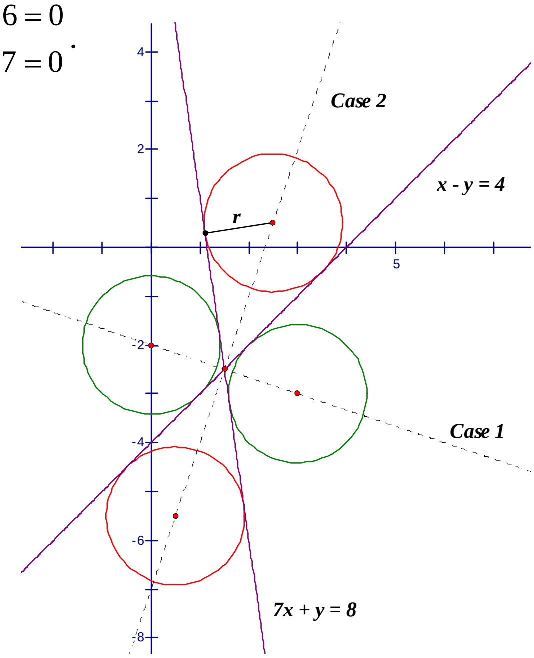
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2014 SOLUTION KEY

Team Round – continued

C) The center of circle O lies on one of the angle bisectors of a pair of vertical angles formed by the given lines. The distance from the point (h,k) to the line $Ax+By+C=0$ is given by $\dfrac{|Ah+Bk+C|}{\sqrt{A^2+B^2}}$. Using the point-to-line distance formula, we have

$$\frac{|7x+y-8|}{\sqrt{7^2+1^2}}=\frac{|x-y-4|}{\sqrt{1^2+(-1)^2}} \Leftrightarrow \frac{|7x+y-8|}{5\cancel{\sqrt{2}}}=\frac{|x-y-4|}{\cancel{\sqrt{2}}} \Leftrightarrow |7x+y-8|=5|x-y-4|$$

$$7x+y-8=\pm 5(x-y-4) \Leftrightarrow \begin{cases} 2x+6y+12=0 \\ 12x-4y-28=0 \end{cases} \Leftrightarrow \begin{cases} x+3y+6=0 \\ 3x-y-7=0 \end{cases}.$$



Case 1: Center on $x + 3y + 6 = 0$ or $y=\dfrac{-x-6}{3}$

Assume the coordinates of the center is $(h,k)=\left(h,\dfrac{-h-6}{3}\right)$

Using $7x + y - 8 = 0$ and $\left(h,\dfrac{-h-6}{3}\right)$, we have

$$\frac{\left|7h+\dfrac{-h-6}{3}-8\right|}{5\sqrt{2}}=\sqrt{2} \Leftrightarrow \left|\frac{20h}{3}-10\right|=10$$

$$\Leftrightarrow h=\frac{3(10\pm 10)}{20} \Rightarrow h=3,\ 0$$

Thus, $(h,k)=(3,-3),\ (0,-2) \Rightarrow h+k=$ **<u>0, –2</u>**.

Case 2: Center on $3x - y - 7 = 0$ or $y = 3x - 7$

Assume the coordinates of the center is $(h,k)=(h,3h-7)$.

Using $7x + y - 8 = 0$ and $(h,3h-7)$, we have

$$\frac{|7h+(3h-7)-8|}{5\sqrt{2}}=\sqrt{2} \Leftrightarrow |10h-15|=10 \Leftrightarrow |2h-3|=2 \Leftrightarrow 2h=3\pm 2 \Rightarrow h=\frac{5}{2},\ \frac{1}{2}.$$

Thus, $(h,k)=\left(\dfrac{5}{2},\dfrac{1}{2}\right),\ \left(\dfrac{1}{2},-\dfrac{11}{2}\right) \Rightarrow h+k=$ **<u>3, –5</u>**.