Snail parameters with g3 correction

$$\hbar g_3 = \frac{1}{6}\frac{\tilde{c}_3}{\tilde{c}_2}\frac{\hbar\omega_a}{2}\left(\frac{4E_c}{\hbar\omega_a}\right)^{1/2}$$

$$\hbar g_4 = \frac{1}{24}\frac{\tilde{c}_4}{\tilde{c}_2}\frac{\hbar\omega_a}{2}\left(\frac{4E_c}{\hbar\omega_a}\right)$$

$$\tilde{c}_2 = \frac{p}{M}c_2$$

$$\tilde{c}_3 = \frac{p^3}{M^2}c_3$$

$$\tilde{c}_4 = \frac{p^4}{M^3}\left[c_4 - \frac{3c_3^2}{c_2 + Mx_J}\right]$$

where $x_J = \frac{L_J}{L}$

$$p = \frac{Mx_J}{c_2 + Mx_J} = \frac{M\frac{L_J}{L}}{c_2 + M\frac{L_J}{L}}$$

$$\Rightarrow Mx_J = pc_2 + pMx_J$$

$$\Rightarrow x_J = \frac{pc_2}{(M - pM)} = \frac{pc_2}{M(1-p)}$$

$$\hbar g_3 = \frac{1}{6}\frac{\frac{\cancel{p^3}^{\,2}}{\cancel{M^2}}c_3}{\frac{\cancel{p}}{\cancel{M}}c_2}\frac{\hbar\omega_a}{\cancel{2}}\left(\frac{\cancel{4}E_c}{\hbar\omega_a}\right)^{1/2}$$

$$\boxed{\Rightarrow \hbar g_3 = \frac{1}{6}\frac{p^2}{M}\frac{c_3}{c_2}\sqrt{\hbar\omega_a E_c}}$$

$$\hbar g_4 = \frac{1}{24}\frac{\frac{\cancel{p^4}^{\,3}}{\cancel{M^3}^{\,2}}\left[c_4 - \frac{3c_3^2}{c_2+Mx_J}\right]}{\frac{\cancel{p}}{\cancel{M}}c_2}\frac{\cancel{\hbar\omega_a}}{\cancel{2}}\left(\frac{\cancel{4}^{\,2}E_c}{\cancel{\hbar\omega_a}}\right)$$

$$= \frac{1}{24}\frac{p^3}{M^2c_2}\left(c_4 - \frac{3c_3^2}{c_2 + Mx_J}\right)2E_c$$

$$= \frac{1}{12}\frac{p^3}{M^2}\frac{1}{c_2}\left(c_4 - \frac{3c_3^2}{\left(c_2 + \cancel{M}\frac{p}{\cancel{M}}\frac{c_2}{(1-p)}\right)}\right)E_c$$

$$\frac{1}{1+\frac{p}{1-p}} = \frac{1}{\frac{1\cancel{-p}\cancel{+p}}{1-p}} = 1-p$$

$$= \frac{1}{12}\frac{p^3}{M^2}\left(c_4 - \frac{3c_3^2}{c_2}\frac{1}{1+\frac{p}{(1-p)}}\right)\frac{E_c}{c_2}$$

$$\boxed{\hbar g_4 = \frac{1}{12}\frac{p^3}{M^2}\left(c_4 - \frac{3c_3^2}{c_2}(1-p)\right)\frac{E_c}{c_2}}$$

$$K = 6g_4 - \frac{30g_3^2}{\omega_a}$$

$$= 6\,\frac{1}{12}\frac{p^3}{M^2}\left(c_4 - \frac{3c_3^2}{c_2}(1-p)\right)\frac{E_c}{c_2\hbar} - \frac{30}{\cancel{\omega_a}}\left(\frac{1}{6}\frac{p^2}{M\hbar}\frac{c_3}{c_2}\sqrt{E_c\hbar\cancel{\omega_a}}\right)^2$$

