# KEYPOINT SUMMARY

*Note: This summary is neither exhaustive nor formal. It is meant to give the intuition behind the mathmatical concepts, and extract core idea for essential proofs.*

## I. CARDINALITY

1. $A$ is a set, $\overline{\overline{A}}$ denote the cardinality of $A$.
   - $\overline{\overline{A}} = \overline{\overline{B}}$ if there exists a bijection $A \sim B$.
   - $\overline{\overline{A}} \le \overline{\overline{B}}$ if there exists an injection $A \hookrightarrow B$.
   - $\overline{\overline{\emptyset}} = 0$, $\overline{\overline{\{1, 2, \dots, n\}}} = n$, $\overline{\overline{\mathbb{N}}} = d$, $\overline{\overline{\mathbb{R}}} = c$.
2. Schröder-Bernstein Theorem:
   - $A \hookrightarrow B,\ B \hookrightarrow A \Rightarrow A \sim B$
   - $\overline{\overline{A}} \le \overline{\overline{B}},\ \overline{\overline{B}} \le \overline{\overline{A}} \Rightarrow \overline{\overline{A}} = \overline{\overline{B}}$
3. $A$ is finite/denumerable/countable:

$$\left.\begin{array}{r} \left.\begin{array}{r} A \sim \emptyset \\ A \sim \{1,2,\dots,n\} \end{array}\right\} \text{finite} \\ A \sim N \qquad \text{denumerable} \end{array}\right\} \text{countable}$$

   Otherwise, $A$ is uncountable.
4. Important results on cardinality of sets:
   - $\mathbb{N}^k \sim \mathbb{N}$

     *Proof.* Consider $f(n_1, \dots, n_k) = p_1^{n_1} \cdots p_k^{n_k}$.
   - $\mathbb{N}^\mathbb{N} \sim \mathbb{R}$

     *Proof.* Code $\mathbb{N}^\mathbb{N}$ into binary strings and show it is uncountable. Consider $f(n_1, n_2, \dots) = \frac{1}{10^{n_1}} + \frac{1}{10^{n_1+n_2}} + \cdots$.
   - $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$

     *Proof.* $\mathcal{P}(\mathbb{N}) \to \mathbb{R}$: $(n_1, n_2, n_3, \dots) \to 0.b_1b_2b_3\dots$.
     $\mathbb{R} \to \mathcal{P}(\mathbb{Q})$: $\{a \in \mathbb{Q} : a < x, x \in \mathbb{R}\}$.
   - $\mathbb{R} \times \mathbb{R} \sim R$, $\mathbb{R}^k \sim \mathbb{R}$

     *Proof.* $(0.a_1a_2\dots, 0.b_1b_2\dots) \to (0.a_1b_1a_2b_2\dots)$.
   - The set of all real valued functions on $[0, 1] \sim 2^c$

     *Proof.* Note $A \hookrightarrow \mathcal{P}([0, 1] \times \mathbb{R})$.
   - $A$ is any set, $\overline{\overline{A}} < \overline{\overline{\mathcal{P}(A)}}$.

     *Proof.* Clearly there exists $f : A \hookrightarrow \mathcal{P}(A)$. But $f$ cannot be surjective. Consider $X = \{a \in A : a \notin f(a)\}$.
   - The union of a countable family of countable sets is countable.
   - The union of a cardinality $c$ family of sets each with cardinality $c$ has cardinality $c$.

     *Proof.* Consider $\{A_\alpha\}_{\alpha\in S}$. There exists a bijection $f_\alpha : A_\alpha \leftrightarrow \mathbb{R}$. Define $f : A \hookrightarrow S \times \mathbb{R}$ as $f(x) = (\alpha, f_\alpha(x))$ where $x \in A_\alpha$.

## II. VECTOR SPACES

1. A vector space (linear space) over $\mathbb{R}$ is a set $V$ with two operations *addition* and *scalar multiplication* such that
   (a) $u + v = v + u$
   (b) $(u + v) + w = u + (v + w)$
   (c) $\exists 0 \in V$, $0 + v = v$
   (d) $(\alpha + \beta)u = \alpha u + \beta u$
   $\alpha(u + v) = \alpha u + \alpha v$
   (e) $(\alpha\beta)u = \alpha(\beta u)$
   (f) $1u = u$

   where $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$.
   - A vector space is a space closed under addition and scalar multiplication, i.e. it is a space that allows linear operations.
   - A set of vectors $\{v_1, v_2, \dots, v_n\}$ in $V$ is called linearly independent if $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \Rightarrow a_1 = a_2 = \cdots = a_n = 0$.
2. A normed vector space is a vector space $V$ over $\mathbb{R}$ with a function $\|\cdot\| : V \to \mathbb{R}$ such that
   (a) $\|u\| \ge 0$, $\|u\| = 0$ iff $u = 0$
   (b) $\|\alpha u\| = |\alpha| \|u\|$
   (c) $\|u + v\| \le \|u\| + \|v\|$
   - A normed vector space is a vector space where the length of vectors can be measured.
   - Euclidean norm: $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$
   - Infinity norm: $\|x\|_\infty = \max_{1\le i\le n} |x_i|$
   - $p$-norm: $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$

