

KEYPOINT SUMMARY

Note: This summary is neither exhaustive nor formal. It is meant to give the intuition behind the mathematical concepts, and extract core idea for essential proofs.

I. CARDINALITY

1. A is a set, $\overline{\overline{A}}$ denote the cardinality of A .
 - $\overline{\overline{A}} = \overline{\overline{B}}$ if there exists a bijection $A \sim B$.
 - $\overline{\overline{A}} \leq \overline{\overline{B}}$ if there exists an injection $A \hookrightarrow B$.
 - $\overline{\overline{\emptyset}} = 0$, $\overline{\overline{\{1, 2, \dots, n\}}} = n$, $\overline{\overline{\mathbb{N}}} = d$, $\overline{\overline{\mathbb{R}}} = c$.
2. Schröder-Bernstein Theorem:
 - $A \hookrightarrow B$, $B \hookrightarrow A \Rightarrow A \sim B$
 - $\overline{\overline{A}} \leq \overline{\overline{B}}$, $\overline{\overline{B}} \leq \overline{\overline{A}} \Rightarrow \overline{\overline{A}} = \overline{\overline{B}}$
3. A is finite/denumerable/countable:

$$\left. \begin{array}{l} A \sim \emptyset \\ A \sim \{1, 2, \dots, n\} \\ A \sim \mathbb{N} \end{array} \right\} \begin{array}{l} \text{finite} \\ \text{denumerable} \end{array} \Bigg\} \text{countable}$$

Otherwise, A is uncountable.

4. Important results on cardinality of sets:
 - $\mathbb{N}^k \sim \mathbb{N}$
Proof. Consider $f(n_1, \dots, n_k) = p_1^{n_1} \cdots p_k^{n_k}$.
 - $\mathbb{N}^{\mathbb{N}} \sim \mathbb{R}$
Proof. Code $\mathbb{N}^{\mathbb{N}}$ into binary strings and show it is uncountable. Consider $f(n_1, n_2, \dots) = \frac{1}{10^{n_1}} + \frac{1}{10^{n_1+n_2}} + \dots$.
 - $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$
Proof. $\mathcal{P}(\mathbb{N}) \rightarrow \mathbb{R}: (n_1, n_2, n_3, \dots) \rightarrow 0.b_1b_2b_3\dots$
 $\mathbb{R} \rightarrow \mathcal{P}(\mathbb{Q}): \{a \in \mathbb{Q} : a < x, x \in \mathbb{R}\}$.
 - $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$, $\mathbb{R}^k \sim \mathbb{R}$
Proof. $(0.a_1a_2\dots, 0.b_1b_2\dots) \rightarrow (0.a_1b_1a_2b_2\dots)$.
 - The set of all real valued functions on $[0, 1] \sim 2^c$
Proof. Note $A \hookrightarrow \mathcal{P}([0, 1] \times \mathbb{R})$.
 - A is any set, $\overline{\overline{A}} < \overline{\overline{\mathcal{P}(A)}}$.
Proof. Clearly there exists $f : A \hookrightarrow \mathcal{P}(A)$. But f cannot be surjective. Consider $X = \{a \in A : a \notin f(a)\}$.
 - The union of a countable family of countable sets is countable.
 - The union of a cardinality c family of sets each with cardinality c has cardinality c .
Proof. Consider $\{A_\alpha\}_{\alpha \in S}$. There exists a bijection $f_\alpha : A_\alpha \leftrightarrow \mathbb{R}$. Define $f : A \hookrightarrow S \times \mathbb{R}$ as $f(x) = (\alpha, f_\alpha(x))$ where $x \in A_\alpha$.

II. VECTOR SPACES

1. A vector space (linear space) over \mathbb{R} is a set V with two operations *addition* and *scalar multiplication* such that

- (a) $u + v = v + u$
- (b) $(u + v) + w = u + (v + w)$
- (c) $\exists 0 \in V$, $0 + v = v$
- (d) $(\alpha + \beta)u = \alpha u + \beta u$
 $\alpha(u + v) = \alpha u + \alpha v$
- (e) $(\alpha\beta)u = \alpha(\beta u)$
- (f) $1u = u$

where $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$.

