

Calculating π in Parallel Using MPI

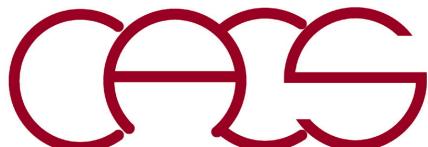
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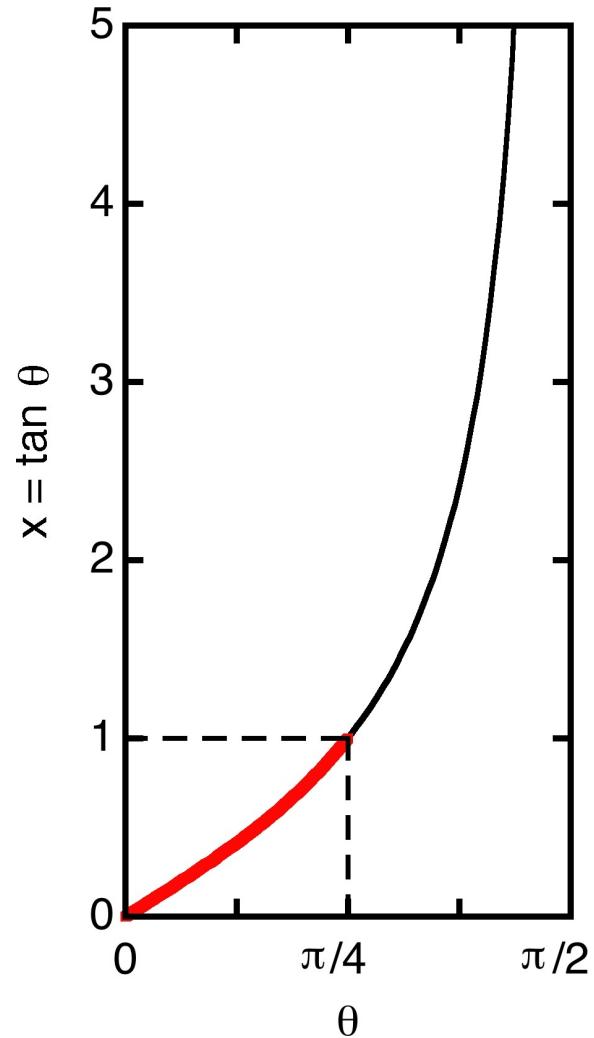
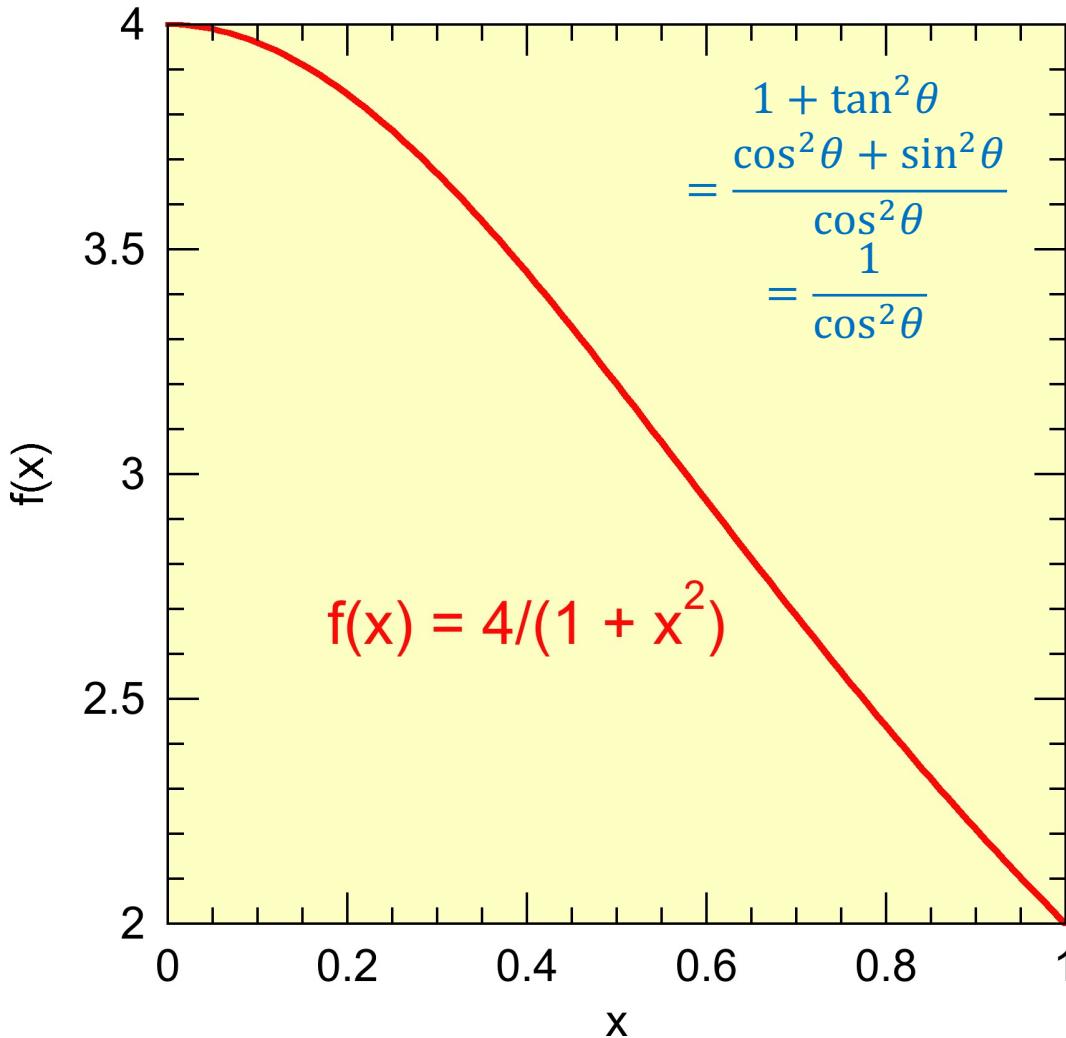
Objectives

