

# Metropolis Monte Carlo Simulation: Q & A

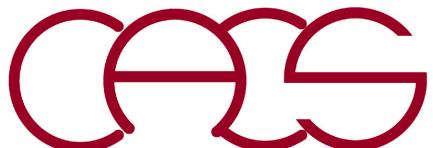
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# Q: What Is $\alpha_{mn}$ in Ising MC?

**States:**  $m, n \in \left\{ s^N = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_N \end{pmatrix} \middle| s_i = \uparrow, \downarrow; i = 1, \dots, N \right\}$

$\pi_{m,n} = \overbrace{\min\left(\frac{\rho_m}{\rho_n}, 1\right)}^{\text{accept/reject}} \overbrace{\tilde{\alpha}_{m,n}}^{\text{attempt}}$

**Attempt matrix:**  $\alpha_{m,n} = \begin{cases} 1/N & \text{Hamming\_distance}(m, n) = 1 \\ 0 & \text{else} \end{cases}$

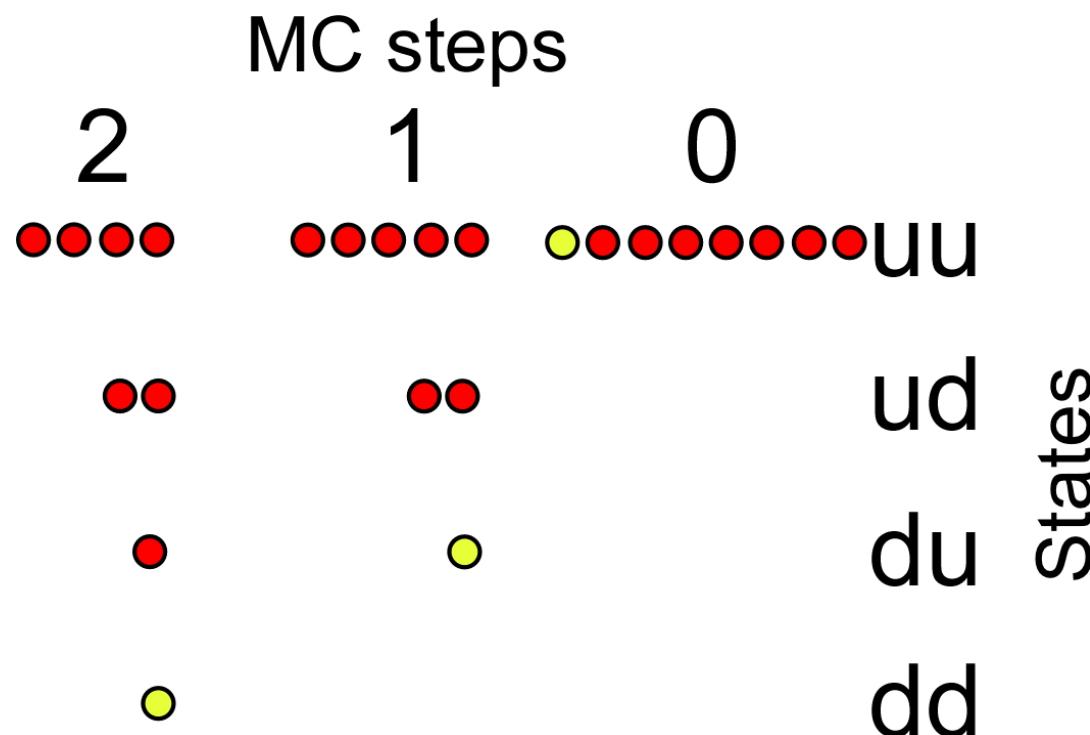
**Example:  $N = 3$  ( $2^N = 8$  states)**

$$\begin{pmatrix} \uparrow \uparrow \uparrow \\ \uparrow \uparrow \downarrow \\ \uparrow \downarrow \uparrow \\ \uparrow \downarrow \downarrow \\ \downarrow \uparrow \uparrow \\ \downarrow \uparrow \downarrow \\ \downarrow \downarrow \uparrow \\ \downarrow \downarrow \downarrow \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 & 1/3 & & & \\ 1/3 & & 1/3 & 1/3 & & \\ 1/3 & & 1/3 & & 1/3 & \\ & 1/3 & 1/3 & & & 1/3 \\ 1/3 & & & 1/3 & 1/3 & \\ & 1/3 & & 1/3 & & 1/3 \\ & & 1/3 & 1/3 & & \\ & & & 1/3 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} \uparrow \uparrow \uparrow \\ \uparrow \uparrow \downarrow \\ \uparrow \downarrow \uparrow \\ \uparrow \downarrow \downarrow \\ \downarrow \uparrow \uparrow \\ \downarrow \uparrow \downarrow \\ \downarrow \downarrow \uparrow \\ \downarrow \downarrow \downarrow \end{pmatrix}$$

