Class Notes on Macroeconomic Theory

Zeming Wang[1]

[1]Australian National University

Contents

1 Dynamic Programming

Consider a typical problem (social planner's problem): a social planner plans for the economy how much to consume and invest for each period in time to maximize the lifetime utility. Mathematically, the social planner wants to solve the optimisation problem:

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to the budget constraint:

$$c_t + i_t = f(k_t)$$

and the law of motion of capital:

$$k_{t+1} = (1-\delta)k_t + i_t$$

Assume k_0 is given. β the discount factor satisfies $0 < \beta < 1$.

We also assume, throughout this note, the utility function U satisfies the following properties:

- U is a twice continuously differentiable real-valued function;
- U is strictly increasing and strictly concave;
- Inada condition: $\lim_{c \to 0} U'(c) = \infty$, $\lim_{c \to \infty} U'(c) = 0$.

The production technology f satisfies the properties:

- f is twice continuously differentiable;
- f is homogeneous of degree one in its input;
- The output of f is increasing in each input at a decreasing rate;
- Inada condition: $\lim_{k \to 0} f'(k) = \infty$, $\lim_{k \to \infty} f'(k) = 0$.

For simplicity, in this notes, we assume capital depreciates fully by the end of each period, i.e. $\delta = 1$. And we substitute the budget constraint into the objective function to avoid solving optimisation problem with constraints. Therefore, the problem can be reduced to:

$$\max_{\{k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1})$$

How do we solve a sequence of $\{k_{t+1}\}_{t=0}^{\infty}$? In paticular, how do we maximize over infinite time horizon? To shed some light on the technique, we begin with somewhat more approachable finite horizon prblem, and extend it to infinite horizon.

1.1 Finite Horizon

Suppose the time ends at T, we want to solve:

$$\max_{\{k_{t+1}\}} \sum_{t=0}^{T} \beta^t U(f(k_t) - k_{t+1})$$

1.1.1 Solve by brutal force

Set up a Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t U(f(k_t) - k_{t+1})$$

The first-order condition (with respect to k_{t+1}) is:

$$-\beta^t U'(c_t) + \beta^{t+1} U'(c_{t+1}) f'(k_{t+1}) = 0$$

Rearrange it a little bit:

$$\underbrace{\frac{U'(c_t)}{\beta U'(c_{t+1})}}_{\text{MRS: marginal rate of substitution}} = \underbrace{f'(k_{t+1})}_{\text{MRT: marginal rate of transformation}}$$

This intertemporal first-order condition is called the **Euler equation**.

Additionally, a boundary condition needs to be satisfied:

$$k_{T+1} = 0$$

There are $T+1$ unknowns $\{k_{t+1}\}_{t=0}^{T}$ and $T+1$ equations, we should be able to solve all $\{k_{t+1}\}_{t=0}^{T}$.

1.1.2 Discover the recursive nature of the problem

If we can discover the recursive nature of the problem, we can hopefully simplify it. Let $V_T(k_0)$ be the **value function**.

$$\begin{aligned}
V_T(k_0) &= \max \sum_{t=0}^{T} \beta^t U(f(k_t) - k_{t+1}) \\
&= \max\{U(f(k_0) - k_1) + \beta \sum_{t=0}^{T-1} \beta^t U(f(k_t) - k_{t+1})\} \\
&= \max\{U(f(k_0) - k_1) + \beta V_{T-1}(k_1)\}
\end{aligned}$$

If V_{T-1} is known, we will be able to solve it just like solving one-choice-variale optimisation!

The clue is from the boundary condition. Since $k_{T+1} = 0$, $V_{-1}(k_{T+1}) = 0$ is known. Then we will be able to solve backward: $V_0, V_1, ..., V_T$.

$$V_0(k_T) = \max_{k_{T+1}}\{U(f(k_T) - k_{T+1}) + \beta \underbrace{V_{-1}(k_{T+1})}_{=0}\}$$

The optimal choice is of course $k_{T+1} = 0$, and the value function becomes $V_0(k_T) = U(f(k_T))$.

Now $V_0(k_T)$ is known, we can use it to solve $V_1(k_{T-1})$:

$$V_1(k_{T-1}) = \max_{k_T}\{U(f(k_{T-1}) - k_T) + \beta \underbrace{V_0(k_T)}_{\text{known}}\}$$

Solve this maximization problem, we we will get

$$k_T = g_{T-1}(k_{T-1})$$

i.e. choice variable k_T as a function of the given state k_{T-1}. g_{T-1} is called the **policy function** at date $T-1$. It tells us the optimal choice at $T-1$. Substitute in $k_T = g_{T-1}(k_{T-1})$, we value function is given by

$$V_1(k_{T-1}) = U(f(k_{T-1}) - g_{T-1}(k_{T-1})) + \beta V_0(k_T)$$

Now $V_1(k_{T-1})$ is known, we can continue to solve $V_2(k_{T-2})$.

$$\vdots$$

Keep going with this process, we will solve a sequence of functions: $\{V_{-1}, V_0, V_1, ..., V_T\}$ and $\{g_T, g_{T-1}, ..., g_0\}$, which are the solutions to the finite horizon problem.

1.2 Infinite Horizon

Exploring the finite horizon problem enlightens us to solve the the infinite horizon problem recursively. It turns out solving the infinite horizon problem recursively is even easier than the finite horizon problem.

It can be proved, as $T \to \infty$, we have $V_T \to V$, i.e. the value function will have the same form for all period t.

We can thus write the recursive optimization problem as:

$$V(k) = \max_{k'} \{U(f(k) - k') + \beta V(k')\}$$

This is called the **Bellman equation**. The solution to the Bellman equation is the value function V.

To solve the value function, define

$$(TV)(k) = \max_{k'} \{U(f(k) - k') + \beta V(k')\}$$

T is a function that maps a given function V to a new V. If we can find the fixed point of T, i.e. $(TV)(k) = V(k)$, then it is the solution to our Bellman equation.

But how do we know such a fixed point exists, and how can we find it? To answer these questions, we need a small technical detour on Banach fixed point theorem.

1.2.1 Banach fixed point theorem

Banach fixed point theorem: If (S, d) is a *complete* metric space and $T : S \to S$ is a *contraction*, then there is a unique fixed point for T.

Blackwell's sufficient condition for a contraction: Let $M : B(X) \to B(X)$ be any map satisfying

- Monotonicity: For any $v, w \in B(X)$ such that $w \geq v \implies Mw \geq Mv$.
- Discounting: There exists a $0 \leq b < 1$ such that $M(w + c) = Mw + bc$, for all $w \in B(X)$ and $c \in \mathbb{R}$. (Define $(f + c)(x) = f(x) + c$)

Then M is a contraction with modulus β.

1.2.2 Fixed point for the infinite horizon problem

Given the assumptions on U and f, the value function is continuous, and the state space $S \ni k$ is bounded (there is an upper bound for capital accumulation). $T : C_b(S) \to C_b(S)$ is a mapping from the space of continuous bounded functions ($C_b(S)$) to itself. It can be proved the space $C_b(S)$ is complete when equipped with the uniform metric $d_u(v, w) = \sup_{x \in S} |v(x) - w(x)|$.

