

# Chapter 10: Free type constructions

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# The interpreter pattern I. Expression trees

Main idea: Represent a program as a data structure, run it later

- Example: a simple DSL for complex numbers

```
val a = "1+2*i".toComplex      Conj(  
val b = a * "3-4*i".toComplex  Mul(  
b.conj                        Str("1+2*i"), Str("3-4*i")  
                                ))
```

- *Unevaluated* operations `Str`, `Mul`, `Conj` are defined as case classes:

```
sealed trait Prg  
case class Str(s: String) extends Prg  
case class Mul(p1: Prg, p2: Prg) extends Prg  
case class Conj(p: Prg) extends Prg
```

- An *interpreter* will “run” the program and return a complex number

```
def run(prg: Prg): (Double, Double) = ...
```

- Benefits: programs are data, can compose & transform before running
- Shortcomings: this DSL works only with simple expressions
  - ▶ Cannot represent variable binding and conditional computations
  - ▶ Cannot use any non-DSL code (e.g. a numerical algorithms library)

# The interpreter pattern II. Variable binding

## A DSL with variable binding and conditional computations

- Example: imperative API for reading and writing files
  - ▶ Need to bind a *non-DSL variable* to a value computed by DSL
  - ▶ Later, need to use that non-DSL variable in DSL expressions
  - ▶ The rest of the DSL program is a (Scala) function of that variable

```
val p = path("/file")
val str: String = read(p)
if (str.nonEmpty)
  read(path(str))
else "Error: empty path"

Bind(
  Read(Path(Literal("/file"))),
  { str => // read value 'str'
    if (str.nonEmpty)
      Read(Path(Literal(str)))
    else Literal("Error: empty path")
  })
```

- Unevaluated operations are implemented via case classes:

```
sealed trait Prg
case class Bind(p: Prg, f: String => Prg) extends Prg
case class Literal(s: String) extends Prg
case class Path(s: Prg) extends Prg
case class Read(p: Prg) extends Prg
```

- Interpreter: `def run(prg: Prg): String = ...`

# The interpreter pattern III. Type safety

- So far, the DSL has no type safety: every value is a `Prg`
  - ▶ We want to avoid errors, e.g. `Read(Read(...))` should not compile
- Let `Prg[A]` denote a DSL program returning value of type `A` *when run*:

```
sealed trait Prg[A]
case class Bind(p: Prg[String], f: String ⇒ Prg[String])
  extends Prg[String]
case class Literal(s: String) extends Prg[String]
case class Path(s: Prg[String]) extends Prg[nio.file.Path]
case class Read(p: Prg[nio.file.Path]) extends Prg[String]
```

- Interpreter: `def run(prg: Prg[String]): String = ...`
- Our example DSL program is type-safe now:

```
val prg: Prg[String] = Bind(
  Read(Path(Literal("/file"))),
  { str: String ⇒
    if (str.nonEmpty)
      Read(Path(Literal(str)))
    else Literal("Error: empty path")
  })
```

# The interpreter pattern IV. Cleaning up the DSL

Our DSL so far:

```
sealed trait Prg[A]
case class Bind(p: Prg[String], f: String ⇒ Prg[String])
  extends Prg[String]
case class Literal(s: String) extends Prg[String]
case class Path(s: Prg[String]) extends Prg[nio.file.Path]
case class Read(p: Prg[nio.file.Path]) extends Prg[String]
```

Problems with this DSL:

- Cannot use `Read(p: nio.file.Path)`, only `Read(p: Prg[nio.file.Path])`
- Cannot bind variables or return values other than `String`

To fix these problems, make `Literal` a fully parameterized operation and replace `Prg[A]` by `A` in case class arguments

```
sealed trait Prg[A]
case class Bind[A, B](p: Prg[A], f: A ⇒ Prg[B]) extends Prg[B]
case class Literal[A](a: A) extends Prg[A]
case class Path(s: String) extends Prg[nio.file.Path]
case class Read(p: nio.file.Path) extends Prg[String]
```

- The type signatures of `Bind` and `Literal` are like `flatMap` and `pure`

# The interpreter pattern V. Define Monad-like methods

- We can actually define the methods `map`, `flatMap`, `pure`:

```
sealed trait Prg[A] {  
  def flatMap[B](f: A ⇒ Prg[B]): Prg[B] = Bind(this, f)  
  def map[B](f: A ⇒ B): Prg[B] = flatMap(this, f andThen Prg.pure)  
}  
object Prg { def pure[A](a: A): Prg[A] = Literal(a) }
```

