Frattini thesis

Link: [Three-wave Mixing in Superconducting Circuits: Stabilizing Cats with SNAILs (bpb-us-w2.wpmucdn.com)]

Notes:

Three-wave mixing: a photon can be split into two correlated photons (or the inverse)

↓

Need 3rd order non-linearity

↓

How: DC flux bias to a superconducting loop with JJs

JJ:

Josephson tunneling energy

superconducting phase across junction

$U_{JJ}(\phi) = -E_J \cos(\varphi)$

↳ potential is an even function of φ ⟷ lowest nonlinearity of the form φ^4

↕

Kerr nonlinearity

engineer interactions between modes !!

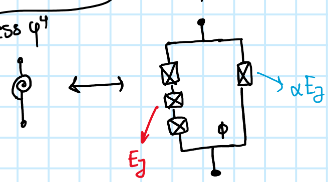
frequency shifts

↳ frequency crowding !!

Solution: use φ^3 and suppress φ^4

↕

SNAIL



Snail

$$U_S(\varphi_S) = -\alpha E_J \cos(\phi_S) - n E_J \cos\left(\frac{\varphi_{ext} - \varphi_S}{n}\right)$$

$$\varphi_{ext} = \frac{2\pi\Phi}{\Phi_0} \quad , \quad \Phi_0 = \frac{h}{2e} = \frac{\Phi_0}{2\pi}$$

$$I_S(\varphi_S) = \frac{1}{\varphi_0}\frac{\partial U_S}{\partial \varphi_S} = \frac{E_J}{\varphi_0}\left[\alpha \sin(\varphi_S) + \sin\left(\frac{\varphi_S - \varphi_{ext}}{n}\right)\right]$$

if $\alpha \sim 0.8$, $\frac{\Phi}{\Phi_0} \sim 0.5$ ⟶ flux qubit ⟶ double potential well

if $\alpha \sim 0.1$, $\frac{\Phi}{\Phi_0} \sim 0.34$ ⟶ snail

Analyzing the snail about one of its equivalent minima φ_{min},

$c_1 = I_S(\varphi_{S,min}) = 0$ (no DC current flowing across the entire dipole)

$$= \alpha \sin(\varphi_{S,min}) + \sin\left(\frac{\varphi_{S,min} - \varphi_{ext}}{n}\right) = 0$$

$\Rightarrow U_{S,eff}(\tilde{\varphi}_S) \quad c_2 \, \tilde{\varphi}_S^2 + c_3 \, \tilde{\varphi}_S^3 + \frac{c_4}{} \tilde{\varphi}_S^4 + \ldots$

$$= \alpha \sin(\varphi_{s,min}) + \sin\left(\frac{\varphi_{s,min} - \varphi_{ext}}{n}\right) = 0$$

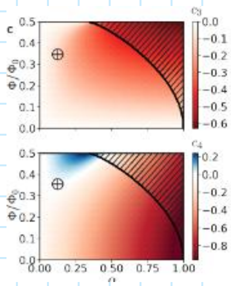
$$\Rightarrow \frac{U_{s,eff}(\tilde{\varphi}_s)}{E_J} = \frac{c_2}{2!}\tilde{\varphi}_s^2 + \frac{c_3}{3!}\tilde{\varphi}_s^3 + \frac{c_4}{4!}\tilde{\varphi}_s^4 + \ldots$$

$2\pi n$ periodic

$c_k(\Phi) = c_k(\Phi + \Phi_0)$

$c_k(\Phi) = (-1)^k c_k(-\Phi)$

3-wave mixing $\rightarrow$ $c_3 \neq 0$



maximize c_3

c_4 changes sign so Kerr non-linearity cancels out

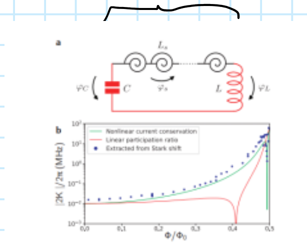
$\alpha \le \frac{1}{4}$ $(n=3$ here$)$

$\alpha = \frac{1}{n^2}$ so that $\Phi|_{c_4=0}$ is furthest from $\frac{\Phi}{\Phi_0} = 0.5$

1 non-linear mode

- In the absence of dissipation and incident microwave radiation

M snails w/ n=3



★ See Vool and Devoret 2017

- Generalized branch phase across each element
  $\varphi_C, \varphi_L, \varphi_{sm}$

phase across the inductive stuff $\Rightarrow$ $\boxed{\varphi = \varphi_L + \sum_{m=1}^{M} \varphi_{sm}}$

- Kirchhoff's voltage law at the loop

$$\boxed{\dot{\varphi}_C = \dot{\varphi}_L + \sum_{m=1}^{M} \dot{\varphi}_{sm} = \dot{\varphi}}$$

because $\quad -V_C + V_L + \sum_{m=1}^{M} V_{sm} = 0$

$$\Rightarrow V_C = V_L + \sum_{m=1}^{M} V_{sm}$$

$$\Rightarrow \dot{\Phi}_C = \dot{\Phi}_L + \sum_{m=1}^{M} \dot{\Phi}_{sm}$$

$$\Rightarrow \frac{\cancel{2e}\dot{\varphi}_C}{\cancel{\hbar}} = \frac{\cancel{2e}\dot{\varphi}_L}{\cancel{\hbar}} + \sum_{m=1}^{M} \frac{\dot{\varphi}_{sm}\cancel{2e}}{\cancel{\hbar}}$$

$$\Rightarrow \dot{\varphi}_C = \dot{\varphi}_L + \sum_{m=1}^{M} \dot{\varphi}_{sm} = \dot{\varphi}$$