The Blackwell's sufficient condition is also satisfied:

- Monotonicity: let $v, w \in C_b(S)$ and $w \geq v$,

$$Tw = \max\{U + \beta w\} \geq \max\{U + \beta v\} = Tv$$

- Discounting: let $w \in C_b(S)$ and let $c \in \mathbb{R}$,

$$T(w + c) = \max\{U + \beta(w + c)\} = \max\{U + \beta w\} + \beta c = Tw + \beta c$$

Therefore, T is contraction on a complete metric space. By Banach fixed point theorem, T has a unique fixed point.

1.2.3 Solving the Bellman equation

The contraction mapping argument also provides us a mechanical way to solve the fixed point for the Bellman equation. We can iterate on an arbitrary initial guess for the value function, even the guess is incorrect, the iteration will eventually converge to the true value function. This is the way how computers solve the problem. In terms of solving the Bellman equation by hand, we usually guess the value function (or policy function) to be a particular form, and solve for the parameters. Specifically, there are mainly two approaches: value function approach and policy function approach.

To demostrate these two approaches, in the following text, we specialize the utility function and the production function to the following form:

$$U(c) = \ln(c)$$
$$f(k) = k^\alpha$$

Value function (Bellman operator) approach:

Guess $V(k) = A + B\ln(k)$. Substitute this form into the Bellman equation:

$$A + B\ln(k) = \max_{k'}\{\ln(k^\alpha - k') + \beta(A + B\ln(k'))\}$$

Solve for k':

$$k' = \frac{\beta B}{1 + \beta B} k^\alpha$$

Substitute k' into the value function:

$$A + B\ln(k) = \ln(k^\alpha - \frac{\beta B}{1 + \beta B} k^\alpha) + \beta(A + B\ln(\frac{\beta B}{1 + \beta B} k^\alpha))$$

Rearrange the equation:

$$A + B\ln(k) = \beta A + \ln(\frac{1}{1 + \beta B}) + \beta B\ln(\frac{\beta B}{1 + \beta B}) + (1 + \beta B)\alpha\ln(k)$$

To make LHS = RHS, we require

$$B = (1 + \beta B)\alpha$$

$$A = \beta A + \ln(\frac{1}{1 + \beta B}) + \beta B \ln(\frac{\beta B}{1 + \beta B})$$

Therefore,

$$B = \frac{\alpha}{1 - \alpha\beta}$$

$$A = \frac{1}{1 - \beta}\left[\ln(1 - \alpha\beta) + \frac{\alpha\beta}{1 - \alpha\beta}\ln(\alpha\beta)\right]$$

Therefore,

$$k' = g(k) = \alpha\beta k^{\alpha}$$

$$V(k) = \frac{1}{1 - \beta}\left[\ln(1 - \alpha\beta) + \frac{\alpha\beta}{1 - \alpha\beta}\ln(\alpha\beta)\right] + \frac{\alpha}{1 - \alpha\beta}\ln(k)$$

Policy function (Euler operator) approach

Guess $k' = g(k) = \theta k^{\alpha}$, then $c = (1 - \theta)k^{\alpha}$. Substitute this into the Euler equation:

$$\frac{1}{(1 - \theta)k^{\alpha}} = \alpha\beta \frac{1}{(1 - \theta)k'^{\alpha}} k'^{\alpha - 1} = \alpha\beta \frac{1}{(1 - \theta)k'}$$

$$\frac{1}{(1 - \theta)k^{\alpha}} = \alpha\beta \frac{1}{(1 - \theta)\theta k^{\alpha}}$$

$$\frac{1}{1 - \theta} = \alpha\beta \frac{1}{\theta(1 - \theta)}$$

Therefore, $\theta = \alpha\beta$.

In both approaches, we can see the optimal savings rate is a fixed proportion of the total income. The fixed proportion $\alpha\beta$ is called the **Golden rule** of savings.

2 Deterministic Competitive Equilibrium

In stead of modelling the economy from a social planner's perspective, it is more realistic to model the economy in a competitive equilibrium setting, in which households maximise their utilities and firms maximise their profits. In the following context, we assume homogeneous households and firms, i.e. the economy can be represented by only one household and one firm that behave competitively to each other.

2.1 Arrow-Debreu Equilibrium

In an Arrow-Debreu setting, there is a centralised market that opens only at the very beginning of time. In this market, agents contract on how much to transfer to each other in all dates in the future, once and for all. All future plans and transfers are then determined. Agents trade to trade to maximise their lifetime utilities and profits.

Let q_t^0 be the relative price of a date-t claim to (consumption or investment) goods in units of date-0 goods.

Household's problem:

$$\max_{\{c_t,n_t,i_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t,n_t) - \mu \underbrace{\left[\sum_{t=0}^{\infty} q_t^0(c_t+i_t) - \sum_{t=0}^{\infty} q_t^0(r_t k_t + w_t n_t) - \Pi_0\right]}_{\text{lifetime budget constraint}}$$

$$\text{subject to } k_{t+1} = (1-\delta)k_t + i_t$$

The first-order conditions are:

$$\begin{aligned}
c_t &: \beta^t U_c - \mu q_t^0 = 0 \\
n_t &: \beta^t U_n + \mu q_t^0 w_t = 0 \\
k_{t+1} &: q_t^0 - q_{t+1}^0(1-\delta) - q_{t+1}^0 r_{t+1} = 0
\end{aligned}$$

We can further derive:

$$\begin{aligned}
-\frac{U_n(c_t,n_t)}{U_c(c_t,n_t)} &= w_t \\
\frac{U_c(c_t,n_t)}{\beta U_c(c_{t+1},n_{t+1})} &= \frac{q_t^0}{q_{t+1}^0} \\
\frac{q_t^0}{q_{t+1}^0} &= 1-\delta+r_{t+1}
\end{aligned}$$

These conditions says: wage is the price for leisure in terms of consumption; the relative price cross two periods is equal to the marginal rate of substitution of consumption between the periods and the gross return through the time.

Firm's problem:

$$\Pi_0 = \max_{\{K_t,N_t\}} \sum_{t=0}^{\infty} q_t^0 [F(K_t,N_t) - r_t K_t - w_t N_t]$$

The first-order condition gives:

$$F_K(K_t, N_t) = r_t$$
$$F_N(K_t, N_t) = w_t$$

Arrow-Debreu equilibrium

An Arrow-Debreu equilibrium is a date-contingent allocation $\{C_t, N_t, K_{t+1}\}$ and prices $\{q_t^0, q_t^0 w_t, q_t^0 r_t\}$ such that

- Given prices, $\{c_t, n_t, k_{t+1}\}$ is optimal for the household;
- Given prices, $\{K_t, N_t\}$ is optimal for the firm;
- Market clears for all dates: $c_t = C_t, k_t = K_t, i_t = I_t, n_t = N_t$; $K_{t+1} = (1-\delta)K_t + I_t$; $C_t + I_t = F(K_t, N_t)$.

