

# Chapter 9: Traversable functors

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# Motivation for the `traverse` operation

- Consider data of type  $\text{List}^A$  and processing  $f : A \Rightarrow \text{Future}^B$
- Typically, we want to wait until the entire data set is processed
- What we need is  $\text{List}^A \Rightarrow (A \Rightarrow \text{Future}^B) \Rightarrow \text{Future}^{\text{List}^B}$
- Generalize:  $L^A \Rightarrow (A \Rightarrow F^B) \Rightarrow F^{L^B}$  for some type constructors  $F, L$
- This operation is called `traverse`
  - ▶ How to implement it: for example, a 3-element list is  $A \times A \times A$
  - ▶ Consider  $L^A \equiv A \times A \times A$ , apply map  $f$  and get  $F^B \times F^B \times F^B$
  - ▶ We will get  $F^{L^B} \equiv F^{B \times B \times B}$  if we can apply `zip` as  $F^B \times F^B \Rightarrow F^{B \times B}$
- So we need to assume that  $F$  is applicative
- In Scala, we have `Future.traverse()` that assumes  $L$  to be a sequence
  - ▶ This is the easy-to-remember example that fixes the requirements
- Questions:
  - ▶ Which functors  $L$  can have this operation?
  - ▶ Can we express `traverse` through a simpler operation?
  - ▶ What are the required laws for `traverse`?
  - ▶ What about contrafunctors or profunctors?

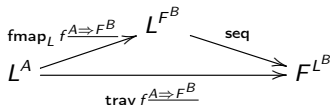
# Deriving the `sequence` operation

- The type signature of `traverse` is a complicated “lifting”
  - ▶ A “lifting” is often equivalent to a simpler natural transformation
- To derive it, ask: what is missing from `fmap` to do the job of `traverse`?

$$\text{fmap}_L : (A \Rightarrow F^B) \Rightarrow L^A \Rightarrow L^{F^B}$$

- We need  $F^{L^B}$ , but the `traverse` operation gives us  $L^{F^B}$  instead
  - ▶ What’s missing is a natural transformation `sequence` :  $L^{F^B} \Rightarrow F^{L^B}$
- The functions `traverse` and `sequence` are computationally equivalent:

$$\text{trav } f \xrightarrow{A \Rightarrow F^B} = \text{fmap}_L f \circ \text{seq}$$



Here  $F$  is an *arbitrary* applicative functor

- ▶ Keep in mind the example `Future.sequence` :  $\text{List}^{\text{Future}^X} \Rightarrow \text{Future}^{\text{List}^X}$
- ▶ Examples:  $L^A \equiv A \times A \times A$ ;  $L^A = \text{List}^A$ ; finite trees
- ▶ Non-traversable:  $L^A \equiv R \Rightarrow A$ ; lazy list (“infinite product”)
  - ★ Note: We *cannot* have the opposite transformation  $F^{L^B} \Rightarrow L^{F^B}$

# Polynomial functors are traversable

- Generalize from the example  $L^A \equiv A \times A \times A$  to other polynomials
- Polynomial functors have the form

$$L^A \equiv Z \times A \times \dots \times A + Y \times A \times \dots \times A + \dots + Q \times A + P$$

- To implement  $\text{seq} : L^{F^B} \Rightarrow F^{L^B}$ , consider monomial  $L^A \equiv Z \times A \times \dots \times A$
- We have  $L^{F^B} = Z \times F^B \times \dots \times F^B$ ; apply `zip` and get  $Z \times F^{B \times \dots \times B}$
- Lift  $Z$  into the functor  $F$  using  $Z \Rightarrow F^A \Rightarrow F^{Z \times A}$  (or with  $F.\text{pure}$ )
- The result is  $F^{Z \times B \times \dots \times B} \equiv F^{L^B}$ 
  - ▶ For a polynomial  $L^A$ , do this to each monomial, then lift to  $F^{L^B}$
  - ▶ Note that we could apply `zip` in various different orders
- The traversal order is arbitrary, may be application-specific
- Non-polynomial functors are not traversable (see [Bird et al., 2013](#))
  - ▶ Example:  $L^A \equiv E \Rightarrow A$ ;  $F^A \equiv 1 + A$ ; can't have  $\text{seq} : L^{F^B} \Rightarrow F^{L^B}$
- All polynomial functors are traversable, and usually in several ways
  - ▶ It is still useful to have a type class for traversable functors