     *Proof.* Minkowski's inequality:
     $\left(\sum_k^n |x_k + y_k|^p\right)^{\frac{1}{p}} \le \left(\sum_k^n |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_k^n |y_k|^p\right)^{\frac{1}{p}}$
   - If $1 \le p \le q \le \infty$, then $\|x\|_p \ge \|x\|_q$.

     *Proof.* Normalize $x$ to $\frac{x}{\|x\|_p}$ so that $\|x\|_p = 1$. Then it is easy to see $\|x\|_q \le 1$ because for each element $|x_i|^q \le |x_i|^p$.
3. An inner product space is a vector space $V$ over $\mathbb{R}$ with a function $\cdot : V \times V \to \mathbb{R}$ such that
   (a) $u \cdot u \ge 0$, $u \cdot u = 0$ iff $u = 0$
   (b) $u \cdot v = v \cdot u$
   (c) $(u + v) \cdot w = u \cdot w + v \cdot w$
   (d) $(\alpha u) \cdot v = \alpha(u \cdot v)$
   - Angle $\theta$ between $u, v$: $u \cdot v = \cos\theta \|u\| \|v\|$.
   - $u, v$ are orthogonal if $u \cdot v = 0$.
   - Every inner product space is a normed space if define $\|u\| = (u \cdot u)^{\frac{1}{2}}$ as the norm.
   - Cauchy-Schwarz Inequality: $|u \cdot v| \le \|u\| \|v\|$

     *Proof.* $f(\lambda) = (u - \lambda v) \cdot (u - \lambda v) \ge 0$, $\forall\lambda$.
     Substitute in $\lambda = \frac{u\cdot v}{\|v\|^2}$.

## III. METRIC SPACES

1. A metric space $(X, d)$ is a set $X$ together with a function $d : X \times X \to \mathbb{R}$ such that
   (a) $d(x, y) \geq 0$, $d = 0$ iff $x = y$
   (b) $d(y, x) = d(x, y)$
   (c) $d(x, y) \leq d(x, z) + d(z, y)$
   - A metric space is a set for which distances between all members of the set are defined.
   - Any normed space $(V, \|\cdot\|)$ is a metric space.
2. Suppose $(X, d)$ is a metric space, and $S \subset X$. $(S, d_S)$ is a metric subspace if we define $d_S(x, y) = d(x, y)$ for $x, y \in S$.
   - $\forall a \in S$, $B_r^S(a) = S \cap B_r^X(a)$
   - $A$ is open in $S \Leftrightarrow A = S \cap U$, $U$ is open in $X$;
     $A$ is closed in $S \Leftrightarrow A = S \cap C$, $C$ is closed in $X$.
     *Proof.* $A$ open in $S \Rightarrow A = \cup_{x \in A} B^S_{r_x}(x) = \cup_{x \in A}(S \cap B_{r_x}(x)) = S \cap (\cup_{x \in A} B_{r_x}(x))$. Thus $U = \cup_{x \in A} B_{r_x}(x)$.
3. Limit and Isolated Points
   - $x$ is a limit point in $A$ if every $B_r(x)$ contains points of $A$ other than $x$.
   - $x$ is a limit point iff $\exists (x_n) \subset A$ and $x_n \to x$.
   - $x$ is a isolated point if $\exists r$ such that $B_r(x) \cap A = \{x\}$.
4. Interior, Exterior and Boundary
   - $x \in \operatorname{int} A$ if $\exists r (B_r(x) \subset A)$
   - $x \in \operatorname{ext} A$ if $\exists r (B_r(x) \subset A^C)$
   - $\operatorname{ext} A = \operatorname{int} A^C$, $\operatorname{int} A = \operatorname{ext} A^C$
   - $x \in \partial A$ (boundary of $A$) if any $B_r(x)$ contains both points of $A$ and points of $A^C$.
   - $X = \operatorname{int} A \cup \operatorname{ext} A \cup \partial A$
5. Open Sets
   - $A$ is open if $A = \operatorname{int} A$.
   - $A$ is open $\Rightarrow A = \cup_{x \in A} B_{r_x}(x)$
   - If $A_i$ are open, $\cap_{i=1}^k A_i$ is open;
     If $A_i$ are open, $\cup_{i \in I} A_i$ is open.
   - $\operatorname{int} A$ is open; $\operatorname{ext} A$ is open.
6. Closed Sets
   - $A$ is closed if $A^C$ is open.
   - $A$ is closed iff $\overline{A} = A$.
   - $A$ is closed iff $A$ contains all its limit points.
   - $A \subset R^k$ is closed iff $A$ is complete.
   - If $B_i$ are closed, $\cup_{i=1}^k B_i$ is closed;
     If $B_i$ are closed, $\cap_{i \in I} B_i$ is closed.
   - Closure of a set is closed.
   - Closed does *not* imply bounded.
7. Closure
   - $\overline{A} = A \cup \{\text{limit point of } A\}$
   - $\overline{A} = \operatorname{int} A \cup \partial A$
   - $\overline{A} = A \cup \partial A$
   - $\overline{A} = (\operatorname{ext} A)^C$
   - $x \in \overline{A}$ iff every $B_r(x)$ contains a point of $A$.
   - $x \in \overline{A}$ iff there exists $(x_n) \subset A$ with $x_n \to x$.