2.2 Sequential Competitive Equilibrium

In a sequential competitive equilibrium modelling, households and firms maximise trade for each period, therefore the budget constraint has to hold for every period (instead of a single lifetime budge constraint). We assume households own all the capital, and firms pay rent to use the capital. There is an alternative model in which firms own capital and households own shares of the firms, but we skip this model in this note.

Household's problem

$$\max \sum_{t=0}^{\infty} \left[\beta^t U(c_t, n_t) - \mu_t(c_t + i_t - r_t k_t - w_t n_t - \pi_t)\right]$$
$$\text{subject to } k_{t+1} = (1-\delta)k_t + i_t$$

The first-order conditions are:

$$c_t : \beta^t U_c - \mu\mu_t^0 = 0$$
$$n_t : \beta^t U_n + \mu\mu_t^0 w_t = 0$$
$$k_{t+1} : \mu_t^0 - \mu_{t+1}^0(1-\delta) - \mu_{t+1}^0 r_{t+1} = 0$$

Therefore we have:

$$-\frac{U_n(c_t, n_t)}{U_c(c_t, n_t)} = w_t$$
$$\frac{U_c(c_t, n_t)}{\beta U_c(c_{t+1}, n_{t+1})} = \frac{\mu_t^0}{\mu_{t+1}^0}$$
$$\frac{\mu_t^0}{\mu_{t+1}^0} = 1 - \delta + r_{t+1}$$

Firm's problem

The firm maximizes its profit for each period t:

$$\pi_t = \max_{\{K_t, N_t\}} [F(K_t, N_t) - r_t K_t - w_t N_t]$$

The first-order conditions are as usual:

$$F_K(K_t, N_t) = r_t$$
$$F_N(K_t, N_t) = w_t$$

Sequential competitive equilibrium

A sequential competitive equilibrium is an allocation $\{C_t, N_t, K_{t+1}\}$ and prices $\{w_t, r_t\}$ such that

- Given prices, the plan $\{c_t, n_t, k_{t+1}\}$ is optimal for the household;
- Given prices, $\{N_t, K_t\}$ maximize the firm's prifit;
- Market clears for every period t.

We can see the equilibrium system in a sequential competitive setting is in effect identical to that in the Arrow-Debreu setting. $\frac{q_t^0}{q_{t+1}^0}$ is just a special case for $\frac{\mu_t}{\mu_{t+1}}$.

We can also find that the equilibrium allocations implied by a competitive market are identical to the social planner's allocations, which means the competitive equilibrium allocations are Pareto efficient (First Welfare theorem). Conversely, we can also find equilibrium prices to support a Pareto optimal allocation for different market arrangement (Second Welfare theorem).

2.3 Recursive Competitive Equilibrium

If the equilibrium decision rules are time invariant, we can describe a competitive equilibrium in a recursive fashion, in which the households maximise:

$$V(k, K) = \max_{c,n,k'} \{U(c, n) + \beta V(k', K')\}$$

A recursive competitive equilibrium in an economy is

- a value function: $V(k, K)$
- decision functions: $c = g^c(k, K)$, $n = g^n(k, K)$, $k' = g^k(k, K)$
- prices: $r(K)$, $w(K)$
- law of motion: $K' = G(K)$

such that

- given prices, g^c, g^n, g^k solve the household's Bellman equation;
- $r(K)$ and $w(K)$ satisfy the firm's optimization condition;
- market clears: $k = K$, $g^k(k, K) = G(K)$, $g^c(k, K) = C$, $g^n(k, K) = N$, $K' - (1 - \delta)K + C = F(K, N)$.

Solving for a recursive competitive equilibrium involves more work than solving for a sequential equilibrium, since it involves solving for the value function and policy functions, which specify not only equilibrium behaviors but also "off-equilibrium" behaviors: what the agents could do if he deviates from the representative agent.

3 Fiscal Policy and Optimal Taxation

Suppose a tax τ_t is levied on households' income (rent and wage) in each period t. Suppose tax is collected to form public capital G in production. Assume labor supply $L_t = 1$, and $\{\tau_t\}_{t=0}^{\infty}$ are exogenously given. We specialize the utility function

$$U(c) = \ln c$$

and the production function

$$F(K, G) = AG^{\theta}K^{\alpha}$$

The household's problem becomes:

$$\max_{\{c_t, k_{t+1}\}} \left\{ \sum_{t=0}^{\infty} \beta^t \ln(c_t) : c_t + k_{t+1} = (1-\tau_t)(w_t + r_t k_t + \pi_t) \right\}$$

The firm's problem is as usual:

$$\pi_t = \max\{F(K_t, G_t) - r_t K_t - w_t\}$$

What is the recursive competitive equilibrium in the presence of taxation?

3.1 Optimal Savings under Taxation

Rewrite the household's problem to a Bellman equation:

$$V^h(k_t, G_t) = \max_{c_t, k_{t+1}} \{\ln(c_t) + \beta V^h(k_{t+1}) : c_t + k_{t+1} = (1-\tau_t)F(k_t, G_t)\}$$

Guess the solution having form $V^h(k, G) = A + B\ln(k) + C\ln(G)$. Solving the Bellman equation gives the policy functions:

$$k_{t+1} = (1-\tau_t)(\alpha\beta)y_t$$
$$c_t = (1-\tau_t)(1-\alpha\beta)y_t$$

where $y_t = AG_t^{\theta}k_t^{\alpha}$.

Therefore the optimal decision is still to save a fixed proportion $(\alpha\beta)$ of after-tax income.

3.2 Optimal Taxation

What is the optimal tax rates $\{\tau_t\}_{t=0}^{\infty}$ in an economy where agents best respond according to their competitive equilibrium behavior? We model this decision problem (optimal taxation) as a government's Bellman equation:

$$V^G(k_t, G_t) = \max_{\tau_t}\{\ln[\underbrace{(1-\tau_t)(1-\alpha\beta)y_t}_{\text{best choice of household}}] + \beta V^G(k_{t+1}, G_{t+1})\}$$

$$\text{subject to } k_{t+1} = (1-\tau_t)\alpha\beta y_t, G_{t+1} = \tau_t y_t$$

Guess $V^G(k,G) = A + B\ln(k) + C\ln(G)$.

We will solve

$$V^G(k,G) = A + \frac{\alpha}{1-(\alpha+\theta)\beta}\ln(k) + \frac{\theta}{1-(\alpha+\theta)\beta}\ln(G)$$

The optimal tax decision is: $\tau_t = \beta\theta$.

So the optimal tax plan is to tax private-sector income at exactly the elasticity of private income with respect to the public capital input.

4 Stochastic Dynamic Programming

4.1 Stochastic Production Technology

Suppose we introduce a random "shock" to out production output:

$$y_t = \theta_t f(k_t)$$

where $\{\theta_t\}$ is a sequence of $i.i.d.$ random variables.