Helpfull note from Vool 2017

- Josephson element has generalized flux Φ_J
  $$\Phi_J(t) = \int_{-\infty}^{t} v_J(t')\,dt'$$
  where v_J is voltage across element
- gauge invariant phase difference is
  $$\varphi = \frac{2e\Phi_J}{\hbar}$$
- $|\Phi_J| \ll \Phi_0 \Rightarrow$ tunnel behaves like an inductance with $L_J = \frac{\Phi_0}{I_0}$

kinetic energy

- $$\mathcal{L}_{M+1} = \frac{C\varphi_0^2}{2}\dot{\varphi}^2 - U_{M+1}(\varphi, \varphi_{s1}, \ldots \varphi_{sM})$$

$$U_{M+1}(\varphi, \varphi_{s1}, \cdots, \varphi_{sM}) = \frac{1}{2}E_L\left(\varphi - \sum_{m=1}^{M}\varphi_{s,m}\right)^2 + \sum_{m=1}^{M} U_s(\varphi_{sm})$$

$$E_L = \frac{\varphi_0^2}{L}$$

- Exclude capacitors across each snail
- Low-frequencies for lumped-element approximation to hold

$$-\frac{\partial \mathcal{L}_{M+1}}{\partial \varphi_{s,m}} = 0 \rightarrow M \text{ Euler-Lagrange equations}$$

$$\Rightarrow \frac{\partial U_{M+1}}{\partial \varphi_{s,m}} = -E_L\left(\varphi - \sum_{m=1}^{M}\varphi_{s,m}\right) + \frac{dU_s}{d\varphi_{s,m}} = 0$$

↳ require that: $\frac{dU_s}{d\varphi_{s,m}} = \frac{dU_s}{d\varphi_{s,m+1}}$

equivalent to $I_s(\varphi_{s,m}) = I_s(\varphi_{s,m+1})$ ⟷ current conservation ↕ Kirchhoff's current law

↓

Solving this enforces $\varphi_{s,m} = \varphi_{s,m+1}$ ⟷ We assumed identical snails with identical current-phase relations

- Reduced Lagrangian: $\mathcal{L}_2 = \frac{C\varphi_0^2}{2}\dot{\varphi}^2 - U_2(\varphi,\varphi_s)$

$$U_2(\varphi,\varphi_s) = \frac{1}{2}E_L(\varphi - \underbrace{M\varphi_s}_{\text{total phase drop across the array of snails}})^2 + \underbrace{MU_s(\varphi_s)}_{\text{total potential energy of array of snails}}$$

- Minimize potential $U_2(\varphi,\varphi_s)$

$$\frac{\partial U_2}{\partial \varphi_s} = 0$$

$$\Rightarrow ME_L(M\varphi_s - \varphi) + M\frac{dU_s}{d\varphi_s} = 0$$

$$\Rightarrow x_J(M\varphi_s - \varphi) + \alpha\sin(\varphi_s) + \sin\left(\frac{\varphi_s - \varphi_{ex}}{n}\right) = 0$$

where $x_J = \frac{L_J}{L} = \frac{E_L}{E_J}$

- The Lagrangian with a single degree of freedom:

$$\mathcal{L}_1 = \frac{C\varphi_0^2}{2}\dot{\varphi}^2 - U_1(\varphi)$$

$$U_1(\varphi) = \frac{1}{2}E_L(\varphi - M\varphi_s[\varphi])^2 + MU_s(\varphi_s[\varphi])$$

for the entire SNAIL array

- Taylor expanding the potential in the regime of small phase fluctuations about the minimum φ_{min}

$$\tilde{c}_k = \frac{1}{E_J}\left(\frac{d^kU_1}{d\varphi^k}\right)\bigg|_{\varphi=\bar{\varphi}_{min}}$$

  - $\bar{\varphi}_{min}$ comes from the condition $\tilde{c}_1 = 0$
  - Using current conservation,

$$\tilde{c}_1 = x_J(\bar{\varphi}_{min} - M\varphi_s[\bar{\varphi}_{min}])$$

$$\tilde{c}_2 = x_J\left(1 - M\frac{d\varphi_s}{d\varphi}[\bar{\varphi}_{min}]\right)$$

$$\tilde{c}_3 = -Mx_J\frac{d^2\varphi_s}{d\varphi^2}[\bar{\varphi}_{min}]$$

$$\tilde{c}_4 = -Mx_J\frac{d^3\varphi_s}{d\varphi^3}[\bar{\varphi}_{min}]$$

$$\vdots$$

- The series linear inductor does not change the location of the potential minimum $\varphi_s[\bar{\varphi}_{min}] = \varphi_{s,min}$