# Q: Where Is Matrix-Vector Multiplication?

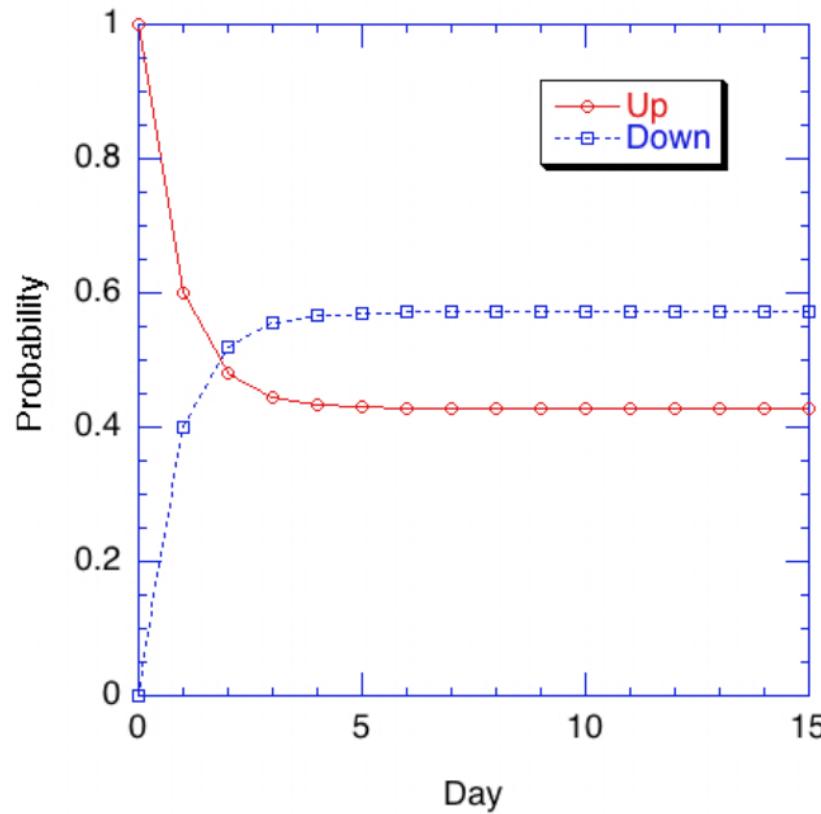
**A:** The probability density vector is replaced by an ensemble of individual MC sequences in Markov-chain MC; the ensemble average is then replaced by time average.

$$\rho^{(t+1)} = \Pi \rho^{(t)}$$



# Example: Two-Level System

$$\Pi = \begin{pmatrix} \uparrow & \downarrow \\ \uparrow & \begin{pmatrix} a & 1-b \\ 1-a & b \end{pmatrix} \\ \downarrow & \end{pmatrix} = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix} \quad (a = 0.6, b = 0.7)$$



$$\begin{pmatrix} p_{\uparrow}^{(t)} \\ p_{\downarrow}^{(t)} \end{pmatrix} = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow[t \rightarrow \infty]{} \begin{pmatrix} 0.4286 \\ 0.5714 \end{pmatrix} \begin{matrix} 3/7 \\ 4/7 \end{matrix}$$

# A Metropolis Monte Carlo

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Your only knowledge = equilibrium probability distribution

$$\rho = \begin{pmatrix} 3/7 \\ 4/7 \end{pmatrix}$$

A choice of attempt matrix

$$\alpha_{\uparrow\downarrow} = \alpha_{\downarrow\uparrow} = 1$$

Detailed-balanced transition-probability matrix

$$\begin{aligned}\Pi &= \begin{pmatrix} \pi_{\uparrow\uparrow} & \pi_{\uparrow\downarrow} \\ \pi_{\downarrow\uparrow} & \pi_{\downarrow\downarrow} \end{pmatrix} = \begin{pmatrix} 1 - \alpha_{\downarrow\uparrow} & \alpha_{\uparrow\downarrow}(\rho_{\uparrow}/\rho_{\downarrow}) \\ \alpha_{\downarrow\uparrow} & 1 - \alpha_{\uparrow\downarrow}(\rho_{\uparrow}/\rho_{\downarrow}) \end{pmatrix} \\ &= \begin{pmatrix} 1 - 1 & 1 \cdot 3/4 \\ 1 & 1 - 1 \cdot 3/4 \end{pmatrix} = \begin{pmatrix} 0 & 3/4 \\ 1 & 1/4 \end{pmatrix}\end{aligned}$$

Q: How to represent the probability distribution?