The social planner's problem is now to maximize the lifetime expected utility:

$$\max\left\{\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t U(c_t) : c_t + k_{t+1} = \theta_t f(k_t)\right\}$$

where $\mathbb{E}_t$ is the expectation conditioned on information available at date t.

The household's Bellman equation is:

$$V(k_t,\theta_t) = \max_{c_t,k_{t+1}}\{U(c_t) + \beta\mathbb{E}_t V(k_{t+1},\theta_{t+1}) : c_t + k_{t+1} = \theta_t f(k_t)\}$$

If we specify $f(k_t) = \theta_t k_t^\alpha$, $U(c_t) = \ln(c_t)$, and further assume

$$\ln(\theta_{t+1}) = \rho\ln(\theta_t) + \epsilon_{t+1}$$

where $\epsilon_{t+1} \sim N(0,\sigma^2)$. What is the household's optimal consumption and savings plan?

Follow the usual steps to solve the Bellman equation by conjecture:

$$V(k,\theta) = A + B\ln(k) + C\ln(\theta)$$

Substitute it into the Bellman equation:

$$\begin{aligned} T\circ V(k,\theta) &= \max\{\ln(\theta k^\alpha - k') + \beta\mathbb{E}[A + B\ln(k') + C\ln(\theta')|\theta]\} \\ &= \max\{\ln(\theta k^\alpha - k') + \beta A + \underbrace{\beta B\ln(k')}_{k_{t+1}\text{ is known at date }t} + \underbrace{\beta C\rho\ln(\theta)}_{\mathbb{E}[\ln(\theta')]=\mathbb{E}[\rho\ln(\theta)+\epsilon']=\rho\ln(\theta)}\} \end{aligned}$$

We will solve out the value function:

$$V(k,\theta) = A + \frac{\alpha}{1-\alpha\beta}\ln(k) + \frac{1}{(1-\beta)(1-\beta\rho)}\ln(\theta)$$

and the policy functions:

$$\begin{aligned} k' &= \alpha\beta k^{\alpha}\theta \\ c' &= (1-\alpha\beta)k^{\alpha}\theta \end{aligned}$$

So the optimal savings plan is unchanged in face of stochastic technological shocks — save a fixed proportion of total income for every period.

4.2 Stochastic Technology with Markov Properties

Suppose the technology in the production function can take only finitely many possible states: $\{z_1, z_2, ..., z_n\}$. Suppose the stochastic process $\{z_t\}$ is governed by a time homogeneous Markov chain, i.e. there is a predetermined probability $\mathbb{P}(z_j|z_i)$ associated with every $z_i, z_j \in \{z_1, ..., z_n\}$.

The social planner's problem becomes:

$$V(k, z_i) = \max_{k'} \left\{ U(c) + \beta \sum_{z_j} V(k', z_j)\mathbb{P}(z_j|z_i) \right\}$$

Note that for each k, there is a vector of value functions:

$$\mathbf{V}(k) = (V(k, z_1), ..., V(k, z_n)) = (V_1(k), ..., V_n(k))$$

The Bellman operator now operates on vectors:

$$\mathbf{T} \circ \mathbf{V}(k) = \begin{bmatrix} T_1 \circ V(k, z_1) \\ T_2 \circ V(k, z_2) \\ \vdots \\ T_n \circ V(k, z_n) \end{bmatrix} = \begin{bmatrix} \max \left\{ U(c) + \beta \sum_{j=1}^n V(k', z_j)\mathbb{P}(z_j|z_1) \right\} \\ \max \left\{ U(c) + \beta \sum_{j=1}^n V(k', z_j)\mathbb{P}(z_j|z_2) \right\} \\ \vdots \\ \max \left\{ U(c) + \beta \sum_{j=1}^n V(k', z_j)\mathbb{P}(z_j|z_n) \right\} \end{bmatrix}$$

Therefore, $\mathbf{T}$ is a mapping from $[C_b(K)]^n$ to itself. Since each component T_i of $\mathbf{T}$ is a contraction mapping, $\mathbf{T}$ is also a contraction mapping. So the Banach fixed point theorem applies. We can solve for $\mathbf{T}$ iteratively as we do in single function case.

5 Equilibrium in Complete Markets

What does the equilibrium look like in the presence of stochastic states in an economy? In this section, we will explore a pure exchange economy with state-dependent endowments and consumptions in a complete market. A **complete market** is a market where there is a price for every asset in every possible states of the world. State-dependent endowments for individuals generate uncertainty and risk for individuals. We will see how a complete market secures individual risks.

Some notations used in this section:

- Stochastic events: $s_t \in S = \{s_1, ..., s_n\}$
- History of events: $h_t = (s_0, s_1, ..., s_t)$
- Probability of events (history): $\pi(s_t)$, $\pi(h_t)$
- Agents: $I = \{1, 2, ..., m\}$

Endowment for individual i at time t is denoted as $y_t^i(h_t)$.

The initial state s_0 is assumed to be given, i.e. $\pi(s_0) = 1$.

5.1 Social Planner's Problem

The social planner maximises the expected lifetime utility for all agents in the economy:

$$\max \sum_{t=0}^{\infty} \sum_{h_t} \left\{ \sum_{i \in I} \lambda_i \beta^t U(c_t^i(h_t)) \pi_t(h_t) + \theta_t(h_t) \sum_{i \in I} [y_t^i(h_t) - c_t^i(h_t)] \right\}$$

λ_i is the weight that the planner assigns to each agient i.

Note the Lagrangian multiplier $\theta_t(h_t)$ is also history-contingent. The constraint

$$\sum_{i \in I} [y_t^i(h_t) - c_t^i(h_t)] \geq 0$$

requires that the aggregrate consumption must not exceed the aggregate endowment.

The first-order condition is:

$$\lambda_i \beta^t U'(c_t^i(h_t)) \pi_t(h_t) - \theta_t(h_t) = 0$$

Therefore,

$$\frac{U'(c_t^i(h_t))}{U'(c_t^j(h_t))} = \frac{\lambda_j}{\lambda_i}$$

$$c_t^i(h_t) = U'^{-1} \left[\frac{\lambda_j}{\lambda_i} U'(c_t^j(h_t)) \right]$$

The budge constraint holds:

$$\sum_{i \in I} y_t^i(h_t) = \sum_{i \in I} c_t^i(h_t)$$

Sum up c_t^i for all i:

$$\sum_{i \in I} U'^{-1}\left[\frac{\lambda_j}{\lambda_i} U'(c_t^j(h_t))\right] = \sum_{i \in I} y_t^i(h_t)$$

Given the assumption on U, we can uniquely solve for c_t^j. Once we solve c_t^j, every c_t^i $(i \neq j)$ is uniquely determined. Therefore, the optimal allocation is only a function of realised aggregate endowment and does not depend on the particular history h_t leading to this outcome or the realisation of individual endowment.

5.2 Arrow-Debreu Equilibrium

Suppose the market opens only at date $t = 0$. In the centralised market, agents exchange claims on time-t consumption contingent on history h_t at price $q_t^0(h_t)$. Assume there is a complete set of securities for all possible h_t.