↳ solving $\tilde{c}_1 = 0 \longrightarrow \bar{\varphi}_{min} = M\varphi_{s,min}$

$$\tilde{c}_2 = \frac{p}{M}c_2$$

$$\tilde{c}_3 = \frac{p^3}{M^3}c_3$$

$$\tilde{c}_4 = \frac{p^4}{M^3}\left[c_4 - \frac{3c_3^2}{c_2}(1-p)\right]$$

$$\vdots$$

where $p = \frac{ML_s}{L+ML_s} = \frac{Mx_J}{c_2+Mx_J}$ → inductive participation ratio

$$L_s = L_s(\Phi) = \frac{L_J}{c_2} = \frac{L_J}{c_2(\Phi)}$$

Expansion

$$U_1(\varphi) = \frac{1}{2}E_L\left[\varphi - M\varphi_s[\varphi]\right]^2 + MU_s\left[\varphi_s[\varphi]\right]$$

Taylor expansion:

φ_{min} is the minimum

$$U_1(\varphi) = U_1(\varphi_{min}) + \cancel{\frac{dU_1}{d\varphi}\bigg|_{\varphi=\varphi_{min}}(\varphi-\varphi_{min})}^{\,0} + \frac{1}{2!}\frac{d^2U_1}{d\varphi^2}\bigg|_{\varphi=\varphi_{min}}(\varphi-\varphi_{min})^2 + \frac{1}{3!}\frac{d^3U_1}{d\varphi^3}\bigg|_{\varphi=\varphi_{min}}(\varphi-\varphi_{min})^3 + \ldots$$

- $1-p$ is the proportion of the total inductance that comes from the linear inductor

- Quantization

$$\mathcal{L}_1 = \frac{C\varphi_0^2}{2}\dot{\varphi}^2 - U_1(\varphi)$$

$$U_1(\varphi) = \frac{1}{2}E_L\left(\varphi - M\varphi_s[\varphi]\right)^2 + MU_s(\varphi_s[\varphi])$$

Legendre transformation

$$\mathcal{H}_1 = 4E_C N^2 + U_1(\varphi)$$

$$U_1(\varphi) = \frac{1}{2}E_L\left(\varphi - M\varphi_s[\varphi]\right)^2 + MU_s(\varphi_s[\varphi])$$

- $E_C = \frac{e^2}{2C}$ is the charging energy
- N is the conjugate momentum of φ and counts the charge (in cooper pairs) across the capacitance
- $[\varphi, N] = i$

expanding Hamiltonian about $\bar{\varphi}_{min}$

$$\mathcal{H}_1 = 4E_C N^2 + E_J\left(\frac{\tilde{c}_2}{2!}\tilde{\varphi}^2 + \frac{\tilde{c}_3}{3!}\tilde{\varphi}^3 + \frac{\tilde{c}_4}{4!}\tilde{\varphi}^4 + \dots\right)$$

where $\tilde{\varphi} = \varphi - \bar{\varphi}_{min}$ phase fluctuations about the potential minimum

and $[\tilde{\varphi}, N] = i$

- Introduce $\hat{a}^+, \hat{a}$ with $[\hat{a}^+, \hat{a}] = 1$ that diagonalize the quadratic part of the Hamiltonian in the excitation number basis.

$$\tilde{\varphi} = \varphi_{zpf}(\hat{a} + \hat{a}^+)$$

where $\varphi_{zpf} = \left(\frac{2E_C}{\tilde{c}_2 E_J}\right)^{1/4} = \left(\frac{1}{8\left(\frac{\hbar}{4e^2}\right)^2}\frac{L + ML_s}{C}\right)^{1/4}$

is the zero-point fluctuations of the phase

$$\tilde{c}_2 = \frac{pc_2}{M}$$

$$N = iN_{zpf}(\hat{a}^+ - \hat{a}) \quad \text{with } N_{zpf} = \frac{1}{2}\varphi_{zpf}$$

Then, the Hamiltonian is

$$\boxed{\frac{\mathcal{H}_1}{\hbar} = \omega_a \hat{a}^+\hat{a} + g_3(\hat{a} + \hat{a}^+)^3 + g_4(\hat{a} + \hat{a}^+)^4 + \dots}$$

where $\omega_a = \frac{1}{\hbar}\sqrt{8\tilde{c}_2 E_J E_C} = \frac{1}{\sqrt{C(L + ML_s)}}$

and the non-linear parameters are $\hbar g_k = \frac{\tilde{c}_k}{k!}E_J(\varphi_{zpf})^k$

but $\varphi_{zpf} = \sqrt{\frac{4E_C}{\hbar\omega_a}} = \sqrt{\frac{\hbar\omega_a}{2\tilde{c}_2 E_J}}$

so $\hbar g_k = \frac{1}{k!}\frac{\tilde{c}_k}{\tilde{c}_2}\frac{\hbar\omega_a}{2}(\varphi_{zpf})^{k-2} = \frac{1}{k!}\frac{\tilde{c}_k}{\tilde{c}_2}\frac{\hbar\omega_a}{2}\left(\frac{4E_C}{\hbar\omega_a}\right)^{\frac{k}{2}-1}$

