KEYPOINT SUMMARY

Note: This summary is neither exhaustive nor formal. It is meant to give the intuition behind the mathmatical concepts, and extract core idea for essential proofs.

I. CARDINALITY

1. A is a set, $\overline{\overline{A}}$ denote the cardinality of A.
   - $\overline{\overline{A}} = \overline{\overline{B}}$ if there exists a bijection $A \sim B$.
   - $\overline{\overline{A}} \le \overline{\overline{B}}$ if there exists an injection $A \hookrightarrow B$.
   - $\overline{\overline{\emptyset}} = 0$, $\overline{\overline{\{1, 2, \dots, n\}}} = n$, $\overline{\overline{\mathbb{N}}} = d$, $\overline{\overline{\mathbb{R}}} = c$.
2. Schröder-Bernstein Theorem:
   - $A \hookrightarrow B,\ B \hookrightarrow A \Rightarrow A \sim B$
   - $\overline{\overline{A}} \le \overline{\overline{B}},\ \overline{\overline{B}} \le \overline{\overline{A}} \Rightarrow \overline{\overline{A}} = \overline{\overline{B}}$
3. A is finite/denumerable/countable:

$$\left.\begin{array}{r} \left.\begin{array}{r} A \sim \emptyset \\ A \sim \{1, 2, \dots, n\} \end{array}\right\} \text{finite} \\ A \sim N \qquad \text{denumerable} \end{array}\right\} \text{countable}$$

   Otherwise, A is uncountable.
4. Important results on cardinality of sets:
   - $\mathbb{N}^k \sim \mathbb{N}$

     Proof. Consider $f(n_1, \dots, n_k) = p_1^{n_1} \cdots p_k^{n_k}$.
   - $\mathbb{N}^{\mathbb{N}} \sim \mathbb{R}$

     Proof. Code $\mathbb{N}^{\mathbb{N}}$ into binary strings and show it is uncountable. Consider $f(n_1, n_2, \dots) = \frac{1}{10^{n_1}} + \frac{1}{10^{n_1+n_2}} + \cdots$.
   - $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$

     Proof. $\mathcal{P}(\mathbb{N}) \to \mathbb{R}$: $(n_1, n_2, n_3, \dots) \to 0.b_1b_2b_3\dots$.
     $\mathbb{R} \to \mathcal{P}(\mathbb{Q})$: $\{a \in \mathbb{Q} : a < x, x \in \mathbb{R}\}$.
   - $\mathbb{R} \times \mathbb{R} \sim R$, $\mathbb{R}^k \sim \mathbb{R}$

     Proof. $(0.a_1a_2\dots, 0.b_1b_2\dots) \to (0.a_1b_1a_2b_2\dots)$.
   - The set of all real valued functions on $[0, 1] \sim 2^c$

     Proof. Note $A \hookrightarrow \mathcal{P}([0, 1] \times \mathbb{R})$.
   - A is any set, $\overline{\overline{A}} < \overline{\overline{\mathcal{P}(A)}}$.

     Proof. Clearly there exists $f : A \hookrightarrow \mathcal{P}(A)$. But f cannot be surjective. Consider $X = \{a \in A : a \notin f(a)\}$.
   - The union of a countable family of countable sets is countable.
   - The union of a cardinality c family of sets each with cardinality c has cardinality c.

     Proof. Consider $\{A_\alpha\}_{\alpha \in S}$. There exists a bijection $f_\alpha : A_\alpha \leftrightarrow \mathbb{R}$. Define $f : A \hookrightarrow S \times \mathbb{R}$ as $f(x) = (\alpha, f_\alpha(x))$ where $x \in A_\alpha$.

II. VECTOR SPACES

1. A vector space (linear space) over $\mathbb{R}$ is a set V with two operations *addition* and *scalar multiplication* such that
   (a) $u + v = v + u$
   (b) $(u + v) + w = u + (v + w)$
   (c) $\exists 0 \in V$, $0 + v = v$
   (d) $(\alpha + \beta)u = \alpha u + \beta u$
   $\alpha(u + v) = \alpha u + \alpha v$
   (e) $(\alpha\beta)u = \alpha(\beta u)$
   (f) $1u = u$

