Topology

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Conventions

$\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.
$\mathbb{N}$ denotes the set $\{1, 2, 3, \dots\}$ of natural numbers (excluding 0).
Inner products are taken to be linear in the first argument and conjugate linear in the second.
The Einstein summation convention is used for tensors unless otherwise specified.

1 Topological Spaces

Definition 1.1. Topological Spaces

Let X be a non-empty set and τ is a collection of subsets U of X. If

1. $\varnothing, X \in \tau$
2. $\bigcup_\alpha U_\alpha \in \tau$
3. $\bigcap_i U_i \in \tau$

Then we refer to τ as the **topology** on X, and the pair (X, τ) is called a **topological space**. For simplicity, we may just refer to X itself as the topological space.

Remarks. τ is a subset of $\mathcal{P}(X)$, and we call the subset U in τ the **open set**. So the three conditions above can be reclaimed by

1. $\varnothing$, X are open.
2. arbitrary union of open sets is open.
3. finite intersection of open sets is open.

Definition 1.2. Discrete and Indiscrete Topology

Let X be any non-empty set and τ be the collection of subsets of X.

1. If $\tau = \mathcal{P}(X)$, then τ is called the **discrete topology**, and X is called a **discrete space**.
2. If $\tau = \{\varnothing, X\}$, then τ is called the **indiscrete topology**, and X is called a **indiscrete space**.

Proposition 1.3

Suppose X is a topological space. If for every $x \in X$, the singleton $\{x\} \in \tau$, then τ is the discrete topology.

Proof. Since $\tau \subseteq \mathcal{P}(X)$, we only need to show $\mathcal{P}(X) \subseteq \tau$.
Take any subset $U \in \mathcal{P}(X)$, then U can be expressed as

$$U = \bigcup_{x \in S} \{x\} \tag{1}$$

Since every singletons are in τ, so U is also in τ. Thus $\mathcal{P}(X) \subseteq \tau$.

Definition 1.4. Closed Set

Let X be a topological space. A subset S of X is said to be a **closed set**, if its complement, namely $X \setminus S$, is open in X.

Proposition 1.5

Let X be a topological space, then

1. $\varnothing$, X are closed.
2. finite union of closed sets is closed.
3. arbitrary intersection of closed sets is closed.

Definition 1.6. Clopen

A subset S of topological space X is called clopen if it is both open and closed.

Remarks.
1. In every topological space, both $\varnothing$ and X are clopen.
2. In the discrete space, all subsets of X are clopen.
3. In the indiscrete space, the only clopen sets are $\varnothing$ and X.

Definition 1.7. Cofinite Topology

Let X be nonempty set. A topology τ on X is called the **finite-closed topology** or the **cofinite topology** if the closed subsets are X and all finite subsets of X. In other words, the open sets are $\varnothing$ and all subsets which have finite complements.