$$\boxed{\hbar g_3 = \frac{1}{6}\frac{p^2}{M}\frac{c_3}{c_2}\sqrt{E_C\hbar\omega_a}}$$

$$\boxed{\hbar g_4 = \frac{1}{12}\frac{p^3}{M^2}\left[c_4 - \frac{3c_3^2}{c_2}(1-p)\right]\frac{1}{c_2}E_C}$$

this works as a renormalization of the potential due to high-frequency degrees of freedom that are inevitably part of the system (eg. plasma freq. of each junction)

can't measure but w low energy (anharmonicity) but they affect 2K

- Truncation valid for a critical number of photons:

$$n_{crit} = 15\left(\frac{\tilde{c}_4}{\tilde{c}_{\dots}}\frac{p^2}{M^2}\right)\frac{M^2}{p^2}\frac{1}{\varphi_{zpf}^2}$$

- Truncation valid for a critical number of photons: but they

$$n_{crit} = 15 \left(\frac{\tilde{C}_4}{C_6} \frac{p^2}{M^2} \right) \frac{M^2}{p^2} \frac{1}{\varphi^2_{zpf}}$$

- Perturbation theory in $\frac{p\varphi_{zpf}}{M}$

$$\boxed{\frac{H_{1,eff}}{\hbar} = (\omega_a + \Delta)\, \hat{a}^+\hat{a} + K\hat{a}^{+2}\hat{a}^2 + K'\hat{a}^{+3}\hat{a}^3 + \dots}$$

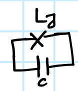
Lamb shift

Kerr non-linearity
$= 2 \cdot$ anharmonicity
and anharmonicity $= -E_c = 2\hbar K$

- Designing non-linearity

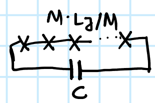
$\boxed{\omega_a = \frac{1}{\sqrt{CL_J}}}$, anharmonicity $= -E_c = 2\hbar K$

Choice #1



$\hookrightarrow n_{crit} \propto \frac{1}{\varphi^2_{zpf}} = \frac{\hbar\omega_a}{E_c} = \frac{2E_J}{\hbar\omega_a}$, $2K\hbar = -E_c$ anharmonicity

Choice #2



Introducing an array of JJs decouples L_J and non-linearity
Then, we replace the junction w/ M Junctions each w/ M times larger critical current.
The canonical mode phase splits equally among the junctions so the anharmonicity is

$$2\hbar K = -12 \frac{ME_J}{4!} \sum_{m=1}^{M} \left(\frac{\varphi_{zpf}}{M} \right)^4 = -\frac{E_c}{M^2}$$

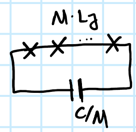
Resonant frequency and impedance remain independent of M. and $n_{crit} \propto \frac{M^2}{\varphi^2_{zpf}}$ matches the scaling of the non-linearity $\left(\propto \frac{1}{M^2}\right)$.

Limitations of this approach: in Fab with fixed critical current density (like our fab), reducing the junction inductance to $\frac{L_J}{M}$ requires M times larger area junction.

The finite size of junctions and their leads contribute to the linear inductance and cannot be ignored. !!

Winner!

Choice #3



Array fixed-E_J junctions ($L_J >$ parasitic inductance).
L_J grows as $\propto M$
C is adjusted by $\propto \frac{1}{M}$ to keep frequency constant
$\rightarrow \propto M$ increase in impedance
$\rightarrow \propto \sqrt{M}$ increase in φ_{zpf}

$\Downarrow$

$$2\hbar K = -12 \frac{E_J}{4!} \sum_{m=1}^{M} \left(\frac{\varphi_{zpf}}{M} \right)^4 = -\frac{E_c}{M}$$

and $n_{crit} \propto \frac{M^2}{\varphi^2} = \frac{2ME_J}{\dots}$

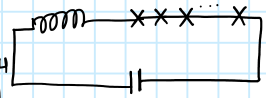
$$2\hbar K = -12\,\frac{E_J}{4!}\sum_{m=1}\left(\frac{\varphi_{zpf}}{M}\right)^4 = -\frac{E_C}{M}$$

and $n_{crit} \propto \frac{N^2}{\varphi_{zpf}^2} = \frac{2ME_J}{\hbar\omega_a}$

NOTE: • the non-linearity scales with ME_J like it would for a single junction

• easier now to realize the capacitance while adding minimal linear inductance

Choice #4



• $2K\hbar = -12\,\frac{E_J}{4!}\sum_{m=1}^{M}\left(p\,\frac{\varphi_{zpf}}{M}\right)^4$

$= -\frac{p^2}{M}E_c$

• $n_{crit} \propto \frac{N^2}{p^2\varphi_{zpf}^2} = \frac{2ME_J}{p\hbar\omega_a}$

• when the linear inductance is appreciable,

↳ $p \ll 1 \approx \frac{ML_J}{L}$

↳ $2\hbar K \approx -ME_C\left(\frac{L_J}{L}\right)^2 \propto M$

↳ $n_{crit} \propto \frac{2E_J}{\hbar\omega_a}\,\frac{L}{L_J}$

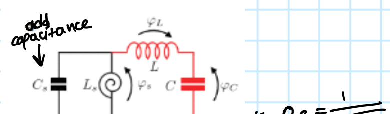
⇓

K scales with M but n_{crit} does not !!