A: An ensemble of many samples

# Ensemble-Average MC

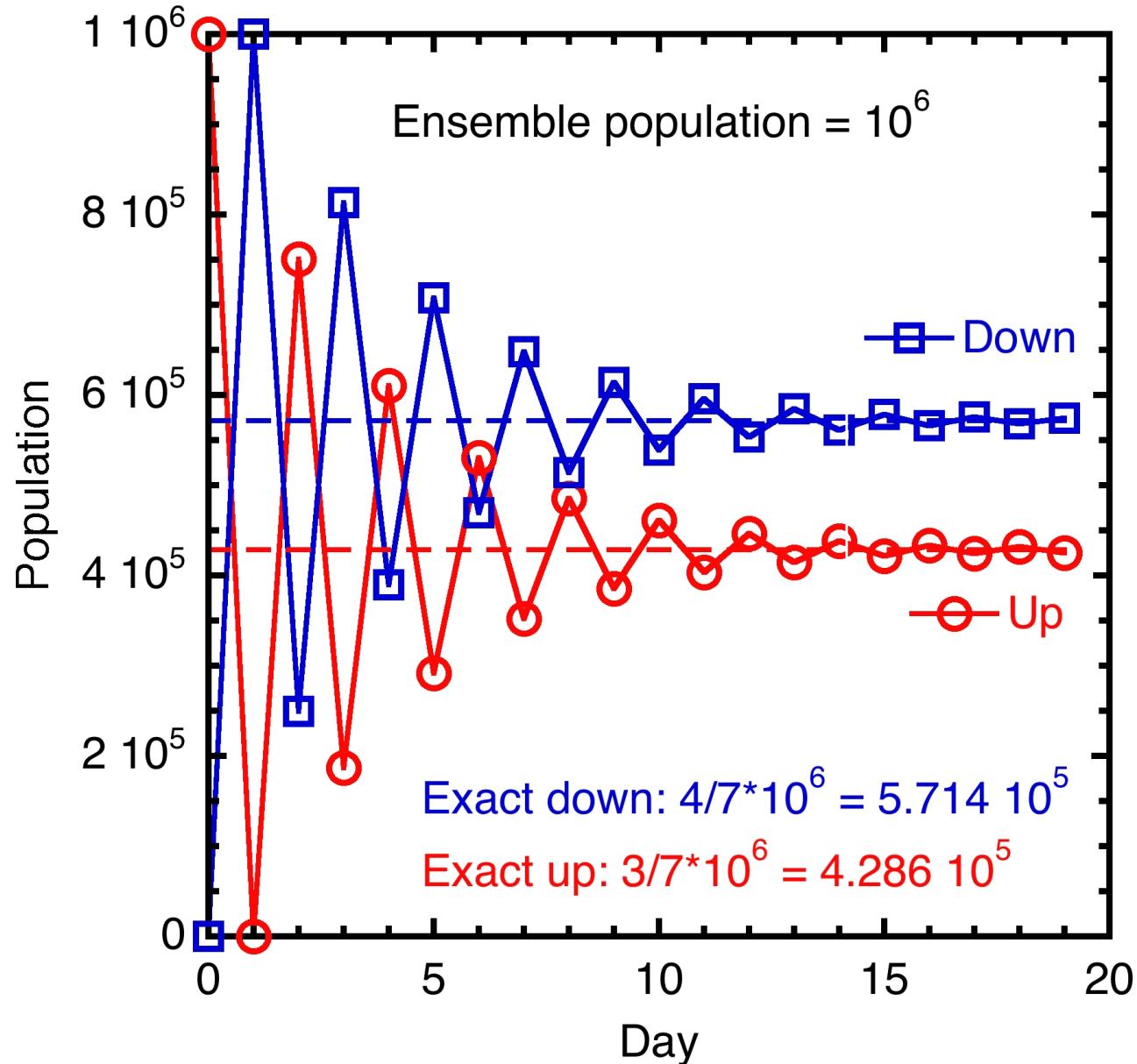
```
#include <stdio.h>
#include <stdlib.h>
#include <time.h>
#define NTRY 20 /* # of MC trials */
#define NENS 1000000 /* ensemble size */
#define TRNS 3.0/4.0 /* up-to-down conditional probability */

int main() {
    int s; /* spin state: 0 = up; 1 = down */
    int hist[NTRY][2]; /* histgram */
    int try,walker;

    srand((unsigned)time((long *)0));
    for (try=0; try<NTRY; try++) for (s=0; s<2; s++) hist[try][s] = 0;

    for (walker=0; walker<NENS; walker++) {
        s = 0; /* up on day 0 */
        ++(hist[0][s]);
        for (try=1; try<NTRY; try++) {
            if (s == 0) s = 1; /* unconditional down move */
            else if (rand()/(double)RAND_MAX < TRNS) s = 0; /* conditional up move */
            ++(hist[try][s]); /* accumulate the average */
        }
    }
    for (try=0; try<NTRY; try++) printf("%d %d %d\n",try,hist[try][0],hist[try][1]);
    return 0;
}
```