1. Task decomposition (parallel programming = who does what)
2. Scalability analysis



Integral Representation of π

$$\int_0^1 dx \frac{4}{1+x^2} = \int_0^{\pi/4} \frac{d\theta}{\cos^2 \theta} \frac{4}{1+\tan^2 \theta} = \int_0^{\pi/4} 4d\theta = \pi \quad \begin{aligned} x &= \tan \theta \\ \frac{dx}{d\theta} &= \frac{1}{\cos^2 \theta} \end{aligned}$$



Numerical Integration of π

- Integration

$$\int_0^1 dx \frac{4}{1+x^2} = \pi$$

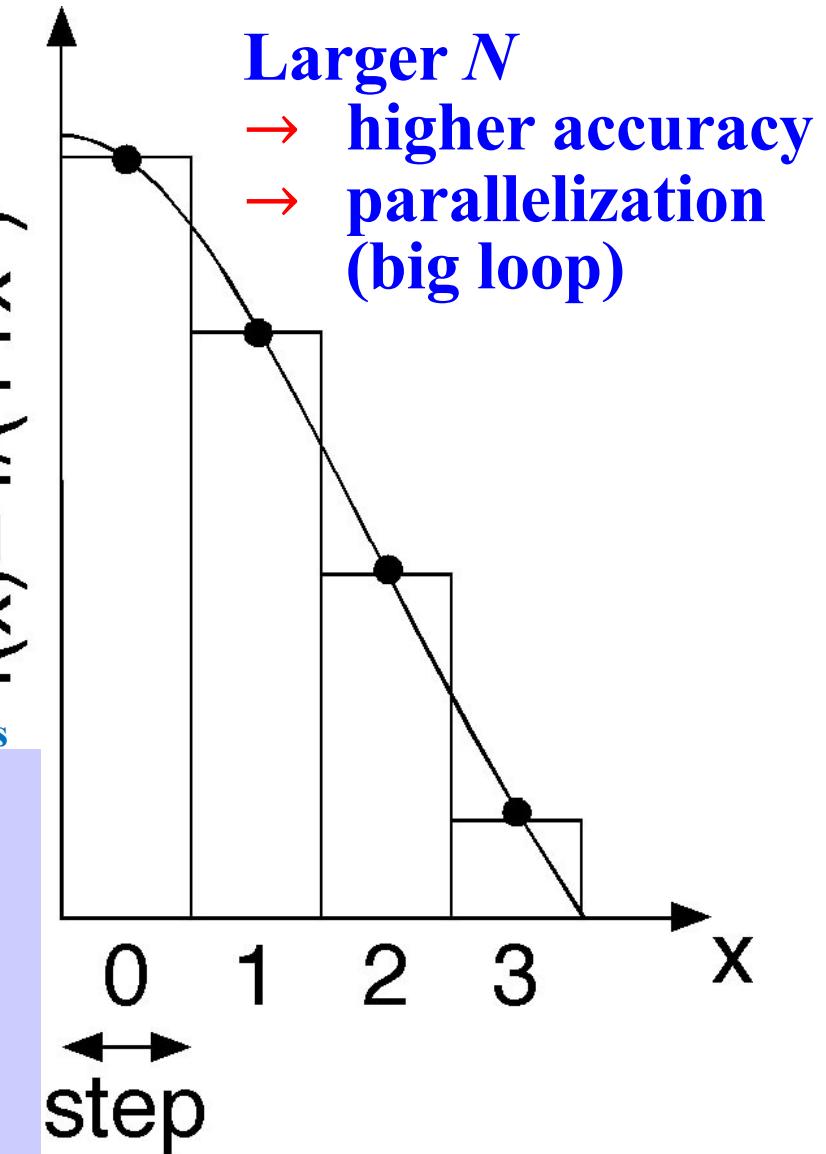
- Discretization:

$$\Delta = 1/N: \text{step} = 1/\text{NBIN}$$
$$x_i = (i+0.5)\Delta \quad (i=0, \dots, N-1)$$

$$\sum_{i=0}^{N-1} \frac{4}{1+x_i^2} \Delta \cong \pi$$

```
#include <stdio.h>
#define NBIN 100000000
void main() {
    long long i; double step,x,sum=0.0,pi;
    step = 1.0/NBIN;
    for (i=0; i<NBIN; i++) {
        x = (i+0.5)*step;
        sum += 4.0/(1.0+x*x);
    }
    pi = sum*step;
    printf("PI = %f\n",pi);
}
```

64-bit integer to handle large indices



See https://en.wikipedia.org/wiki/Chronology_of_computation_of_%CF%80

Parallelization: Who Does What?

Interleaved assignment of quadrature points (bins) to MPI processes

Single program multiple data (SPMD)