• Renormalization of Kerr

• high frequency modes that cannot be probed directly in low-energy experiments have an effect on anharmonicity

↳ Example



• the snail plasma frequency is $\Omega_S = \frac{1}{\sqrt{L_S C_S}}$

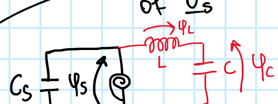
and $\Omega_S \gg \omega_a = \frac{1}{\sqrt{C(L+L_S)}}$

our operating frequency

use Born-Oppenheimer method to integrate out this higher degree of freedom

the effective Kerr nonlinearity K is still 2.53 in thesis and does not depend on the precise value of Ω_S

How?



BBQ method

1. Seperate the linear inductance L_S and the non-linear part

↳ Snail is partitioned as two elements in parallel

linear potential

non-linear potential

$c_2 E_J \tilde{\varphi}_S^{\,2}$

$U_{NL}(\tilde{\varphi}_S) = U_S(\tilde{\varphi}_S + \varphi_{s\,min}) - \frac{c_2}{2}E_J\tilde{\varphi}_S^{\,2}$

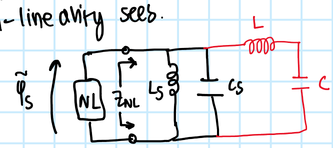
potential

potential

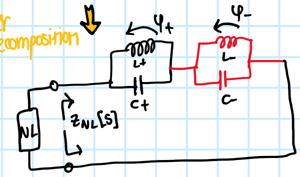
$$\frac{C_2 E_J \tilde{\varphi}_s^2}{2} \qquad U_{NL}(\tilde{\varphi}_s) = U_s(\tilde{\varphi}_s + \varphi_{s\,min}) - \frac{C_2}{2} E_J \tilde{\varphi}_s^2$$

↳ 0 applied flux in the newly formed look

2. Compute the impedance that the non-linearity sees.



Foster Decomposition



$$Z_{NL}[s] = sL_s \frac{\omega_+^2 \omega_-^2}{\omega_+^2 - \omega_-^2} \left[\frac{1 - \omega_-^2 LC}{s^2 + \omega_-^2} - \frac{1 - \omega_+^2 LC}{s^2 + \omega_+^2} \right]$$

* s is the Laplace transform
$s = j\omega$ recovers the fourier transform

poles at $\to j\omega_\pm$, $\to -j\omega_\pm$

$$\omega_\pm^2 = \frac{L + L_s}{2LL_sC_s} + \frac{1}{2LC} \pm \sqrt{\left(\frac{L+L_s}{2LL_sC_s} + \frac{1}{2LC}\right)^2 - \frac{1}{LL_sCC_s}}$$

$$= \frac{\Omega_s^2}{2(1-p)} \left(1 + \frac{\omega_a^2}{\Omega_s^2}\right) \left[1 \pm \sqrt{1 - 4(1-p) \frac{\omega_a^2/\Omega_s^2}{(1+\omega_a^2/\Omega_s^2)^2}} \right]$$

$p = \frac{L_s}{L + L_s}$ and $1 - p = LC\omega_a^2$

when $\Omega_s \gg \omega_a$, $\omega_- \approx \omega_a = \frac{1}{\sqrt{C(L+L_s)}}$

$$\omega_+ \approx \frac{\Omega_s}{\sqrt{1-p}} = \frac{1}{\sqrt{C_s \frac{L_s L}{(L+L_s)}}}$$

→ Equivalent to calculating the residue for each pole with characteristic impedances seen by the non-linearity

$$Z_- = L_s\omega_- \left[\frac{1 - \left(\frac{\omega_-}{\omega_a}\right)^2 (1-p)}{1 - \left(\frac{\omega_-}{\omega_+}\right)^2} \right] \approx pL_s\omega_- = p^2\sqrt{\frac{L+L_s}{C}}$$

$$Z_+ = L_s\omega_+ \left(\frac{\omega_-}{\omega_a}\right)^2 \left[\frac{1 - p - \left(\frac{\omega_a}{\omega_+}\right)^2}{1 - \left(\frac{\omega_-}{\omega_+}\right)^2} \right]$$

$$\approx (1-p) L_s \omega_+ = p(1-p)\frac{\omega_+}{\omega_a}\sqrt{\frac{L+L_s}{C}}$$

$$Z_\pm = \sqrt{\frac{L_\pm}{C_\pm}}$$

$Z_\pm = \frac{\omega_\pm}{C_\pm}$ and $\Omega_s \gg \omega_a$

$$L_- = L_s \left[\frac{1 - \left(\frac{\omega_-}{\omega_a}\right)^2 (1-p)}{1 - \left(\frac{\omega_-}{\omega_+}\right)^2} \right] \approx pL_s = p^2 (L + L_s)$$

