Set Theory and Logic

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August 24, 2024

Contents

Conventions

$\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.
$\mathbb{N}$ denotes the set $\{1, 2, 3, \ldots\}$ of natural numbers (excluding 0).
Inner products are taken to be linear in the first argument and conjugate linear in the second.
The Einstein summation convention is used for tensors unless otherwise specified.

1 Language of set theory

1.1 Basic Set-Building Axioms

We construct a formal language suitable for describing sets. The language consists of some mathematical symbols as well as purely logical symbols.

The complete list of symbols of language is as below:

Definition 1.1. Symbols in LOST(Language of Set Theory)

1. variable: $v_0, v_1, v_2, \ldots$
2. equality: $=$
3. membership: $\in$
4. connectives: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$
5. quantifiers: $\forall, \exists$
6. parentheses: $($, $)$

Remarks. Bounded set quantifiers shall be used. You can abbreviate the formula:

1. $\forall x(x \in y \Rightarrow x \notin a)$ by $(\forall x \in y)(x \notin a)$.
2. $\exists x(x \in y \wedge x \notin a)$ by $(\exists x \in y)(x \notin a)$.

Definition 1.2. Zermelo-Fraenkel Axioms

1. **Extensionality Axiom.**
   Two sets are equal iff they have the same elements.

   $$\forall A \forall B(A = B \Leftrightarrow \forall x(x \in A \Leftrightarrow x \in B))$$

2. **Empty Set Axiom.**
   There is a set with no elements.

   $$\exists A \forall x(x \notin A)$$

3. **Subset Axiom.**
   Let $\varphi(x)$ be a formula. For every set A there exists a set S that consists of all $x \in A$ with $\varphi(x)$ holds.

   $$\forall A \exists S \forall x(x \in S \Leftrightarrow (x \in A \wedge \varphi(x)))$$

4. **Pairing Axiom.**
   For every u and v there is a set that consists of just u and v.

   $$\forall u \forall v \exists S \forall x(x \in S \Leftrightarrow (x = u \vee x = v))$$

5. **Union Axiom.**
   For every set $\mathcal{F}$ there exists a set U that consists of all elements that belong to at least one set in $\mathcal{F}$.

   $$\forall \mathcal{F} \exists U \forall x(x \in U \Leftrightarrow \exists F \in \mathcal{F}(x \in F))$$

6. **Power Set Axiom.**
   For every set A there is a set $\mathcal{P}$ that consists of all subsets of A.

   $$\forall A \exists \mathcal{P} \forall P(P \in \mathcal{P} \Leftrightarrow P \subseteq A)$$

Remarks.

1. The empty set axiom defines the **empty set** denoted by $\varnothing$.
2. The subset axiom defines the set denoted by $\{x \in A : \varphi(x)\}$.
3. The pairing axiom defines the **unordered pair** denoted by $\{u, v\}$. If $u = v$, then the set $\{u\}$ is referred to as a **singleton**.
4. The union axiom defines the **union** of $\mathcal{F}$ denoted by $\bigcup \mathcal{F}$.
5. The power set axiom defines the **power set** of A denoted by $\mathcal{P}(A) = \{X : X \subseteq A\}$.

Definition 1.3. Class

We shall refer to any collection of the form $\{x : \varphi(x)\}$ as a **class**. We call it **proper class**, when the class is not a set, such as $\{x : x = x\}$. Sometimes we also call it **unbounded collection.**

Theorem 1.4. Sufficient condition for class to be a set

Let $\varphi(x)$ be a formula. Suppose that there is a set A such that for every x, if $\varphi(x)$, then $x \in A$. Then there is a unique set S such that for all x, $x \in S \Leftrightarrow \varphi(x)$.

$$\exists A \forall x(\varphi(x) \Rightarrow x \in A) \Rightarrow \exists! S \forall x(x \in S \Leftrightarrow \varphi(x))$$

In other words, the class $\{x : \varphi(x)\}$ is equal to the set S.