- A vector space is a space closed under addition and scalar multiplication, i.e. it is a space that allows linear operations.
 - A set of vectors $\{v_1, v_2, \dots, v_n\}$ in V is called linearly independent if $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0 \Rightarrow a_1 = a_2 = \dots = a_n = 0$.
2. A normed vector space is a vector space V over \mathbb{R} with a function $\|\cdot\| : V \rightarrow \mathbb{R}$ such that
 - (a) $\|u\| \geq 0$, $\|u\| = 0$ iff $u = 0$
 - (b) $\|\alpha u\| = |\alpha| \|u\|$
 - (c) $\|u + v\| \leq \|u\| + \|v\|$
 - A normed vector space is a vector space where the length of vectors can be measured.
 - Euclidean norm: $\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$
 - Infinity norm: $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$
 - p -norm: $\|x\|_p = (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}}$
Proof. Minkowski's inequality:
 $(\sum_k^n |x_k + y_k|^p)^{\frac{1}{p}} \leq (\sum_k^n |x_k|^p)^{\frac{1}{p}} + (\sum_k^n |y_k|^p)^{\frac{1}{p}}$
 3. An inner product space is a vector space V over \mathbb{R} with a function $\cdot : V \times V \rightarrow \mathbb{R}$ such that
 - (a) $u \cdot u \geq 0$, $u \cdot u = 0$ iff $u = 0$
 - (b) $u \cdot v = v \cdot u$
 - (c) $(u + v) \cdot w = u \cdot w + v \cdot w$
 - (d) $(\alpha u) \cdot v = \alpha(u \cdot v)$
 - Angle θ between u, v : $u \cdot v = \cos \theta \|u\| \|v\|$.
 - u, v are orthogonal if $u \cdot v = 0$.
 - Every inner product space is a normed space if define $\|u\| = (u \cdot u)^{\frac{1}{2}}$ as the norm.
 - Cauchy-Schwarz Inequality: $|u \cdot v| \leq \|u\| \|v\|$
Proof. $f(\lambda) = (u - \lambda v) \cdot (u - \lambda v) \geq 0, \forall \lambda$.
Substitute in $\lambda = \frac{u \cdot v}{\|v\|^2}$.

III. MATRIX SPACES

1. A metric space (X, d) is a set X together with a function $d : X \times X \rightarrow \mathbb{R}$ such that
 - (a) $d(x, y) \geq 0$, $d = 0$ iff $x = y$
 - (b) $d(y, x) = d(x, y)$

$$(c) \quad d(x, y) \leq d(x, z) + d(z, y)$$

- A metric space is a set for which distances between all members of the set are defined.
 - Any normed space $(V, \|\cdot\|)$ is a metric space.
2. Suppose (X, d) is a metric space, and $S \subset X$. (S, d_S) is a metric subspace if we define $d_S(x, y) = d(x, y)$ for $x, y \in S$.
- $\forall a \in S, B_r^S(a) = S \cap B_r^X(a)$
 - A is open in $S \Leftrightarrow A = S \cap U, U$ is open in X ;
 A is closed in $S \Leftrightarrow A = S \cap C, C$ is closed in X .
Proof. A open in $S \Rightarrow A = \bigcup_{x \in A} B_{r_x}^S(x) = \bigcup_{x \in A} (S \cap B_{r_x}(x)) = S \cap (\bigcup_{x \in A} B_{r_x}(x))$. Thus $U = \bigcup_{x \in A} B_{r_x}(x)$.

3. Limit and Isolated Points

- x is a limit point in A if every $B_r(x)$ contains points of A other than x .
- x is a limit point iff $\exists (x_n) \subset A$ and $x_n \rightarrow x$.
- x is a isolated point if $\exists r$ such that $B_r(x) \cap A = \{x\}$.