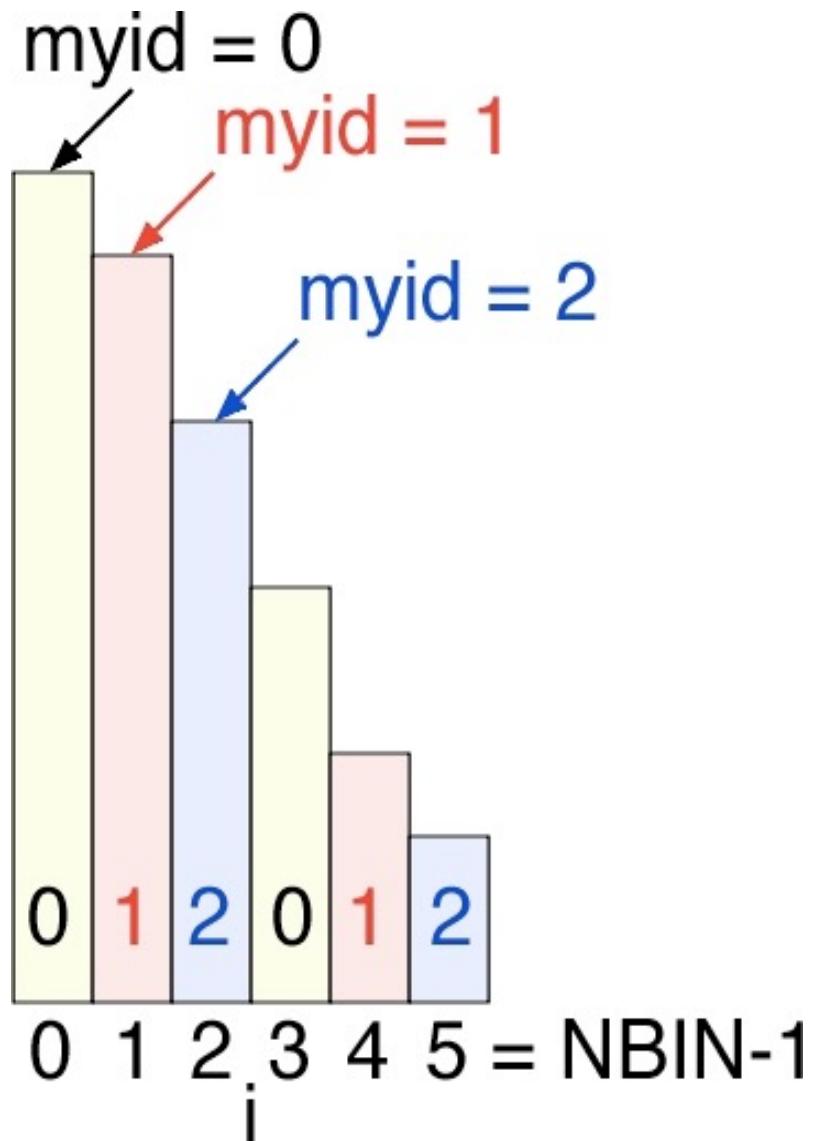
```
...
for (i=myid; i<NBIN; i+=nprocs)
{
    x = (i+0.5)*step;
    sum += 4.0/(1.0+x*x);
}
partial = sum*step;
pi = global_sum(partial);
...
```

`MPI_Comm_rank(MPI_COMM_WORLD, &myid)`

myid = MPI rank

nprocs = Number of MPI processes

`MPI_Comm_size(MPI_COMM_WORLD, &nprocs)`



- Use `double MPI_Wtime()` to measure the running time in seconds

Measuring Runtime

double MPI_Wtime(): Returns the elapsed wall-clock time in seconds since some time in the past. The “time in the past” is guaranteed not to change during the lifetime of the process

See p. 9 in <https://aiichironakano.github.io/cs596/02MPI.pdf>

```
int main() {
    double cpu1, cpu2;

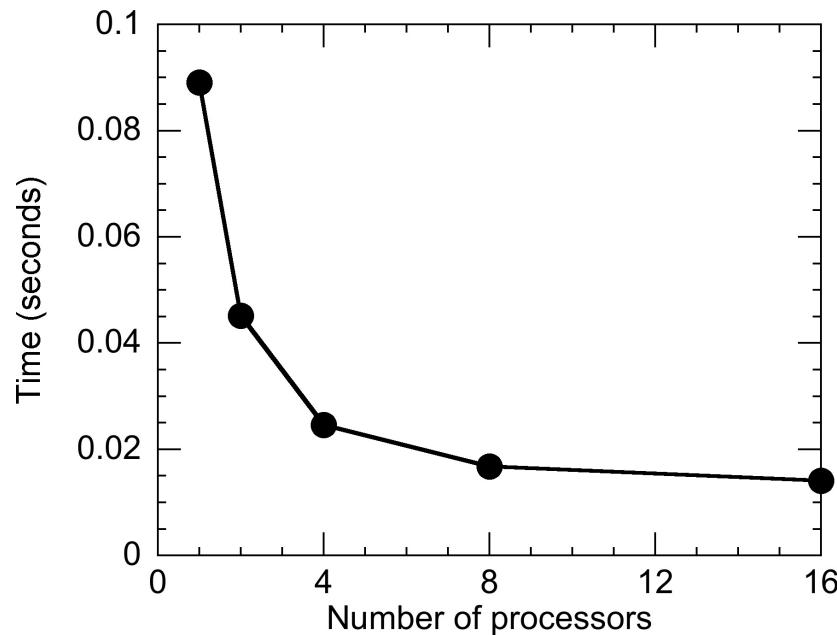
    MPI_Init(&argc, &argv);
    ...
    cpu1 = MPI_Wtime();
    /* Do computation here */
    cpu2 = MPI_Wtime();
    ...
    if (myid == 0) {
        printf("Nprocs & Global sum = %d %le\n", nprocs, pi);
        printf("Execution time (s) = %le\n",cpu2-cpu1);
    }
    ...
    MPI_Finalize();
    return 0;
}
```



Parallel Running Time

global_pi.c: NBIN = 10^7 , on Discovery

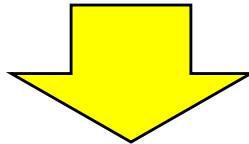
```
#SBATCH --nodes=16
#SBATCH --ntasks-per-node=1
...
srun -n $SLURM_NTASKS ./global_pi
srun -n 8 ./global_pi
srun -n 4 ./global_pi
srun -n 2 ./global_pi
srun -n 1 ./global_pi
```



Q: How Efficient Is the Parallel Program?

Scalability Analysis

- Parallel computing = Solving a big problem (W) in a short time (T) using many processors (P)



- How W , T & P scale with each other?
- How to define the efficiency of a parallel program?

See [Gramma et al.](#),
Chap. 5—Analytical modeling of parallel programs

Parallel Efficiency

- Execution time: $T(W,P)$
 W : Workload
 P : Number of processors

- Speed: $S(W,P) = \frac{W}{T(W,P)}$

- Speedup: $S_P = \frac{S(W_P, P)}{S(W_1, 1)} = \frac{W_P T(W_1, 1)}{W_1 T(W_P, P)}$

- Efficiency: $E_P = \frac{S_P}{P} = \frac{W_P T(W_1, 1)}{P W_1 T(W_P, P)}$

Ideal speedup

How to scale W_P with P ?

