MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2017 SOLUTION KEY

Team Round

A) Note that the coefficients of *x*, *y*, and *z* are each $a+b+c$ when the 3 equations are added.

Therefore, $\begin{cases}(a+b+c)(x+y+z)=360\\ x+y+z=5\\ a:b:c=1:2:3\end{cases} \Rightarrow (n+2n+3n)=72 \Rightarrow n=12 \Rightarrow (a,b,c)=(12,24,36)$.

Dividing through by 12, the 3 equations become $\begin{cases}(1)\quad x+2y+3z=18\\ (2)\ 2x+3y+\ \ z=5\\ (3)\ 3x+\ \ y+2z=7\end{cases}$

Subtracting $x+y+z=5$ from each equation, $\begin{cases}(4)\quad y+2z=13\\ (5)\quad x+2y=0\\ (6)\ 2x+\ \ z=2\end{cases}$

(6) $\Rightarrow\ z=2-2x$ (7)

Substituting for *z* in (4), $y+4-4x=13 \Rightarrow y=4x+9$ (8)

Substituting for *x* in (5), $x+2(4x+9)=0 \Rightarrow x=-2$. Substituting in (7) and (8), $(x,y,z)=\underline{(\mathbf{-2,1,6})}$.

B) Since $\left(1+\sqrt{2}\right)^2=3+2\sqrt{2}$, it follows that $\sqrt{3+2\sqrt{2}}=1+\sqrt{2}$.

Alternately, to extract the square root in the numerator, $3+2\sqrt{2}$ must be a perfect square.

So, assume there are integers *a* and *b* for which $\left(a+b\sqrt{2}\right)^2=3+2\sqrt{2}$.

Expanding and equating the rational and irrational coefficients, $\begin{cases}a^2+2b^2=3\\ 2ab=2\end{cases}$

Clearly, $(a,b)=(1,1)$ or $(-1,-1)$ satisfy both of these equations, but, since $-1-\sqrt{2}<0$ it is rejected, and we have the same result.

$\dfrac{\sqrt{3+2\sqrt{2}}}{2\sqrt{1+\sqrt{2}}}=\dfrac{1+\sqrt{2}}{2\sqrt{1+\sqrt{2}}}=\underline{\dfrac{\sqrt{\mathbf{1+\sqrt{2}}}}{\mathbf{2}}}$ The numerator $\sqrt{1+\sqrt{2}}$ cannot be further simplified.

Suppose it could.

Then we would be able to write $1+\sqrt{2}$ as $\left(A+B\sqrt{2}\right)^2$, for rational numbers *A* and *B*.

$\Rightarrow 1+\sqrt{2}=A^2+2B^2+2AB\sqrt{2} \Rightarrow \begin{cases}A^2+2B^2=1\\ 2AB=1\end{cases}$

Solving the second equation for *B* and substituting in the first, $A^2+\dfrac{1}{2A^2}=1 \Rightarrow 2A^4-2A^2+1=0$.

This equation has no real (let alone rational) roots.

Therefore, the numerator $\sqrt{1+\sqrt{2}}$ cannot be further simplified.