

Advanced: Quantum Dynamics Simulation

We will perform quantum dynamics (QD) simulation on a quantum computer for the transverse-field Ising model (TFIM) Hamiltonian for two spins,

$$H = -J\sigma_0^z\sigma_1^z - B\sum_{j=0}^1\sigma_j^x, \quad (1)$$

where σ_j^z and σ_j^x are Pauli Z and X matrices acting on the j -th spin, J is the exchange coupling, and B is the magnetic field along the x axis.

Time evolution of a two-spin wave function, $|\Psi(t)\rangle = |\psi_0(t)\rangle|\psi_1(t)\rangle$ ($|\psi_j(t)\rangle$ is the wave function of the j -th spin at time t), for small time step Δt is governed by (cf. <https://aiichironakano.github.io/phys516/03QD.pdf>)

$$|\Psi(t + \Delta t)\rangle = \exp(-iH\Delta t)|\Psi(t)\rangle \quad (2)$$

in the atomic unit. Using Trotter expansion, the time-propagation operator is approximated as

$$\exp(-iH\Delta t) = \exp(i\Delta t J\sigma_0^z\sigma_1^z)\exp(i\Delta t B\sigma_0^x)\exp(i\Delta t B\sigma_1^x) + O(\Delta t^2). \quad (3)$$

Let us first consider the transverse-field propagator $\exp(i\Delta t B\sigma_j^x)$ acting on the j -th spin independent of the other spin. We use the eigendecomposition (see [Appendix A](#)) of Pauli X matrix,

$$\sigma^x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (4)$$

Note that

$$\sigma^x H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H\sigma^z, \quad (5)$$

where H is the Hadamard gate (which is column-aligned eigenvectors $(1/\sqrt{2}, \pm 1/\sqrt{2})^T$ of σ^x with respective eigenvalues ± 1), or equivalently

$$\sigma^x = H\sigma^z H, \quad (6)$$

where we have used the fact H is a symmetric orthogonal matrix, *i.e.*, $H^{-1} = H^T = H$ and thus

$$H^2 = I \quad (7)$$

(I is the identity matrix).

Using Taylor expansion of the time propagator and Eqs. (6) and (7) (the procedure is called telescoping),

$$\begin{aligned} \exp(i\Delta t B\sigma^x) &= \sum_{n=0}^{\infty} \frac{(i\Delta t B)^n}{n!} \sigma^{xn} = \sum_{n=0}^{\infty} \frac{(i\Delta t B)^n}{n!} (H\sigma^z H)^n = \\ &= \sum_{n=0}^{\infty} \frac{(i\Delta t B)^n}{n!} \overbrace{H\sigma^z H H\sigma^z H \cdots H\sigma^z H}^{n \text{ times}} \text{ (every internal HH product becomes } I \text{)} = \\ &= H \sum_{n=0}^{\infty} \frac{(i\Delta t B)^n}{n!} \sigma^{zn} H = H \sum_{n=0}^{\infty} \frac{(i\Delta t B)^n}{n!} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^n H = H \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(i\Delta t B)^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{(-i\Delta t B)^n}{n!} \end{pmatrix} H = \\ &= H \begin{pmatrix} e^{i\Delta t B} & 0 \\ 0 & e^{-i\Delta t B} \end{pmatrix} H = H R_z(-2\Delta t B) H = \frac{1}{2} \begin{pmatrix} e^{i\Delta t B} + e^{-i\Delta t B} & e^{i\Delta t B} - e^{-i\Delta t B} \\ e^{i\Delta t B} - e^{-i\Delta t B} & e^{i\Delta t B} + e^{-i\Delta t B} \end{pmatrix} = \\ &= \begin{pmatrix} \cos(\Delta t B) & i\sin(\Delta t B) \\ i\sin(\Delta t B) & \cos(\Delta t B) \end{pmatrix} = R_x(-2\Delta t B). \end{aligned} \quad (8)$$

In terms of the native gates on IBM Q computers, Eq. (8) can be implemented using either rotation around the z axis, $R_z(\theta)$, along with Hadamard gate H , or solely using rotation around the x axis, $R_x(\theta)$. Here, R_z and R_x gates are defined as

$$R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}, \quad (9)$$

$$R_x(\theta) = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \quad (10)$$

(see https://github.com/Qiskit/qiskit-tutorials/blob/master/tutorials/circuits/3_summary_of_quantum_operations.ipynb).