Fixed Problem-Size Scaling

$W_P = W$ —constant (strong scaling)

- **Speedup:** $S_P = \frac{T(W,1)}{T(W,P)}$
- **Efficiency:** $E_P = \frac{T(W,1)}{PT(W,P)}$

$$S_P = \frac{S(W_P, P)}{S(W_1, 1)} = \frac{W_P T(W_1, 1)}{W_1 T(W_P, P)}$$
$$E_P = \frac{S_P}{P} = \frac{W_P T(W_1, 1)}{P W_1 T(W_P, P)}$$

Solving the same problem faster using more processors!

- **Amdahl's law:** f (= sequential fraction of the workload) limits the asymptotic speedup

$$S_P = \frac{T(W,1)}{T(W,P)} \leq P$$

$$T(W,P) = fT(W,1) + \frac{(1-f)T(W,1)}{P}$$
$$\therefore S_P = \frac{T(W,1)}{T(W,P)} = \frac{1}{f + (1-f)/P}$$
$$\therefore S_P \rightarrow \frac{1}{f} \quad (P \rightarrow \infty)$$

Isogranular Scaling

$W_P = Pw$ (weak scaling)

w = constant workload per processor (granularity)

- Speedup:
$$S_P = \frac{S(P \bullet w, P)}{S(w, 1)} = \frac{P \bullet w / T(P \bullet w, P)}{w / T(w, 1)} = \frac{P \bullet T(w, 1)}{T(P \bullet w, P)}$$

- Efficiency:
$$E_P = \frac{S_P}{P} = \frac{T(w, 1)}{T(P \bullet w, P)}$$

$$S_P = \frac{S(W_P, P)}{S(W_1, 1)} = \frac{W_P T(W_1, 1)}{W_1 T(W_P, P)}$$

$$E_P = \frac{S_P}{P} = \frac{W_P T(W_1, 1)}{P W_1 T(W_P, P)}$$

*Solving larger problems within the same time
using more processors!*

$$E_P = \frac{T(w, 1)}{T(Pw, P)} \leq 1$$

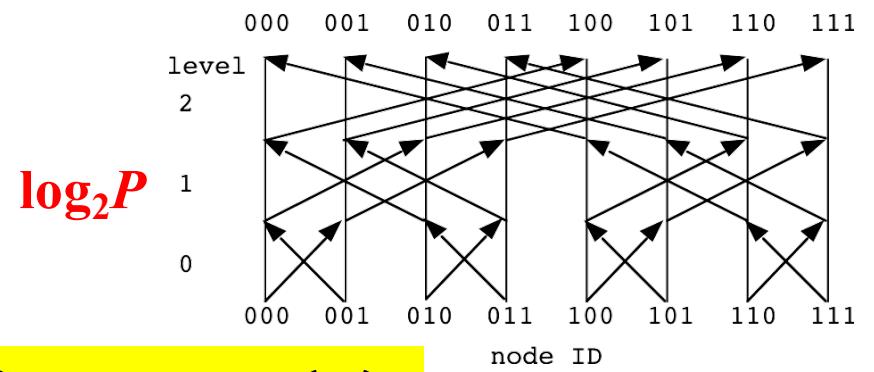
Analysis of Global_Pi Program

- Workload \propto Number of quadrature points, N (or **NBIN** in the program)
- Parallel execution time on P processors:
 - > Local computation $\propto N/P$

```
for (i=myid; i<N; i+=P){  
    x = (i+0.5)*step; partial += 4.0/(1.0+x*x);  
}
```

- > Butterfly computation/communication in `global()` $\propto \log P$

```
for (l=0; l<log2P; ++l) {  
    partner = myid XOR 2l;  
    send mydone to partner;  
    receive hisdone from partner;  
    mydone += hisdone  
}
```



$$\begin{aligned} T(N, P) &= T_{\text{comp}}(N, P) + T_{\text{global}}(P) \\ &= \alpha \frac{N}{P} + \beta \log P \end{aligned}$$

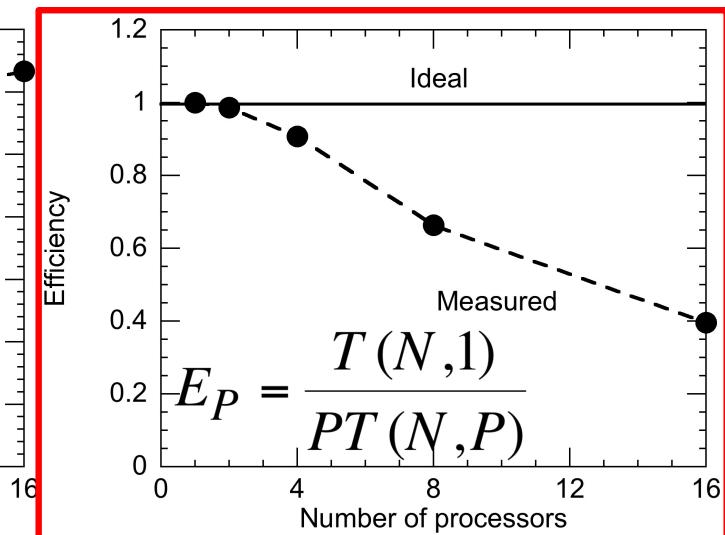
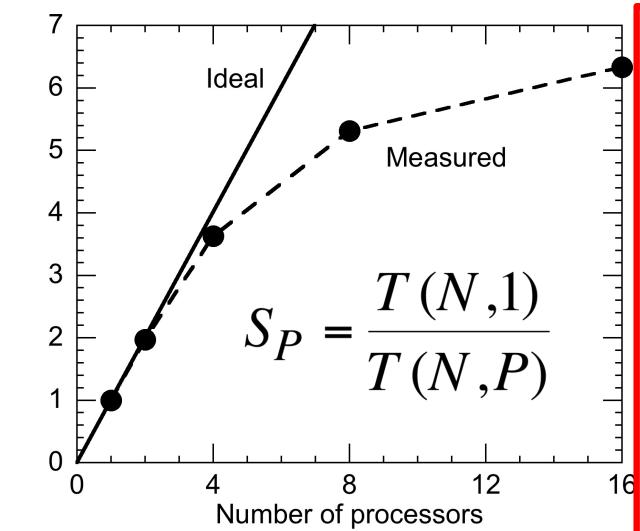
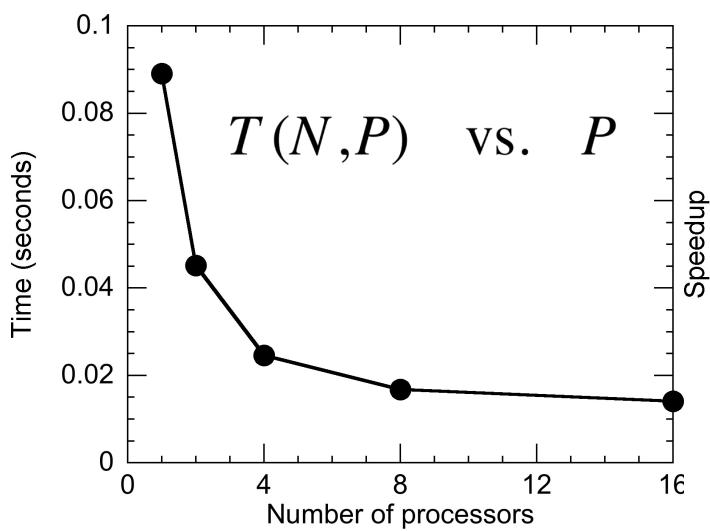
Fixed Problem-Size (Strong) Scaling

