# Markovian State and Action Abstractions for MDPs via Hierarchical MCTS

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# State Abstraction

# State Abstraction

- Ground MDP: $\langle S, A, T, R \rangle$
- Group a set of states as a unit
  - Abstract states: $X = \{x_1, x_2, \cdots\}$
  - A partition on $S$
  - Abstraction function: $\phi : S \rightarrow X$
- Advantages:
  - Reduced abstract state space size
  - Reduced stochastic branching factors

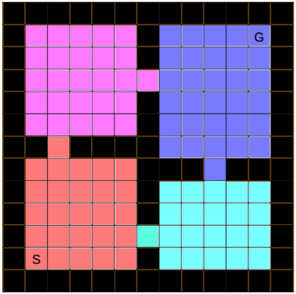



Figure 1: Rooms domain

# Non-Markovianess

- The transition system in abstract state space is non-Markovian
  - $\Pr(x'|x, a) = \sum_{s' \in x'} \sum_{s \in x} T(s'|s, a) \Pr(s|x)$
  - Occupancy probability: $\Pr(s|x)$
  - Bayesian update: $\Pr(s'|hax) = \eta \mathbf{1}[\phi(s') = x] \sum_{s \in S} T(s'|s, a) \Pr(s|h)$
    - Where, $h = x_0 a_0 \cdots x_n$
    - Dependent on the history, or in other words, the policy being executed/computed!

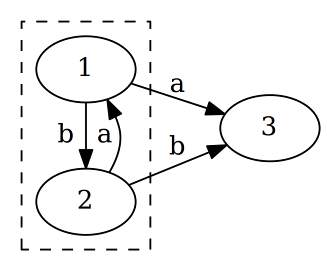



Figure 2: A 3-state MDP

# State Abstraction in the Literature

- Safe abstraction
  - Ignore only irrelevant state variables (Dietterich, 1999; Andre & Russell, 2002)
  - Exploit particular symmetric structure (e.g. bisimulation or homomorphism) in the transition function (Dearden & Boutilier, 1997; Givan et al., 2003; Jiang et al., 2014; Anand et al., 2015)
  - Not always possible

# State Abstraction in the Literature (cont'd)

- Weighting function/aggregation probability: $w(s, x) \approx \Pr(s|x)$ (Bertsekas et al., 1995; Singh et al., 1995; Li et al., 2006; Hostetler et al., 2014)
  - $T_\phi(x'|x, a) = \sum_{s' \in x'} \sum_{s \in x} T(s'|s, a) w(s, x)$
  - $R_\phi(x, a) = \sum_{s \in x} R(s, a) w(s, x)$
- Abstract MDP: $\langle X, A, T_\phi, R_\phi \rangle$
- Abstract policy: $\pi_\phi : X \to A$
- Problem: $\Pr(s|x)$ is non-stationary, which can not be well approximated by a constant weighting function

# State Abstraction from a POMDP Perspective

- State abstraction introduces partial observability
  - Abstract states as observations
- Doing state abstraction $\phi$ over an MDP $M = \langle S, A, R, T \rangle$ creates a POMDP: $M|_\phi = \langle S, A, X, T, R, \Omega \rangle$
  - Observation function: $\Omega(x|s) = \mathbf{1}[x = \phi(s)]$
  - Belief state $b(s)$ gives the occupancy probability

# Weighting Function from a POMDP Perspective

- Approximate belief state using a constant distribution $w(s, x)$ for each observation $x$
- Find a memory-less policy $\pi_\phi : X \to A$
  - Let $\pi$ be the optimal policy for $M$
  - Let $\pi_{M|_\phi}$ be the optimal policy for $M|_\phi$
  - Performance: $\pi_\phi \prec \pi_{M|_\phi} \prec \pi$
- The weighting function approach doesn't seem to be well motivated!