$$L_+ = L_s \left(\frac{\omega_-}{\omega_a}\right)^2 \left[\frac{1 - p - \left(\frac{\omega_a}{\omega_+}\right)^2}{1 - \left(\frac{\omega_-}{\omega_+}\right)^2} \right] \approx (1-p) L_s$$

$$= \frac{LL_s}{L + L_s} = p(1-p)(L + L_s)$$

$$L_+ = L_S \left(\frac{\omega_a}{\ }\right) \left[\frac{\ }{1-\left(\frac{\omega_-}{\omega_+}\right)^2} \right]$$

$$= \frac{L L_S}{L + L_S} = p(1-p)(L+L_S)$$

$$C_- = C\left(\frac{\omega_a}{\omega_-}\right)^2 \frac{1}{p}\left[\frac{1-\left(\frac{\omega_-}{\omega_+}\right)^2}{1-\left(\frac{\omega_-}{\omega_a}\right)^2(1-p)}\right] \approx \frac{1}{p^2} C$$

$$C_+ = C_S \left(\frac{\Omega_s}{\omega_+}\right)^2 \left(\frac{\omega_a}{\omega_-}\right)^2 \left[\frac{1-\left(\frac{\omega_-}{\omega_+}\right)^2}{1-p-\left(\frac{\omega_a}{\omega_+}\right)^2}\right] \approx C_S$$

$$H_2 = \sum_{\sigma=\pm} 4 E C_\sigma N_\sigma^2 + \frac{1}{2} E L_\sigma \varphi_\sigma^2 + U_{NL}(\varphi_+ + \varphi_-)$$

$$\varphi_{-,zpf} = \sqrt{p}\sqrt{\frac{1-\left(\frac{\omega_-}{\omega_a}\right)^2(1-p)}{1-\left(\frac{\omega_-}{\omega_+}\right)^2}}\ \varphi_{zpf} \approx p\,\varphi_{zpf}$$

$$\varphi_{+,zpf} = \sqrt{p\frac{\omega_+}{\omega_a}}\left(\frac{\omega_-}{\omega_a}\right)\sqrt{\frac{1-p-\left(\frac{\omega_a}{\omega_+}\right)^2}{1-\left(\frac{\omega_-}{\omega_+}\right)^2}}\ \varphi_{zpf}$$

$$\approx \sqrt{p(1-p)\frac{\omega_+}{\omega_a}}\ \varphi_{zpf}$$

$$\varphi, \varphi_s \xrightarrow[\text{transformation}]{\text{canonical}} \varphi_+, \varphi_-$$

$$\varphi_- = \varphi_{-,zpf}(\hat{a}+\hat{a}^\dagger) \qquad [\hat{a},\hat{a}^\dagger]=1$$

$$\varphi_+ = \varphi_{+,zpf}(\hat{s}+\hat{s}^\dagger) \qquad [\hat{s},\hat{s}^\dagger]=1$$

$$H_2 = \hbar\omega_-\hat{a}^\dagger\hat{a} + \hbar\omega_+\hat{s}^\dagger\hat{s} + H_{NL}$$

$$H_{NL} = \frac{c_3}{3!}E_J\left(\varphi_{-,zpf}(\hat{a}+\hat{a}^\dagger)+\varphi_{+,zpf}(\hat{s}+\hat{s}^\dagger)\right)^3 + \frac{c_4}{4!}E_J\left(\varphi_{-,zpf}(\hat{a}+\hat{a}^\dagger)+\varphi_{+,zpf}(\hat{s}+\hat{s}^\dagger)\right)^4 + \dots$$

- truncation of the series (aka setting higher order terms to 0 requires $\varphi_{-,zpf} \ll 1$, $\varphi_{+,zpf} \ll 1$

$$H_{NL,eff} = \underbrace{K_{aa}}_{\text{eff. self kerr}}\hat{a}^{\dagger 2}\hat{a}^2 + \underbrace{K_{as}}_{\text{eff. cross-kerr}}\hat{a}^\dagger\hat{a}\,\hat{s}^\dagger\hat{s} + K_{ss}\hat{s}^{\dagger 2}\hat{s}^2 + \dots$$

Assume: $\langle s^\dagger s\rangle = 0$ in experiments because Ω_s large and the environment is cold

$\Downarrow$

ignore K_{ss}

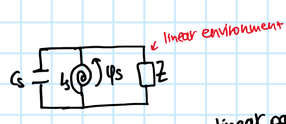
Then,
$$\boxed{2\hbar K_{aa} = \frac{p^3}{c_2}\left[c_4 - \frac{3c_3^2}{c_2}(1-p) - \frac{5}{3}\frac{c_3^2}{c_2}\right]E_J}$$

BBQ for SNAILs

Goal: given a JJ with linear environment and simulated impedance $Z(s)$, find the Hamiltonian [Nigg et al 2012]

OR

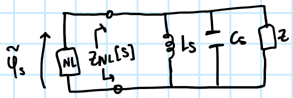
instead of $Z[s]$ which can be numerically tedious, use energy participation ratios BBQ [Miner et al 2021]