- Speedup:

$$S_P = \frac{T(N, 1)}{T(N, P)} = \frac{\alpha N}{\alpha \frac{N}{P} + \beta \log P} = \frac{P}{1 + \frac{\beta P \log P}{\alpha N}}$$

- Efficiency:

$$E_P = \frac{S_P}{P} = \frac{1}{1 + \frac{\beta P \log P}{\alpha N}}$$

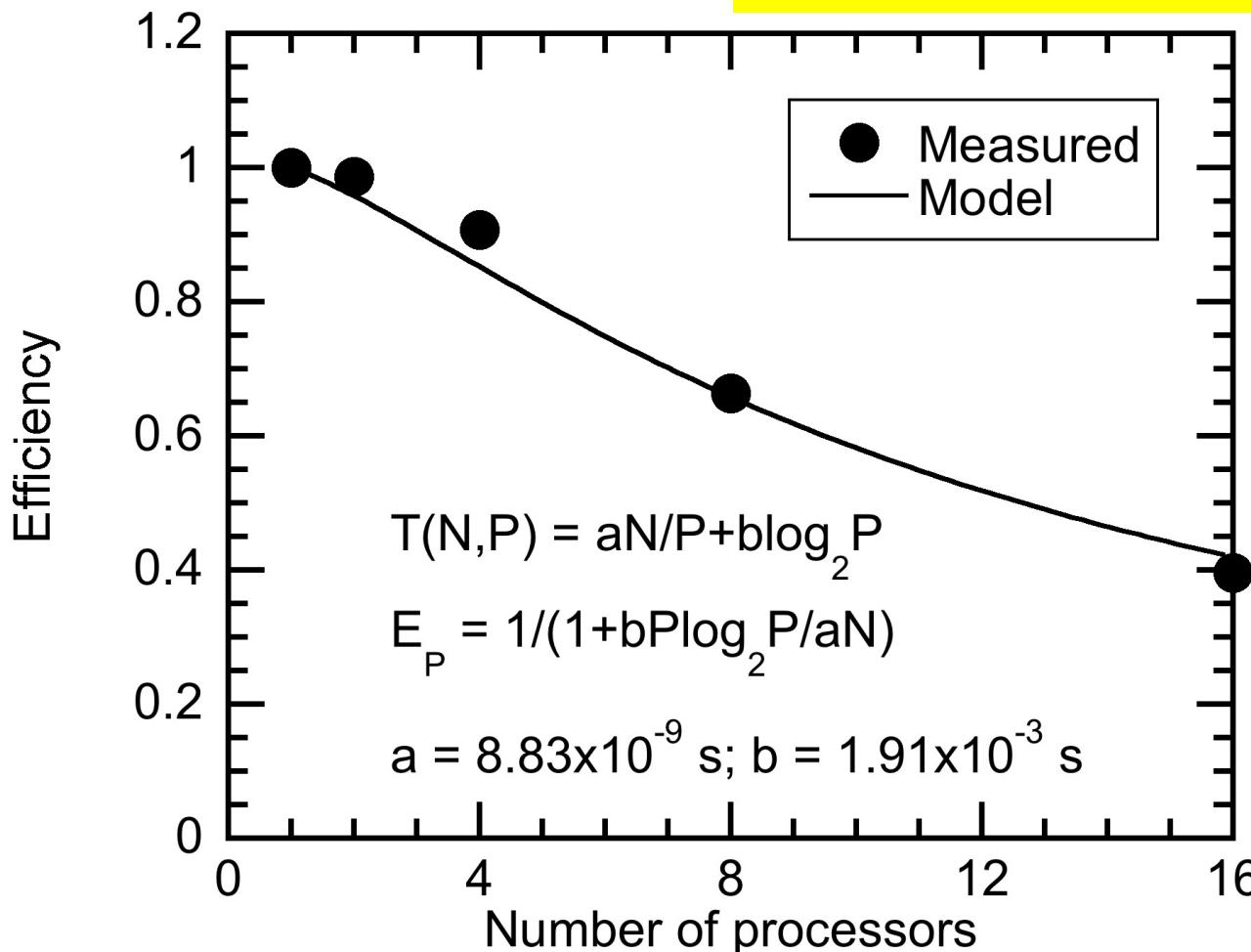


global_pi.c: $N = 10^7$, on Discovery

Fixed Problem-Size Scaling

- Speedup model:

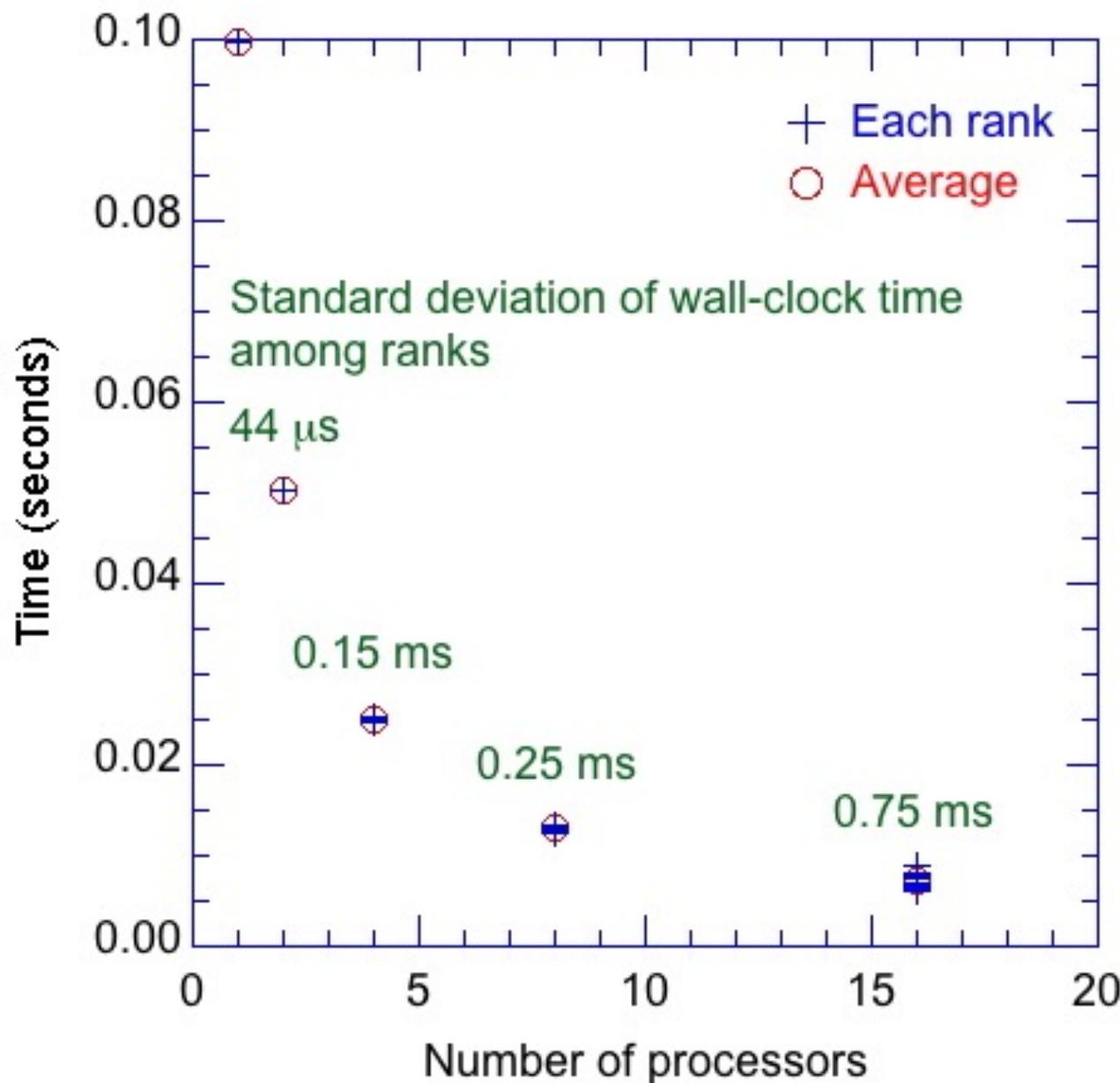
$$E_P = \frac{S_P}{P} = \frac{1}{1 + \frac{\beta}{\alpha} \frac{P \log P}{N}}$$



Computation \sim ns
Communication \sim ms

global_pi.c: $N = 10^7$, on HPC (predecessor of Discovery)