# Overcome Non-Markovianess via a POMDP Formulation

- Exactly solving $M|_\phi$ via dynamic programming is intractable
  - Continuous belief space
- The search tree in $M|_\phi$ has lower branching factor than in $M$
  - Let $\mathcal{T}(s_0)$ be the search tree in $M$ starting from $s_0$
  - Let $\mathcal{T}(b_0)$ be the search tree in $M|_\phi$ starting from $b_0$, where $b_0(s) = \mathbf{1}[s = s_0]$
  - Branching factor: $|A||X| \ll |A||S|$
- Solving $M|_\phi$ via point-based method is possible
  - Find the near-optimal action for $b_0$ by building an expectimax tree

# Solving $\mathcal{M}|_\phi$ via Monte Carlo Tree Search

- MCTS$|_\phi$: run POMCP in $\mathcal{M}|_\phi$
  - Build a search tree in the history space via sampling
  - Use particles to approximate the belief at the root node
    - Not necessary, since the ground state at the root node can be observed
- Advantages of MCTS$|_\phi$:
  - Only a simulator for the ground MDP is needed
  - Can be applied to continuous state-space MDPs

# Action Abstraction

# Action Abstraction in $M|_\phi$

- What we have now:
  - $\text{MCTS}|_\phi$: a MCTS algorithm with Markovian state abstraction for MDPs via a POMDP formulation
- Action abstraction decomposes the overall problem into a hierarchy of sub-problems
  - Abstract actions/options/subtasks/HAMs
- The transitions between abstract states forms a natural hierarchical structure in $M|_\phi$
  - Abstract actions as transitions between abstract states

# Value Function Decomposition

- Let $\alpha \in \mathcal{A}$ be an abstract action: $\alpha = \langle x \in X, y \in X, A, \pi \rangle$
- Let $\pi$ be a hierarchical policy: $\pi = \{\pi_0, \pi_1, \pi_2, \cdots\}$
  - Where, $\pi_0 : \mathcal{H} \to \mathcal{A}$ is the root task
- Hierarchical decomposition:
  - $V(\pi, h) = Q(\pi, h, \pi_0(h))$
  - $Q(\pi, h, \alpha) =$ 
    $V(\alpha, h) + \sum_{h' \in \mathcal{H}} \gamma^{|h'|-|h|} \Pr(h'|h, \alpha) V(\pi, h')$
  - $V(\alpha, h) = Q(\alpha, h, \pi_\alpha(h))$
  - $Q(\alpha, h, a) = R(h, a) + \gamma \sum_{x \in X} \Pr(x|h, a) V(\alpha, hax)$

# Semi-Markovianess

- Executing an abstract action takes a random number of steps
  - $N = |h'| - |h|$
  - $h' \sim \Pr(h'|h, \alpha)$ — terminating distribution of $\alpha$
  - Terminating distributions are unknown in advance
    - They are dependent on local policies
    - Local policies, as well as the high-level policy, are being computed!

# Overcome Semi-Markovianess via Monte-Carlo Learning

- High-level:
  - Learn $\pi_0$ by running MCTS over abstract actions
- Low-level:
  - Learn $\pi_\alpha$ by running MCTS over primitive actions
    * $\pi_\alpha$ converges given infinite samples
    * $\Pr(h'|h, \alpha)$ also converges as $\pi_\alpha$ converges
- Action values are backuped according to the hierarchical decomposition
- The hierarchical policy $\pi$ converges to a recursively optimal hierarchical policy

# The Overall Algorithm

- MCTS$|_{\phi,\mathcal{A}}$: a MCTS algorithm with Markovian state and action abstractions for MDPs via a POMDP formulation
  - Overcome non-Markovianess resulting from doing state abstraction via a POMDP formulation
  - Overcome semi-Markovianess resulting from doing action abstraction via a Monte Carlo learning process