Next, we consider the exchange-coupling propagator $\exp(i\Delta t J \sigma_0^z \sigma_1^z)$. We first consider a tensor product of operators multiplied by a scalar constant,

$$i\Delta t J \sigma_0^z \otimes \sigma_1^z = i\Delta t J \begin{pmatrix} 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & -1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} i\Delta t J & 0 & 0 & 0 \\ 0 & -i\Delta t J & 0 & 0 \\ 0 & 0 & -i\Delta t J & 0 \\ 0 & 0 & 0 & i\Delta t J \end{pmatrix}. \quad (11)$$

Since this is a diagonal matrix, it can be exponentiated element by element as

$$\exp(i\Delta t J \sigma_0^z \sigma_1^z) = \begin{pmatrix} \exp(i\Delta t J) & 0 & 0 & 0 \\ 0 & \exp(-i\Delta t J) & 0 & 0 \\ 0 & 0 & \exp(-i\Delta t J) & 0 \\ 0 & 0 & 0 & \exp(i\Delta t J) \end{pmatrix} = \begin{pmatrix} R_z(-2\Delta t J) & 0 \\ 0 & R_z(2\Delta t J) \end{pmatrix}. \quad (12)$$

Now consider the following sequence of quantum gates operating on two qubits, q_0 and q_1 ,

$$G = CX(q_0, q_1) \cdot R_1^z(-2\Delta t J) \cdot CX(q_0, q_1), \quad (13)$$

where

$$CX(q_0, q_1) = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} \quad (14)$$

is the controlled X (CNOT) gate, with q_0 and q_1 being the control and target bits, and R_1^z is the R^z gate acting on q_1 . When operating on two qubits, R_1^z signifies a tensor product,

$$I \otimes R^z(-2\Delta t J) = \begin{pmatrix} 1 \cdot R^z(-2\Delta t J) & 0 \cdot R^z(-2\Delta t J) \\ 0 \cdot R^z(-2\Delta t J) & 1 \cdot R^z(-2\Delta t J) \end{pmatrix} = \begin{pmatrix} R^z(-2\Delta t J) & 0 \\ 0 & R^z(-2\Delta t J) \end{pmatrix}. \quad (15)$$

Substituting Eqs. (14) and (15) in Eq. (13), we obtain

$$G = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} R^z(-2\Delta t J) & 0 \\ 0 & R^z(-2\Delta t J) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} R^z(-2\Delta t J) & 0 \\ 0 & X R^z(-2\Delta t J) X \end{pmatrix}. \quad (16)$$

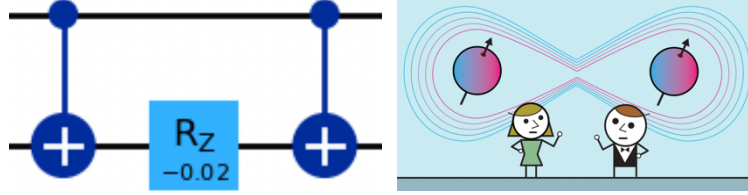
Here,

$$\begin{aligned} X R^z(-2\Delta t J) X &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \exp(i\Delta t J) & 0 \\ 0 & \exp(-i\Delta t J) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} 0 & \exp(-i\Delta t J) \\ \exp(i\Delta t J) & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \exp(-i\Delta t J) & 0 \\ 0 & \exp(i\Delta t J) \end{pmatrix} = R^z(2\Delta t J). \end{aligned} \quad (17)$$

Substituting Eq. (17) in Eq. (16) and compare the result with Eq. (12), we arrive at the identity,

$$G = CX(q_0, q_1)R_1^z(-2\Delta tJ)CX(q_0, q_1) = \begin{pmatrix} R^z(-2\Delta tJ) & 0 \\ 0 & R^z(2\Delta tJ) \end{pmatrix} = \exp(i\Delta tJ\sigma_0^z\sigma_1^z). \quad (18)$$

where the last equality results from Eq. (12). Namely, $G = CX(q_0, q_1) \cdot R_1^z(-2\Delta tJ) \cdot CX(q_0, q_1)$ is a quantum-gate implementation of the exchange-coupling propagator $\exp(i\Delta tJ\sigma_0^z\sigma_1^z)$.