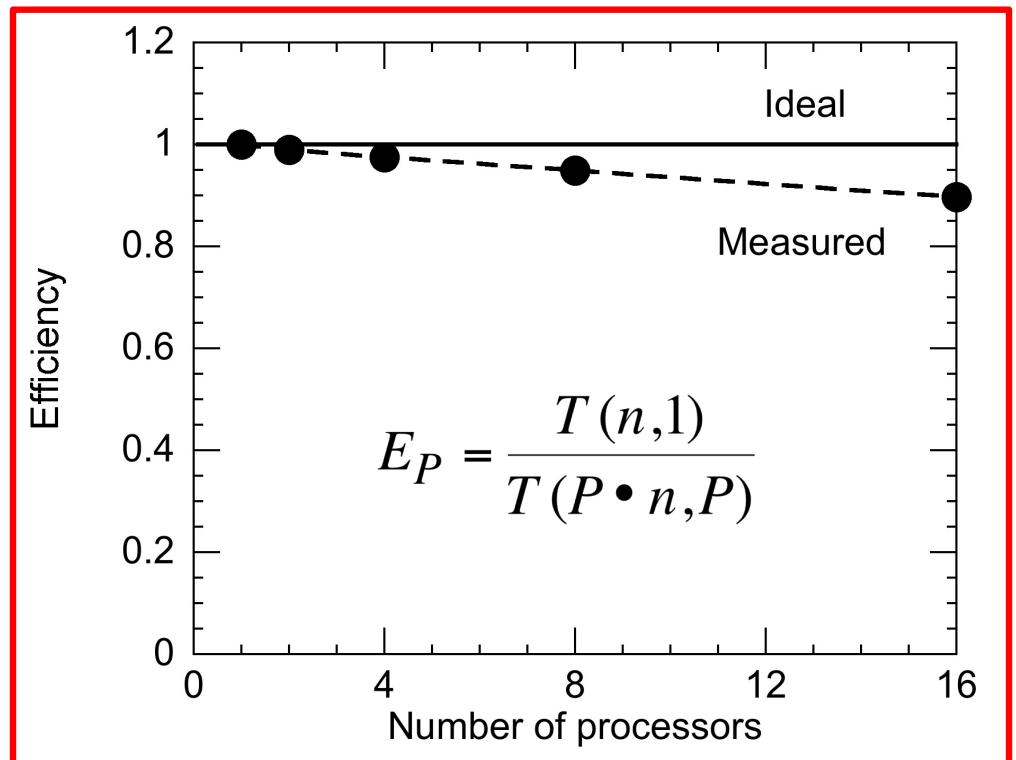
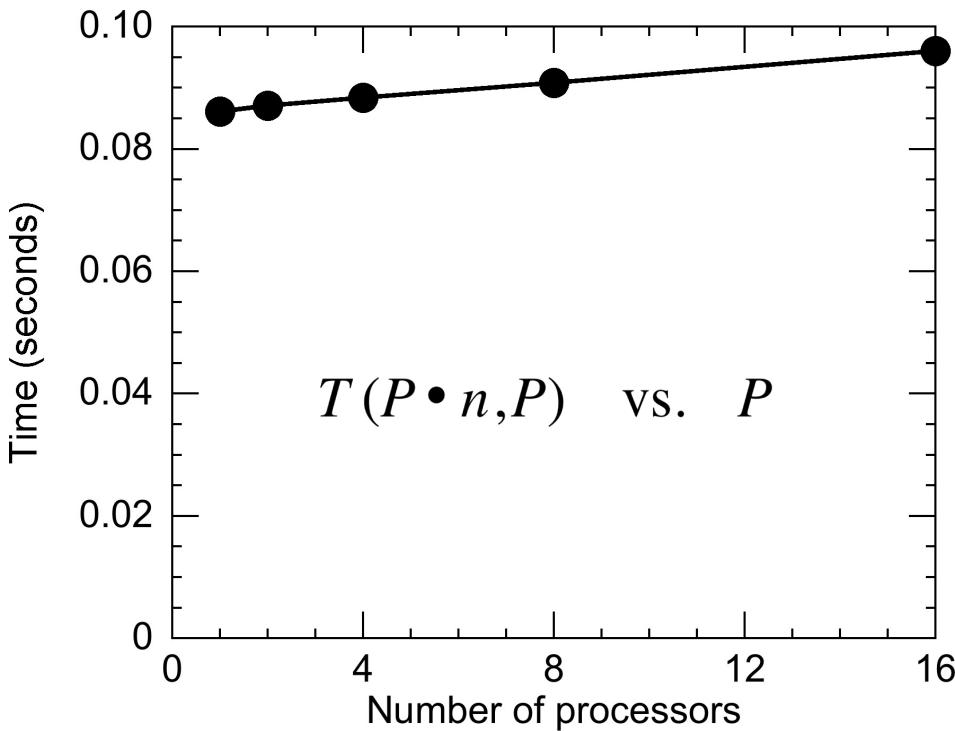
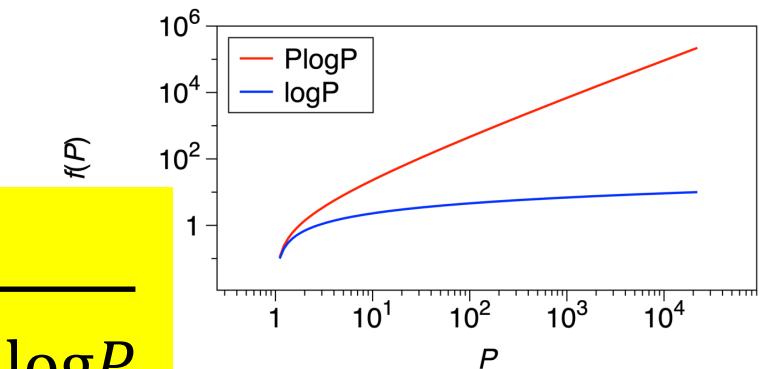
Runtime Variance among Ranks



Isogranular (Weak) Scaling

- $n = N/P = \text{constant}$
- Efficiency:

$$E_P = \frac{T(n, 1)}{T(nP, P)} = \frac{\alpha n}{\alpha n + \beta \log P} = \frac{1}{1 + \frac{\beta}{\alpha n} \log P}$$