4. Interior, Exterior and Boundary

- $x \in \text{int } A$ if $\exists r(B_r(x) \subset A)$
- $x \in \text{ext } A$ if $\exists r(B_r(x) \subset A^C)$
- $\text{ext } A = \text{int } A^C, \text{int } A = \text{ext } A^C$
- $x \in \partial A$ (boundary of A) if any $B_r(x)$ contains both points of A and points of A^C .
- $X = \text{int } A \cup \text{ext } A \cup \partial A$

5. Open Sets

- A is open if $A = \text{int } A$.
- A is open $\Rightarrow A = \bigcup_{x \in A} B_{r_x}(x)$
- If A_i are open, $\bigcap_{i=1}^k A_i$ is open;
 If A_i are open, $\bigcup_{i \in I} A_i$ is open.
- $\text{int } A$ is open; $\text{ext } A$ is open.

6. Closed Sets

- A is closed if A^C is open.
- A is closed iff $\overline{A} = A$.
- A is closed iff A contains all its limit points.
- $A \subset \mathbb{R}^k$ is closed iff A is complete.
- If B_i are closed, $\bigcup_{i=1}^k B_i$ is closed;
 If B_i are closed, $\bigcap_{i \in I} B_i$ is closed.
- Closure of a set is closed.
- Closed does *not* imply bounded.

7. Closure

- $\overline{A} = A \cup \{\text{limit point of } A\}$
- $\overline{A} = \text{int } A \cup \partial A$
- $\overline{A} = A \cup \partial A$
- $\overline{A} = (\text{ext } A)^C$
- $x \in \overline{A}$ iff every $B_r(x)$ contains a point of A .
- $x \in \overline{A}$ iff there exists $(x_n) \subset A$ with $x_n \rightarrow x$.

IV. SEQUENCES AND CONVERGENCE

1. (x_n) converges to x if $\forall \epsilon, \exists N, [\forall n > N \Rightarrow d(x, x_n) < \epsilon]$.
2. (x_n) is Cauchy if $\forall \epsilon, \exists N, [\forall m, n > N \Rightarrow d(x_m, x_n) < \epsilon]$.
 · Every convergent/Cauchy sequence is bounded.
Proof. convergence $\Rightarrow (x_n)$ is bounded after some N , left only finite elements.
 · \mathbb{R}^k : sequence (x_n) converges $\Leftrightarrow (x_n)$ is Cauchy.
 · X : sequence (x_n) converges $\not\Leftrightarrow (x_n)$ is Cauchy.
3. A metric space is complete if every Cauchy sequence converges in itself.
 · $S \subset \mathbb{R}^k$ is complete *iff* it is closed.
4. Monotone Convergence Theorem: if a sequence is increasing (decreasing) and bounded by a supremum (infimum), it will converge to the supremum (infimum).
Proof. Let $c = \sup_n \{a_n\}$. $\forall \epsilon > 0, \exists N$ s.t. $c - \epsilon < a_N \leq a_n \leq c, \forall n > N$. As $\epsilon \rightarrow 0, a_n \rightarrow c$.
5. Banach Fixed-Point Theorem: If (X, d) is a *complete* metric space, and $f : X \rightarrow X$ is a *contraction*, i.e. $\exists \lambda \in [0, 1)$ such that $d(f(x), f(y)) \leq \lambda d(x, y)$, then there exists a unique fixed point $f(x) = x$.
Proof. First show (x_n) is Cauchy, then prove $d(x, f(x)) \rightarrow 0$.

V. SEQUENCES AND COMPACTNESS

1. Bolzano-Weierstrass Theorem: Every bounded sequence in \mathbb{R}^k has a convergent subsequence.
Proof. Geometric intuition: if a sequence in \mathbb{R}^k is bounded, we can always trap it in a whatever small subspace. Construct a convergent subsequence by trapping it a smaller and smaller subspace.
2. A metric space is (sequentially) compact if every sequence has a convergent subsequence.
 · \mathbb{R}^k : compact \Leftrightarrow closed, bounded (Heine-Borel Thm)
 · X : compact $\not\Leftrightarrow$ closed, bounded.
Proof. Consider $X = C[0, 1]$. $\overline{B_1(\mathbf{0})} \subset X$ is closed and bounded, but *not* compact.
 · Compactness is sort of a topological generalization of finiteness. For example, if a set A is finite then every function $f : A \rightarrow \mathbb{R}$ is bounded and has max/min. If A is compact, the every *continuous* function $f : A \rightarrow \mathbb{R}$ is bounded and has max/min.