- These methods don't run anything, only create unevaluated structures
- DSL programs can now be written as functor blocks and composed:

```
def readPath(p: String): Prg[String] = for {  
  path ← Path(p)  
  str  ← Read(path)  
} yield str
```

```
val prg: Prg[String] = for {  
  str ← readPath("/file")  
  result ← if (str.nonEmpty)  
    readPath(str)  
    else Prg.pure("Error: empty path")  
} yield result
```

- Interpreter: `def run[A](prg: Prg[A]): A = ...`

# The interpreter pattern VI. Refactoring to an abstract DSL

- Write a DSL for complex numbers in a similar way:

```
sealed trait Prg[A] { def flatMap ... } // no code changes
case class Bind[A, B](p: Prg[A], f: A⇒Prg[B]) extends Prg[B]
case class Literal[A](a: A) extends Prg[A]
type Complex = (Double, Double) // custom code starts here
case class Str(s: String) extends Prg[Complex]
case class Mul(c1: Complex, c2: Complex) extends Prg[Complex]
case class Conj(c: Complex) extends Prg[Complex]
```

- Refactor this DSL to separate common code from custom code:

```
sealed trait DSL[F[_], A] { def flatMap ... } // no code changes
type Prg[A] = DSL[F, A] // just for convenience
case class Bind[A, B](p: Prg[A], f: A⇒Prg[B]) extends Prg[B]
case class Literal[A](a: A) extends Prg[A]
case class Ops[A](f: F[A]) extends Prg[A] // custom operations here
```

- Interpreter is parameterized by a “value extractor”

$$\text{Ex}^F \equiv \forall A. (F^A \Rightarrow A)$$

```
def run[F[_], A](ex: Ex[F])(prg: DSL[F, A]): A = ...
```

## The interpreter pattern VII. Handling errors

- To handle errors, we want to evaluate `DSL[F[_], A]` to `Either[Err, A]`
- Suppose we have a value extractor of type  $\text{Ex}^F \equiv \forall A. (F^A \Rightarrow \text{Err} + A)$
- The code of the interpreter is almost unchanged:

```
def run[F[_], A](extract: Ex[F])(prg: DSL[F, A]): Either[Err, A] =  
  prg match {  
    case b: Bind[F, _, A]  $\Rightarrow$  b match { case Bind(p, f)  $\Rightarrow$   
      run(extract)(p).flatMap(f andThen run(extract))  
    }    // Here, the .flatMap is from Either.  
    case Literal(a)  $\Rightarrow$  Right(a) // pure: A  $\Rightarrow$  Err + A  
    case Ops(f)  $\Rightarrow$  extract(f)  
  }
```

- The code of `run` only uses `flatMap` and `pure` from `Either`
- We can generalize to any other monad  $M^A$  instead of `Either[Err, A]`

The resulting construction:

- Start with an “operations type constructor”  $F^A$  (often not a functor)
- Use  $\text{DSL}^{F,A}$  and interpreter  $\text{run} : (\forall X. F^X \Rightarrow M^X) \Rightarrow \text{DSL}^{F,A} \Rightarrow M^A$
- Create a DSL program  $\text{prg} : \text{DSL}^{F,A}$  and an extractor  $\text{ex}^X : F^X \Rightarrow M^X$
- Run the program with the extractor:  $\text{run}(\text{ex})(\text{prg})$ ; get a value  $M^A$



## The interpreter pattern VIII. Monadic DSLs: summary

- Begin with a number of operations, which are typically functions of fixed known types such as  $A_1 \Rightarrow B_1$ ,  $A_2 \Rightarrow B_2$  etc.
- Define a type constructor (typically not a functor) encapsulating all the operations as case classes, with or without type parameters

```
sealed trait F[A]
case class Op1(a1: A1) extends F[B1]
case class Op2(a1: A2) extends F[B2]
```

- Use `DSL[F,A]` with this `F` to write monadic DSL programs `prg: DSL[F,A]`
- Choose a target monad `M[A]` and implement an extractor `ex: F[A]  $\Rightarrow$  M[A]`
- Run the program with the extractor, `val res: M[A] = run(ex)(prg)`

Further directions (out of scope for this chapter):

- May choose another monad `N[A]` and use interpreter `M[A]  $\Rightarrow$  N[A]`
  - ▶ E.g. transform into another monadic DSL to optimize, test, etc.
- Since `DSL[F,A]` has a monad API, we can use monad transformers on it
- Can combine two or more DSLs in a disjunction: `DSLF+G+H,A`