# Motivation for the laws of the `traverse` operation

- The “**law of traversals**” paper (2012) argues that `traverse` should “visit each element” of the container  $L^A$  exactly once, and evaluate each corresponding “effect”  $F^B$  exactly once; then they formulate the laws
- To derive the laws, use the “lifting” intuition for `traverse`,

$$\text{trav} : (A \Rightarrow F^B) \Rightarrow L^A \Rightarrow F^{L^B}$$

Look for “identity” and “composition” laws:

- ① “Identity” as `pure` :  $A \Rightarrow F^A$  must be lifted to `pure` :  $L^A \Rightarrow F^{L^A}$
- ② “Identity” as  $\text{id}^{A \Rightarrow A}$  with  $F^A \equiv A$  (identity functor) lifted to  $\text{id}^{L^A \Rightarrow L^A}$
- ③ “Compose”  $f : A \Rightarrow F^B$  and  $g : B \Rightarrow G^C$  to get  $h : A \Rightarrow F^{G^C}$ , where  $F, G$  are applicative; a traversal with  $h$  maps  $L^A$  to  $F^{G^{L^C}}$  and must be equal to the composition of traversals with  $f$  and then with  $g^{F^\uparrow}$

Questions:

- Are the laws for the `sequence` operation simpler?
- Are all these laws independent?
- What functors  $L$  satisfy these laws *for all* applicative functors  $F$ ?

# Formulation of the laws for `traverse`

- Identity law: For any applicative functor  $F$ ,

$$\text{trav}(\text{pure}) = \text{pure}$$

$$L^A \xrightarrow[\text{trav}(\text{pure}^{A \Rightarrow F^A})]{\text{pure}^{L^A \Rightarrow F^{L^A}}} F^{L^A}$$

- ▶ Second identity law:  $\text{trav}^{\text{Id}}(\text{id}^A) = \text{id}^{L^A}$  is a consequence with  $F = \text{Id}$

★ So, we need only one identity law

- Composition law: For any  $f^{A \Rightarrow F^B}$  and  $g^{B \Rightarrow G^C}$ , & applicative  $F$  and  $G$ ,

$$\text{trav } f \circ (\text{trav } g)^{F\uparrow} = \text{trav} (f \circ g^{F\uparrow})$$

$$\begin{array}{ccc} & & F^{L^B} \\ \text{trav}^F f^{A \Rightarrow F^B} \nearrow & & \searrow \text{fmap}_F(\text{trav}^G g)^{L^B \Rightarrow G^{L^C}} \\ L^A & \xrightarrow{\quad\quad\quad} & F^{G^{L^C}} \\ & \text{trav}^{FG} h^{A \Rightarrow F^{G^C}} \nearrow & \end{array}$$

where  $h^{A \Rightarrow F^{G^C}} \equiv f \circ g^{F\uparrow}$ . (Note:  $H^A \equiv F^{G^A}$  is applicative!)

# Derivation of the laws for **sequence**

Express  $\text{trav } f = f^{L\uparrow} \circ \text{seq}$  and substitute into the laws for  $\text{trav}$ :

- Identity law:  $\text{trav } (\text{pure}) = \text{pure}^{L\uparrow} \circ \text{seq} = \text{pure}$

$$\begin{array}{ccc}
 & L^{FA} & \\
 \text{fmap}_L \text{ pure}^A \nearrow & & \searrow \text{seq} \\
 L^A & \xRightarrow{\text{pure}^{LA}} & F^{LA}
 \end{array}$$

Naturality law:  $\text{seq} \circ g^{F\uparrow L\uparrow} = g^{L\uparrow F\uparrow} \circ \text{seq}$  with  $g^{A \Rightarrow B}$ , mapping  $L^{FA} \Rightarrow F^{LB}$