Proof. Let $S = \{x \in A \colon \varphi(x)\}$ which is uniquely defined by subset axiom.
$(\Rightarrow)$
$x \in S \Rightarrow x \in A \wedge \varphi(x) \Rightarrow \varphi(x)$.
$(\Leftarrow)$
$\varphi(x) \Rightarrow x \in A \Rightarrow x \in A \wedge \varphi(x) \Rightarrow x \in S$. 😎

Remarks. Furthermore, the sufficient and necessary condition for class to be a set is as follows:

$$\exists S \forall x(x \in S \Leftrightarrow \varphi(x)) \Leftrightarrow \exists A \forall x(\varphi(x) \Rightarrow x \in A)$$

Corollary 1.5

Let $\varphi(x)$ be a formula. Then $\{x : \varphi(x)\}$ is a proper class iff for every set A there is a x s.t. $\varphi(x)$ and $x \notin A$.

By the Theorem 1.4, Some sets operations below are well defined and can be used to create new sets.

Corollary 1.6

Union, Intersection and Difference of sets can be defined by Theorem 1.4. For any set A, B and nonempty collection $\mathcal{F}$:

1. Union: $A \cup B = \{x : x \in A \vee x \in B\}$
2. Intersection: $A \cap B = \{x : x \in A \wedge x \in B\}$
3. Difference: $A \setminus B = \{x : x \in A \wedge x \notin B\}$
4. Union of $\mathcal{F}$: $\bigcup \mathcal{F} = \{x : \exists F \in \mathcal{F}(x \in F)\}$
5. Intersection of $\mathcal{F}$: $\bigcap \mathcal{F} = \{x : \forall F \in \mathcal{F}(x \in F)\}$

Remark. If collection $\mathcal{F}$ is empty, then

1. $\bigcup \varnothing = \varnothing$
2. $\bigcap \varnothing$ is not well defined as set. Because the statement x belongs to every members of empty set is vacuously true. So all x will belongs to it, which is a contradiction.

Theorem 1.7

Suppose nonempty sets $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F} \subseteq \mathcal{G}$. Then $\bigcup \mathcal{F} \subseteq \bigcup \mathcal{G}$ and $\bigcap \mathcal{G} \subseteq \bigcap \mathcal{F}$.

1.2 Operations on Sets

There are some other important ways to build new sets by axioms in previous section.

Lemma 1.8

Let A and $\mathcal{F}$ be sets. Then there is a unique set $\mathcal{C}$ such that for all X,

$$X \in \mathcal{C} \Leftrightarrow X = A \setminus F \text{ for some } F \in \mathcal{F}$$

or,

$$X \in \mathcal{C} \Leftrightarrow \exists F \in \mathcal{F}(X = A \setminus F)$$

We shall denote $\mathcal{C}$ by $\{A \setminus F : F \in \mathcal{F}\}$.

Proof. $\exists F \in \mathcal{F}(X = A \setminus F) \Rightarrow X \subseteq A \Rightarrow X \in \mathcal{P}(A)$
by Theorem 1.4, set $\mathcal{C}$ is uniquely constructed.

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Theorem 1.9. De Morgan's Laws

If A is a set and $\mathcal{F}$, then

$$A \setminus \bigcup \mathcal{F} = \bigcap \{A \setminus F : F \in \mathcal{F}\}$$
$$A \setminus \bigcap \mathcal{F} = \bigcup \{A \setminus F : F \in \mathcal{F}\}$$

Proof. Proof the first one.
($\Rightarrow$)

$$\begin{aligned} x \in A \setminus \bigcup \mathcal{F} &\Rightarrow x \in A \wedge x \notin \bigcup \mathcal{F} \\ &\Rightarrow x \in A \wedge x \notin F \text{ for all } F \in \mathcal{F} \\ &\Rightarrow x \in A \setminus F \text{ for all } F \in \mathcal{F} \\ &\Rightarrow x \in \bigcap \{A \setminus F : F \in \mathcal{F}\} \end{aligned}$$

($\Leftarrow$)

$$\begin{aligned} & x \in \bigcap \{A \setminus F : F \in \mathcal{F}\} \\ & \Rightarrow x \in A \setminus F \text{ for all } F \in \mathcal{F} \\ & \Rightarrow x \in A \wedge (x \notin F \text{ for all } F \in \mathcal{F}) \\ & \Rightarrow x \in A \wedge x \notin \bigcup \mathcal{F} \\ & \Rightarrow x \in A \setminus \bigcup \mathcal{F} \end{aligned}$$

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2 Relations and Functions