For each individual i, s/he solve the problem:

$$\max \left\{ \sum_{t=0}^{\infty} \sum_{h_t} \beta^t U(c_t^i(h_t))\pi_t(h_t) + \mu_i \sum_{t=0}^{\infty} \sum_{h_t} q_t^0(h_t)[y_t^i(h_t) - c_t^i(h_t)] \right\}$$

Note that trades happen at date-0 once and for all implies that each individual faces only one lifetime budge constraint.

The first-order condition is:

$$\beta^t U'(c_t^i(h_t))\pi_t(h_t) - \mu_i q_t^0(h_t) = 0$$

Therefore,

$$\frac{U'(c_t^i(h_t))}{U'(c_t^j(h_t))} = \frac{\mu_i}{\mu_j}$$

The equilibrium condition here is actually equivalent to the first-order condition of the social planner's problem only by letting $\mu_i = \lambda_i^{-1}$. So we still have the similar result: the Arrow-Debreu allocation is a function of the realised aggregate endowment and does not depend on either the realisation of individual endowment or the particular history leading up to that outcome.

The implication is, since agents can trade contracts on all possible contingencies in the future, the market aggregate the risks and eliminate individual risks. Or we could say, the individual risks are fully secured by the complete market.

5.2.1 Asset pricing implications

In a complete market, every contingency is priced by $q_t^0(h_t)$. So any other asset is "redundant". Any other asset can be priced with a combination of $q_t^0(h_t)$.

Suppose an asset pays a stream of dividents $\{d_t(h_t)\}_{t\geq 0}$, then the asset has a price at date-0:

$$p^0(h_t) = \sum_{t=0}^{\infty} \sum_{h_t} q_t^0(h_t) d_t(h_t)$$

The price of the asset at date-t is given by:

$$p^t(h_{t+k}) = \sum_{k\geq 0} \sum_{h_{t+k}|h_t} q_{t+k}^t(h_{t+k}) d_{t+k}(h_{t+k})$$

where $q_{t+k}^t(h_{t+k})$ is the relative price:

$$q_{t+k}^t(h_{t+k}) = \beta^k \frac{U'(c_{t+k}^i(h_{t+k}))}{U'(c_t^i(h_t))} \pi_t(h_{t+k}|h_t)$$

5.3 Sequential Market with Arrow Securities

Now suppose at each date t with history h_t, traders meet to trade for contingent goods deliverable at $t+1$.

Let $a_t^i(h_t)$ be the total **asset** or **wealth** owned by individual i at date t.

Individual i solves:

$$\max \sum_{t=0}^{\infty} \sum_{h_t} \left\{ \beta^t U(c_t^i(h_t)) \pi_t(h_t) + \eta_t^i(h_t) \left[y_t^i(h_t) + a_t^i(h_t) - c_t^i(h_t) - \sum_{s_{t+1}} a_{t+1}^i(s_{t+1}, h_t) Q_t(s_{t+1}|h_t) \right] \right\}$$

where $Q_t(s_{t+1}|h_t)$ is called the intertemporal pricing kernel.

Note that a_{t+1}^i can be negative, i.e. agents can borrow assets. In principle, we should restrict that each agent cannot borrow more than the maximal amount that he could repay — the present value of all his future endowment:

$$-a_{t+1}^i(s_{t+1}) \leq \sum_{k\geq 1} \sum_{h_{t+k}|h_{t+1}} q_{t+k}^{t+1}(h_{t+k}) y_{t+k}^i(h_{t+k})$$

But given the Inada condition, the debt constraint will never bind. If the constraint binds, consumption from $t+1$ on has to be zero forever. But this implies that the consumer has infinite marginal utility for future consumptions, s/he could be better off by just saving a little from today for future consumption.

The first-order conditions are:

$$c_t^i : \beta^t U(c_t^i(h_t)) \pi_t(h_t) - \eta_t^i(h_t) = 0$$
$$a_{t+1}^i : -\eta_t^i(h_t) Q_t(s_{t+1}|h_t) + \eta_t^i(s_{t+1}, h_t) = 0$$

We can solve out the pricing kernel:

$$Q_t(s_{t+1}|h_t) = \beta \frac{U'(c_{t+1}^i(h_{t+1}))}{U'(c_t^i(h_t))} \pi_t(h_{t+1}|h_t)$$

We will show that, with an appropriate choice of initial wealth distribution, there is an equivalence in allocations under Arrow-Debreu equilibrium and sequential market equilibrium.

5.3.1 Equivalence of equilibriums

If we define wealth as the present value of all the household's owned *current and future net claims on consumption* (Ω_t^i), i.e.

$$\begin{aligned} a_t^i(h_t) = \Omega_t^i(h_t) &= \sum_{k\geq 0} \sum_{h_{t+k}|h_t} q_{t+k}^t(h_{t+k})[c_{t+k}^i(h_{t+k}) - y_{t+k}^i(h_{t+k})] \\ &= \sum_{k\geq 0} \sum_{h_{t+k}|h_t} \beta^k \frac{U'(c_{t+k}^i(h_{t+k}))}{U'(c_t^i(h_t))} \pi_t(h_{t+k}|h_t)[c_{t+k}^i(h_{t+k}) - y_{t+k}^i(h_{t+k})] \end{aligned}$$

Then we can rewrite the expected future wealth (the last term in the constraint) as:

$$\sum_{s_{t+1}} a_{t+1}^i(s_{t+1}, h_t) Q_t(s_{t+1}|h_t) = \sum_{k\geq 1} \sum_{h_{t+k}|h_t} q_{t+k}^t(h_{t+k})[c_{t+k}^i(h_{t+k}) - y_{t+k}^i(h_{t+k})]$$

At $t = 0$, the budget constaint becomes

$$c_0^i(s_0) + \sum_{k\geq 1} \sum_{h_k} q_k^0(h_k)[c_k^i(h_k) - y_k^i(h_k)] = y_0^i(s_0) + a_0^i(s_0)$$

The budget constraint is exactly the same as in an Arrow-Debreu setting if we set $a_0^i(s_0) = 0$.

In all consecutive future period, $t > 0$, if we replace

$$\sum_{s_{t+1}} a_{t+1}^i(s_{t+1}, h_t) Q_t(s_{t+1}|h_t) = \Omega_t^i(h_t) - [c_t^i(h_t) - y_t^i(h_t)]$$

in the budget constraint, we will get exactly the same c_t^i for each t as in an Arrow-Debreu equilibrium.

5.4 Recursive Competitive Equilibrium

If the state s_t is governed by a Markov chain,

$$\pi_t(h_t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2}) \cdots \pi(s_1|s_0)\pi(s_0)$$

then the allocation for each period depend only on the current state:

$$y_t^i(h_t) = y^i(s_t)$$

$$c_t^i(h_t) = c^i(s_t)$$

$$Q_t(s_{t+1}|h_t) = Q(s_{t+1}|s_t)$$

This enables us to restate the households' utility maximisation problem recursively.

