right) \end{cases}$$

Adding, $2h^2 = 25 + a^2 + b^2 - 2\left(\frac{9a + 16b}{5}\right)$ and we have a much different-looking result for $\frac{1}{h^2}$.

You could even find the numerical value of $\frac{1}{h^2}$, namely $\frac{13}{64}\left(\frac{4}{\sqrt{3}} + 1\right)$, and then determined coefficients to write this value as a linear combination of $a = \frac{5}{13}\left(4\sqrt{3} - 3\right)$ and $b = \frac{20}{13}\left(4 - \sqrt{3}\right)$.

It's quite challenging to show that all these variants are equivalent for the known values of (h, a, b).