# Monad laws for DSL programs

Monad laws hold for DSL programs only after evaluating them

- Consider the law  $\text{flm}(\text{pure}) = \text{id}$ ; both functions  $\text{DSL}^{F,A} \Rightarrow \text{DSL}^{F,A}$
- Apply both sides to some  $\text{prg} : \text{DSL}^{F,A}$  and get the new value

```
prg.flatMap(pure) == Bind(prg, a  $\Rightarrow$  Literal(a))
```

- This new value is *not equal* to `prg`, so this monad law fails!
  - ▶ Other laws fail as well because operations never reduce anything
- After interpreting this program into a target monad  $M^A$ , the law holds:

```
run(ex)(prg).flatMap((a  $\Rightarrow$  Literal(a)) andThen run(ex))  
  == run(ex)(prg).flatMap(a  $\Rightarrow$  run(ex)(Literal(a))  
  == run(ex)(prg).flatMap(a  $\Rightarrow$  pure(a))  
  == run(ex)(prg)
```

- ▶ Here we have assumed that the laws hold for  $M^A$
- ▶ All other laws also hold after interpreting into a lawful monad  $M^A$

The monad law violations are “not observable”

# Free constructions in mathematics: Example I

- Consider the Russian letter  $\mathfrak{u}$  (tsè) and the Chinese word 水 (shuǐ)
- We want to *multiply*  $\mathfrak{u}$  by 水. Multiply how?
- Say, we want an associative (but noncommutative) product of them
  - ▶ So we want to define a *semigroup* that *contains*  $\mathfrak{u}$  and 水 as elements
    - ★ while we still know nothing about  $\mathfrak{u}$  and 水
- Consider the set of all *unevaluated expressions* such as  $\mathfrak{u} \cdot \text{水} \cdot \text{水} \cdot \mathfrak{u} \cdot \text{水}$ 
  - ▶ Here  $\mathfrak{u} \cdot \text{水}$  is different from  $\text{水} \cdot \mathfrak{u}$  but  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- All these expressions form a **free semigroup** generated by  $\mathfrak{u}$  and 水
  - ▶ This is the most unrestricted semigroup that contains  $\mathfrak{u}$  and 水
- Example calculation:  $(\text{水} \cdot \text{水}) \cdot (\mathfrak{u} \cdot \text{水}) \cdot \mathfrak{u} = \text{水} \cdot \text{水} \cdot \mathfrak{u} \cdot \text{水} \cdot \mathfrak{u}$

How to represent this as a data type:

- **Tree encoding**: the full expression tree:  $((\text{水}, \text{水}), (\mathfrak{u}, \text{水})), \mathfrak{u})$ 
  - ▶ Implement the operation  $a \cdot b$  as pair constructor (easy)
- **Reduced encoding**, as a “smart” structure:  $\text{List}(\text{水}, \text{水}, \mathfrak{u}, \text{水}, \mathfrak{u})$ 
  - ▶ Implement  $a \cdot b$  by concatenating the lists (more expensive)

# Free constructions in mathematics: Example II

- Want to define a product operation for  $n$ -dimensional vectors:  $\mathbf{v}_1 \otimes \mathbf{v}_2$
- The  $\otimes$  must be linear and distributive (but not commutative):

$$\mathbf{u}_1 \otimes \mathbf{v}_1 + (\mathbf{u}_2 \otimes \mathbf{v}_2 + \mathbf{u}_3 \otimes \mathbf{v}_3) = (\mathbf{u}_1 \otimes \mathbf{v}_1 + \mathbf{u}_2 \otimes \mathbf{v}_2) + \mathbf{u}_3 \otimes \mathbf{v}_3$$

$$\mathbf{u} \otimes (a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2) = a_1 (\mathbf{u} \otimes \mathbf{v}_1) + a_2 (\mathbf{u} \otimes \mathbf{v}_2)$$

$$(a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2) \otimes \mathbf{u} = a_1 (\mathbf{v}_1 \otimes \mathbf{u}) + a_2 (\mathbf{v}_2 \otimes \mathbf{u})$$

- ▶ We have such a product for 3-dimensional vectors only; ignore that
- Consider *unevaluated expressions* of the form  $\mathbf{u}_1 \otimes \mathbf{v}_1 + \mathbf{u}_2 \otimes \mathbf{v}_2 + \dots$ 
  - ▶ A free vector space generated by pairs of vectors
- Impose the equivalence relationships shown above
  - ▶ The result is known as the **tensor product**
- Tree encoding: full unevaluated expression tree
  - ▶ A list of any number of vector pairs  $\sum_i \mathbf{u}_i \otimes \mathbf{v}_i$
- Reduced encoding: an  $n \times n$  matrix
  - ▶ Reduced encoding requires proofs and more complex operations