How
1) Consider 

2) Break this into linear and non-linear parts

How 1) Consider 

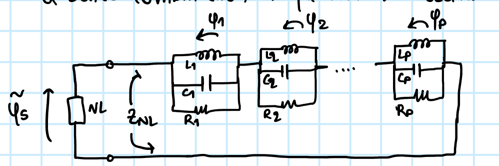
2) Break this into linear and non-linear parts



$$U_{NL}(\tilde{\varphi}_s) = U_s(\tilde{\varphi}_s + \varphi_{s\,min}) - \frac{C_2}{2} E_J \tilde{\varphi}_s^2$$

$$\tilde{\varphi}_s = \varphi_s - \varphi_{s\,min}$$

3) Use foster's theorem to decompose Z_{NL} into a series combination of parallel LCR oscillators



* requires high frequency cut-off Ω_{max} approximation

$P \leftarrow$ number of modes in the environment

$$Z_{NL}[s] = \sum_{p=1}^{P} \left(sC_p + \frac{1}{sL_p} + \frac{1}{R_p}\right)^{-1} = \sum_{p=1}^{P} Z_p \frac{\frac{s}{\omega_p}}{\left(\frac{s}{\omega_p}\right)^2 + \frac{s}{\omega_p} Q_p + 1}$$

pair of poles

Where $\omega_p = \frac{1}{\sqrt{L_p C_p}}$ → resonance freq.

$Z_p = \sqrt{\frac{L_p}{C_p}}$ → characteristic impedance

$Q_p = \frac{R_p}{Z_p}$ → quality factor

and $\tilde{\varphi}_s = \sum_{p=0}^{P} \varphi_p$

Note on Foster's theorem

- The reactance of a passive, loss-less two-terminal network always strictly increases monotonically with frequency.

↳ get a driving-point impedance by

① in series parallel LC circuits

② in parallel series LC circuits

Problem: the enviroment could have a lot (maybe even infinite) modes so

4) Introduce a high-frequency cutoff (Ω_{max})

I) Select modes into $P_{max} < P$ which will appear in final Hamiltonian each with $\omega_p < \Omega_{max}$

II) Summarize the impedance at low frequencies of the remaining $P - P_{max}$ modes as $\frac{sL_p}{1 + \frac{L_p s}{R_p}} \approx sL_p$

implies

$R_p \gg L_p \Omega_{max}$ or high Quality factor limit

$Q_p = \omega_p R_p C_p$ for parallel RLC

$\Rightarrow \frac{1}{R_p} = \frac{\omega_p C_p}{Q_p}$

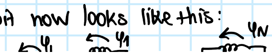
so $\frac{sL_p}{1 + L_p s \cancel{\frac{\omega_p C_p}{Q_p}}} \approx sL_p$

III) With the cutoff

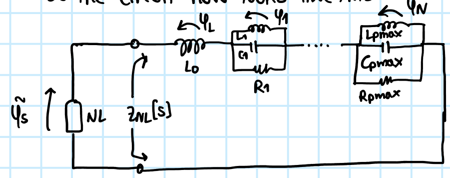
$$Z_{NL}[s] = sL_0 + \sum_{p=1}^{P_{max}} Z_p \frac{\frac{s}{\omega_p}}{\left(\frac{s}{\omega_p}\right)^2 + \frac{s}{\omega_p} Q_p + 1}$$

where $L_0 = \sum_{p=P_{max}+1}^{P} L_p = L_s - \sum_{p=1}^{P_{max}} L_p$

So the circuit now looks like this:



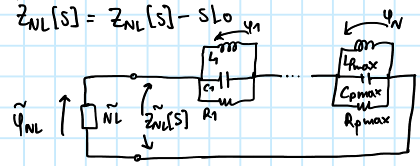
So the circuit now looks like this:



5) L_0 does not have it's own capacitance so it is not an independent degree of freedom

↳ Renormalize non-linearity by L_0

$$\tilde{Z}_{NL}[s] = Z_{NL}[s] - sL_0$$



$$\tilde{U}_{NL}(\tilde{\varphi}_{NL}, \varphi_s) = \frac{1}{2} E_{L0} (\tilde{\varphi}_{NL} - \tilde{\varphi}_s)^2 + U_{NL}(\tilde{\varphi}_s)$$

$$\tilde{\varphi}_{NL} = \tilde{\varphi}_s - \varphi_L$$

We eliminate $\tilde{\varphi}_s$ using current conservation at node between L_0 and nonlinearity.

$$0 = \frac{\partial \tilde{U}_{NL}}{\partial \tilde{\varphi}_s} = E_{L0}(\tilde{\varphi}_s - \tilde{\varphi}_{NL}) + \frac{dU_{NL}}{d\tilde{\varphi}_s}$$