$$= \frac{1}{2}\frac{p^3}{M^2}\left(c_4 - \frac{3c_3^2}{c_2}(1-p)\right)\frac{E_c}{c_2\hbar} - \frac{5}{6}\frac{p^4}{M^2}\frac{c_3^2}{c_2^2}E_c\frac{1}{\hbar}$$

$$\Rightarrow 2\hbar K = \frac{p^3}{M^2}\left(c_4 - \frac{3c_3^2}{c_2}(1-p)\right)\frac{E_c}{c_2} - \frac{5}{3}\frac{p^4}{M^2}\frac{c_3^2}{c_2^2}E_c$$

$$= \frac{p^3}{M^2}\left[c_4 - \frac{3c_3^2}{c_2}(1-p) - \frac{5}{3}p\frac{c_3^2}{c_2}\right]\underbrace{\frac{E_c}{c_2}}_{\text{Krbbq}}$$

! anharmonicity = 2K !

$$\frac{V}{E_J} = -\cos\varphi = -\left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!}\right)$$

$$2\hbar K = \frac{p^3}{\ldots}$$

$$2\hbar K = \frac{P^3}{M^2}$$

Vatsan's note for the 2nd order perturbation theory

$$\hat{H} = \sum_{p=1}^{P_{cut}} \hbar\omega_p \left(\hat{a}_p^+ \hat{a}_p\right) + \overbrace{\underbrace{E_J^b \sum_{n=3}^{\infty} \frac{\tilde{C}_n}{n!}\left[\sum_{p=1}^{P_{cut}} \phi_p^{zpf}\left(\hat{a}_p + \hat{a}_p^+\right)\right]^n}_{\Downarrow\; g_3(\hat{a}+\hat{a}^+)^3 + g_4(\hat{a}+\hat{a}^+)^4 + \dots}}^{H_{NL}}$$

- $p = \omega_a$ on the Harini thesis
  $= \frac{1}{\hbar}\sqrt{8\tilde{C}_2 E_J E_C} = \frac{1}{\sqrt{C(L+MLS)}}$
- $\frac{\tilde{C}_n}{n!} E_J \left(\phi^{zpf}\right)^n = \hbar g_n$
- $\varphi_{zpf} = \sqrt{\frac{4E_c}{\hbar\omega_a}} = \sqrt{\frac{\hbar\omega_a}{2\tilde{C}_2 E_J}}$

Standard perturbation theory

$$|\psi_i\rangle = |n_0\, n_1 \dots n_{pcut}\rangle = |\{n_k\}\rangle$$

$$E_i = E_i^0 + \langle\psi_i|\hat{H}_{NL}|\psi_i\rangle + \sum_{k\neq i}\frac{\langle\psi_i|\hat{H}_{NL}|E_k^0\rangle\langle E_k^0|\hat{H}_{NL}|\psi_i\rangle}{(E_i^0 - E_k^0)}$$

$$\simeq E_i^0 + \underbrace{\langle\psi_i|\hat{H}_{NL}^{(4)}|\psi_i\rangle}_{\delta E^{(4)}} + \underbrace{\sum_{k\neq i}\frac{\langle\psi_i|\hat{H}_{NL}^{(3)}|E_k^0\rangle\langle E_k^0|H_{NL}^{(3)}|\psi_i\rangle}{(E_i^0 - E_k^0)}}_{\delta E^{(3)}}$$

because

$$\langle\psi_i| E_J^b \sum_{n=3}^{\infty}\frac{\tilde{C}_n}{n!}\left[\sum_{p=1}^{P_{cut}}\phi_p^{zpf}\left(\hat{a}_p + \hat{a}_p^+\right)\right]^n|\psi_i\rangle$$

$$= \langle\psi_i|E_J^b \frac{C_3}{3!}\left(\sum_{p=1}^{P_{cut}}\phi_p^{zpf}\left(\hat{a}_p + \hat{a}_p^+\right)\right)^3|\psi_i\rangle$$