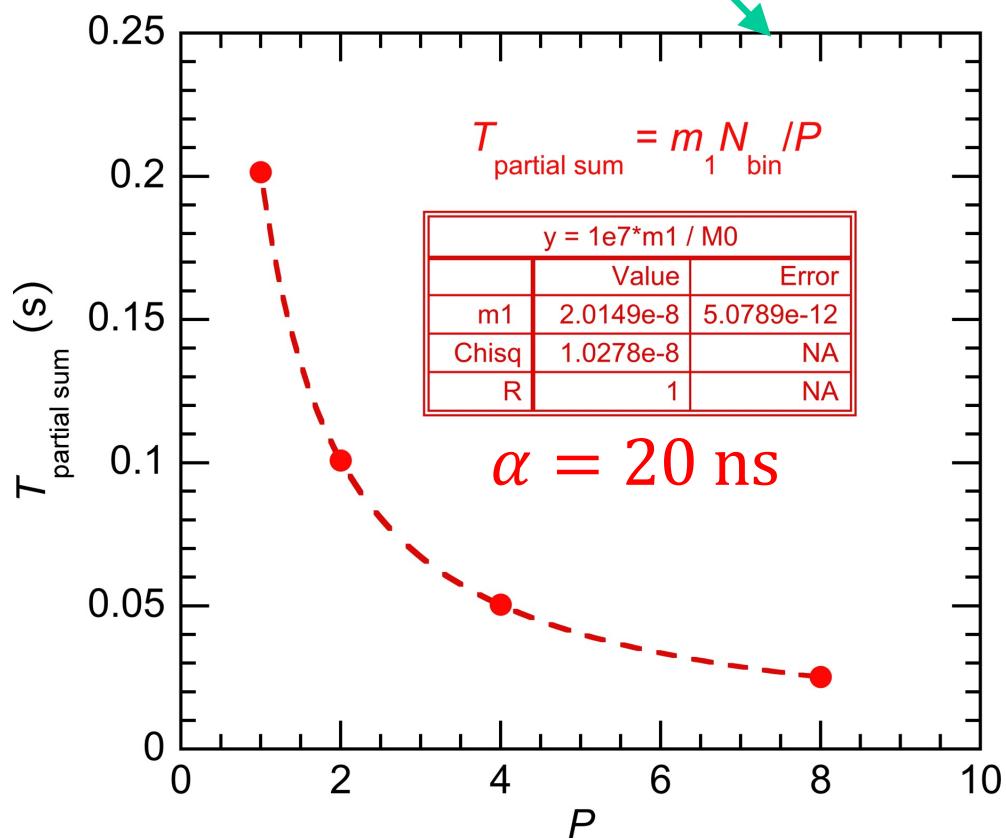


global_pi_iso.c: $N/P = 10^7$, on Discovery

Fitting Fixed Problem-Size Scaling

$$T(N, P) = \alpha \frac{N}{P} + \beta \log_2 P$$

```
for(i=myid;i<N;i+=P){...}
```



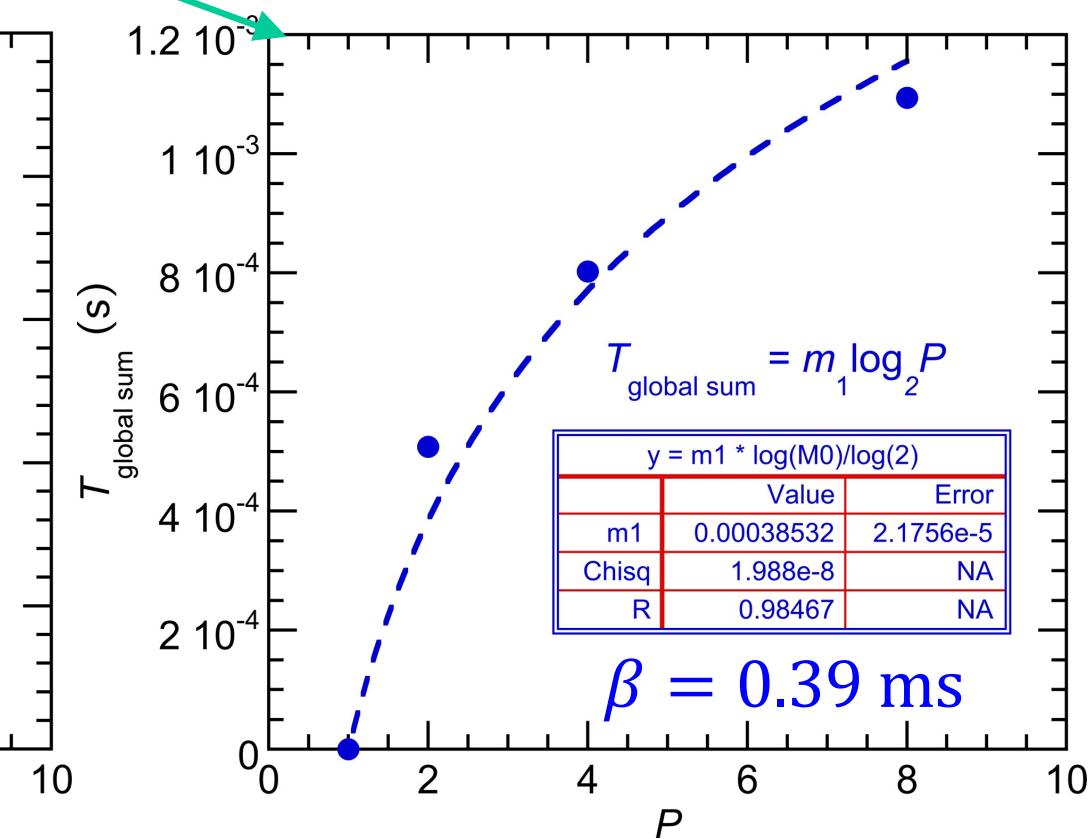
Computation \sim ns

global_pi_breakdown.c: $N = 10^7$, on 3.0 GHz Xeon

cf.

[discovery]\$ ping hpc-transfer.usc.edu
time=0.046 ms

```
pi=global_sum(partial);
```



Communication \sim ms

Scalability Analysis: Example

Computer Physics Communications 219 (2017) 246–254

A derivation and scalable implementation of the synchronous parallel kinetic Monte Carlo method for simulating long-time dynamics

Hyekyung Suk Byun ^a, Mohamed Y. El-Naggar ^{a,b,c}, Rajiv K. Kalia ^{a,d,e,f}, Aiichiro Nakano ^{a,b,d,e,f,*},
Priya Vashishta ^{a,d,e,f}

$$T(N, P) = T_{\text{comp}}(N, P) + T_{\text{comm}}(N, P) + T_{\text{global}}(P) \\ = aN/P + b(N/P)^{2/3} + c \log P. \quad (6)$$

For isogranular scaling, the number of atoms per processor, $N/P = n$, is constant, and the isogranular parallel efficiency is

$$E_P = \frac{T(n, 1)}{T(nP, P)} = \frac{an}{an + bn^{2/3} + c \log P} \\ = \frac{1}{1 + \frac{b}{a}n^{-1/3} + \frac{c}{an} \log P}. \quad (7)$$

For fixed problem-size scaling, the global number of hemes, N , is fixed, and the speedup is given by

$$S_P = \frac{T(N, 1)}{T(N, P)} = \frac{aN}{aN/P + b(N/P)^{2/3} + c \log P} \\ = \frac{P}{1 + \frac{b}{a}\left(\frac{P}{N}\right)^{1/3} + \frac{c}{a}\frac{P \log P}{N}}, \quad (8)$$

and the parallel efficiency is

$$E_P = \frac{S_P}{P} = \frac{1}{1 + \frac{b}{a}\left(\frac{P}{N}\right)^{1/3} + \frac{c}{a}\frac{P \log P}{N}}. \quad (9)$$