- Composition law:

$$\begin{aligned}
 \text{trav } f \circ (\text{trav } g)^{F\uparrow} &= f^{L\uparrow} \circ \text{seq} \circ (g^{L\uparrow} \circ \text{seq})^{F\uparrow} \\
 &= f^{L\uparrow} \circ \text{seq} \circ g^{L\uparrow F\uparrow} \circ \text{seq}^{F\uparrow} = f^{L\uparrow} \circ g^{F\uparrow L\uparrow} \circ \text{seq} \circ \text{seq}^{F\uparrow} \\
 \text{trav } (f \circ g^{F\uparrow}) &= (f \circ g^{F\uparrow})^{L\uparrow} \circ \text{seq} = f^{L\uparrow} \circ g^{F\uparrow L\uparrow} \circ \text{seq}
 \end{aligned}$$

Now omit the common prefix  $f \cdots \circ g \cdots$  and obtain:  $\text{seq} \circ \text{seq}^{F\uparrow} = \text{seq}$

$$\begin{array}{ccc}
 & F^{LG^C} & \\
 \text{seq}^F \nearrow & & \searrow (\text{seq}^G)^{F\uparrow} \\
 L^{FG^C} & \xRightarrow{\text{seq}^{FG}} & F^{GL^C}
 \end{array}$$

# Constructions of traversable and bitraversable functors

Constructions of traversable functors:

- ①  $L^A \equiv Z$  (constant functor) and  $L^A \equiv A$  (identity functor)
  - ②  $L^A \equiv G^A \times H^A$  for any traversable  $G^A$  and  $H^A$
  - ③  $L^A \equiv G^A + H^A$  for any traversable  $G^A$  and  $H^A$
  - ④  $L^A \equiv S^{A,L^A}$  (recursive) for a bitraversable bifunctor  $S^{A,B}$ 
    - If  $L^A$  is infinite, laws will appear to hold but `seq` will not terminate
- A bifunctor  $S^{A,B}$  is **bitraversable** if `bisequence` exists such that

$$\text{biseq} : S^{F^A, F^B} \Rightarrow F^{S^{A,B}}$$

for any applicative functor  $F$ ; the analogous laws must hold

Constructions of bitraversable bifunctors:

- ①  $S^{A,B} \equiv Z$ ,  $S^{A,B} \equiv A$ , and  $S^{A,B} = B$
- ②  $S^{A,B} \equiv G^{A,B} \times H^{A,B}$  for any bitraversable  $G$  and  $H$
- ③  $S^{A,B} \equiv G^{A,B} + H^{A,B}$  for any bitraversable  $G$  and  $H$
- All polynomial bifunctors are bitraversable
- All polynomial functors, including recursive functors, are traversable



# Foldable functors: traversing with respect to a monoid

- Take  $F^A \equiv Z$  where  $Z$  is a monoid
  - ▶ The `zip` operation is the monoid operation  $\oplus$
- The type signature of `traverse` becomes  $(A \Rightarrow Z) \Rightarrow L^A \Rightarrow Z$ 
  - ▶ This method is called `foldMap`
- The type signature of `seq` becomes  $L^Z \Rightarrow Z$ 
  - ▶ This is called `mconcat` – combines all values in  $L^Z$  with  $Z$ 's  $\oplus$
- It is convenient to define the `Foldable` type class
  - ▶ But it has no laws any more
  - ▶ All traversable functors are also foldable
- The `foldLeft` method can be defined via `foldMap` with  $Z \equiv (B \Rightarrow B)$ :

$$\text{foldl} : (A \Rightarrow B \Rightarrow B) \Rightarrow L^A \Rightarrow B \Rightarrow B$$

# Traversable contrafunctors and profunctors are not useful

Traversing profunctors with respect to functors  $F$ : effects of  $F$  are ignored

- All contrafunctors  $C^A$  are traversable w.r.t. applicative profunctors  $F^A$ ,

$$\text{seq} : C^{F^A} \Rightarrow F^{C^A} \equiv \text{pure}^{C\downarrow} \circ \text{pure}$$

$$C^{F^A} \xrightarrow{\text{cmap}_{C\text{pure}_F^A}} C^A \xrightarrow{\text{pure}_F^{C^A}} F^{C^A}$$

- But not profunctors that are neither functors not contrafunctors
- Counterexample:  $P^A \equiv A \Rightarrow A$ ; need  $\text{seq} : (F^A \Rightarrow F^A) \Rightarrow F^{A \Rightarrow A}$ ; we can't get an  $A \Rightarrow A$ , so the only implementation is to return  $\text{pure}_F(\text{id})$ , which ignores its argument and so will fail the identity law