$$+ \langle\psi_i|E_J^b \frac{C_4}{4!}\left(\sum_{p=1}^{P_{cut}}\phi_p^{zpf}\left(\hat{a}_p + \hat{a}_p^+\right)\right)^4|\psi_i\rangle$$

but $\langle\psi_i|\left(\hat{a}_p + \hat{a}_p^+\right)^3|\psi_i\rangle = \langle\psi_i|\left(\hat{a}_p + \hat{a}_p^+\right)\left(\hat{a}_p + \hat{a}_p^+\right)^2|\psi_i\rangle$

$$= \langle\psi_i|\left(\hat{a}_p + \hat{a}_p^+\right)\left(\hat{a}_p^2 + \hat{a}_p\hat{a}_p^+ + \hat{a}_p^+\hat{a}_p + \hat{a}_p^{+2}\right)|\psi_i\rangle$$

$$= \langle\psi_i|\left(\hat{a}_p^3 + \hat{a}_p^2\hat{a}_p^+ + \hat{a}_p\hat{a}_p^+\hat{a}_p + \hat{a}_p\hat{a}_p^{+2} + \hat{a}_p^+\hat{a}_p^2 + \hat{a}_p^+\hat{a}_p\hat{a}_p^+ + \hat{a}_p^{+2}\hat{a}_p + \hat{a}_p^3\right)|\psi_i\rangle = 0$$

$\Rightarrow$ this is why we use only $H_{NL}^{(4)}$ on our first term

and $\langle\psi_i|\left(\hat{a}_p + \hat{a}_p^+\right)^4|\psi_i\rangle = \langle\psi_i|\left(\hat{a}_p + \hat{a}_p^+\right)^2\left(\hat{a}_p + \hat{a}_p^+\right)^2|\psi_i\rangle$

$$= \langle\psi_i|\left(\hat{a}_p^2 + \hat{a}_p^{+2} + \hat{a}_p\hat{a}_p^+ + \hat{a}_p^+\hat{a}_p\right)\left(\hat{a}_p^2 + \hat{a}_p^{+2} + \hat{a}_p\hat{a}_p^+ + \hat{a}_p^+\hat{a}_p\right)|\psi_i\rangle$$

$$= \langle\psi_i|\left(\cancel{\hat{a}_p^4} + \hat{a}_p^2\hat{a}_p^{+2} + \cancel{\hat{a}_p^3\hat{a}_p^+} + \cancel{\hat{a}_p^2\hat{a}_p^+\hat{a}_p}\right.$$

$$+ \hat{a}_p^{+2}\hat{a}_p^2 + \cancel{\hat{a}_p^{+4}} + \cancel{\hat{a}_p^{+2}\hat{a}_p\hat{a}_p^+} + \cancel{\hat{a}_p^{+3}\hat{a}_p}$$

$$= \langle\psi_i|\Big(\cancel{\hat{a}_p^4} + \hat{a}_p^2\hat{a}_p^{+2} + \cancel{\hat{a}_p^3\hat{a}_p^+} + \cancel{\hat{a}_p^2\hat{a}_p^+\hat{a}_p}$$
$$+ \hat{a}_p^{+2}\hat{a}_p^2 + \cancel{\hat{a}_p^{+4}} + \cancel{\hat{a}_p^{+2}\hat{a}_p\hat{a}_p^+} + \cancel{\hat{a}_p^{+3}\hat{a}_p}$$
$$+ \cancel{\hat{a}_p\hat{a}_p^+\hat{a}_p^2} + \cancel{\hat{a}_p\hat{a}_p^{+3}} + \boxed{\hat{a}_p\hat{a}_p^+\hat{a}_p\hat{a}_p^+} + \boxed{\hat{a}_p\hat{a}_p^{+2}\hat{a}_p}$$
$$+ \cancel{\hat{a}_p^+\hat{a}_p^3} + \cancel{\hat{a}_p^+\hat{a}_p\hat{a}_p^{+2}} + \boxed{\hat{a}_p^+\hat{a}_p^2\hat{a}_p^+} + \boxed{\hat{a}_p^+\hat{a}_p\hat{a}_p^+\hat{a}_p}\Big)|\psi_i\rangle$$