   where $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$.
   - A vector space is a space closed under addition and scalar multiplication, i.e. it is a space that allows linear operations.
   - A set of vectors $\{v_1, v_2, \dots, v_n\}$ in V is called linearly independent if $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \Rightarrow a_1 = a_2 = \cdots = a_n = 0$.
2. A normed vector space is a vector space V over $\mathbb{R}$ with a function $\|\cdot\| : V \to \mathbb{R}$ such that
   (a) $\|u\| \ge 0$, $\|u\| = 0$ iff $u = 0$
   (b) $\|\alpha u\| = |\alpha| \, \|u\|$
   (c) $\|u + v\| \le \|u\| + \|v\|$
   - A normed vector space is a vector space where the length of vectors can be measured.
   - Euclidean norm: $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$
   - Infinity norm: $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$
   - p-norm: $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$

     Proof. Minkowski's inequality:
     $\left(\sum_k^n |x_k + y_k|^p\right)^{\frac{1}{p}} \le \left(\sum_k^n |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_k^n |y_k|^p\right)^{\frac{1}{p}}$
   - If $1 \le p \le q \le \infty$, then $\|x\|_p \ge \|x\|_q$.

     Proof. Normalize x to $\frac{x}{\|x\|_p}$ so that $\|x\|_p = 1$. Then it is easy to see $\|x\|_q \le 1$ because for each element $|x_i|^q \le |x_i|^p$.
3. An inner product space is a vector space V over $\mathbb{R}$ with a function $\cdot : V \times V \to \mathbb{R}$ such that
   (a) $u \cdot u \ge 0$, $u \cdot u = 0$ iff $u = 0$
   (b) $u \cdot v = v \cdot u$
   (c) $(u + v) \cdot w = u \cdot w + v \cdot w$
   (d) $(\alpha u) \cdot v = \alpha(u \cdot v)$
   - Angle θ between u, v: $u \cdot v = \cos\theta \, \|u\| \, \|v\|$.
   - u, v are orthogonal if $u \cdot v = 0$.
   - Every inner product space is a normed space if define $\|u\| = (u \cdot u)^{\frac{1}{2}}$ as the norm.
   - Cauchy-Schwarz Inequality: $|u \cdot v| \le \|u\| \, \|v\|$

     Proof. $f(\lambda) = (u - \lambda v) \cdot (u - \lambda v) \ge 0$, $\forall \lambda$.
     Substitute in $\lambda = \frac{u \cdot v}{\|v\|^2}$.

III. METRIC SPACES

1. A metric space (X, d) is a set X together with a function $d : X \times X \to \mathbb{R}$ such that
   (a) $d(x, y) \geq 0$, $d = 0$ iff $x = y$
   (b) $d(y, x) = d(x, y)$
   (c) $d(x, y) \leq d(x, z) + d(z, y)$
   - A metric space is a set for which distances between all members of the set are defined.
   - Any normed space $(V, \|\cdot\|)$ is a metric space.
2. Suppose (X, d) is a metric space, and $S \subset X$. (S, d_S) is a metric subspace if we define $d_S(x, y) = d(x, y)$ for $x, y \in S$.
   - $\forall a \in S$, $B_r^S(a) = S \cap B_r^X(a)$
   - A is open in $S \Leftrightarrow A = S \cap U$, U is open in X;
     A is closed in $S \Leftrightarrow A = S \cap C$, C is closed in X.
     Proof. A open in $S \Rightarrow A = \cup_{x \in A} B_{r_x}^S(x) = \cup_{x \in A}(S \cap B_{r_x}(x)) = S \cap (\cup_{x \in A} B_{r_x}(x))$. Thus $U = \cup_{x \in A} B_{r_x}(x)$.
3. Limit and Isolated Points
   - x is a limit point in A if every $B_r(x)$ contains points of A other than x.
   - x is a limit point iff $\exists (x_n) \subset A$ and $x_n \to x$.
   - x is a isolated point if $\exists r$ such that $B_r(x) \cap A = \{x\}$.
4. Interior, Exterior and Boundary
   - $x \in \operatorname{int} A$ if $\exists r (B_r(x) \subset A)$
   - $x \in \operatorname{ext} A$ if $\exists r (B_r(x) \subset A^C)$
   - $\operatorname{ext} A = \operatorname{int} A^C$, $\operatorname{int} A = \operatorname{ext} A^C$
   - $x \in \partial A$ (boundary of A) if any $B_r(x)$ contains both points of A and points of A^C.
   - $X = \operatorname{int} A \cup \operatorname{ext} A \cup \partial A$
5. Open Sets
   - A is open if $A = \operatorname{int} A$.
   - A is open $\Rightarrow A = \cup_{x \in A} B_{r_x}(x)$
   - If A_i are open, $\cap_{i=1}^k A_i$ is open;
     If A_i are open, $\cup_{i \in I} A_i$ is open.
   - $\operatorname{int} A$ is open; $\operatorname{ext} A$ is open.
6. Closed Sets
   - A is closed if A^C is open.
   - A is closed iff $\overline{A} = A$.
   - A is closed iff A contains all its limit points.
   - $A \subset R^k$ is closed iff A is complete.
   - If B_i are closed, $\cup_{i=1}^k B_i$ is closed;
     If B_i are closed, $\cap_{i \in I} B_i$ is closed.
   - Closure of a set is closed.
   - Closed does *not* imply bounded.
7. Closure
   - $\overline{A} = A \cup \{\text{limit point of } A\}$
   - $\overline{A} = \operatorname{int} A \cup \partial A$
   - $\overline{A} = A \cup \partial A$
   - $\overline{A} = (\operatorname{ext} A)^C$
   - $x \in \overline{A}$ iff every $B_r(x)$ contains a point of A.
   - $x \in \overline{A}$ iff there exists $(x_n) \subset A$ with $x_n \to x$.

