MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2008 SOLUTION KEY

Team Round

A) $g(x) = Ax^3 \rightarrow f(g(x)) = 8(Ax^3) = 27x^3 \rightarrow A = 27/8$

$h(x) = Bx^5 \rightarrow g(h(x)) = \dfrac{27}{8}(Bx^5)^3 = \dfrac{27B^3}{8}x^{15} \rightarrow f(g(h(x))) = 27B^3x^{15} = 27x^{15} \rightarrow B = 1$

$g(x) = \dfrac{27}{8}x^3 \rightarrow g^{-1}(x) = \left(\dfrac{8x}{27}\right)^{1/3} = \dfrac{2}{3}x^{1/3}$ and $h(x) = x^5 \rightarrow h^{-1}(x) = x^{1/5}$

$\therefore (1024)^{1/5} \cdot (2/3)(-1)^{1/3} = 4(2/3)(-1) = \underline{\mathbf{-\dfrac{8}{3}}}$

B) The prime factorization of $N! = 2^x5^y$(a bunch of irrelevant primes), where $x > y$ since every other factor is even and only every 5^{th} is a multiple of 5. Thus, the number of factors of 10 is determined by y. Let $[A]$ denote the largest integer less than or equal to A.

$\left[\dfrac{N}{5}\right]$ represents the number of multiples of 5 in the first N positive integers.

Each contains one factor of 5. However, some may contain more than one.

$\left[\dfrac{N}{25}\right]$ represents the number of multiples of 25 in the first N positive integers.

Each contains two factors of 5. However, some may contain more than 2.

By adding, $\left[\dfrac{N}{5}\right] + \left[\dfrac{N}{25}\right] + \left[\dfrac{N}{125}\right] + \left[\dfrac{N}{625}\right] + \ldots$, we eliminate any duplication and count the exact number of 5s in the integers from 1 to N inclusive.

Note that eventually a term will become 0 and all subsequent terms will be 0 insuring that even though there are an infinite number of terms, a finite total will always exist.

$1000! \rightarrow \left[\dfrac{1000}{5}\right] + \left[\dfrac{1000}{25}\right] + \left[\dfrac{1000}{125}\right] + \left[\dfrac{1000}{625}\right] + \ldots = 200 + 40 + 8 + 1 + 0 + 0 + \ldots = 249$ zeros

Therefore, 8000! would be expected to end in about 2000 zeros.

Specifically, 8000! ends in 1998 zeros. ($8000 \rightarrow 1600 + 320 + 64 + 12 + 2 + 0 = 1998$)

8004! ends in the same number of zeros, since none of the additional factors of 8001, 8002, 8003 and 8004 introduce any more factors of 5. Additional factors of 5 are introduced by each of these 9 factors : 8005, 8010, 8015, 8020, 8025, 8030, 8035, 8040 and 8045.

Since 8025 is a multiple of 25, it introduces 2 factors of 5.

Thus, $N =$ **<u>8045</u>** is the smallest value for which $N!$ ends in exactly 2008 zeros.