$$\langle\psi_i|H_{NL}^{(4)}|\psi_i\rangle = \langle\psi_i|\, E_J^b\frac{\tilde{c}_4}{4!}\Big[\sum_{p=1}^{P_{out}}\phi_p^{zpf}(\hat{a}_p+\hat{a}_p^+)\Big]^4|\psi_i\rangle$$
$$= E_J^b\frac{\tilde{c}_4}{4!}\sum_{p_1,p_2,p_3,p_4}\phi_{p_1}^{zpf}\phi_{p_2}^{zpf}\phi_{p_3}^{zpf}\phi_{p_4}^{zpf}\langle\{n_k\}|(\hat{a}_{p_1}+\hat{a}_{p_1}^+)(\hat{a}_{p_2}+\hat{a}_{p_2}^+)$$
$$(\hat{a}_{p_3}+\hat{a}_{p_3}^+)(\hat{a}_{p_4}+\hat{a}_{p_4}^+)|\{n_k\}\rangle$$
$$= E_J^b\frac{\tilde{c}_4}{4!}\phi_{S,J}^{zpf\,4}\sum_{pqrs}\beta_p\beta_q\beta_r\beta_s\,\langle\{n_k\}|(\hat{a}_p+\hat{a}_p^+)(\hat{a}_q+\hat{a}_q^+)$$
$$(\hat{a}_r+\hat{a}_r^+)(\hat{a}_s+\hat{a}_s^+)|\{n_k\}\rangle$$
$$= \hbar g_4\sum_{pqrs}\beta_p\beta_q\beta_r\beta_s\,\langle\{n_k\}|(\hat{a}_p+\hat{a}_p^+)(\hat{a}_q+\hat{a}_q^+)$$
$$(\hat{a}_r+\hat{a}_r^+)(\hat{a}_s+\hat{a}_s^+)|\{n_k\}\rangle$$

but $(\hat{a}_p+\hat{a}_p^+)(\hat{a}_q+\hat{a}_q^+)(\hat{a}_r+\hat{a}_r^+)(\hat{a}_s+\hat{a}_s^+)$

$$= (\hat{a}_p\hat{a}_q + \hat{a}_p\hat{a}_q^+ + \hat{a}_p^+\hat{a}_q + \hat{a}_p^+\hat{a}_q^+)(\hat{a}_r\hat{a}_s + \hat{a}_r\hat{a}_s^+ + \hat{a}_r^+\hat{a}_s + \hat{a}_r^+\hat{a}_s^+)$$
$$= (\underline{\hat{a}_p\hat{a}_q\hat{a}_r\hat{a}_s} + \underline{\hat{a}_p\hat{a}_q\hat{a}_r\hat{a}_s^+} + \underline{\hat{a}_p\hat{a}_q\hat{a}_r^+\hat{a}_s} + \hat{a}_p\hat{a}_q\hat{a}_r^+\hat{a}_s^+$$
$$+ \hat{a}_p\hat{a}_q^+\hat{a}_r\hat{a}_s + \hat{a}_p\hat{a}_q^+\hat{a}_r\hat{a}_s^+ + \hat{a}_p\hat{a}_q^+\hat{a}_r^+\hat{a}_s + \hat{a}_p\hat{a}_q^+\hat{a}_r^+\hat{a}_s^+$$
$$+ \hat{a}^+\hat{a}_q\hat{a}_r\hat{a}_s + \hat{a}_p^+\hat{a}_q\hat{a}_r\hat{a}_s^+ + \hat{a}_p^+\hat{a}_q\hat{a}_r^+\hat{a}_s + \hat{a}_p^+\hat{a}_q\hat{a}_r^+\hat{a}_s^+$$
$$+ \hat{a}_p^+\hat{a}_q^+\hat{a}_r\hat{a}_s + \hat{a}_p^+\hat{a}_q^+\hat{a}_r\hat{a}_s^+ + \hat{a}_p^+\hat{a}_q^+\hat{a}_r^+\hat{a}_s + \underline{\hat{a}_p^+\hat{a}_q^+\hat{a}_r^+\hat{a}_s^+})$$

