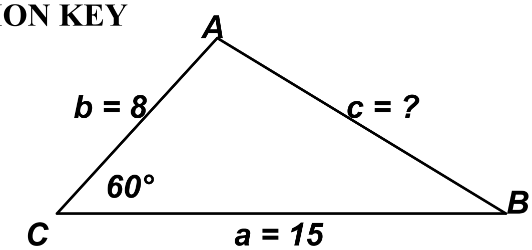
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010 SOLUTION KEY

Round 1

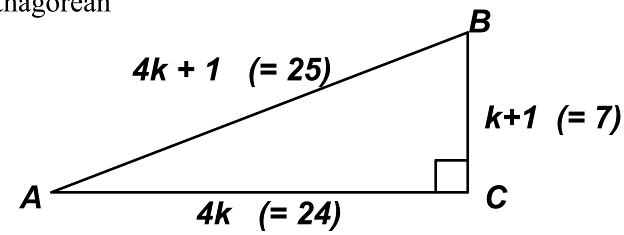


A) Using the laws of Cosines, $c^2 = a^2 + b^2 - 2ab\cos C$

→ $c^2 = 8^2 + 15^2 - 2(8)(15)\cos 60°$

= 4 + 225 – 120 = 169 → c = **<u>13</u>**

B) Assuming $4k + 1$ is the hypotenuse, and applying the Pythagorean theorem, $(k+1)^2 + (4k)^2 = (4k+1)^2$.



Expanding and canceling, $k^2 + 2k = 8k$

→ $k^2 - 6k = k(k - 6) = 0$ → $k = 6$.

→ 7 – 24 – 25 right triangle

→ a sum of 31.

But was $4k + 1$ necessarily the hypotenuse???

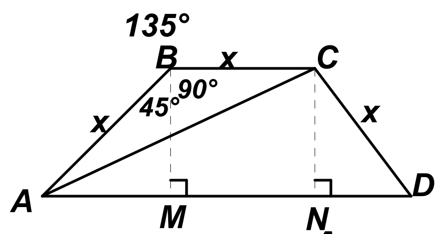
For $k > 0$ (to insure that $4k$ is positive), $4k + 1 > 4k$.

Also $k > 0$ → $3k > 0$

→ $3k + 1 > 1$ (adding 1 to both sides of the inequality)

→ $4k + 1 > k + 1$ (adding k to both sides of the inequality)

→ $4k + 1$ is the longest side and must be the hypotenuse. Thus, **<u>31</u>** is the only possible sum.



C) $AM = \dfrac{x}{\sqrt{2}}$ → $AD = \dfrac{x\sqrt{2}}{2} + x + \dfrac{x\sqrt{2}}{2} = x\left(\sqrt{2}+1\right)$

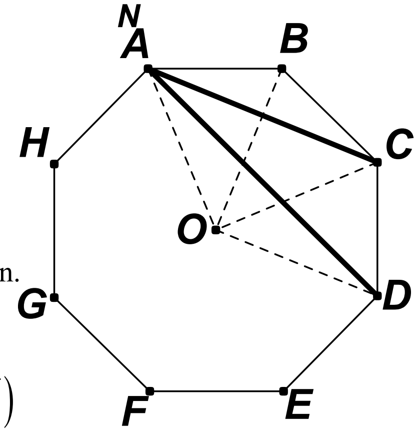
Using the law of cosines on ΔABC,

→ $AC^2 = x^2\left(2+\sqrt{2}\right)$.

$AC^2 = AD$ → $x^2\left(2+\sqrt{2}\right) = x\left(\sqrt{2}+1\right)$

Dividing by x ($\neq 0$), $x\left(2+\sqrt{2}\right) = \sqrt{2}+1$

→ $AB = x = \dfrac{\sqrt{2}+1}{2+\sqrt{2}} \cdot \dfrac{2-\sqrt{2}}{2-\sqrt{2}} = \dfrac{2\sqrt{2}-2+2-\sqrt{2}}{2} = \underline{\dfrac{\sqrt{2}}{2}}$



Alternate Solution #1 (Tuan Le)

Let O be the center of the circle of radius R circumscribed about the regular octagon.

Now, m$\angle AOB$ = 45° → m$\angle AOC$ = 90° → ΔAOC is an isosceles right triangle with legs of length R; hence, $AC = R\sqrt{2}$ or $AC^2 = 2R^2$.

Applying the Law of Cosines to ΔAOD, $AD^2 = 2R^2 - 2R^2\cos 135° = R^2\left(2+\sqrt{2}\right)$

The given $AC^2 = AD$ implies $2R^2 = R\sqrt{2+\sqrt{2}}$ → $R = \dfrac{\sqrt{2+\sqrt{2}}}{2}$ → $R^2 = \dfrac{2+\sqrt{2}}{4}$

Applying the Law of Cosines to ΔAOB, $AB^2 = 2R^2 - 2R^2\cos 45° = R^2\left(2-\sqrt{2}\right)$

Substituting, $AB^2 = \dfrac{2+\sqrt{2}}{4}\cdot\left(2-\sqrt{2}\right) = \dfrac{4-2}{4} = \dfrac{1}{2}$ → $AB = \underline{\dfrac{\sqrt{2}}{2}}$