IV. SEQUENCES AND CONVERGENCE

1. (x_n) converges to x if $\forall \epsilon, \exists N, [\forall n > N \Rightarrow d(x, x_n) < \epsilon]$.
2. (x_n) is Cauchy if $\forall \epsilon, \exists N, [\forall m, n > N \Rightarrow d(x_m, x_n) < \epsilon]$.
   - Every convergent/Cauchy sequence is bounded.
     Proof. convergence $\Rightarrow$ (x_n) is bounded after some N, left only finite elements.
   - $\mathbb{R}^k$: sequence (x_n) converges $\Leftrightarrow$ (x_n) is Cauchy.
   - X: sequence (x_n) converges $\not\Leftarrow\!\!\!\Rightarrow$ (x_n) is Cauchy.
3. A metric space is complete if every Cauchy sequence converges in itself.
   - $S \subset \mathbb{R}^k$ is complete *iff* it is closed.
4. Monotone Convergence Theorem: if a sequence is increasing (decreasing) and bounded by a supremum (infimum), it will converge to the supremum (infimum).
   Proof. Let $c = \sup_n\{a_n\}$. $\forall \epsilon > 0, \exists N$ s.t. $c - \epsilon < a_N \leq a_n \leq c$, $\forall n > N$. As $\epsilon \to 0$, $a_n \to c$.
5. Banach Fixed-Point Theorem: If (X, d) is a *complete* metric space, and $f : X \to X$ is a *contraction*, i.e. $\exists \lambda \in [0, 1)$ such that $d(f(x), f(y)) \leq \lambda d(x, y)$, then there exists a unique fixed point $f(x) = x$.
   Proof. First show (x_n) is Cauchy, then prove $d(x, f(x)) \to 0$.

V. SEQUENCES AND COMPACTNESS

1. Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.
   Proof. Geometric intuition: if a sequence in $\mathbb{R}^k$ is bounded, we can always trap it in a whatever small subspace. Construct a convergent subsequence by trapping it a smaller and smaller subspace.
2. A metric space is (sequentially) compact if every sequence has a convergent subsequence.
   - $\mathbb{R}^k$: compact $\Leftrightarrow$ closed, bounded (Beine-Borel Thm)
   - X: compact $\not\Leftarrow\!\!\!\Rightarrow$ closed, bounded.
     Proof. Consider $X = C[0, 1]$. $\overline{B_1(\mathbf{0})} \subset X$ is closed and bounded, but *not* compact.
   - Compactness is sort of a topological generalization of finiteness. For example, if a set A is finite then every function $f : A \to \mathbb{R}$ is bounded and has max/min. If A is compact, the every *continuous* function $f : A \to \mathbb{R}$ is bounded and has max/min.