- $\hat{a}_p^+\hat{a}_q^+\hat{a}_r^+\hat{a}_s^+$
- $\hat{a}_p\hat{a}_q\hat{a}_r\hat{a}_s$

$$[\hat{a}_r,\hat{a}_s^+] = \hat{a}_r\hat{a}_s^+ - \hat{a}_s^+\hat{a}_r$$
$$[\hat{a}_q,\hat{a}_s^+] = \hat{a}_q\hat{a}_s^+ - \hat{a}_s^+\hat{a}_q$$
$$[\hat{a}_p,\hat{a}_s^+] = \hat{a}_p\hat{a}_s^+ - \hat{a}_s^+\hat{a}_p$$

- $\hat{a}_p\hat{a}_q\hat{a}_r\hat{a}_s^+ = \hat{a}_p\hat{a}_q([\hat{a}_r,\hat{a}_s^+] + \hat{a}_s^+\hat{a}_r) = \hat{a}_p\hat{a}_q[\hat{a}_r,\hat{a}_s^+] + \hat{a}_p\hat{a}_q\hat{a}_s^+\hat{a}_r$

$$= \hat{a}_p\hat{a}_q[\hat{a}_r,\hat{a}_s^+] + \hat{a}_p\big([\hat{a}_q,\hat{a}_s^+] + \hat{a}_s^+\hat{a}_q\big)\hat{a}_r$$
$$= \hat{a}_p\hat{a}_q[\hat{a}_r,\hat{a}_s^+] + \hat{a}_p[\hat{a}_q,\hat{a}_s^+]\hat{a}_r + \hat{a}_p\hat{a}_s^+\hat{a}_q\hat{a}_r$$
$$= \hat{a}_p\hat{a}_q[\hat{a}_r,\hat{a}_s^+] + \hat{a}_p[\hat{a}_q,\hat{a}_s^+]\hat{a}_r + [\hat{a}_p,\hat{a}_s^+]\hat{a}_q\hat{a}_r + \hat{a}_s^+\hat{a}_p\hat{a}_q\hat{a}_r$$

- $\hat{a}_p\hat{a}_q\hat{a}_r^+\hat{a}_s = \hat{a}_p([\hat{a}_q,\hat{a}_r^+] + \hat{a}_r^+\hat{a}_q)\hat{a}_s = \hat{a}_p[\hat{a}_q,\hat{a}_r^+]\hat{a}_s + \hat{a}_p\hat{a}_r^+\hat{a}_q\hat{a}_s$

$$= \hat{a}_p[\hat{a}_q,\hat{a}_r^+]\hat{a}_s + [\hat{a}_p,\hat{a}_r^+]\hat{a}_q\hat{a}_s + \hat{a}_r^+\hat{a}_p\hat{a}_q\hat{a}_s$$

$$\langle\psi_i|H_{NL}^{(4)}|\psi_i\rangle = \hbar g_4\sum_{pqrs}\beta_q\beta_q\beta_r\beta_s\langle\{n_k\}|\Big[\cancel{\hat{a}_p^+\hat{a}_q^+\hat{a}_r^+\hat{a}_s^+} + \cancel{4\hat{a}_p^+\hat{a}_q^+\hat{a}_r^+\hat{a}_s}$$
$$+ 6\hat{a}_p^+\hat{a}_q^+\hat{a}_r\hat{a}_s + \cancel{4\hat{a}_p^+\hat{a}_q\hat{a}_r\hat{a}_s} + \cancel{\hat{a}_p\hat{a}_q\hat{a}_r\hat{a}_s}\Big]|\{n_k\}\rangle$$