# Worked example: Free semigroup

Implement a free semigroup `FSIS` generated by two types `Int` and `String`

- A value of `FSIS` can be an `Int`; it can also be a `String`
- If `x, y` are of type `FSIS` then so is `x |+| y`

```
sealed trait FSIS // tree encoding: full expression tree
case class Wrap1(x: Int) extends FSIS
case class Wrap2(x: String) extends FSIS
case class Comb(x: FSIS, y: FSIS) extends FSIS
```

- Short type notation:  $FSIS \equiv Int + String + FSIS \times FSIS$
- For a semigroup  $S$  and given  $Int \Rightarrow S$  and  $String \Rightarrow S$ , map  $FSIS \Rightarrow S$
- Simplify and generalize this construction by setting  $Z = Int + String$ 
  - ▶ The tree encoding is  $FS^Z \equiv Z + FS^Z \times FS^Z$

```
def |+|(x: FS[Z], y: FS[Z]): FS[Z] = Comb(x, y)
def run[S: Semigroup, Z](extract: Z  $\Rightarrow$  S): FS[Z]  $\Rightarrow$  S = {
  case Wrap(z)  $\Rightarrow$  extract(z)
  case Comb(x, y)  $\Rightarrow$  run(extract)(x) |+| run(extract)(y)
} // Semigroup laws will hold after applying run().
```

- The reduced encoding is  $FSR^Z \equiv Z \times List^Z$  (non-empty list of  $Z$ 's)
  - ▶ `x |+| y` requires concatenating the lists, but `run()` is faster

# Worked example: Free monoid

Implement a free monoid `FM[Z]` generated by type `Z`

- A value of `FM[Z]` can be the empty value; it can also be a `Z`
- If `x, y` are of type `FM[Z]` then so is `x |+| y`

```
sealed trait FM[Z] // tree encoding
case class Empty[Z]() extends FM[Z]
case class Wrap[Z](z: Z) extends FM[Z]
case class Comb[Z](x: FM[Z], y: FM[Z]) extends FM[Z]
```

- Short type notation:  $FM^Z \equiv 1 + Z + FM^Z \times FM^Z$
- For a monoid  $M$  and given  $Z \Rightarrow M$ , map  $FM^Z \Rightarrow M$

```
def |+|(x: FM[Z], y: FM[Z]): FM[Z] = Comb(x, y)
def run[M: Monoid, Z](extract: Z ⇒ M): FM[Z] ⇒ M = {
  case Empty() ⇒ Monoid[M].empty
  case Wrap(z) ⇒ extract(z)
  case Comb(x, y) ⇒ run(extract)(x) |+| run(extract)(y)
} // Monoid laws will hold after applying run().
```

- The reduced encoding is  $FMR^Z \equiv List^Z$  (list of `Z`'s)
  - ▶ Implementing `|+|` requires concatenating the lists
- Reduced encoding and tree encoding give identical results after `run()`

# Mapping a free semigroup to different targets

What if we interpret  $\text{FS}^X$  into *another* free semigroup?

- Given  $Y \Rightarrow Z$ , can we map  $\text{FS}^Y \Rightarrow \text{FS}^Z$ ?
  - Need to map  $\text{FS}^Y \equiv Y + \text{FS}^Y \times \text{FS}^Y \Rightarrow Z + \text{FS}^Z \times \text{FS}^Z$
  - This is straightforward since  $\text{FS}^X$  is a functor in  $X$ :

```
def fmap[Y, Z](f: Y => Z): FS[Y] => FS[Z] = {  
  case Wrap(y) => Wrap(f(y))  
  case Comb(a, b) => Comb(fmap(f)(a), fmap(f)(b))  
}
```

- Now we can use `run` to interpret  $\text{FS}^X \Rightarrow \text{FS}^Y \Rightarrow \text{FS}^Z \Rightarrow S$ , etc.
  - Functor laws hold for  $\text{FS}^X$ , so `fmap` is composable as usual
  - The “interpreter” commutes with `fmap` as well (naturality law):

$$\begin{array}{ccc} & \text{fmap } f^{X \Rightarrow Y} & \text{FS}^Y \\ \text{FS}^X & \nearrow & \searrow \text{run}_g^{S, Y \Rightarrow S} \\ & \text{run}_g^{S, (f \circ g)^{X \Rightarrow S}} & S \end{array}$$

- Combine two free semigroups:  $\text{FS}^{X+Y}$ ; inject parts:  $\text{FS}^X \Rightarrow \text{FS}^{X+Y}$