Traversing profunctors  $L$  with respect to profunctors  $F$ : effects are ignored

- Counterexample 1: contrafunctor  $L^A \equiv A \Rightarrow R$  and contrafunctor  $F^A \equiv A \Rightarrow S$ , a `seq` of type  $L^{F^A} \Rightarrow F^{L^A}$  must return  $1 + 0$
- Counterexample 2: contrafunctor  $F^A \equiv (R \Rightarrow A) \Rightarrow S$  and functor  $L^A \equiv 1 + A$ ; `seq` must return  $1 + 0$
- So, the result is trivial and probably not useful
  - ▶ Laws of traversables allow ignoring the effects of  $F$

# Examples of usage

- 1 Convert foldable data structures to list
- 2 Fold a tree to aggregate data
- 3 Decorate a tree with Depth-First traversal order labels
- 4 Implement `scanMap` and `scanLeft` as `traverse` with a state monad
- 5 Traversal for a non-monadic “rigid” tree  $T^A \equiv A + T^{A \times A}$ 
  - The corresponding construction is  $L^A \equiv S^{A, L^{R^A}}$  where  $R$  is applicative and traversable and  $S$  is bitraversable

# Naturality with respect to applicative functor as parameter

- The `traverse` method must be “generic in the functor  $F$ ”:

$$\text{trav}^{F,A,B} : (A \Rightarrow F^B) \Rightarrow L^A \Rightarrow F^{L^B}$$

Which means: The code of `traverse` can only use `pure` and `zip` from  $F$

- A functor  $F^A$  is “generic in  $A$ ”: have  $\text{fmap} : (A \Rightarrow B) \Rightarrow F^A \Rightarrow F^B$
- “Generic in  $F$ ” means mapping  $(F \Rightarrow G) \Rightarrow \text{trav}^F \Rightarrow \text{trav}^G$  in some way

Mathematical formulation:

- For any natural transformation  $F^A \Rightarrow G^A$  between applicative functors  $F$  and  $G$  such that  $F.\text{pure}$  and  $F.\text{zip}$  are mapped into  $G.\text{pure}$  and  $G.\text{zip}$ , the result of transforming  $\text{trav}^F$  is  $\text{trav}^G$ 
  - ▶ Such a natural transformation is a morphism of applicative functors
  - ▶ Category theory can describe  $(F \Rightarrow G) \Rightarrow \text{trav}^F \Rightarrow \text{trav}^G$  as a “lifting”
  - ▶ Use a more general definition of category than what we had so far (morphisms between type constructors)

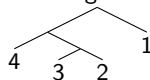
# Exercises

- 1 Show that any traversable functor  $L$  admits a method

$$\text{consume} : (L^A \Rightarrow B) \Rightarrow L^{F^A} \Rightarrow F^B$$

for any applicative functor  $F$ . Show that `traverse` and `consume` are equivalent.

- 2 Show that  $\text{seq} : L^{F^A} \Rightarrow F^{L^A} = \text{id}$  if we choose  $F^A \equiv A$  as the identity functor.
- 3 Show that the identity law is not satisfied by an implementation of  $\text{seq} : L^{F^A} \Rightarrow F^{L^A}$  for  $F^A \equiv 1 + A$  when  $\text{seq}$  always returns an empty option.
- 4 Show that  $K^A \equiv G^{H^A}$  is traversable if  $G$  and  $H$  are traversable.
- 5 Show that all the bitraversable laws hold for the bifunctor  $S^{A,B} \equiv A \times B$ .
- 6 For the tree-like type defined as  $T^A \equiv 1 + A \times T^A \times T^A$ , define a traversable instance. Verify that the laws hold, using a suitable bifunctor  $S^{X,Y}$ .
- 7 For a tree type  $T^A$  of your choice, implement a traversable instance for right-to-left traversal order, and test that the tree is decorated with labels as



- 8 \* Can we use `traverse` to implement breadth-first traversal for a tree type?
- 9 Implement a traversable instance for a “rose tree”  $T^A \equiv A + \text{List } T^A$ . Represent  $T^A$  via a suitable bifunctor  $S^{X,Y}$  and show that  $S$  is bitraversable (use constructions).
- 10 Is the recursive type constructor  $L^A \equiv A + L^{\text{List } A}$  traversable? Explain what sort of data container it is.