$$\Rightarrow \langle\psi_i|H_{NL}^{(4)}|\psi_i\rangle = 6g_4\hbar\sum_{pqrs}\beta_p\beta_q\beta_r\beta_s\underbrace{\langle\{n_k\}|\hat{a}_p^+\hat{a}_q^+\hat{a}_r\hat{a}_s|\{n_k\}\rangle}_{n_p n_q(\delta_{pr}\delta_{ps}+\delta_{ps}\delta_{qr})}$$

$$\Rightarrow \langle\psi_i|H_{NL}^{(4)}|\psi_i\rangle = 6 g_4 \hbar \sum_{pqrs} \beta_p\beta_q\beta_r\beta_s \underbrace{\langle\{n_k\}|\hat{a}_p^+\hat{a}_q^+\hat{a}_r\hat{a}_s|\{n_k\}\rangle}_{n_p n_q(\delta_{pr}\delta_{ps}+\delta_{ps}\delta_{qr})}$$

$$= 6 g_4 \hbar \sum_{p,q} \beta_p^2 \beta_q^2 n_p n_q$$

$$\langle\psi_i|\hat{H}_{NL}^{(3)}|E_k^0\rangle = \langle\psi_i|\frac{\hbar c_3}{3!}\sum_{p=1}^{\#} \left(\phi_p^{ZPF}(\hat{a}_p+\hat{a}_p^+)\right)^3|E_k^0\rangle$$

$$= \hbar g_3 \sum_{pqr} \beta_p\beta_q\beta_r (\hat{a}_p+\hat{a}_p^+)(\hat{a}_q+\hat{a}_q^+)(\hat{a}_r+\hat{a}_r^+)|E_k^0\rangle$$

but $(\hat{a}_p+\hat{a}_p^+)(\hat{a}_q+\hat{a}_q^+)(\hat{a}_r+\hat{a}_r^+)$

$$= (\hat{a}_p\hat{a}_q + \hat{a}_p\hat{a}_q^+ + \hat{a}_p^+\hat{a}_q + \hat{a}_p^+\hat{a}_q^+)(\hat{a}_r+\hat{a}_r^+)$$

$$= \hat{a}_p\hat{a}_q\hat{a}_r + \hat{a}_p\hat{a}_q\hat{a}_r^+ + \hat{a}_p\hat{a}_q^+\hat{a}_r + \hat{a}_p\hat{a}_q^+\hat{a}_r^+ + \hat{a}_p^+\hat{a}_q\hat{a}_r + \hat{a}_p^+\hat{a}_q\hat{a}_r^+ + \hat{a}_p^+\hat{a}_q^+\hat{a}_r + \hat{a}_p^+\hat{a}_q^+\hat{a}_r^+$$

b/c $[\hat{a}_p, a_q^+] = \delta_{pq}$

$$= \hat{a}_p\hat{a}_q\hat{a}_r + \hat{a}_p^+\hat{a}_q\hat{a}_r + (\delta_{pq}+\hat{a}_q^+\hat{a}_p)\hat{a}_r + \hat{a}_p^+\hat{a}_q^+\hat{a}_r + \boxed{\hat{a}_p\hat{a}_q\hat{a}_r^+} + \hat{a}_p^+(\delta_{rq}+\hat{a}_r^+\hat{a}_q) + \boxed{\hat{a}_p\hat{a}_q^+\hat{a}_r^+} + \hat{a}_p^+\hat{a}_q^+\hat{a}_r^+$$

$$\hat{a}_p\hat{a}_q\hat{a}_r^+ = \hat{a}_p(\hat{a}_r^+\hat{a}_q + \delta_{rq})$$
$$= \hat{a}_p\delta_{rq} + (\delta_{pr}+\hat{a}_r^+\hat{a}_p)\hat{a}_q$$

$$\hat{a}_p\hat{a}_q^+\hat{a}_r^+ = (\delta_{pq}+a_q^+\hat{a}_p)\hat{a}_r^+ = \delta_{pq}\hat{a}_r^+ + \hat{a}_q^+\hat{a}_p\hat{a}_r^+$$
$$= \delta_{pq}\hat{a}_r^+ + \hat{a}_q^+(\delta_{pr}+\hat{a}_r^+\hat{a}_p)$$
$$= \delta_{pq}\hat{a}_r^+ + \hat{a}_q^+\delta_{pr} + \hat{a}_q^+\hat{a}_r^+\hat{a}_p$$