Combining Eqs. (8) and (18) for the transverse-field and exchange-coupling time propagators, respectively, quantum-circuit implementation for a single time step of time evolution for the TFIM model, Eq. (1), is given by

$$\exp(-iH\Delta t) = \exp(i\Delta tJ\sigma_0^z\sigma_1^z)\exp(i\Delta tB\sigma_0^x)\exp(i\Delta tB\sigma_1^x) = CX(q_0, q_1)R_1^z(-2\Delta tJ)CX(q_0, q_1)R_0^x(-2\Delta tB)R_1^x(-2\Delta tB). \quad (18)$$

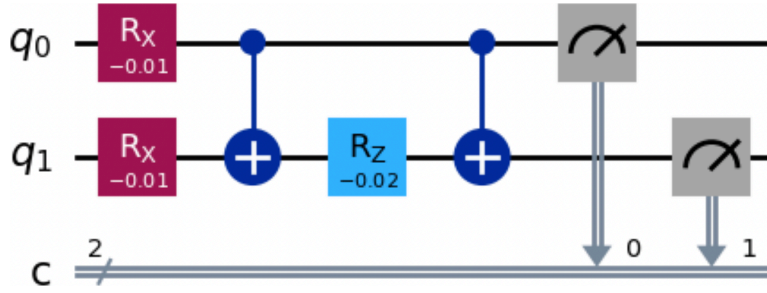
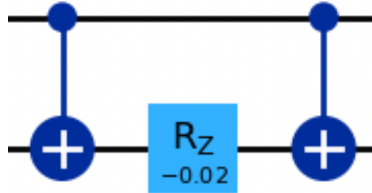


Fig. 1: Quantum circuit for time evolution of TFIM in IBM Quantum Lab.

In summary, time evolution is simply a rotation around an eigen-axis of the Hamiltonian operator, *e.g.*,

$$R_x(-0.01),$$

or concerted rotations of multiple qubits mediated by, *e.g.*, the CNOT gate,



Hands-on Exercise (try it in Jupyter Notebook using Qiskit AerSimulator)

Execute the following Qiskit program to perform a single time step of QD simulation. Here, we have used model parameters, $J = 1$, $B = 0.5$ and $\Delta t = 0.01$, in atomic units. [Appendix B](#) shows how to set up an environment for the exercise.

```
##### Single step of Trotter propagation in transverse-field Ising model #####
import numpy as np

# Import standard Qiskit libraries
from qiskit import QuantumCircuit
from qiskit_aer import AerSimulator
from qiskit.visualization import *

### Physical parameters (atomic units) ###
J = 1.0      # Exchange coupling
B = 0.5      # Transverse magnetic field
dt = 0.01    # Time-discretization unit

### Build a circuit ###
circ = QuantumCircuit(2, 2) # 2 quantum & 2 classical registers

circ.rx(-2*dt*B, 0) # Transverse-field propagation of spin 0
circ.rx(-2*dt*B, 1) # Transverse-field propagation of spin 1
circ.cx(0, 1)       # Exchange-coupling time propagation (1)
circ.rz(-2*dt*J, 1) # (2)
circ.cx(0, 1)       # (3)
circ.measure(range(2), range(2)) # Measure both spins
circ.draw('mpl')
```

This will build a circuit and draw it, which should then be transpiled and run on a simulator as follows.

```
### Simulate on Aer backend ###

# Use Aer simulator
backend = AerSimulator()

# Transpile the quantum circuit to low-level QASM instructions
from qiskit import transpile
circ_compiled = transpile(circ, backend)

# Execute the circuit on the simulator, repeating 1024 times
job_sim = backend.run(circ_compiled, shots=1024)

# Grab the results from the job
result_sim = job_sim.result()

# Get the result
counts = result_sim.get_counts(circ_compiled)

# Plot histogram
from qiskit.visualization import plot_histogram
plot_histogram(counts)
```

Table I: Qiskit program for single-time-step QD simulation of TFIM: tfim-1step.qiskit
(<https://aiichironakano.github.io/phys516/src/QComp/tfim-1step.qiskit>).

