Set Theory and Logic

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Conventions

$\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.
$\mathbb{N}$ denotes the set $\{1, 2, 3, \ldots\}$ of natural numbers (excluding 0).
Inner products are taken to be linear in the first argument and conjugate linear in the second.
The Einstein summation convention is used for tensors unless otherwise specified.

1 Language of set theory

We construct a formal language suitable for describing sets. The language consists of some mathematical symbols as well as purely logical symbols.

The complete list of symbols of language is as below:

Definition 1.1. Symbols in LOST(Language of Set Theory)

1. variable: $v_0, v_1, v_2, \ldots$
2. equality: $=$
3. membership: $\in$
4. connectives: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$
5. quantifiers: $\forall, \exists$
6. parentheses: (,)

Remark. Bounded set quantifiers shall be used. Thus, we can abbreviate the formula
$\forall x(x \in y \Rightarrow x \notin a)$ by $(\forall x \in y)(x \notin a)$.
$\exists x(x \in y \wedge x \notin a)$ by $(\exists x \in y)(x \notin a)$.

Definition 1.2. Zermelo-Fraenkel Axioms

1. **Extensionality Axiom.**
   Two sets are equal iff they have the same elements.

   $\forall A \forall B\,(A = B \Leftrightarrow \forall x\,(x \in A \Leftrightarrow x \in B))$

2. **Empty Set Axiom.**
   There is a set with no elements.

   $\exists A \forall x(x \notin A)$

3. **Subset Axiom.**
   Let $\varphi(x)$ be a formula. For every set A there exists a set S that consists of all $x \in A$ with $\varphi(x)$ holds.

   $\forall A \exists S \forall x(x \in S \Leftrightarrow (x \in A \wedge \varphi(x)))$

4. **Pairing Axiom.**
   For every u and v there is a set that consists of just u and v.

   $\forall u \forall v \exists S \forall x(x \in S \Leftrightarrow (x = u \vee x = v))$

5. **Union Axiom.**
   For every set $\mathcal{F}$ there exists a set U that consists of all elements that belong to at least one set in $\mathcal{F}$.

   $\forall \mathcal{F} \exists U \forall x(x \in U \Leftrightarrow \exists F(x \in F \wedge F \in \mathcal{F}))$

6. **Power Set Axiom.**
   For every set A there is a set P that consists of all subsets of A.

   $\forall A \exists \mathcal{P} \forall P(P \in \mathcal{P} \Leftrightarrow P \subseteq A)$

Remark. All sets guaranteed by axioms 2-6 are unique.

1. The empty set axiom defines the empty set denoted by $\varnothing$.
2. The subset axiom defines the set denoted by $\{x \in A : \varphi(x)\}$.
3. The pairing axiom defines the unordered pair set denoted by $\{u, v\}$.
4. The union axiom defines the union of $\mathcal{F}$ denoted by $\cup\mathcal{F}$.

5. The power set axiom defines the power set denoted by $\mathcal{P}(A) = \{X : X \subseteq A\}$.

Definition 1.3. Class

We shall refer to any collection of the form $\{x : \varphi(x)\}$ as a **class**. We call it **proper class**, when the class is not a set, such as $\{x : x = x\}$. Sometimes we also call it unbounded collection.

Lemma 1.4. Condition for class to be a set

Let $\varphi(x)$ be a formula. Suppose that there is a set A s.t. for every x, if $\varphi(x)$, then $x \in A$.

Then there is a unique set S s.t. for all x, $x \in S \Leftrightarrow \varphi(x)$. In other words, the class $\{x : \varphi(x)\}$ is equal to the set S.

Proof. Let $S = \{x \in A \colon \varphi(x)\}$ which is uniquely defined by subset axiom.
$(\Rightarrow)$
$x \in S \Rightarrow x \in A \wedge \varphi(x) \Rightarrow \varphi(x)$.
$(\Leftarrow)$
$\varphi(x) \Rightarrow x \in A \Rightarrow x \in A \wedge \varphi(x)$.
$x \in S$ sunglasses

Corollary 1.5

1. Intersection: $A \cap B = \{x : x \in A \wedge x \in B\}$

2. Difference: $A \setminus B = \{x : x \in A \wedge x \notin B\}$

By the previous theorem, these set operations are well defined and create new sets.