# Ensemble-Average MC Result



# Time-Average MC

```
#include <stdio.h>
#include <stdlib.h>
#include <time.h>
#define NTRY 1000 /* ensemble size */
#define TRNS 3.0/4.0 /* up-to-down conditional probability */

int main() {
    int s; /* spin state: 0 = up; 1 = down */
    int hist[NTRY][2]; /* histogram */
    int try,i;

    srand((unsigned)time((long *)0));
    for (try=0; try<NTRY; try++) for (s=0; s<2; s++) hist[try][s] = 0;

    s = 0; /* up on day 0 */
    ++(hist[0][s]);
    for (try=1; try<NTRY; try++) {
        if (s == 0) s = 1; /* unconditional down move */
        else if (rand()/(double)RAND_MAX < TRNS) s = 0; /* conditional up move */
        for (i=0; i<2; i++) hist[try][i] = hist[try-1][i];
        ++(hist[try][s]); /* accumulate the average */
    }

    for (try=0; try<NTRY; try++)
        printf("%d %d %d\n",try,hist[try][0],hist[try][1]);

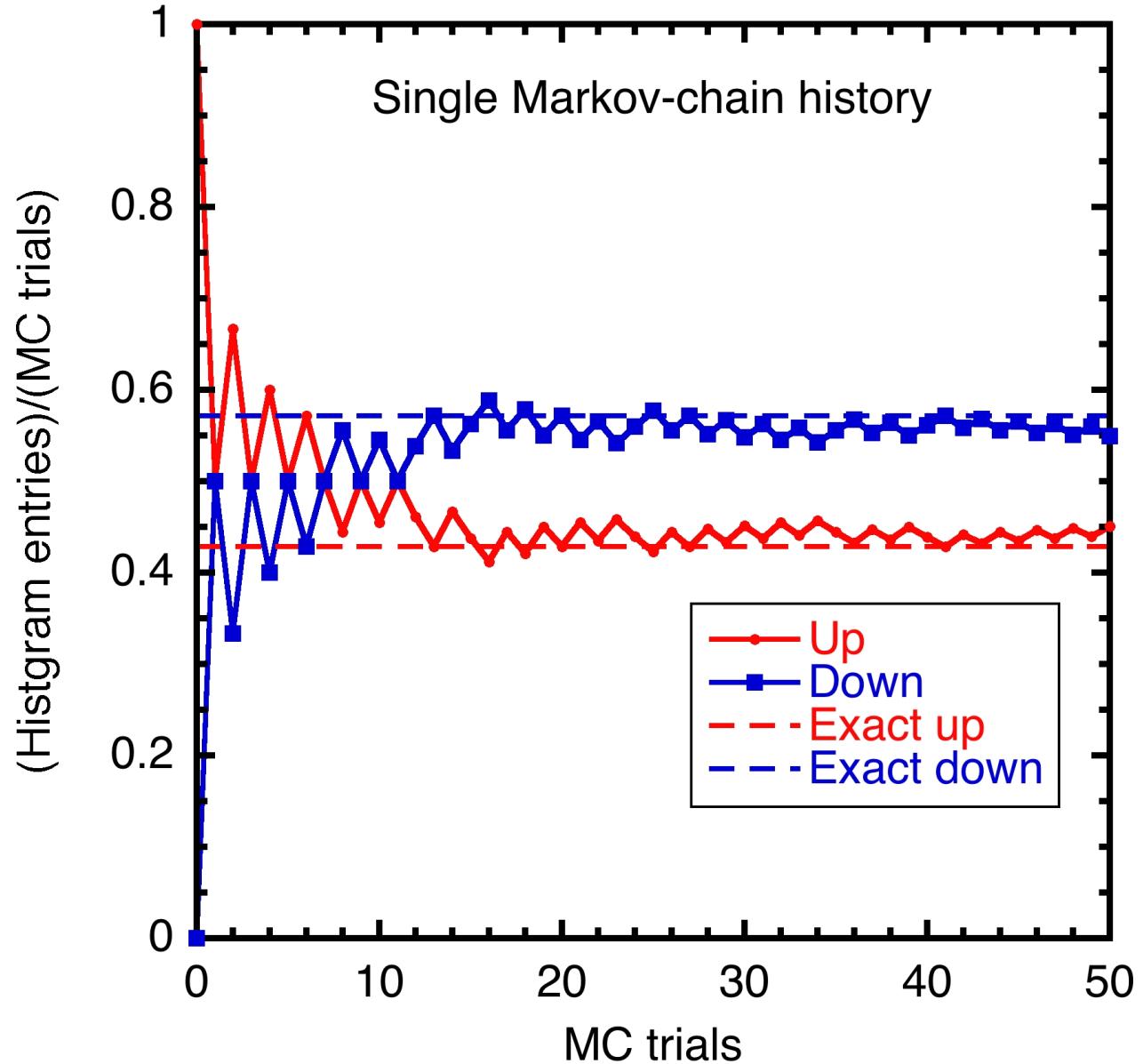
    return 0;
}
```