$$\hat{H}_{NL}^{(3)} = \hbar g_3 \sum_{pqr} \beta_p\beta_q\beta_r (\hat{a}_p+\hat{a}_p^+)(\hat{a}_q+\hat{a}_q^+)(\hat{a}_r+\hat{a}_r^+)$$

$$= g_3\hbar \sum_{pqr} \beta_p\beta_q\beta_r \left[3\delta_{pq}(\hat{a}_r+\hat{a}_r^+) + \{ 3\hat{a}_p^+\hat{a}_q\hat{a}_r + 3\hat{a}_p^+\hat{a}_q^+\hat{a}_r + \hat{a}_p^+\hat{a}_q^+\hat{a}_r^+ + \hat{a}_p\hat{a}_q\hat{a}_r \} \right]$$

$$\delta E^{(3)} = \sum_{k\neq i} \frac{\langle\psi_i|\hat{H}_{NL}^{(3)}|E_k^0\rangle\langle E_k^0|\hat{H}_{NL}^{(3)}|\psi_i\rangle}{(E_i^0 - E_k^0)}$$

$$= \sum_{k\neq i} \frac{\langle\{n_i\}|\hat{H}_{NL}^{(3)}|E_k^0\rangle\langle E_k^0|\hat{H}_{NL}^{(3)}|\{n_i\}\rangle}{(E_i^0 - E_k^0)}$$

$$= \hbar^2 \sum_{pqr}\sum_{p'q'r'} g_3^2 \beta_p\beta_q\beta_r\beta_{p'}\beta_{q'}\beta_{r'} \frac{\langle\{n_i\}|\tilde{O}_{pqr}^{(3)}|E_k\rangle\langle E_k|\tilde{O}_{p'q'r'}^{(3)}|\{n_i\}\rangle}{(E_i^0 - E_k^0)}$$

$$= \frac{\sum_{pqr} g_3^2 \beta_p\beta_q\beta_r \langle\{n_i\}|\tilde{O}_{pqr}^{(3)}|E_k\rangle\langle E_k|\sum_{p'q'r'}\beta_{p'}\beta_{q'}\beta_{r'}\tilde{O}_{p'q'r'}^{(3)}|\{n_i\}\rangle}{(E_i^0 - E_k^0)}$$

$$\hat{a}_r^{+} |n_r\rangle = \sqrt{n_r+1}\, |n_r+1\rangle, \qquad E = \hbar\omega$$

$$\delta E^{(3)} = \sum_{pqr} g_3^2 \beta_p^2 \beta_q^2 \beta_r^2 \Big\{ g\,\delta_{pq} \left[\frac{(n_r+1)^2}{-\omega_r} + \frac{(n_r)^2}{\omega_r} \right] + \frac{n_p n_q n_r}{(\omega_p+\omega_q+\omega_r)} 6 + g \frac{n_p n_q (n_r+1)}{(\omega_p+\omega_q-\omega_r)} 2$$

$$+ g \frac{n_p (n_q+1)(n_r+1)}{(\omega_p-\omega_q-\omega_r)} 2 + \frac{(n_p+1)(n_q+1)(n_r+1)}{-(\omega_p+\omega_q+\omega_r)} 6 \Big\}$$

$$\delta E^{(3)} = \sum_{pqr} g_3^2 \beta_p^2 \beta_q^2 \beta_r^2 \Big\{ 6\frac{n_p n_q n_r}{(\omega_p+\omega_q+\omega_r)} - 6 \frac{(n_p n_q + n_p + n_q + 1)(n_r+1)}{(\omega_p+\omega_q+\omega_r)} + 18 \frac{n_p (n_q+1)(n_r+1)}{(\omega_p-\omega_q-\omega_r)} + 18 \frac{n_p n_q (n_r+1)}{(\omega_p+\omega_q-\omega_r)} \Big\}$$

