MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016 SOLUTION KEY

Round 4

A) $x^2 = 38x - N \Leftrightarrow x^2 - 38x + \square = 0$.

To have equal roots, this trinomial must be a perfect square (like $(x+3)^2 = x^2 + 6x + 9$).

Note that to have equal roots, the value in the box must be the square of half the coefficient of the middle (i.e. linear) term.

$\Rightarrow N = 361$.

Check: $(x - 19)^2 = x^2 - 38x +$ **361** $= 0$ has exactly one root, namely 19.

B) Let $y = x + \frac{4}{x}$. Then the equation becomes

$y^4 - 17y^2 + 16 = 0 \Rightarrow (y^2 - 1)(y^2 - 16) \Rightarrow y = \pm 1, \pm 4$.

$x + \frac{4}{x} = \pm 1 \Leftrightarrow x^2 \pm x + 4 = 0$ which has no rational solutions.

$x + \frac{4}{x} = \pm 4 \Leftrightarrow x^2 \pm 4x + 4 = (x \pm 2)^2 = 0 \Rightarrow x = \pm 2$.

$(-2)^3 \cdot (2)^3 = \underline{\mathbf{-64}}$.

C) $x^2 - kx + 2k = 2x \Leftrightarrow x^2 - (k+2)x + 2k = 0$

Let $R_1 = A$ and $R_2 = A - 3$. Then: $\begin{cases} 2A - 3 = k + 2 \\ A(A-3) = 2k \end{cases}$

Substituting $k = 2A - 5$ in the second equation,

$A^2 - 3A = 2(2A - 5) \Leftrightarrow A^2 - 7A + 10 = (A - 5)(A - 2) = 0 \Rightarrow (A, k) = (5,5), (2,-1)$.

Thus, $(R_1, R_2, k) =$ **(5, 2, 5)**, **(2, –1, –1)**.

Alternately, $x^2 - kx + 2k = 2x \Leftrightarrow x^2 - (k+2)x + 2k = (x-k)(x-2) = 0$.

Therefore, the roots are 2 and k, which implies k must be $2 \pm 3 = \begin{cases} -1 \\ 5 \end{cases}$.