# Theoretical Results

# Aggregation Error

- The aggregation error of state abstraction $\phi$ is $e$, if for all $x \in X$ and $s \in x$, $\exists \mathring{a} \in A$, such that $V^*(s) - Q^*(s, \mathring{a}) \leqslant e$, where $V^*$ and $Q^*$ are optimal value and action-value functions for the ground MDP $M$
  - $e = 0$ implies that all ground states within the same abstract state share the same optimal action
- The bounded aggregation error requires that the action value $Q^*(s, \mathring{a})$ of $\mathring{a}$ is close to the optimal value $V^*(s)$ for all ground states $s \in x$ within abstract state $x$
  - Measures the quality of the state abstraction

# Optimality Results with State Abstraction

Theorem

*(In short) The performance of the optimal policy* $\pi_{M|_\phi}$ *of* $M|_\phi$ *is bounded by a constant multiple of the state aggregation error, comparing with the optimal policy* $\pi$ *of* $M$

# Convergence Results with Action Abstraction

Theorem

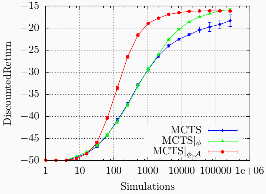
*$MCTS|_{\Phi,\mathcal{A}}$ converges to a recursively optimal hierarchical policy for $\mathrm{M}|_{\Phi}$ over the hierarchy defined by abstract actions with probability 1*

# Empirical Evaluation

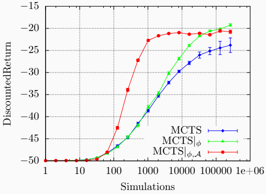
# The Rooms Domain

- The ROOMS$[m, n, k]$ problem:
  - A robot navigates in a $m \times n$ grid map containing $k$ rooms
  - Primitive actions: $E, S, W, N$
  - Probability 0.2 of moving into perpendicular positions
  - Abstract states: rooms
  - Abstract actions: transitions between rooms

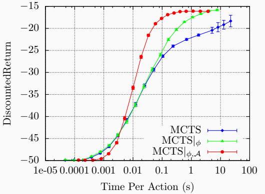
# Experimental Results



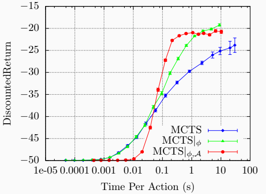

(a) ROOMS[17, 17, 4]




(b) ROOMS[25, 13, 8]




(c) ROOMS[17, 17, 4]



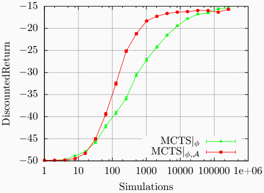

(d) ROOMS[25, 13, 8]

Figure 3: Empirical results on the rooms rooms domain.

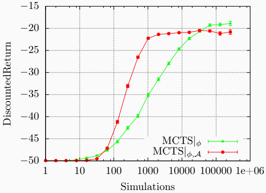
# The Continuous Rooms Domain

- The C-ROOMS$[m, n, k, w]$ problem:
  - Each grid has a size of $w \times w$ ($m^2$)
  - The position of the robot: continuous $(x, y)$
  - Primitive actions: $E, S, W, N$
    - Movement is augmented with a Gaussian error
    - Move in stochastic directions by a distance of $w$ in expectation
  - MCTS in such continuous domain reduces to depth-1 search

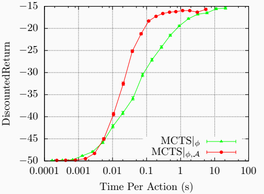
# Experimental Results



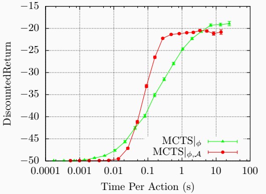

(a)
C-ROOMS[17, 17, 4, 1]




(b)
C-ROOMS[25, 13, 8, 1]




(c)
C-ROOMS[17, 17, 4, 1]




(d)
C-ROOMS[25, 13, 8, 1]