VI. LIMITS AND CONTINUITY

1. Let $f : X \to Y$. The following are equivalent:
   (a) $\lim_{x \to a} f(x) = b$;
   (b) $x_n \to a \Rightarrow f(x_n) \to b$;
   (c) $x \in B_\delta^X(a) \setminus \{a\} \Rightarrow f(x) \in B_\epsilon^Y(b)$.

Proof. (ii)⇒(iii): Suppose the opposite. Let $x_n \in B^X_{1/n}(a)$, then $x_n \to a$ and $f(x_n) \to b$, but by assumption $\exists\epsilon$ s.t. $f(x_n) \notin B^Y_\epsilon(b)$. contradiction. (iii)⇒(ii): Suppose $x_n \to a$. (iii) $\Rightarrow \forall\epsilon\ \exists\delta\ \exists x_n \in B^X_\delta(a) \Rightarrow f(x_n) \in B^Y_\epsilon(b) \Rightarrow f(x_n) \to b$.

2. $f : X \to Y$ is continuous at $a \in X$ if a is an isolated point, or $\lim_{x\to a} f(x) = f(a)$.
   - Every function is continuous at isolated points.
   - Intuitively, a continuous function is a function for which sufficiently small changes in the input result in arbitrarily small changes in the output.
3. Quantitative Meansures of Continuity
   - $f : X \to Y$ is Lipschitz continuous if there exists a constant M such that $d_Y(f(x_1), f(x_2)) \leq M d_X(x_1, x_2)$. M is called the Lipschitz constant.
   - $f : X \to Y$ is Hölder continuous with exponent $\alpha \in (0, 1]$ if there exists a constant M such that $d_Y(f(x_1), f(x_2)) \leq M\,[d_X(x_1, x_2)]^\alpha$.
4. Continuity and Compactness: $f : X \to Y$ is continuous, if $K \subset X$ is compact, then $f(K)$ is compact in Y.

   Proof. $\forall(y_n) \subset f(K)$, $\exists(x_n) : f(x_n) = y_n$. K compact $\Rightarrow \exists(x_{n_j}) \to x$; f continuous $\Rightarrow f(x_{n_j}) \to f(x) \Rightarrow y_{n_j} \to f(x)$.

   Cor. Continuous function on a compact set is bounded.
5. Extreme Value Theorem: Each continuous function on a compact set attains its maximum and minimum.

   Proof. K compact $\Rightarrow f(K)$ compact $\Rightarrow f(K)$ closed and bounded ⇒ exist least upper bound γ and $\gamma \in f(K)$ (take a sequence approaching γ and extract its convergent subsequence). Therefore, $\exists x_0 \in K$ s.t. $\gamma = f(x_0)$.
6. Continuity and Open Sets: The following statements are equivalent:

   (a) $f : X \to Y$ is continuous on X;

   (b) $f^{-1}(E)$ is open whenever E is an open set in Y;

   (c) $f^{-1}(E)$ is closed whenever E is a closed set in Y.

   Proof. f continuous $\Rightarrow \forall\epsilon\ \exists\delta\ f(B_\delta(x)) \subset B_\epsilon(f(x)) \subset E \Rightarrow B_\delta(x) \subset f^{-1}(E)$. On the other hand, $f^{-1}(B_\epsilon(f(x)))$ is an open set $\Rightarrow \exists B_\delta(x) \subset f^{-1}(B_\epsilon(f(x))) \Rightarrow f(B_\delta(x)) \subset B_\epsilon(f(x)) \Rightarrow f$ is continuous.

   Note. Continuous functions do *not* necessarily map open sets to open sets, or closed sets to closed sets.

   Cor. If $f : X \to \mathbb{R}$ is continuous, $\{x : f(x) < 0\}$ is open.
7. The function $f : X \to Y$ is uniformly continuous if for each $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \epsilon$, forall $x, x' \in X$.
   - In general continuity, δ depends on both ϵ and x.
   - In uniform continuity, δ depends only on ϵ, not on x.
   - If f is continuous and X is *compact*, then f is uniformly continuous.
   - A continuous real-valued functions defined on closed and bounded subset of $\mathbb{R}^n$ is uniformly continuous.
8. Uniform continuity and Cauchy sequences: If $f : X \to Y$ is uniformly continuous, then (x_n) is Cauchy in $X \Rightarrow (f(x_n))$ is Cauchy in Y. If f is only continuous, $(f(x_n))$ may not converge (consider $f(x) = \frac{1}{x}$ on $(0, \infty)$).