Method

Homework № 4.

Chapter 9.14 p 146-148 ex 2ab, 4, 5.

Ex. 2a Let $X_1, \dots X_n \sim$ Uniform (a, b) where a and b are unknown parameters and $a < b$

(a) Find the method of moments estimators for a and b.

$X_1, \dots X_n =$ Uniform (a, b)

So, first moment is:

$$E(X) = \frac{a+b}{2}$$

Sample first moment:

$$m_1 = \frac{X_1 + \dots + X_n}{n}.$$

$$\frac{a+b}{2} = m_1$$

$$a + b = 2m_1$$

$a = 2m_1 - b$ Method of Moments for a.

$$E(X^2) = Var(X) + E(X)^2$$

$$E(X^2) = \frac{(b-a)^2}{12} + \left(\frac{a+b}{2}\right)^2$$

Second sample moment:

$$m_2 = \frac{X_1^2 + \dots + X_n^2}{n}$$

$$\frac{(b-a)^2}{12} + \left(\frac{a+b}{2}\right)^2 = m_2$$

Подставляем $a = 2m_1 - b$.

$$\frac{(b - 2m_1 + b)^2}{12} + \left(\frac{2m_1 - b + b}{2}\right)^2 = m_2$$

$$\frac{(2b - 2m_1)^2}{12} + \left(\frac{2m_1}{2}\right)^2 = m_2$$

$$\frac{(2b - 2m_1)^2}{12} = m_2 - m_1^2$$

$$\frac{\cancel{4}(b - m_1)^2}{1\cancel{2}\,3} = m_2 - m_1^2$$

$$\frac{(b - m_1)^2}{3} = m_2 - m_1^2$$

$$b - m_1 = \sqrt{3(m_2 - m_1^2)}$$

$$\boxed{b = \sqrt{3(m_2 - m_1^2)} + m_1}$$

Method of moments for b.

Answer: MoM for $a = 2m_1 - b$

$a = 2m_1 - \sqrt{3(m_2 - m_1^2)} - m_1$

$a = m_1 - \sqrt{3(m_2 - m_1^2)}$

$b = \sqrt{3(m_2 - m_1^2)} + m_1$.

Ex 2b. Let $X_1, \dots X_n \sim$ Uniform (a, b) where a and b are unknown parameters and $a < b$

(b) Find the MLE $\hat{a}$ and $\hat{b}$.

PDF of Uniform (a, b)

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } x \in [a, b] \\ 0, & \text{otherwise} \end{cases}$$

$$L(a, b) = f(X_1) \dots f(X_n) = \frac{1}{b-a} \cdot \dots \cdot \frac{1}{(b-a)^n}$$

for $a \leq X_i \leq b$.

log-likelihood function:

$$\mathcal{L}(a, b) = \log L(a, b) = -n \cdot \log(b-a) + 1$$

if $a \leq X_i \leq b$ $\quad i = 1, 2, \dots, n$

$= -n \cdot \log(b-a) + 0$, otherwise.

So,

$$\max(X_1, \dots X_n) <= b$$
$$\min(X_1 \dots X_n) \geq a$$

So,

$$\hat{a} = \min(X_1, \dots X_n)$$
$$\hat{b} = \max(X_1, \dots X_n)$$

Ex 5 Let $X_1, \dots X_n \sim \text{Poisson}(\lambda)$. Find the method of moments estimator, the maximum likelihood estimator and the Fisher information $I(\lambda)$.

(a) Method of moments estimator

First moment.

$$E(X) = \lambda$$

Sample moment $\quad m_1 = \dfrac{X_1 + \dots + X_n}{n}$

(b) Maximum likelihood estimator

PDF of Poisson (λ)

$$f(X=n) = e^{-\lambda} \cdot \frac{\lambda^n}{n!}$$

$$L(\lambda) = f(X_1) \cdot \ldots \cdot f(X_n) = e^{-\lambda} \cdot \frac{\lambda^1}{1!} \cdot \ldots$$

$$\ldots e^{-\lambda} \cdot \frac{\lambda^n}{n!}$$

log-likelihood function:

$$L(\lambda) = \log L(\lambda) = -\log(e) \cdot \lambda + n \cdot \log(\lambda) -$$

$$-\log(n!) = n \cdot \log(\lambda) - \log(e)\,\lambda - \log(n!)$$

MLE:

$$\frac{d}{d\lambda} = \frac{n-\lambda}{\lambda} = \frac{n}{\lambda} - 1 = \frac{X_i}{\lambda} - n$$

$$\frac{X_i}{\lambda} - n = 0$$

$$\lambda = \frac{X_i}{n} = \text{MME}$$

(c) Fisher information:

Score function: $\lambda = \frac{X_1 + \ldots X_n}{}$

$$I(\lambda) = E_\lambda\left(\sum_{i=1}^{n} \frac{X_1 + \ldots X_n}{n}\right) = \sum_{i=1}^{n} E_\lambda\left(\frac{X_1 + \ldots + X_n}{n}\right)$$

$$\frac{\partial^2}{\partial^2 \lambda}\left(\frac{X_1 + \ldots + X_n}{n}\right) = -\frac{X_1 + \ldots + X_n}{\lambda^2}$$

$$-E\left(\frac{X_1 + \ldots + X_n}{\lambda^2}\right) = -\frac{n}{\lambda}$$

$$I(\lambda) = \frac{n}{\lambda}$$

Ex 4. Let $X_1, \dots X_n \sim Uniform(0, \theta)$. Show that the MLE is consistent. Hint: Let $Y = \max(X_1, \dots X_n)$. For any c, $P(Y < c) =$
$= P(X_1 < c, X_2 < c, \dots, X_n < c) = P(X_1 < c) P(X_2 < c) \dots P(X_n < c)$

PDF:

$$f(x) = \begin{cases} \frac{1}{\theta}, & \text{if } 0 \leq X \leq \theta \\ 0, & \text{otherwise} \end{cases}$$

Consistency of MLE: $\hat{\theta} \to \theta$ as $n \to \infty$

In our case

$\hat{\theta} \to Y$

We will use Chebyshev's inequality

$$P(|Z - E(Z)| \geq \varepsilon) \leq \frac{Var(Z)}{\varepsilon^2}$$

In our case:

$$P(|Y-\theta| \geq \varepsilon) \leq \frac{Var(Y)}{\varepsilon^2}$$

$$Var(Y) = n \cdot Var(X) = n \cdot \frac{\theta^2}{12}$$

$$P(|Y-\theta| \geq \varepsilon) \leq \frac{n \cdot \frac{\theta^2}{12}}{\varepsilon^2}$$

Divide both sides by $n \cdot \theta^2$ and take limit as $n \to \infty$

$$\lim P\left(\frac{|Y-\theta|}{\theta} \geq \frac{\varepsilon}{\theta}\right) \leq \lim \frac{1}{12} \cdot \left(\frac{\varepsilon}{\theta}\right)^2 = 0$$

So, therefore, as $n \to \infty$, we have $P\left(\frac{|Y-\theta|}{\theta} \geq \frac{\varepsilon}{\theta}\right) \to 0$ which means that $\hat{\theta} \to \theta$ in probability. Thus, the MLE is consistent.