MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2011 SOLUTION KEY

Team Round

Substituting $\sqrt{c(c-b)}$ for PB in $R\left(\dfrac{PB-R}{PB+R}\right)$ is tedious, so we revert to using the first expression for R.

$$PB^2 = R^2 + \left(\frac{a+c-b}{2}\right)^2 = \left(\frac{ab}{a+b+c}\right)^2 + \left(\frac{a+c-b}{2}\right)^2 = \frac{4(ab)^2 + \left((a+c-b)(a+c+b)\right)^2}{4(a+b+c)^2}$$

$$= \frac{4(ab)^2 + \left((a+c)^2 - b^2\right)^2}{4(a+b+c)^2} = \frac{4(ab)^2 + \left(a^2+c^2-b^2+2ac\right)^2}{4(a+b+c)^2}$$

But since $a^2 + b^2 = c^2$, this simplifies to

$$\frac{4(ab)^2 + \left(2a^2+2ac\right)^2}{4(a+b+c)^2} = \frac{(ab)^2 + a^2(a+c)^2}{(a+b+c)^2} = \frac{a^2\left(b^2+(a+c)^2\right)}{(a+b+c)^2}$$

Thus, $PB = \dfrac{a}{a+b+c}\sqrt{(a+c)^2+b^2} = \dfrac{ab}{b(a+b+c)}\sqrt{(a+c)^2+b^2} = \boxed{\dfrac{R}{b}\sqrt{(a+c)^2+b^2}}$.

Now substitute for PB:

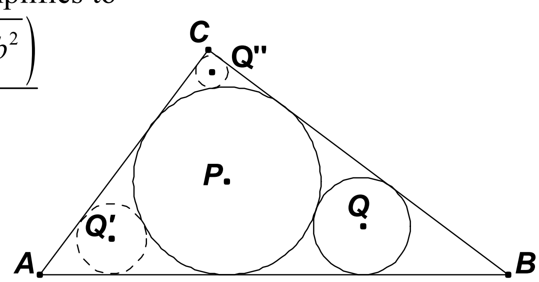
$$r = R\left(\frac{PB-R}{PB+R}\right) = R\left(\frac{\frac{R}{b}\sqrt{(a+c)^2+b^2} - R}{\frac{R}{b}\sqrt{(a+c)^2+b^2} + R}\right) = R\left(\frac{\cancel{\frac{R}{b}}\left(\sqrt{(a+c)^2+b^2} - b\right)}{\cancel{\frac{R}{b}}\left(\sqrt{(a+c)^2+b^2} + b\right)}\right)$$

Rationalizing the denominator, $R\left(\dfrac{\left(\sqrt{(a+c)^2+b^2}-b\right)}{\left(\sqrt{(a+c)^2+b^2}+b\right)}\right)\cdot\dfrac{\left(\sqrt{(a+c)^2+b^2}-b\right)}{\left(\sqrt{(a+c)^2+b^2}-b\right)} = R\dfrac{\left(\sqrt{(a+c)^2+b^2}-b\right)^2}{(a+c)^2+b^2-b^2}$

$$= R\frac{(a+c)^2+2b^2-2b\sqrt{(a+c)^2+b^2}}{(a+c)^2}$$

Now, using the second expression for R, the expression for r simplifies to

$$\frac{(a+b-c)\left((a+c)^2+2b^2-2b\sqrt{(a+c)^2+b^2}\right)}{2(a+c)^2}$$



Q.E.D.

You are invited to verify that

1) for a circle with center at Q' a similar formula for the radius can be derived, namely:

$$\frac{(a+b-c)\left((b+c)^2+2a^2-2a\sqrt{(b+c)^2+a^2}\right)}{2(b+c)^2}$$

2) for the circle with center at Q'', the radius is given by $\left(\dfrac{a+b-c}{2}\right)\left(3-2\sqrt{2}\right)$.

Created with nitroPDF professional
download the free trial online at nitropdf.com/professional