Ergodic hypothesis

# Time-Average MC Result

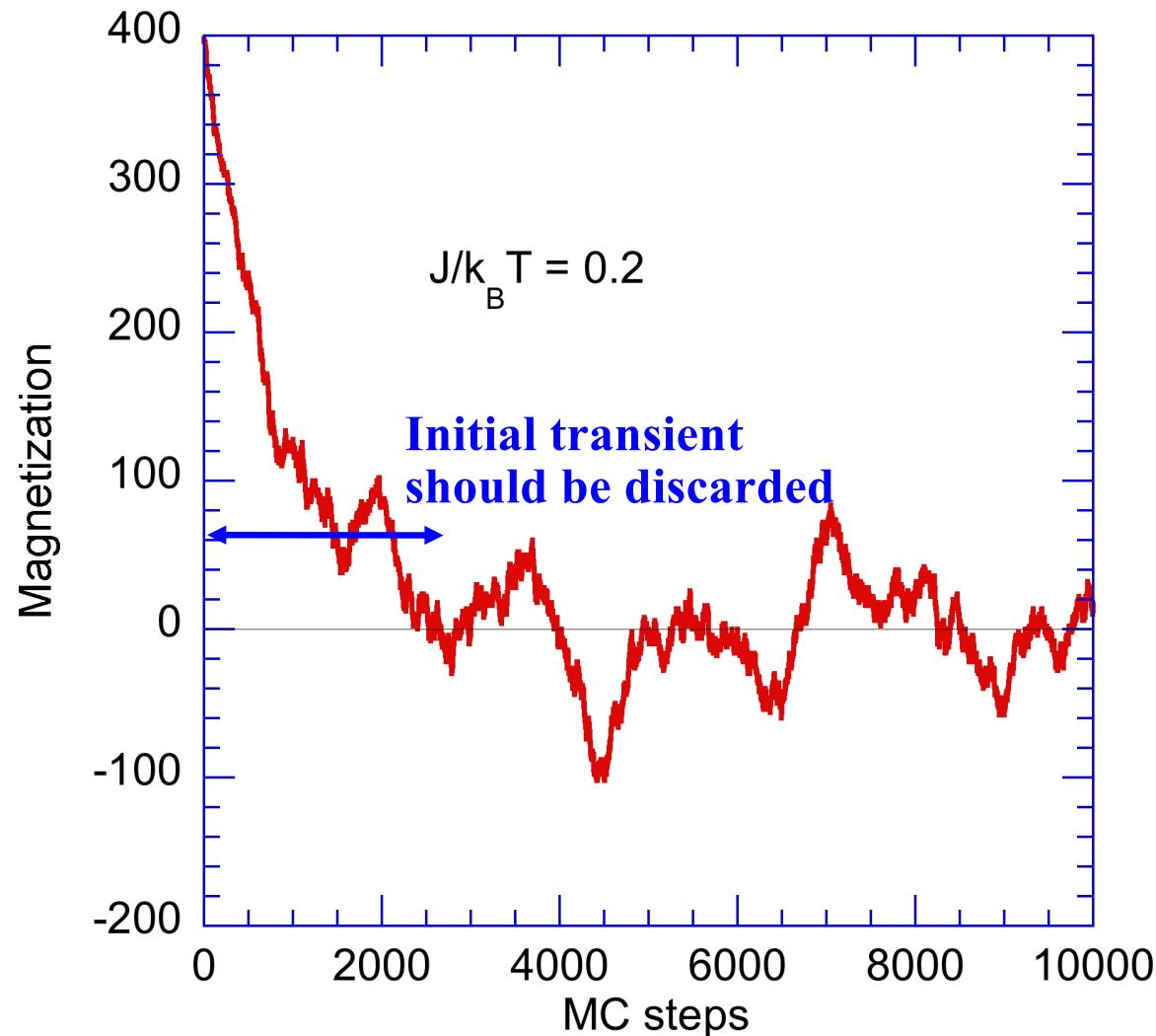
Try	Up	Down
0	1	0
1	1	1
2	2	1
3	2	2
4	3	2
5	3	3
6	4	3
7	4	4
8	4	5
9	5	5
10	5	6
11	6	6
12	6	7
13	6	8
14	7	8
15	7	9
16	7	10
17	8	10
18	8	11
19	9	11
20	9	12

Cumulative  
histogram



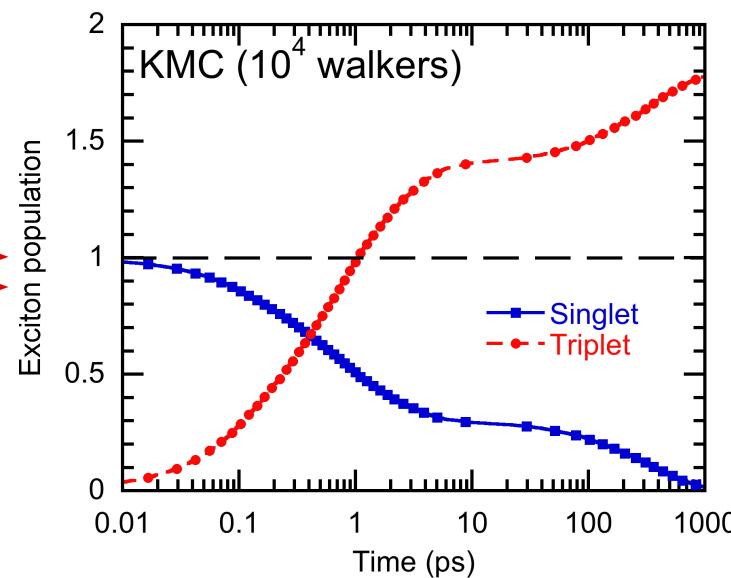
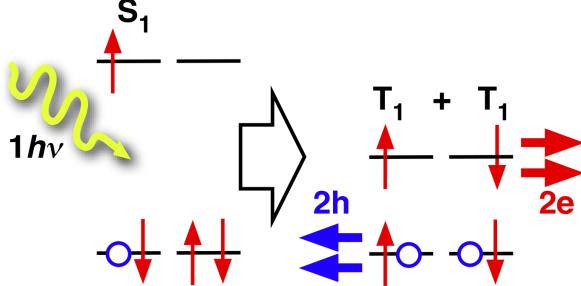
# Q: Need Equilibration Steps?