$$+ \sum_p g_3^2 \beta_p^4 \; g \cdot \left(\frac{\cancel{-n_r^2} - 2n_r - 1 \cancel{+ n_r^2}}{\omega_r} \right)$$

$$= \sum_p g_3^2 \beta_p^4 \left(-18 \frac{n_r}{\omega_r} \right) + \sum_{pqr} g_3^2 \beta_p^2 \beta_r^2 \Big\{ -\frac{6 (n_p n_q + n_p n_r + n_q n_r)}{(\omega_p+\omega_q+\omega_r)} + 18 \frac{n_p n_q (n_r+1)}{(\omega_p+\omega_q-\omega_r)} + 18 \frac{n_p (n_q+1)(n_r+1)}{(\omega_p-\omega_q-\omega_r)} \Big\}$$

$$= 18 g_3^2 \sum_{pq} \beta_p^2 \beta_q^2 n_p n_q \Big\{ \sum_r \beta_r^2 \Big[\frac{1}{(\omega_p+\omega_q-\omega_r)} + \frac{1}{(\omega_p-\omega_q-\omega_r)} + \frac{1}{(-\omega_p+\omega_q-\omega_r)} - \frac{1}{(\omega_p+\omega_q+\omega_r)} \Big] \Big\}$$

<u>Inter-cavity cross-Kerr</u> $\quad H_{eff} = \sum_p \hbar \tilde{\omega}_p n_p + \sum_{pq} K_{pq} n_p n_q$

$$\boxed{K_{pq} = \beta_p^2 \beta_q^2 \Big[12 g_4 + 18 g_3^2 \sum_r \beta_r^2 \Big\{ \frac{1}{\omega_p+\omega_q-\omega_r} + \frac{1}{\omega_p-\omega_q-\omega_r} + \frac{1}{-\omega_r+\omega_q-\omega_r} - \frac{1}{\omega_p+\omega_q+\omega_r} \Big\} \Big]}$$

for modes a, b

$$\boxed{\chi_{ab} = \left(\frac{g_a}{\Delta_a}\right)^2 \left(\frac{g_b}{\Delta_b}\right)^2 \left[24 g_4 + \frac{36 g_3^2}{\tilde{\omega}} \right]}$$

$$\frac{1}{\tilde{\omega}} = \left[\frac{1}{\omega_a+\omega_b-\omega_c} + \frac{1}{\omega_a-\omega_b-\omega_c} + \frac{1}{-\omega_a+\omega_b-\omega_c} - \frac{1}{\omega_a+\omega_b+\omega_c} \right]$$

<u>Self-Kerr</u>

$$K_p = K_{pp} = \beta_p^4 \Big[12 g_4 + 18 g_3^2 \sum_r \beta_r^2 \Big\{ \frac{1}{(\omega_p+\omega_q-\omega_r)} + \frac{1}{(\omega_p-\omega_q-\omega_r)} + \frac{1}{(-\omega_p+\omega_q-\omega_r)} - \frac{1}{(\omega_p+\omega_q+\omega_r)} \Big\} \Big]$$

$$+ \frac{1}{(-\omega_p+\omega_q-\omega_r)} - \frac{1}{(\omega_p+\omega_q+\omega_r)} \Big\}\Big]$$

$$\boxed{K_p \simeq \varphi_p^4 \Big[12 g_4 + 18 g_3^2 \Big\{ \frac{1}{(2\omega_p - \omega_c)} - \frac{1}{(2\omega_p + \omega_c)} - \frac{2}{\omega_c} \Big\}\Big]}$$

<u>Snailmon anharmonicity</u>

$$\alpha = 12 g_4 + 18 g_3^2 \Big\{ \frac{1}{\omega_c} - \frac{1}{3\omega_c} - \frac{2}{\omega_c} \Big\} = \boxed{12 g_4 - \frac{24}{\omega_c} g_3^2 = \alpha}$$