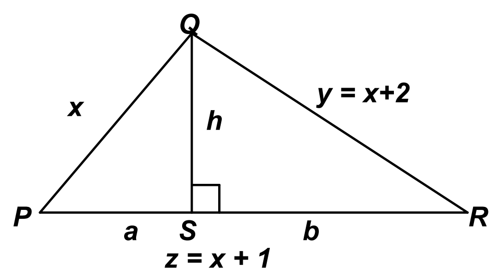
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2009 SOLUTION KEY

Team Round

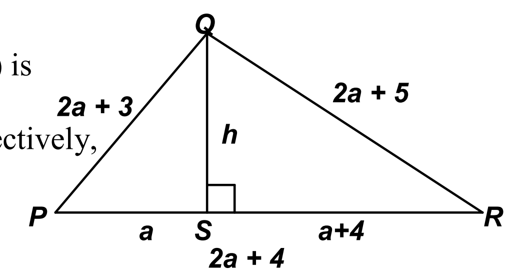
A) 3 faces: corner cubes (at the 8 vertices) → 8
2 faces: edge cubes (12 edges) → 7 x 8: 22, 7 x 9: 24, 8 x 9: 26 → 72
1 face: center cubes (6 faces) → 2[(5)(6) + (5)(7) + (6)(7)] →214
0 faces: interior cubes only 5(6)(7) → 210
This totals 504 unit cubes in total and there should be 7(8)(9) = 504 cubes.
Thus, (8 + 72) : (210 + 214) → **<u>10 : 53</u>**

B) In right triangles PQS and RQS, we have $\begin{cases}(1)\ a^2+h^2=x^2\\(2)\ b^2+h^2=y^2\\(3)\ z=a+b\end{cases}$



Subtracting (2) – (1),
$b^2 - a^2 = (b + a)(b - a) = y^2 - x^2 = (x + 2)^2 - x^2 = 4(x + 1)$
But $a + b = x + 1$!!!
Canceling, $b - a = 4$ → $b = a + 4$
Substituting, $2a + 4 = x + 1$ → $x = 2a + 3$
It remains to find h in terms of a.
In ΔPQS, $a^2 + h^2 = x^2$ → $a^2 + h^2 = (2a + 3)^2$ → $h^2 = 3(a^2 + 4a + 3)$ → $h = \sqrt{3(a+1)(a+3)}$

Clearly, we want values of a for which $3(a+1)(a+3)$ is a perfect square, i.e $(a + 1)$ is a perfect square and $(a + 3)$ is 3 times a perfect square or vice versa.
The first and third term are generated by $a = 5$ and 95 respectively, so we restrict our search to integer values of a between 6 and 94 inclusive.



$a = 24$ → $\sqrt{3\cdot 25\cdot 27} = 45$ BINGO!
Thus, the second term is **<u>(51, 52, 53)</u>**.

Check: ΔPQS: (24, 45, 51) → 3(8, 15, 17) and ΔRQS: (28, 45, 53) → $28^2 + 45^2 = 2809 = 53^2$