A: Yes, statistics should be taken after the memory of the initial configuration is lost

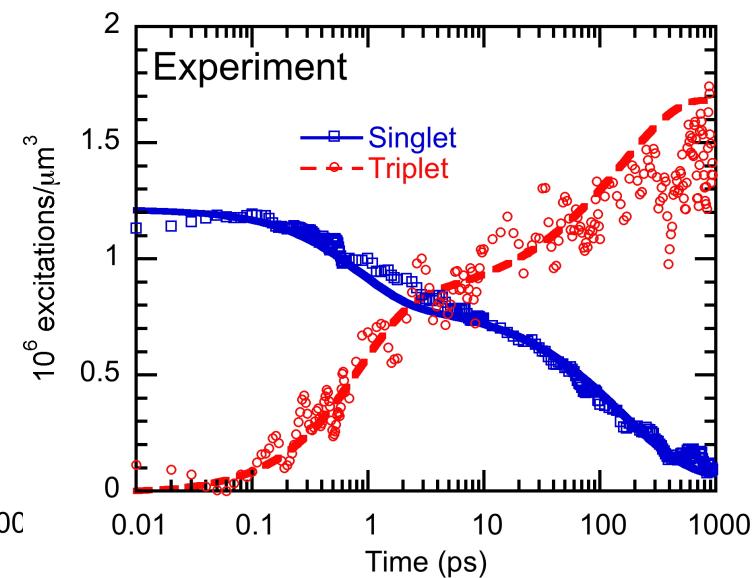


# Metropolis MC vs. Kinetic MC

- **Metropolis MC:** Given probability density  $\rho_\alpha$  ( $\alpha = 1, \dots, N_{\text{states}}$ ) calculate statistical average of a physical quantity as  $\langle A \rangle = \sum_\alpha \rho_\alpha A_\alpha$  where the transition-probability matrix  $\pi_{\alpha\beta}$  is an artifact for importance sampling
- **Kinetic MC:** Given transition-rate matrix  $\pi_{\alpha\beta}$  (calculated, e.g., based on the transition state theory) & initial distribution  $\rho_\alpha(t=0)$ , obtain the time variation of  $\rho_\alpha(t)$  by solving the master equation represented by an ensemble of state samples,  $d\rho_\alpha/dt = -\sum_\beta \pi_{\beta\alpha} \rho_\alpha + \sum_\beta \pi_{\alpha\beta} \rho_\beta$