After creating a new Python notebook, you can copy and paste the above code into a cell in the notebook. Actual quantum dynamics simulation [L. Bassman *et al.*, *Phys. Rev. B* **101**, 184305 ('20)] will iterate this unit-time stepping for many time steps. For Python programming underlying Qiskit, see A. Scopatz and K. D. Huff, *Effective Computation in Physics* (O'Reilly, '15).

Appendix A: Eigendecomposition

For a 2×2 Hermitian matrix,

$$\mathbf{A} = \begin{bmatrix} a & b \\ b^* & a \end{bmatrix}, \quad (\text{A1})$$

where a and b are real and complex numbers, respectively, consider an eigenvalue problem,

$$\begin{bmatrix} a & b \\ b^* & a \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \varepsilon \begin{bmatrix} u \\ v \end{bmatrix}. \quad (\text{A2})$$

or equivalently

$$\begin{bmatrix} \varepsilon - a & -b \\ -b^* & \varepsilon - a \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (\text{A3})$$

For nontrivial solutions (*i.e.*, other than $u = v = 0$), the determinant of the matrix in Eq. (A3) should be zero. (Otherwise, one can invert Eq. (A3) to get $u = v = 0$.) Hence,

$$\begin{vmatrix} \varepsilon - a & -b \\ -b^* & \varepsilon - a \end{vmatrix} = (\varepsilon - a)^2 - |b|^2 = 0, \text{ Secular (characteristic) equation} \quad (\text{A4})$$

which has two solutions,

$$\varepsilon_{\pm} = a \pm |b|. \text{ Eigenvalues} \quad (\text{A5})$$

The corresponding eigenvectors can be obtained by solving Eq. (A3) for these eigenvalues

$$\begin{bmatrix} |b| & -b \\ -b^* & |b| \end{bmatrix} \begin{bmatrix} u_+ \\ v_+ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} -|b| & -b \\ -b^* & -|b| \end{bmatrix} \begin{bmatrix} u_- \\ v_- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{A6})$$

with the answers (note the degeneracy of the two linear equations for each eigenvalue, *e.g.*, $|b|u_+ - bv_+ = 0 \Rightarrow \left(\times \frac{-b^*}{|b|}\right) -b^*u_+ + |b|v_+ = 0$)

$$\mathbf{w}_{\pm} = \begin{bmatrix} u_{\pm} \\ v_{\pm} \end{bmatrix} = \frac{1}{\sqrt{2}|b|} \begin{bmatrix} b \\ \pm |b| \end{bmatrix}. \text{ Eigenvectors} \quad (\text{A7})$$

In Eq. (A7), we have normalized each eigenvector so that

$$\mathbf{w}_{\pm}^{\dagger} \mathbf{w}_{\pm} = \begin{bmatrix} u_{\pm}^* & v_{\pm}^* \end{bmatrix} \begin{bmatrix} u_{\pm} \\ v_{\pm} \end{bmatrix} = \frac{|b|^2}{2|b|^2} = 1, \quad (\text{A8})$$

where $\mathbf{w}_{\pm}^{\dagger}$ denotes the Hermitian conjugate (or conjugate transpose) of \mathbf{w}_{\pm} . Also, the two eigenvectors are orthogonal:

$$\mathbf{w}_{\mp}^{\dagger} \mathbf{w}_{\pm} = \begin{bmatrix} u_{\mp}^* & v_{\mp}^* \end{bmatrix} \begin{bmatrix} u_{\pm} \\ v_{\pm} \end{bmatrix} = \frac{\widetilde{b^*b} - |b|^2}{2|b|^2} = 0. \quad (\text{A9})$$

