

Chapter 11: Computations in a functor context III

Monad transformers

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Computations within a functor context: Combining monads

Programs often need to combine monadic effects

- “Effect” \equiv what else happens in $A \Rightarrow M^B$ besides computing B from A
- Examples of effects for some standard monads:
 - ▶ **Option** – computation will have no result or a single result
 - ▶ **List** – computation will have zero, one, or multiple results
 - ▶ **Either** – computation may fail to obtain its result, reports error
 - ▶ **Reader** – computation needs to read an external context value
 - ▶ **Writer** – some value will be appended to a (monoidal) accumulator
 - ▶ **Future** – computation will be scheduled to run later
- How to combine several effects in the same functor block (**for/yield**)?

```
// This is not valid Scala!           // This is not valid Scala!
val result = for { i ← 1 to n          (1 to n).flatMap { i ⇒
    j ← Future { q(i) }                Future(q(i)).flatMap { j ⇒
    k ← maybeError(j) : Try[Int]        maybeError(j).map { k ⇒
} yield f(k)                           f(k)
// What should be the type of result??   }}
```

- The code will work if we “unify” all effects in a new, larger monad
- Need to compute the type of new monad that contains all given effects

Combining monadic effects I. Trial and error

There are several ways of combining two monads into a new monad:

- If M_1^A and M_2^A are monads then $M_1^A \times M_2^A$ is also a monad
 - ▶ But $M_1^A \times M_2^A$ describes two separate values with two separate effects
- If M_1^A and M_2^A are monads then $M_1^A + M_2^A$ is usually not a monad
 - ▶ If it worked, it would be a choice between two different values / effects
- If M_1^A and M_2^A are monads then one of $M_1^{M_2^A}$ or $M_2^{M_1^A}$ is often a monad
- Examples and counterexamples for functor composition:
 - ▶ Combine $Z \Rightarrow A$ and List^A as $Z \Rightarrow \text{List}^A$
 - ▶ Combine `Future[A]` and `Option[A]` as `Future[Option[A]]`
 - ▶ But `Either[Z, Future[A]]` and `Option[Z \Rightarrow A]` are not monads
 - ▶ Neither `Future[State[A]]` nor `State[Future[A]]` are monads
- The order of effects matters when composition works both ways:
 - ▶ Combine `Either` ($M_1^A = Z + A$) and `Writer` ($M_2^A = W \times A$)
 - ★ as $Z + W \times A$ – either compute result and write a message, or all fails
 - ★ as $(Z + A) \times W$ – message is always written, but computation may fail
- Find a general way of defining a new monad with combined effects
- Derive properties required for the new monad

Combining monadic effects II. Lifting into a larger monad

If a “big monad” `BigM[A]` somehow combines all the needed effects:

```
// This could be valid Scala...           // If we define the various
val result: BigM[Int] = for {              // required “lifting” functions:
  i ← lift1(1 to n)                        def lift1[A]: Seq[A] ⇒ BigM[A] = ???
  j ← lift2(Future{ q(i) })                def lift2[A]: Future[A] ⇒ BigM[A] = ???
  k ← lift3(maybeError(j))                def lift3[A]: Try[A] ⇒ BigM[A] = ???
} yield f(k)
```

- Example 1: combining as `BigM[A] = Future[Option[A]]` with liftings:

```
def lift1[A]: Option[A] ⇒ Future[Option[A]] = Future.successful(_)
def lift2[A]: Future[A] ⇒ Future[Option[A]] = _.map(x ⇒ Some(x))
```

- Example 2: combining as `BigM[A] = List[Try[A]]` with liftings:

```
def lift1[A]: Try[A] ⇒ List[Try[A]] = x ⇒ List(x)
def lift2[A]: List[A] ⇒ List[Try[A]] = _.map(x ⇒ Success(x))
```

Remains to be understood:

- Finding suitable laws for the liftings; checking that the laws hold
- Building a “big monad” out of “smaller” ones, with lawful liftings
 - ▶ Is this always possible? Unique? Are there alternative solutions?
- Ways of reducing the complexity of code; make liftings automatic

Laws for monad liftings I. Identity laws

Whatever identities we expect to hold for monadic programs must continue to hold after lifting M_1 or M_2 values into the “big monad” BigM

- We assume that M_1 , M_2 , and BigM already satisfy all the monad laws

Consider the various functor block constructions containing the liftings:

- Left identity law after lift_1

// Anywhere inside a for/yield:	// Must be equivalent to...
$i \leftarrow \text{lift}_1(M_1.\text{pure}(x))$	$i = x$
$j \leftarrow \text{bigM}(i)$ // Any BigM value.	$j \leftarrow \text{bigM}(x)$

$\text{lift}_1(M_1.\text{pure}(x)).\text{flatMap}(b) = b(x)$ — in terms of Kleisli composition (\diamond):
 $(\text{pure}_{M_1} \circ \text{lift}_1)^{X \Rightarrow \text{BigM}^Y} \diamond b^{X \Rightarrow \text{BigM}^Y} = b$ with $f^{X \Rightarrow M^Y} \diamond g^{Y \Rightarrow M^Z} \equiv x \Rightarrow f(x).\text{flatMap}(g)$