W. Mou et al.  
APL 102, 173301 ('13)

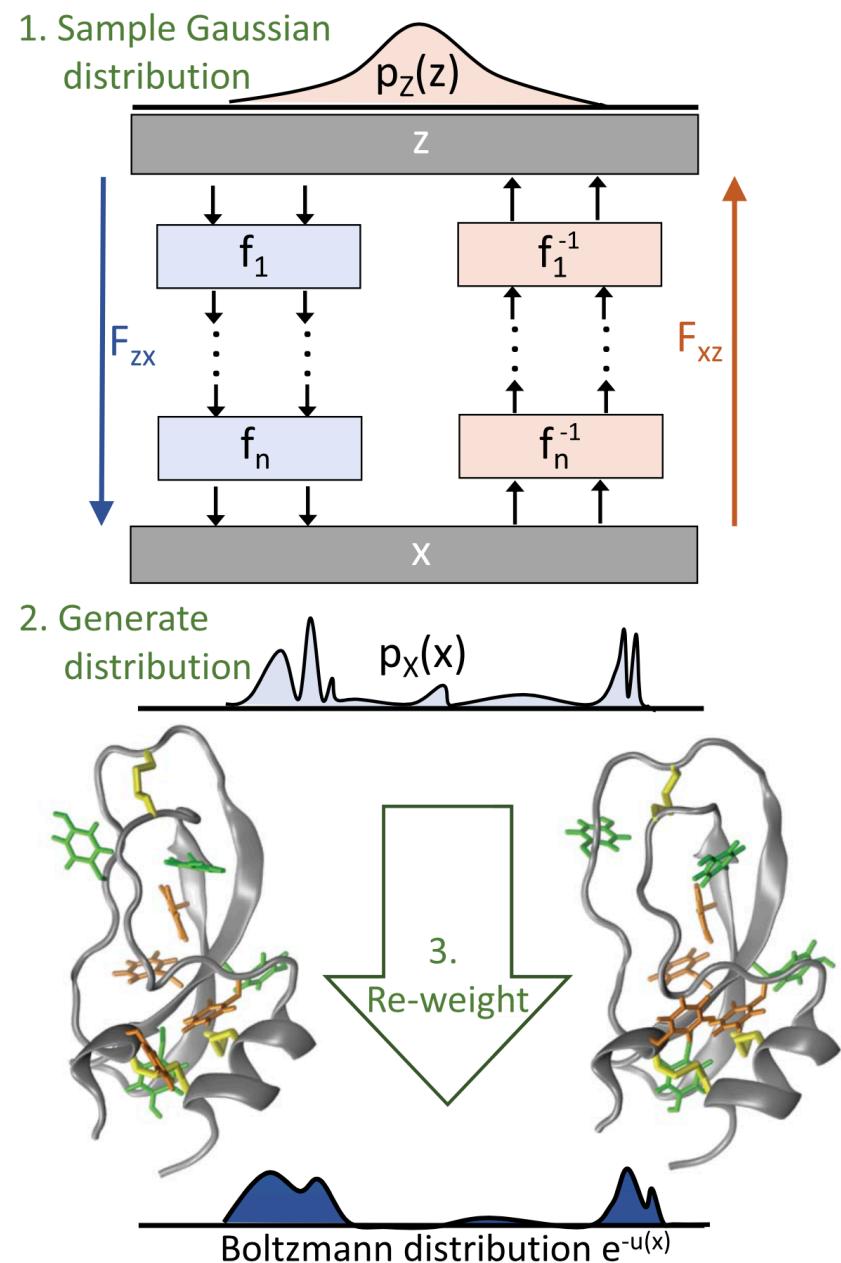


S. T. Roberts et al.  
JACS 134, 6388 ('12)

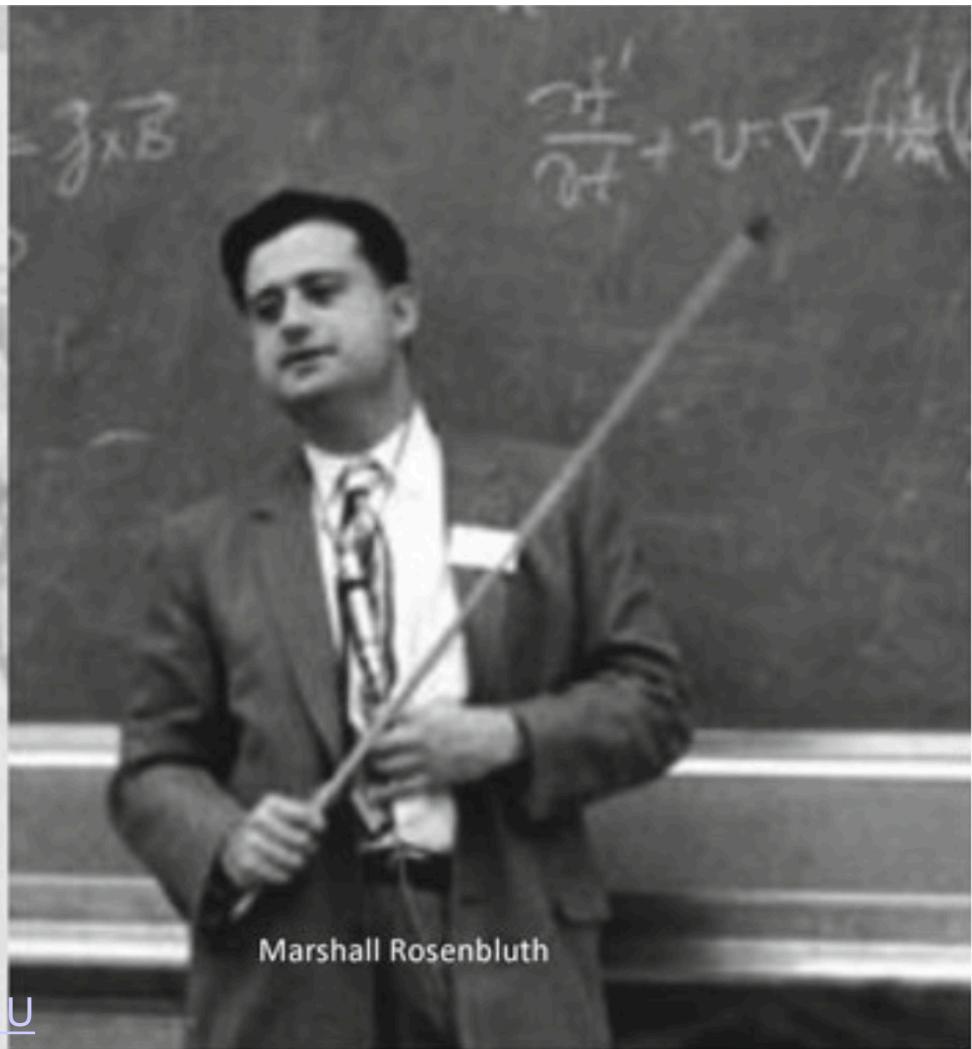
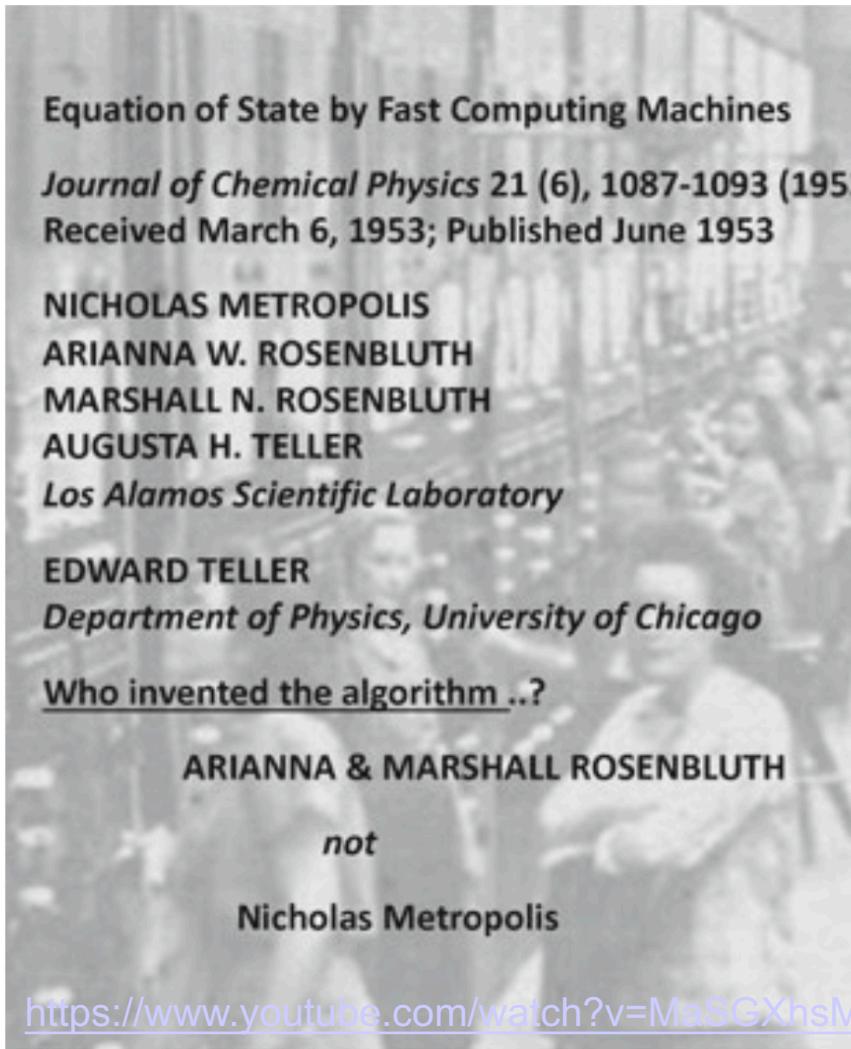
# Coordinate Transformation?

