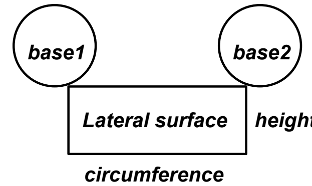
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006 SOLUTION KEY

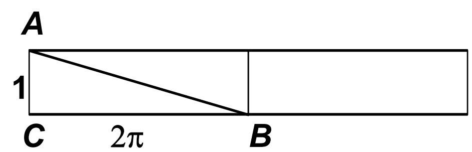


Round 1

A) $C = 2\pi r = 8\pi \rightarrow r = 4$, $V = \pi r^2 h = 48\pi \rightarrow h = 3$,
$TSA = 2(\pi r^2) + (2\pi r)h = 32\pi + 24\pi = \underline{\mathbf{56\pi}}$

B) Let x denote the edge of the original cube. Then $(x - 1)$ denotes the edge of the smaller cube.
The decrease in volume is $x^3 - (x - 1)^3 = 169 \rightarrow x^3 - (x^3 - 3x^2 + 3x - 1) = 169$
$\rightarrow 3x^2 - 3x - 168 = 3(x^2 - x - 56) = 3(x - 8)(x + 7) = 0 \rightarrow x = 8$
Thus, the volume of the smaller cube is $7^3 = \underline{\mathbf{343}}$ cm^3.

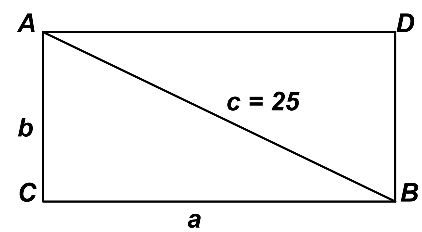
C) A path from point B along the lateral surface from the bottom base to the top base (perpendicularly) and then through the center of the top base to point A has length $1 + 4 = 5$
Cutting the surface of the cylinder along a line through point A perpendicular to a base and rolling out the lateral surface of the cylinder into a plane, we get a rectangle whose width equals the height of the cylinder and whose length equals the circumference of the base of the cylinder. The shortest path along this surface is the hypotenuse of ΔABC.
Using the Pythagorean Theorem, we have $AB = \underline{\sqrt{\mathbf{1+4\pi^2}}}$



Since $5 < \sqrt{1+4\pi^2}$, the shortest distance is $\underline{\mathbf{5}}$
An interesting question is under what circumstances is one path shorter than the other?

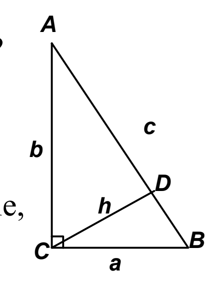
Round 2

A) Method I – Trial and Error
Since $a^2 + b^2 = 25^2 = 625$ and both a and b are integers, it remains to test integers from 1 to 24 inclusive. The only winners are $15^2 + 20^2 = 225 + 400 = 625$ and $7^2 + 24^2 = 49 + 576 = 625$. The corresponding perimeters $= 2(a + b)$ are $2(35) = \underline{\mathbf{70}}$ and $2(31) = \underline{\mathbf{62}}$.



Method II
We all know the ordered triple (3, 4, 5) satisfies the Pythagorean theorem, i.e. are sides of a right triangle. But so are multiples!
$\rightarrow$ (15, 20, 25) $\rightarrow 2(15 + 20) = \underline{\mathbf{70}}$
The Pythagorean Triple (PT) (7, 24, 25) $\rightarrow 2(7 + 24) = \underline{\mathbf{62}}$ $(7^2 + 24^2 = 49 + 576 = 25^2 = 625)$
Some other common Pythagorean triples worth remembering:
(5, 12, 13) *(7,24,25)* (9, 40, 41) (11, 60, 61) (13, 84, 85) … Do you see a pattern?
Another sequence: (8, 15, 17) (12, 35, 37) (16, 63, 65) (20, 99, 101) …

B) Since Area$(\Delta ABC) = \frac{1}{2}ab = \frac{1}{2}hc$, it follows that $h = \frac{ab}{c}$. Using the Pythagorean Theorem, or noting that (20, 21, 29) is a primitive Pythagorean Triple, the hypotenuse $c = 29$. Thus, $h = \boxed{\mathbf{\frac{420}{29}}}$.



C) The fact that $\Delta AEB \sim \Delta CED$ implies $\frac{AE}{CE} = \frac{BE}{DE} = \frac{AB}{CD} = \frac{10}{25} = \frac{2}{5}$ or $d = (5/2)a$ and $c = (5/2)b$. Thus, $AC = a + d = (7/2)a$ and $BD = b + c = (7/2)b$. Requiring the diagonals to have integer lengths implies both a and b must be integers. Since $a^2 + b^2 = 10^2$, it follows that $(a, b) = (6, 8)$ and $(c, d) = (20, 15)$ or vice versa. Thus, the diagonals have lengths $6 + 15 = \underline{\mathbf{21}}$ and $8 + 20 = \underline{\mathbf{28}}$.