- Right identity law after lift_1

// Anywhere inside a for/yield:	// Must be equivalent to...
$x \leftarrow \text{bigM}$ // Any BigM value.	$x \leftarrow \text{bigM}$
$i \leftarrow \text{lift}_1(M_1.\text{pure}(x))$	$i = x$

$b.\text{flatMap}(M_1.\text{pure} \text{ andThen } \text{lift}_1) = b$ — in terms of Kleisli composition:

$$b^{X \Rightarrow \text{BigM}^Y} \diamond (\text{pure}_{M_1} \circ \text{lift}_1)^{Y \Rightarrow \text{BigM}^Y} = b$$

- The same identity laws must hold for M_2 and lift_2 as well

Laws for monad liftings II. Simplifying the laws

$(\text{pure}_{M_1} \circ \text{lift}_1)$ is a unit for the Kleisli composition \diamond in the monad `BigM`

- But the monad `BigM` already has a unit element: $\text{pure}_{\text{BigM}}$
- The two-sided unit element is always unique: $\text{id} = \text{id} \diamond \text{id}' = \text{id}'$
- So the two identity laws for $(\text{pure}_{M_1} \circ \text{lift}_1)$ can be reduced to one law:

$$\text{pure}_{M_1} \circ \text{lift}_1 = \text{pure}_{\text{BigM}}$$

Refactoring a portion of a monadic program under `lift1` gives another law:

// Anywhere inside a for/yield:

`i ← lift1(p)` // Any M_1 value.

`j ← lift1(q(i))` // Any M_1 value.

// Must be equivalent to...

`pq = p.flatMap(q)` // In M_1 .

`j ← lift1(pq)` // Now lift it.

$$\text{lift}_1(p).\text{flatMap}(q \text{ andThen } \text{lift}_1) = \text{lift}_1(p \text{ flatMap } q)$$

- Rewritten equivalently through $\text{flm}_M : (A \Rightarrow M^B) \Rightarrow M^A \Rightarrow M^B$ as

$$\text{lift}_1 \circ \text{flm}_{\text{BigM}}(q \circ \text{lift}_1) = \text{flm}_{M_1} q \circ \text{lift}_1$$

- Rewritten in terms of Kleisli composition:

$$(b : X \Rightarrow M_1^Y \circ \text{lift}_1) \diamond (c : Y \Rightarrow M_1^Z \circ \text{lift}_1) = (b \diamond c) \circ \text{lift}_1$$

- Liftings `lift1` and `lift2` must obey an identity law and a composition law
- The laws say that the liftings **commute with** the monads' operations

Laws for monad liftings III. The naturality law

$\text{lift}_1 : M_1^A \Rightarrow \text{BigM}^A$ is a natural transformation between monads

- It maps pure_{M_1} to $\text{pure}_{\text{BigM}}$ and flm_{M_1} to flm_{BigM}
 - ▶ lift_1 is a **monadic morphism** between monads M_1 and BigM

The (functor) naturality law follows:

$$\text{lift}_1 \circ \text{fmap}_B f^{X \Rightarrow Y} = \text{fmap}_{M_1} f^{X \Rightarrow Y} \circ \text{lift}_1$$

$$\begin{array}{ccc} M_1^X & \xrightarrow{\text{lift}_1} & \text{BigM}^X \\ \text{fmap}_{M_1} f^{X \Rightarrow Y} \downarrow & & \downarrow \text{fmap}_{\text{BigM}} f^{X \Rightarrow Y} \\ M_1^Y & \xrightarrow{\text{lift}_1} & \text{BigM}^Y \end{array}$$

Derivation:

- Express fmap as $\text{fmap}_M f = \text{flm}_M (f \circ \text{pure}_M)$ for both monads
- Given $f^{X \Rightarrow Y}$, use the law $\text{flm}_{M_1} q \circ \text{lift}_1 = \text{lift}_1 \circ \text{flm}_{\text{BigM}} (q \circ \text{lift}_1)$ to compute $\text{flm}_{M_1} (f \circ \text{pure}_{M_1}) \circ \text{lift}_1 = \text{lift}_1 \circ \text{flm}_{\text{BigM}} (f \circ \text{pure}_{M_1} \circ \text{lift}_1) = \text{lift}_1 \circ \text{flm}_{\text{BigM}} (f \circ \text{pure}_{\text{BigM}}) = \text{lift}_1 \circ \text{fmap}_{\text{BigM}} f$

- Combining $Z \Rightarrow A$ and $1 + A$ works only as $Z \Rightarrow 1 + A$
- It is not possible to combine two monads via a natural bifunctor
- But we can fix M_1 (base monad) and let M_2 (foreign monad) vary
- The result is a monad transformer $T_{M_1}^{M_2, A}$ – a natural functor in M_2

Definition:

- Monad transformer for a base monad L is a type constructor $T_L^{M, A}$

Exercises

- 1 Show that the method `pure`: $A \Rightarrow M^A$ is a monad morphism between monads $\text{Id}^A \equiv A$ and M^A .
- 2 Show that $M_1^A + M_2^A$ is *not* a monad when $M_1^A \equiv 1 + A$ and $M_2^A \equiv Z \Rightarrow A$.