For each household i, its Bellman equation is:

$$V^i(a,s) = \max_{c,\hat{a}} \left\{ U(c) + \beta \sum_{s'} V^i(\hat{a}(s'), s')\pi(s'|s) \right\}$$

$$\text{s.t. } c + \sum_{s'} \hat{a}(s')Q(s'|s) \le y^i(s) + a$$

$$-\hat{a}(s') \le \Omega^i(s')$$

The first-order conditions read:

$$c : U'(c) - \mu = 0$$

$$\hat{a} : \beta \frac{\partial V^i}{\partial \hat{a}} \pi(s'|s) - \mu Q(s'|s) = 0$$

where μ the is the Lagrangian multiplier for the first constraint. The debt limit will never bind.

By applying the Envelope theorem,

$$\frac{\partial V^i}{\partial a} = \mu = U'(c)$$

we have

$$\frac{\partial V^i}{\partial \hat{a}} = \mu' = U'(c')$$

Therefore, the pricing kernel can be experssed as:

$$Q(s'|s) = \beta \frac{U'(c')}{U'(c)} \pi(s'|s)$$

6 New Keynesian Model

6.1 Households' Problem

We formulate the household's problem as following:

$$\max \mathbb{E}_t \left\{ \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) \right\}$$
$$\text{s.t. } P_t C_t + Q_{t,t+1} A_{t+1} \leq W_t N_t + A_t + \Pi_t$$

where P_t stands for the aggregate price for consumption goods; A_t for the net asset owned by the houdehold; and $Q_{t,t+1}$ for the relative price of the assets.

The first-order condition gives:

$$-\frac{U_n(C_t, N_t)}{U_c(C_t, N_t)} = \frac{W_t}{P_t}$$
$$Q_{t,t+1} = \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} \right\} = \beta \mathbb{E}_t \left\{ \frac{U_c(C_{t+1}, N_{t+1})}{U_c(C_t, N_t)} \cdot \frac{P_t}{P_{t+1}} \right\}$$

If we specialize the utility function to

$$U(C_t, N_t) = \frac{U_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\gamma}}{1+\gamma}$$

Then $U_c(C_t, N_t) = C_t^{-\sigma}$. Let i_t be the riskless one-period nominal interest rate, i_t satisfies

$$\frac{1}{1+i_t} = Q_{t,t+1} = \beta \mathbb{E}_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right\}$$

Log-linearize the above equation (see What is log-linearization?):

$$-\hat{i}_t = \mathbb{E}_t \left\{ -\sigma(\hat{C}_{t+1} - \hat{C}_t) - (\hat{P}_{t+1} - \hat{P}_t) \right\}$$

Rearrange it,

$$\hat{C}_t = \mathbb{E}_t \hat{C}_{t+1} - \frac{1}{\sigma} [\hat{i}_t - \mathbb{E}_t \pi_{t+1}]$$

where we define $\pi_{t+1} = \hat{P}_{t+1} - \hat{P}_t$ as the inflation rate.

If we assume good markets clear, i.e. $\hat{Y}_t = \hat{C}_t$ where Y_t represents the output, then we can rewrite the above equation as

$$\hat{Y}_t = \mathbb{E}_t \hat{Y}_{t+1} - \frac{1}{\sigma} [\hat{i}_t - \mathbb{E}_t \pi_{t+1}]$$

This is the forward-looking Keynesian IS equation. We will return to this equation later.

To facilitate the analysis of firms' pricing behavior, we assume the aggregated consumption is composed of consumptions on a list of differentiable goods:

$$C_t = \left[\int_0^1 C_t(i)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}}$$

Note that we aggregate the consumption bundle by a CES aggregator.

Similarly, the price is aggregated by

$$P_t = \left[\int_0^1 (P_t(i))^{1-\epsilon} \right]^{\frac{1}{1-\epsilon}}$$

Then we can show the optimal intratemporal allocation of expenditures on each good conforms

$$C_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\epsilon} C_t$$

6.2 Firms' Problem

We assume there is a continuum of firms on $[0, 1]$ each producing a good differentiated by product type i.

We assume a capital-free technology:

$$Y_t(i) = A_t N_t(i)$$

The firm minimizes its cost:

$$\min_{N_t} W_t N_t(i) + MC_t[Y_t(i) - A_t N_t(i)]$$

where MC_t is the marginal cost.

The first-order condition is:

$$W_t = MC_t A_t$$

6.2.1 Marginal cost

Let $mc_t := \dfrac{MC_t}{P_t}$ denote the **real marginal cost** of the firm. Together with the optimal choice from the household, we can derive:

$$mc_t = \frac{MC_t}{P_t} = \frac{W_t}{P_t} \cdot \frac{1}{A_t} = -\frac{U_n(C_t, N_t)}{U_c(C_t, N_t)} \frac{1}{A_t} = \frac{N_t^\gamma}{C_t^{-\sigma}} \frac{1}{A_t}$$

Assume markets clear, $C_t = Y_t = A_t N_t$, we have

$$mc_t = \frac{(Y_t/A_t)^\gamma}{Y_t^{-\sigma}} \frac{1}{A_t} = Y_t^{\sigma+\gamma} A_t^{-\gamma-1}$$

Log-linearize this equation:

$$\hat{mc}_t = (\gamma + \sigma)\hat{Y}_t - (1+\gamma)\hat{A}_t$$

This equation establishes the real marginal cost changes as output or technology changes.

6.2.2 Price stickiness

Now we introduce price stickiness. Suppose each period, there is a constant probability $(1-\theta)$ that firms get to adjust their prices. If firm i gets to reset its product price at time t, the firm knows the probability it will be stuck at that new price for the next k periods ahead will be θ^k.

The firm choose $P_t^*(i)$ to maximize expected profit:

$$\max_{P_t(i)} \mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} C_{t+k}(i)[P_t(i) - MC_{t+k}]$$
$$\text{s.t. } C_{t+k}(i) = \left(\frac{P_t(i)}{P_{t+k}}\right)^{-\epsilon} C_{t+k}$$

The first-order condition gives:

$$P_t^*(i) = \frac{\epsilon}{\epsilon - 1} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} MC_{t+k} C_{t+k}(i)}{\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} C_{t+k}(i)}$$

If we substitute the following into the equation,

$$Q_{t,t+k} = \beta^k \frac{U_c(C_{t+k}}{U_c(C_t)} \frac{P_t}{P_{t+k}} = \beta^k \Gamma_{t,t+k}$$
$$C_{t+k}(i) = \left(\frac{P_t(i)}{P_{t+k}}\right)^{-\epsilon} C_{t+k}$$
$$MC_{t+k} = mc_{t+k} P_{t+k}$$

the condition can be rewritten as:

$$P_t^*(i) = \frac{\epsilon}{\epsilon - 1} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \Gamma_{t,t+k} mc_{t+k} P_{t+k}^{\epsilon+1} C_{t+k}}{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \Gamma_{t,t+k} P_{t+k}^{\epsilon} C_{t+k}}$$