Now, define a 2×2 matrix composed of column aligned eigenvectors,

$$\mathbf{U} = [\mathbf{w}_+ \quad \mathbf{w}_-] = \begin{bmatrix} u_+ & u_- \\ v_+ & v_- \end{bmatrix} = \frac{1}{\sqrt{2}|b|} \begin{bmatrix} b & b \\ |b| & -|b| \end{bmatrix}, \quad (\text{A10})$$

then

$$\mathbf{U}^{\dagger} \mathbf{U} = \begin{bmatrix} \mathbf{w}_+^{\dagger} \\ \mathbf{w}_-^{\dagger} \end{bmatrix} [\mathbf{w}_+ \quad \mathbf{w}_-] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}, \quad (\text{A11})$$

where \mathbf{I} is the 2×2 identity matrix and we have used the orthonormalization relations, Eqs. (A8) and (A9). Using the explicit formula for \mathbf{U} in Eq. (A10), we can also verify that $\mathbf{U}\mathbf{U}^\dagger = \mathbf{I}$ and hence \mathbf{U} is a unitary matrix:

$$\mathbf{U}^\dagger \mathbf{U} = \mathbf{U}\mathbf{U}^\dagger = \mathbf{I}. \text{ Unitary} \quad (\text{A12})$$

The two solutions of Eq. (A2) can now be combined into a matrix form as

$$\begin{cases} \begin{bmatrix} a & b \\ b^* & a \end{bmatrix} \begin{bmatrix} u_+ \\ v_+ \end{bmatrix} = \varepsilon_+ \begin{bmatrix} u_+ \\ v_+ \end{bmatrix} \\ \begin{bmatrix} a & b \\ b^* & a \end{bmatrix} \begin{bmatrix} u_- \\ v_- \end{bmatrix} = \varepsilon_- \begin{bmatrix} u_- \\ v_- \end{bmatrix} \end{cases} \Leftrightarrow \underbrace{\begin{bmatrix} a & b \\ b^* & a \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} u_+ & u_- \\ v_+ & v_- \end{bmatrix}}_{\mathbf{U}} = \underbrace{\begin{bmatrix} u_+ & u_- \\ v_+ & v_- \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} \varepsilon_+ & 0 \\ 0 & \varepsilon_- \end{bmatrix}}_{\mathbf{D}}, \quad (\text{A13})$$

i.e.,

$$\mathbf{A}\mathbf{U} = \mathbf{U}\mathbf{D}, \quad (\text{A14})$$

where we have defined a diagonal matrix,

$$\mathbf{D} = \begin{bmatrix} \varepsilon_+ & 0 \\ 0 & \varepsilon_- \end{bmatrix}. \quad (\text{A15})$$

$$\therefore \begin{bmatrix} u_+ & u_- \\ v_+ & v_- \end{bmatrix} \begin{bmatrix} \varepsilon_+ \\ 0 \end{bmatrix} = \varepsilon_+ \begin{bmatrix} u_+ \\ v_+ \end{bmatrix} \text{ and } \begin{bmatrix} u_+ & u_- \\ v_+ & v_- \end{bmatrix} \begin{bmatrix} 0 \\ \varepsilon_- \end{bmatrix} = \varepsilon_- \begin{bmatrix} u_- \\ v_- \end{bmatrix} \quad \text{1}^{\text{st}} \& \text{2}^{\text{nd}}\text{-column pickers}$$

Multiplying both sides of Eq. (A14) by \mathbf{U}^\dagger from the right hand and using the unitary, Eq. (A12), we obtain

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^\dagger. \text{ Eigendecomposition} \quad (\text{A16})$$

or more explicitly

$$\begin{bmatrix} a & b \\ b^* & a \end{bmatrix} = \frac{1}{\sqrt{2}|b|} \begin{bmatrix} b & b \\ |b| & -|b| \end{bmatrix} \begin{bmatrix} a+|b| & 0 \\ 0 & a-|b| \end{bmatrix} \frac{1}{\sqrt{2}|b|} \begin{bmatrix} b^* & |b| \\ b^* & -|b| \end{bmatrix}. \quad (\text{A17})$$

(Example) Pauli X matrix, *i.e.*, $a = 0$ and $b = 1$

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \mathbf{H}\mathbf{Z}\mathbf{H}. \quad (\text{A18})$$

where \mathbf{H} and \mathbf{Z} are matrix representations of Hadamard and Pauli Z gates.

Appendix B: Setting an Environment

Use a Python notebook environment such as qBraid Lab (<https://account.qbraid.com/>) and open a new Python notebook. Type the following command in the code cell to set up an environment.

```
pip install qiskit qiskit_ibm_runtime qiskit_aer matplotlib pylatexenc
```

In qBraid, you can add a predefined Qiskit environment and open a new Python [Qiskit] notebook. In this way, you can skip the above command.