# Conclusion

# Conclusion

- A hierarchical MCTS algorithm with Markovian state and action abstractions for MDPs
  - Overcome non-Markovianess introduced by state abstraction via a POMDP formulation
  - Overcome semi-Markovianess introduced by action abstraction via a Monte Carlo learning process
  - Find a recursively hierarchical optimal policy bounded by a multiple constant of an aggregation error

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# The Pseudo Code

```
Agent (s_0 : initial ground state)
h ← ∅
P(h) ← {s_0}
repeat
    T ← an empty tree
    a ← OnlinePlanning (h)
    Execute a and observe abstract state x
    h ← hax
    P(h) ← ParticleFilter (P(h), a, x)
until terminating conditions

Rollout (α : task, s : state, h : history, d : depth)
if d ≥ H or α terminates at h then
    return (0, 0, h, s)
else
    a ← GetPrimitive (π_rollout, α, h)
    (s', x, r') ← Simulate (s, a)
    (r'', n, h'', s'') ← Rollout (α, s', hax, d + 1)
    r ← r' + γr''
    return (r, n + 1, h'', s'')

GetGreedyPrimitive (α : task, h : history)
if α is primitive then
    return α
else
    a* ← argmax_a Q(α, h, a)
    return GetGreedyPrimitive (a*, h)

GetPrimitive (π : policy, α : task, h : history)
if α is primitive then
    return α
else
    return GetPrimitive (π, π_α(h), h)

OnlinePlanning (h : history)
repeat
    s ~ P(h)
    Search (root task, s, h, 0)
until resource budgets reached
return GetGreedyPrimitive (root task, h)

Search (α : task, s : state, h : history, d : depth)
if α is primitive then
    (s', x, r) ~ Simulate (s, α)
    return (r, 1, hαx, s')
else
    if d ≥ H or α terminates at h then
        return (0, 0, h, s)
    else
        if node ⟨α, h⟩ is not in tree T then
            Insert node ⟨α, h⟩ to T
            return Rollout (α, s, h, d)
        else
            a* ← argmax_a { Q̃(α, h, a) + c sqrt( logN(α,h) / N(α,h,a) ) }
            (r', n', h', s') ← Search (a*, s, h, d)
            (r'', n'', h'', s'') ← Search (α, s', h', d + n')
            N(α, h) ← N(α, h) + 1
            N(α, h, a*) ← N(α, h, a*) + 1
            r ← r' + γ^{n'} r''
            r̃ ← r' + γ^{n'} r'' + γ^{n'+n''} R̃_α(h'')
            Q(α, h, a*) ← Q(α, h, a*) + (r − Q(α,h,a*)) / N(α,h,a*)
            Q̃(α, h, a*) ← Q̃(α, h, a*) + (r̃ − Q̃(α,h,a*)) / N(α,h,a*)
            return (r, n' + n'', h'', s'')
```

Figure 5: $\text{MCTS}|_{\phi,\mathcal{A}}$ — Monte-Carlo tree search with state and action abstractions for MDPs

# Optimality Results

Theorem

*For state abstraction* $\langle X, \phi \rangle$ *with aggregation error* $e$*, let* $s_0$ *be the current state in the ground MDP* $M$ *and* $h_0$ *with* $\mathcal{P}(h_0) = \{s_0\}$ *be the corresponding history node in POMDP* $M|_\phi$*. Let* $Q^*(s, \cdot)$ *and* $Q^*(h, \cdot)$ *be the optimal action values for* $M$ *and* $M|_\phi$ *respectively. Let* $a^* = \operatorname{argmax}_{a \in A} Q^*(h_0, a)$ *be the optimal action found in* $M|_\phi$ *at history* $h_0$*, and define action-value error as* $E(a^*) = |\max_{a \in A} Q^*(s_0, a) - Q^*(s_0, a^*)|$*. Suppose the maximal planning horizon is* $H$*, then* $E(a^*)$ *is bounded by* $E(a^*) \leqslant 2He$ *if* $\gamma = 1$*, else* $E(a^*) \leqslant 2\gamma \frac{1-\gamma^H}{1-\gamma} e$*.*