# Chapter 10: Free type constructions

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# Free constructions in mathematics: Example I

- Consider the Russian letter ц (tsè) and the Chinese word 水 (shuǐ)
- We want to *multiply* ц by 水. Multiply how?
- Say, we want an associative (but noncommutative) product of them
  - So we want to define a *semigroup* that *contains* ц and 水
    - while we still know nothing about ц and 水
- Consider the set of all *unevaluated expressions* such as ц·水·水·ц·水
  - Here ц·水 is different from 水·ц but $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- All these expressions form a **free semigroup** generated by ц and 水
- Example calculation: (水·水)·(ц·水)·ц = 水·水·ц·水·ц

How to represent this as a data type:

- Tree encoding, as the full expression tree: ((水,水),(ц,水)),ц)
  - Implement the operation $a \cdot b$ as pair constructor (easy and cheap)
- "Smart" encoding, as a reduced structure: List(水,水,ц,水,ц)
  - Implement the opration $a \cdot b$ by concatenating the lists (more expensive)

# Free constructions in mathematics: Example II

- Want to define a product operation for $n$-dimensional vectors: $\mathbf{v}_1 \otimes \mathbf{v}_2$
- The $\otimes$ must be associative and distributive (but not commutative):

$$
\begin{aligned}
\mathbf{u} \otimes (\mathbf{v} \otimes \mathbf{w}) &= (\mathbf{u} \otimes \mathbf{v}) \otimes \mathbf{w} \\
\mathbf{u} \otimes (a_1\mathbf{v}_1 + a_2\mathbf{v}_2) &= a_1 (\mathbf{u} \otimes \mathbf{v}_1) + a_2 (\mathbf{u} \otimes \mathbf{v}_2) \\
(a_1\mathbf{v}_1 + a_2\mathbf{v}_2) \otimes \mathbf{u} &= a_1 (\mathbf{v}_1 \otimes \mathbf{u}) + a_2 (\mathbf{v}_2 \otimes \mathbf{u})
\end{aligned}
$$

- We have such a product for 3-dimensional vectors only; ignore that
- Consider *unevaluated expressions* of the form $\mathbf{u}_1 \otimes \mathbf{v}_1 + \mathbf{u}_2 \otimes \mathbf{v}_2 + ...$

# Exercises

1. Show that