- **Box-Muller algorithm:** For a harmonic oscillator,  $u(x) = Kx^2/2$ , Boltzmann probability density (which is Gaussian  $p(x) \propto \exp(-u(x)/k_B T) = \exp(-Kx^2/2k_B T)$ ) can be generated by coordinate transformation
- **Boltzmann generator:** Machine learning of coordinate transformation such that the probability density is Gaussian in the transformed coordinate system,  $z(x)$ , for complex, multidimensional  $u(x)$

F. Noe et al.  
Science 365, 1001 ('19)



# Metropolis Algorithm?



A slide taken from a recent presentation by Michael Klein, giving proper credit to the creators of the “Metropolis algorithm” (M. Klein)

<https://aiichironakano.github.io/phys516/Battimelli-ComputerMeetsPhysics-Springer20.pdf>, p. 29

# RIP Arianna

## Arianna W. Rosenbluth

From Wikipedia, the free encyclopedia

**Arianna Rosenbluth** (September 15, 1927 – December 28, 2020) was an American [physicist](#) who contributed to the development of the [Metropolis–Hastings algorithm](#). She wrote the first full implementation of the [Markov chain Monte Carlo](#) method.

### Contents [hide]

- [1 Early life and education](#)
- [2 Career](#)
- [3 Personal life](#)
- [4 Death](#)
- [5 References](#)

### Early life and education [edit]

Arianna Rosenbluth (née Wright) was born on September 15, 1927, in [Houston, Texas](#).

**Arianna W. Rosenbluth**



Rosenbluth in 2013

Born

Arianna Wright  
September 15, 1927

### Death [edit]

Arianna died on December 28, 2020 in the greater Los Angeles, California area.

[https://en.wikipedia.org/wiki/Arianna\\_W.\\_Rosenbluth](https://en.wikipedia.org/wiki/Arianna_W._Rosenbluth)