MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2012 SOLUTION KEY

Team Round

E) – continued

In polar coordinates $B(r, 30°)$. $r = \frac{\sqrt{3}}{2} + \sqrt{3} \cdot \frac{1}{2} = \sqrt{3}$

Knowing $OB = \sqrt{3}$, drop a perpendicular from B to $\overrightarrow{OX}$ forming a 30-60-90 right triangle.

The horizontal side is $\frac{3}{2}$, the vertical side is $\frac{\sqrt{3}}{2}$, so the (x, y) coordinates of B are $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$.

In polar coordinates $C(r, 60°)$. $r = \frac{1}{2} + \sqrt{3} \cdot \frac{\sqrt{3}}{2} = 2$

Knowing $OC = 2$, drop a perpendicular from C to $\overrightarrow{OX}$ forming a 30-60-90 right triangle.

The horizontal side is 1, the vertical side is $\sqrt{3}$, so the (x, y) coordinates of C are $\left(1, \sqrt{3}\right)$.

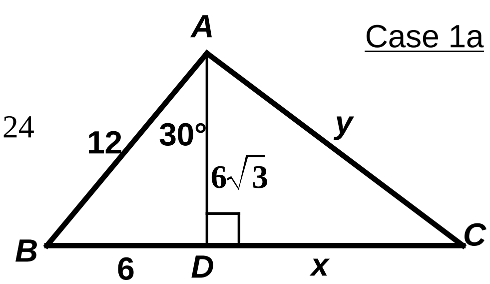
Applying the distance formula, we have $BC = \sqrt{\left(\frac{3}{2} - 1\right)^2 + \left(\frac{\sqrt{3}}{2} - \sqrt{3}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \underline{\mathbf{1}}$.

F) Using the Pythagorean Theorem on ΔADC, we have

$$y^2 - x^2 = \left(6\sqrt{3}\right)^2 = 108$$

Since x and y are integers, we examine the possible factorizations of 108.

$$\begin{cases} y - x = 1 \quad \boxed{2} \quad 3 \quad 4 \quad \boxed{6} \quad 9 \quad 12 \\ y + x = 108 \quad \boxed{54} \quad 36 \quad 27 \quad \boxed{18} \quad 12 \quad 9 \end{cases}$$

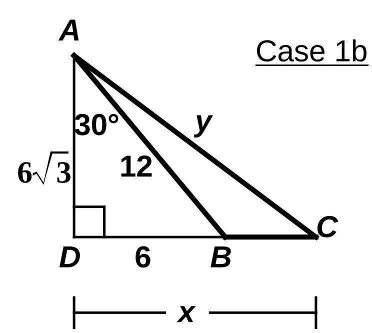


Adding, only two factorizations give integer results: $2y = 56$ or 24

Case 1: $y = 28 \Rightarrow x = 26$ Case 2: $y = 12 \Rightarrow x = 6$

The second result gives us an equilateral triangle of side 12.

The first result gives us a scalene triangle with sides 12, 28 and 32, resulting in a perimeter of **<u>72</u>**.



If B were reflected over $\overline{AD}$, the diagram would also satisfy the given conditions and $BC = 26 - 6 = 20$, resulting in a perimeter of 12 + 28 + 20 = **<u>60</u>**.

<u>FYI</u>:

In case 1, using the Law of Cosines, $\cos(\angle BAC) = \frac{12^2 + 28^2 - 32^2}{2 \cdot 12 \cdot 28} = -\frac{1}{7}$

and $\angle BAC$ must be obtuse (approx. 98°).

In case 2, as the supplement of $\angle ABD$, $m\angle ABC = 120°$ and this is verified by the Law of Cosines,

$\cos(\angle ABC) = \frac{12^2 + 20^2 - 28^2}{2 \cdot 12 \cdot 20} = -\frac{1}{2}$.