# Continuation encoding I. Motivation

- Multiple target semigroups  $S_i$  require many “extractors”  $\text{ex}_i : Z \Rightarrow S_i$
- Refactor extractors  $\text{ex}_i$  into evidence of a typeclass constraint on  $S_i$

// Typeclass `ExZ[S]` has the single method ‘`extract: Z  $\Rightarrow$  S`’.

```
implicit val extractS: ExZ[MySemigroup] = { z  $\Rightarrow$  ... }  
def run[S: ExZ: Semigroup](fm: FM[Z]): S = fm match {  
  case Wrap(z)  $\Rightarrow$  implicitly[ExZ[S]].extract(z)  
  case Comb(x, y)  $\Rightarrow$  run(x) |+| run(y)  
}
```

- Refactor `run` using a helper function `wrap`

```
def wrap[S: ExZ](z: Z): S = implicitly[ExZ[S]].extract(z)
```

- To create free monoid values, define functions with constraint `[S: ExZ]`

```
def x[S: ExZ : Semigroup]: S = wrap(1) |+| wrap(2)
```

- The type of `x` is  $\forall S. (Z \Rightarrow S) \times (S \times S \Rightarrow S) \Rightarrow S$ ; equivalently:

$$\forall S. (Z \Rightarrow S) \times (S \times S \Rightarrow S) \Rightarrow S \cong \forall S. ((Z + S \times S) \Rightarrow S) \Rightarrow S$$

which is similar to the type of the continuation monad,  $(A \Rightarrow R) \Rightarrow R$

- ▶ That’s why I propose to call this a “**continuation encoding**”

- Continuation encoding works for any type:  $A \cong \forall X. (A \Rightarrow X) \Rightarrow X$



# Continuation encoding II. Further examples

- Consider the continuation encoding for the disjunction type  $P + Q$ 
  - The encoding is  $\forall X. (P + Q \Rightarrow X) \Rightarrow X \cong \forall X. (P \Rightarrow X) \times (Q \Rightarrow X) \Rightarrow X$

```
trait Disj[P, Q] { def run[X](cp: P => X)(cq: Q => X): X }
```

- Define some values of this type:

```
def inLeft[P, Q](p: P) = new Disj[P, Q] {  
  def run[X](cp: P => X)(cq: Q => X): X = cp(p)  
}
```

- Now we can implement the analog of the `case` expression simply as

```
val result = disj.run {p => ...} {q => ...}
```

- This works in programming languages that have no disjunction types
- General recipe for continuation encoding:

```
trait Blah { def run[X](cont: ... => X): X }
```

- For convenience, define a type class `Ex` describing the inner function

```
trait Ex[X] { def cp: P => X; def cq: Q => X }
```

- Different methods of this class return `X`; convenient with disjunctions

# General properties of free type constructions

Generalizing from our examples so far:

- We “enriched”  $Z$  to a monoid  $\text{FM}^Z$  and  $F^A$  to a monad  $\text{DSL}^{F,A}$ 
  - ▶ The “enrichment” adds case classes representing the needed operations
  - ▶ Very similar results for a type  $Z$  and for a type constructor  $F^A$
- Obtain a **free type construction**, which performs no computations

Questions:

- Can we promote any type constructor  $F^A$  to any typeclass  $C$ ?
  - ▶ Useful typeclasses: functor, filterable, monad, applicative
- What are the laws for the “free instance of  $C$  generated by  $F$ ”?
  - ▶ For all  $F^A$ , must have `wrap` :  $F^A \Rightarrow \text{FreeC}^{F,A}$
  - ▶ For all  $M^A : C$ , given  $F^A \Rightarrow M^A$ , must have `run` :  $\text{FreeC}^{F,A} \Rightarrow M^A$
  - ▶ The laws of typeclass  $C$  must hold after interpreting into an  $M^A : C$
  - ▶ Given any `f` :  $F^A \Rightarrow G^A$ , must have `fmap(f)` :  $\text{FreeC}^{F,A} \Rightarrow \text{FreeC}^{G,A}$
- Which of the possible encodings to use?
  - ▶ Tree encoding, reduced encodings, continuation encoding
- Look at further examples

# Worked example: free functor

- Generalize

## Worked example: free pointed functor

- Also consider the case when we start from a functor  $F$
- Free monad when starting from a functor  $F$

# Worked example: free filterable

- Generalie

# Worked example: free applicative

- Generalize

- 1 Implement a free semigroup generated by a type parameter  $Z$  in the tree encoding and in the reduced encoding. Show that the semigroup laws hold for the reduced encoding but not for the tree encoding before interpreting into a lawful semigroup  $S$ .