Then, $$\mathcal{L}_{P_{max}} = \sum_{p=1}^{P_{max}} \left(\frac{C_p \varphi_0^2}{2} \dot{\varphi}_p^2 - \frac{1}{2} E_{L_p} \varphi_p^2 \right) - \tilde{U}_{NL}(\tilde{\varphi}_{NL})$$

$$\tilde{U}_{NL}(\tilde{\varphi}_{NL}) = \frac{1}{2} E_{L0} \left(\tilde{\varphi}_{NL} - \tilde{\varphi}_s[\tilde{\varphi}_{NL}]\right)^2 + U_{NL}(\tilde{\varphi}_s[\tilde{\varphi}_{NL}])$$

$$\tilde{\varphi}_{NL} = \sum_{p=1}^{P_{max}} \varphi_p$$

Kerr renormalization

- Taylor expand the full potential about its global minimum which we set at $\varphi_1 = \varphi_2 = \dots = \varphi_p = \dots = \varphi_{pmax} = 0$

So the coefficients are $\tilde{C}_k = \frac{1}{E_J} \left(\frac{d^k \tilde{U}_{NL}}{d\tilde{\varphi}_{NL}^k} \right) \Big|_{\tilde{\varphi}_{NL}=0}$ for $k > 2$

$$\tilde{C}_3 = -x_J \frac{d^2 \tilde{\varphi}_s}{d\tilde{\varphi}_{NL}^2}[0] \qquad x_J = \frac{L_J}{L_0}$$

$$\tilde{C}_4 = -x_J \frac{d^3 \tilde{\varphi}_s}{d\tilde{\varphi}_{NL}^3}[0] \qquad \frac{d\tilde{\varphi}_s}{d\tilde{\varphi}_{NL}} = \left(1 + \frac{1}{E_{L0}} \frac{d^2 U_{NL}}{d\tilde{\varphi}_s^2}\right)^{-1}$$

$$\tilde{C}_5 = -x_J \frac{d^4 \tilde{\varphi}_s}{d\tilde{\varphi}_{NL}^4}[0]$$

$$\tilde{C}_6 = -x_J \frac{d^5 \tilde{\varphi}_s}{d\tilde{\varphi}_{NL}^5}[0]$$

We define the participation ratio as $p = \frac{L_s}{L_s + L_0}$ and then

$$\tilde{C}_3 = C_3$$

$$\tilde{C}_4 = C_4 - 3C_3^2 \frac{E_J}{E_0} = C_4 - \frac{3C_3^2}{C_2}(1-p)$$

$$\tilde{C}_5 = C_5 - \frac{10\, C_4 C_3}{C_2}(1-p) + \frac{15\, C_3^3}{C_2^2}(1-p)^2$$

$$\tilde{C}_6 = C_6 - \frac{10\, C_4^2 + 15\, C_5 C_3}{C_2}(1-p) + \frac{10\, C_4 C_3^2}{C_2^2}(1-p)^2$$
$$- \frac{105\, C_3^4}{} (1-p)^3$$

$$\tilde{c}_6 = c_6 - \frac{10 c_4^2 + 15 c_5 c_3}{c_2}(1-p) + \frac{10 c_4 c_3^2}{c_2^2}(1-p)^2 - \frac{105 c_3^4}{c_2^3}(1-p)^3$$

Quantization

1) Legendre transformation on $\mathcal{L}_{p_{max}}$ to get the Hamiltonian

$$\mathcal{L}_{p_{max}} = \sum_{p=1}^{p_{max}} \left(\frac{C_p \varphi_0^2}{2} \dot{\varphi}_p^2 - \frac{1}{2} E_{Lp} \varphi_p^2 \right) - \tilde{U}_{NL}(\tilde{\varphi}_{NL})$$

$$\downarrow$$

$$H_{p_{max}} = \sum_{p=1}^{p_{max}} \left(4 E_c N_p^2 + \frac{1}{2} E_{Lp} \varphi_p^2 \right) + \tilde{U}_{NL}\left(\sum_{p=1}^{p_{max}} \varphi_p \right)$$

where $[\varphi_p, N_q] = i\delta_{p,q'}$

Note on Legendre transformation

- Used to convert functions of one quantity to functions of the conjugate quantity

In classical mechanics, we change variables from generalized velocities to generalized momenta

$$\mathcal{L}(q, \dot{q}) \Rightarrow L^*(q, p) = H(q, p)$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

2) Introduce the standard bosonic creation and annihilation operators defined as

$$\varphi_p = \varphi_{p,zpf}(\hat{a}_p + \hat{a}_p^+)$$

$$[\hat{a}_p, \hat{a}_q^+] = \delta_{p,q'}$$

$$\varphi_{zpf} = \left(\frac{2E_{cp}}{E_{Lp}} \right)^{1/4} \longrightarrow \text{zero point fluctuations of the phase for mode } p$$

so
$$\boxed{H_{p_{max}} = \sum_{p=1}^{p_{max}} \omega_p \hat{a}_p^+ \hat{a}_p + \tilde{H}_{NL}}$$

$$\boxed{\tilde{H}_{NL} = \frac{\tilde{c}_3}{3!} E_J \left(\sum_{p=1}^{p_{max}} \varphi_{p,zpf}(\hat{a}_p + \hat{a}_p^+) \right)^3 + \frac{\tilde{c}_4}{4!} E_J \left(\sum_{p=1}^{p_{max}} \varphi_{p,zpf}(\hat{a}_p + \hat{a}_p^+) \right)^4 + \dots}$$

which is truncated using $\varphi_{p,zpf} \ll 1$