Log-linearize this equation around its steady state:

$$\begin{aligned}\hat{P}_t^* &= \mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \frac{mc_{ss} P_{ss}^{\epsilon+1} C_{ss}}{S_{1,ss}} (\hat{\Gamma}_{t,t+k} + \hat{mc}_{t+k} + (\epsilon+1)\hat{P}_{t+k} + \hat{C}_{t+k}) \\ &- \mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \frac{P_{ss}^{\epsilon} C_{ss}}{S_{2,ss}} (\hat{\Gamma}_{t,t+k} + \epsilon\hat{P}_{t+k} + \hat{C}_{t+k})\end{aligned}$$

where

$$S_{1,ss} = \sum_{k=0}^{\infty} (\beta\theta)^k mc_{ss} P_{ss}^{\epsilon+1} C_{ss} = mc_{ss} P_{ss}^{\epsilon+1} C_{ss} \sum_{k=0}^{\infty} (\beta\theta)^k = mc_{ss} P_{ss}^{\epsilon+1} C_{ss} \frac{1}{1-\beta\theta}$$
$$S_{2,ss} = \sum_{k=0}^{\infty} (\beta\theta)^k P_{ss}^{\epsilon} C_{ss} = P_{ss}^{\epsilon} C_{ss} \sum_{k=0}^{\infty} (\beta\theta)^k = P_{ss}^{\epsilon} C_{ss} \frac{1}{1-\beta\theta}$$

The log-linearized equation can be simplified to:

$$\hat{P}_t^* = (1-\beta\theta)\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k (\hat{mc}_{t+k} + \hat{P}_{t+k})$$

We can also write it in a recursive form:

$$\hat{P}_t^* = (1-\beta\theta)[\hat{mc}_t + \hat{P}_t] + \beta\theta\mathbb{E}_t\hat{P}_{t+1}^*$$

If we have enough many firms, by Law of Large Numbers, the fraction of firms not allowed to change their prices in one-period is approximately θ. Therefore, the aggregate price index can be expressed as

$$P_t = (\theta P_{t-1}^{1-\epsilon} + (1-\theta)(P_t^*)^{1-\epsilon})^{\frac{1}{1-\epsilon}}$$

The log-linearized form is:

$$\hat{P}_t = \theta\hat{P}_{t-1} + (1-\theta)\hat{P}_t^*$$

Therefore the inflation is:

$$\begin{aligned}
\pi_t = \hat{P}_t - \hat{P}_{t-1} &= (1-\theta)(\hat{P}_t^* - \hat{P}_{t-1}) \\
&= (1-\theta)\left[(1-\beta\theta)(\hat{mc}_t + \hat{P}_t) + \beta\theta\mathbb{E}_t\hat{P}_{t+1}^* - \hat{P}_{t-1}\right] \\
&= (1-\theta)\left[(1-\beta\theta)\hat{mc}_t + \beta\theta\mathbb{E}_t(\hat{P}_{t+1}^* - \hat{P}_t) + \pi_t\right]
\end{aligned}$$

Hence,

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \frac{(1-\theta)(1-\beta\theta)}{\theta}\hat{mc}_t$$

This is called the New-Keynesian Phillips curve. We can read from it that the more sticky prices are (higher θ), the smaller the elasticity of inflation to marginal cost is.

6.3 Equilibrium

6.3.1 Natural level of output, output gaps

The natural level of output is defined as the output level when prices are completely flexible, which implies $\hat{mc}_t = 0$.

Recall

$$\hat{mc}_t = (\gamma + \sigma)\hat{Y}_t - (1+\gamma)\hat{A}_t$$

If $\hat{mc}_t = 0$, we get the natural level of output:

$$\hat{Y}_t^n = \left(\frac{1+\gamma}{\gamma+\sigma}\right)\hat{A}_t$$

i.e. the output change only depends on technological shocks.

Define the output gap, $x_t = \hat{Y}_t - \hat{Y}_t^n$.

$$x_t = \hat{Y}_t - \hat{Y}_t^n = \frac{\hat{mc}_t}{\gamma+\sigma} + \frac{1+\gamma}{\gamma+\sigma}\hat{A}_t - \frac{1+\gamma}{\gamma+\sigma}\hat{A}_t = \frac{\hat{mc}_t}{\gamma+\sigma}$$

Therefore, $\hat{mc}_t = (\gamma+\sigma)x_t$.

6.3.2 New-Keynesian equilibrium

We can rewrite out Phillips curve and IS curve in terms of the output gap.

Recall the New-Keynesian Phillipse curve:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \frac{(1-\theta)(1-\beta\theta)}{\theta}\hat{mc}_t$$

Replace $\hat{mc}_t$ with x_t, we have

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t \tag{1}$$

where $\kappa = \frac{(1-\theta)(1-\beta\theta)(\gamma+\sigma)}{\theta}$.

Recall the Keynesian IS curve:

$$\hat{Y}_t = \mathbb{E}_t \hat{Y}_{t+1} - \frac{1}{\sigma}[\hat{i}_t - \mathbb{E}_t \pi_{t+1}]$$

Replace $\hat{Y}_t$ with x_t:

$$x_t + \hat{Y}_t^n = \mathbb{E}_t(x_{t+1} + \hat{Y}_{t+1}^n) - \frac{1}{\sigma}[\hat{i}_t - \mathbb{E}_t \pi_{t+1}]$$

Rearrange it, we get:

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1} - r_t^n) \tag{2}$$

where $r_t^n = \sigma \mathbb{E}_t(\hat{Y}_{t+1}^n - \hat{Y}_t^n) = \sigma \frac{1+\gamma}{\gamma+\sigma}\mathbb{E}_t(\hat{A}_{t+1} - \hat{A}_t)$ is a representation of expected technology changes.

The equilibrium in this New-Keynesian economy is thus defined as follows: given the stochastic process $\{\hat{A}_t\}$ and a monetary policy plan $\{\hat{i}_t\}$, a **rational expectation equilibrium** (REE) is the set of stochastic processes $\{x_t, \pi_t\}$ satisfying (1) and (2).

6.4 Simple monetary policy

Consider the monetary authority following a simple decision rule specified by the reaction function:

$$\hat{i}_t = \phi_\pi \pi_t$$

Substitute the monetary policy in (2), then the REE can be characterized by

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \underbrace{\frac{1}{\sigma + \kappa\phi_\pi}\begin{bmatrix} \alpha\beta + \kappa & \sigma\kappa \\ 1 - \beta\phi_\pi & \sigma \end{bmatrix}}_{\mathbf{F}} \begin{bmatrix} \mathbb{E}_t \pi_{t+1} \\ \mathbb{E}_t x_{t+1} \end{bmatrix} + \frac{1}{\sigma + \kappa\phi_\pi}\begin{bmatrix} \kappa \\ 1 \end{bmatrix} r_t^n$$

It can be shown if $\phi_\pi \geq 1$, $\mathbf{F}$ is a stable matrix (all its eigenvalues are inside the unit circle), and there exists a unique stable REE $\{x_t, \pi_t\}$ satisfying (1) and (2).