VI. LIMITS AND CONTINUITY

1. Let $f : X \rightarrow Y$. The following are equivalent:

- (i) $\lim_{x \rightarrow a} f(x) = b$;
- (ii) $x_n \rightarrow a \Rightarrow f(x_n) \rightarrow b$;
- (iii) $x \in B_\delta^X(a) \setminus \{a\} \Rightarrow f(x) \in B_\epsilon^Y(b)$.

Proof. (ii) \Rightarrow (iii): Suppose the opposite. Let $x_n \in B_{1/n}^X(a)$, then $x_n \rightarrow a$ and $f(x_n) \rightarrow b$, but by assumption $\exists \epsilon$ s.t. $f(x_n) \notin B_\epsilon^Y(b)$. contradiction. (iii) \Rightarrow (ii): Suppose $x_n \rightarrow a$. (iii) $\Rightarrow \forall \epsilon \exists \delta \exists x_n \in B_\delta^X(a) \Rightarrow f(x_n) \in B_\epsilon^Y(b) \Rightarrow f(x_n) \rightarrow b$.

2. $f : X \rightarrow Y$ is continuous at $a \in X$ if a is an isolated point, or $\lim_{x \rightarrow a} f(x) = f(a)$.

- Every function is continuous at isolated points.
- Intuitively, a continuous function is a function for which sufficiently small changes in the input result in arbitrarily small changes in the output.

3. Quantitative Measures of Continuity

- $f : X \rightarrow Y$ is Lipschitz continuous if there exists a constant M such that $d_Y(f(x_1), f(x_2)) \leq M d_X(x_1, x_2)$. M is called the Lipschitz constant.
- $f : X \rightarrow Y$ is Hölder continuous with exponent $\alpha \in (0, 1]$ if there exists a constant M such that $d_Y(f(x_1), f(x_2)) \leq M [d_X(x_1, x_2)]^\alpha$.

4. Continuity and Compactness: $f : X \rightarrow Y$ is continuous, if $K \subset X$ is compact, then $f(K)$ is compact in Y .

Proof. $\forall (y_n) \subset f(K)$, $\exists (x_n) : f(x_n) = y_n$. K is compact $\Rightarrow \exists (x_{n_j}) \rightarrow x$; f is continuous $\Rightarrow f(x_{n_j}) \rightarrow f(x) \Rightarrow y_{n_j} \rightarrow f(x)$.

Cor. Continuous function on a compact set is bounded.

5. Extreme Value Theorem: Each continuous function on a compact set attains its maximum and minimum.

Proof. K compact $\Rightarrow f(K)$ compact $\Rightarrow f(K)$ closed and bounded \Rightarrow exist least upper bound γ and $\gamma \in f(K)$ (take a sequence approaching γ and extract its convergent subsequence). Therefore, $\exists x_0 \in K$ s.t. $\gamma = f(x_0)$.

6. Continuity and Open Sets: The following statements are equivalent:

- (i) $f : X \rightarrow Y$ is continuous on X ;
- (ii) $f^{-1}(E)$ is open whenever E is an open set in Y ;
- (iii) $f^{-1}(E)$ is closed whenever E is a closed set in Y .

Proof. f continuous $\Rightarrow \forall \epsilon \exists \delta f(B_\delta(x)) \subset B_\epsilon(f(x)) \subset E \Rightarrow B_\delta(x) \subset f^{-1}(E)$. On the other hand, $f^{-1}(B_\epsilon(f(x)))$ is an open set $\Rightarrow \exists B_\delta(x) \subset f^{-1}(B_\epsilon(f(x))) \Rightarrow f(B_\delta(x)) \subset B_\epsilon(f(x)) \Rightarrow f$ is continuous.

Note. Continuous functions do *not* necessarily map open sets to open sets, or closed sets to closed sets.

Cor. If $f : X \rightarrow \mathbb{R}$ is continuous, $\{x : f(x) < 0\}$ is open.