7 Search and Matching

This section presents a simple model that a single unemployed worker is faced with a contemporaneous decision of accepting a stochastic wage offer and be employed immediately, or accepting some unemployment benefit payout and thus remaining unemployed for one more period.

7.1 Wage Offer Distribution

Suppose wage offer is a random variable drawn from a distribution with cumulative probability:

$$\mathbb{P}(w \leq W) = F(W)$$

We assume F satisfying the following properties:

- F is nondecreasing and continuous;
- There exists a finite B such that $F(B) = 1$

Then the expected value of w is:

$$\mathbb{E}(w) = \int_0^B w dF(w) = [wF(w)]_0^B - \int_0^B F(w)dw = B - \int_0^B F(w)dw$$

7.2 One-Sided Job Search

Assume a worker facing a job search decision. Given wage offer w, the decision process is a binary action:

- accept w, and receive w per period forever;
- reject w, receive unemployment benefit c, and wait for next period offer w'.

Let $V(w)$ be the woker's value function:

$$V(w) = \begin{cases} w + \beta w^2 + \cdots = \frac{w}{1-\beta}, & \text{if accept} \\ c + \beta \int_0^B V(w') dF(w') & \text{if reject} \end{cases}$$

The worker's optimal decision is to choose the action that yields the highest payoff. Note that $\frac{w}{1-\beta}$ is an increasing function in w, and $c + \beta \int_0^B V(w')dF(w')$ is a constant determined by the distribution function and is irrespective of w. So there must exists a threshold $\overline{w}$ such that the worker optimally accept the offer if $w \geq \overline{w}$, and reject if $w < \overline{w}$.

Therefore, we can write the worker's Bellman equation as

$$V(w) = \begin{cases} w + \beta w^2 + \cdots = \frac{w}{1-\beta}, & \text{if } w \geq \overline{w} \\ c + \beta \int_0^B V(w') dF(w') = \frac{\overline{w}}{1-\beta} & \text{if } w < \overline{w} \end{cases}$$

$\overline{w}$ is called the **reservation wage**.

7.2.1 Reservation Wage

We argue that given distribution F, the reservation wage $\overline{w}$ is uniquely determined. To prove this argument, we arrange the second the equation in the Bellman equation:

$$
\begin{aligned}
\frac{\overline{w}}{1-\beta} &= c + \beta \int_0^B V(w')dF(w') \\
&= c + \beta \int_0^{\overline{w}} \frac{\overline{w}}{1-\beta} dF(w') + \beta \int_{\overline{w}}^B \frac{w'}{1-\beta} dF(w') \\
&= c + \frac{\beta \overline{w}}{1-\beta} \int_0^{\overline{w}} dF(w') + \beta \int_{\overline{w}}^B \frac{w'}{1-\beta} dF(w')
\end{aligned}
$$

Therefore,

$$
\overline{w} \int_0^{\overline{w}} dF(w') = c + \frac{\beta}{1-\beta} \int_{\overline{w}}^B w' dF(w')
$$

Add $\int_{\overline{w}}^B \overline{w} dF(w')$ on both sides, we have:

$$
\underbrace{\overline{w} - c}_{\text{cost of being unemployed}} = \underbrace{\frac{\beta}{1-\beta} \int_{\overline{w}}^B (w' - \overline{w}) dF(w')}_{\text{expected benefit of searching one more period}} \tag{3}
$$

Let $g(w) = w - c$, and $h(w) = \frac{\beta}{1-\beta} \int_w^B (w' - w) dF(w')$.

$$
\begin{aligned}
h'(w) &= \frac{\beta}{1-\beta} [F(w) - 1] < 0 \\
h''(w) &= \frac{\beta}{1-\beta} F'(w) > 0
\end{aligned}
$$

So $h(w)$ is a decreasing convex function of w, while $g(w)$ is an increasing function of w. Therefore there is a unique $\overline{w}$ that equals $h(w)$ and $g(w)$, which proves the unique existence of the reservation wage.

If the unemployment compensation c increases, $g(w)$ is shifted downwards, it follows $\overline{w}$ increases — the worker's threshold increases and demands higher wage offers.

7.2.2 Mean Preserving Spread

What happened if we change the wage distribution F? Suppose we change the distribution so as to increase the probability of getting higher wages as well as lower wages. Let's call the new distribution F'. F' satisfies the following properties:

$$
\int_0^B [F'(w) - F(w)] dw = 0
$$

$$
F'(w) - F(w) = \begin{cases} \leq 0 \text{ for } w \geq \hat{w} \\ \geq 0 \text{ for } w \leq \hat{w} \end{cases}
$$

Such a redistribution represents a **mean-preserving increase in risk** since it redistribute the mass of probability toward the tails of the distribution while keeping the mean constant.

To see the effect such redistribution, we rearrange Equation (3):

$$\begin{aligned}
\overline{w} - c &= \frac{\beta}{1-\beta}\int_{\overline{w}}^{B}(w' - \overline{w})dF(w') \\
&= \frac{\beta}{1-\beta}\int_{0}^{B}(w' - \overline{w})dF(w') - \frac{\beta}{1-\beta}\int_{0}^{\overline{w}}(w' - \overline{w})dF(w') \\
&= \frac{\beta}{1-\beta}[\mathbb{E}(w) - \overline{w}] - \frac{\beta}{1-\beta}\int_{0}^{\overline{w}}(w' - \overline{w})dF(w') \\
&= \frac{\beta}{1-\beta}[\mathbb{E}(w) - \overline{w}] + \frac{\beta}{1-\beta}\int_{0}^{\overline{w}}F(w')dF(w')
\end{aligned}$$

Therefore,

$$\overline{w} - c = \beta[\mathbb{E}(w) - c] + \beta\int_{0}^{\overline{w}}F(w')dF(w')$$

Let $g(w) = w - c$, and $h(w) = \beta[\mathbb{E}(w) - c] + \beta\int_0^w F(w')dF(w')$.

$$\begin{aligned}
h'(w) &= F(w) > 0 \\
h''(w) &= F'(w) > 0
\end{aligned}$$

Therefore, $h(w)$ is an increasing convex function of w. The intersection of $g(w)$ and $h(w)$ uniquely determines $\overline{w}$.

If we replace F with F', note that the mean-preserving condition implies

$$\int_0^y F'(w)dw \geq \int_0^y F(w)dw \qquad \text{for } 0 \leq y \leq B$$

So $h\,(w)$ is shifted upward. The result is the threshold $\overline{w}$ increases. The intuition is, greater probability of high wage offers increases the value of searching, therefore increases the threshold of accepting an offer.

8 References

1. Ljungqvist, Lars, and Thomas J. Sargent. *Recursive macroeconomic theory*. MIT press, 2012.
2. Miao, Jianjun. *Economic dynamics in discrete time*. MIT press, 2014
3. Kam, Timothy. Dynamic Macroeconomics.