## IV. SEQUENCES AND CONVERGENCE

1. $(x_n)$ converges to $x$ if $\forall \epsilon, \exists N, [\forall n > N \Rightarrow d(x, x_n) < \epsilon]$.
2. $(x_n)$ is Cauchy if $\forall \epsilon, \exists N, [\forall m, n > N \Rightarrow d(x_m, x_n) < \epsilon]$.
   - Every convergent/Cauchy sequence is bounded.
     *Proof.* convergence $\Rightarrow$ $(x_n)$ is bounded after some $N$, left only finite elements.
   - $\mathbb{R}^k$: sequence $(x_n)$ converges $\Leftrightarrow$ $(x_n)$ is Cauchy.
   - $X$: sequence $(x_n)$ converges $\not\Leftarrow\!\Rightarrow$ $(x_n)$ is Cauchy.
3. A metric space is complete if every Cauchy sequence converges in itself.
   - $S \subset \mathbb{R}^k$ is complete *iff* it is closed.
4. Monotone Convergence Theorem: if a sequence is increasing (decreasing) and bounded by a supremum (infimum), it will converge to the supremum (infimum).
   *Proof.* Let $c = \sup_n \{a_n\}$. $\forall \epsilon > 0, \exists N$ s.t. $c - \epsilon < a_N \leq a_n \leq c$, $\forall n > N$. As $\epsilon \to 0$, $a_n \to c$.
5. Banach Fixed-Point Theorem: If $(X, d)$ is a *complete* metric space, and $f : X \to X$ is a *contraction*, i.e. $\exists \lambda \in [0, 1)$ such that $d(f(x), f(y)) \leq \lambda d(x, y)$, then there exists a unique fixed point $f(x) = x$.
   *Proof.* First show $(x_n)$ is Cauchy, then prove $d(x, f(x)) \to 0$.

## V. SEQUENCES AND COMPACTNESS

1. Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.
   *Proof.* Geometric intuition: if a sequence in $\mathbb{R}^k$ is bounded, we can always trap it in a whatever small subspace. Construct a convergent subsequence by trapping it a smaller and smaller subspace.
2. A metric space is (sequentially) compact if every sequence has a convergent subsequence.
   - $\mathbb{R}^k$: compact $\Leftrightarrow$ closed, bounded (Beine-Borel Thm)
   - $X$: compact $\not\Leftarrow\!\Rightarrow$ closed, bounded.
     *Proof.* Consider $X = C[0, 1]$. $\overline{B_1(\mathbf{0})} \subset X$ is closed and bounded, but *not* compact.
   - Compactness is sort of a topological generalization of finiteness. For example, if a set A is finite then every function $f : A \to \mathbb{R}$ is bounded and has max/min. If A is compact, the every *continuous* function $f : A \to \mathbb{R}$ is bounded and has max/min.

## VI. LIMITS AND CONTINUITY

1. Let $f : X \to Y$. The following are equivalent:
   (a) $\lim_{x \to a} f(x) = b$;
   (b) $x_n \to a \Rightarrow f(x_n) \to b$;
   (c) $x \in B^X_\delta(a) \setminus \{a\} \Rightarrow f(x) \in B^Y_\epsilon(b)$.

*Proof.* (ii)⇒(iii): Suppose the opposite. Let $x_n \in B^X_{1/n}(a)$, then $x_n \to a$ and $f(x_n) \to b$, but by assumption $\exists\epsilon$ s.t. $f(x_n) \notin B^Y_\epsilon(b)$. contradiction. (iii)⇒(ii): Suppose $x_n \to a$. (iii) $\Rightarrow \forall\epsilon\ \exists\delta\ \exists x_n \in B^X_\delta(a) \Rightarrow f(x_n) \in B^Y_\epsilon(b) \Rightarrow f(x_n) \to b$.

2. $f : X \to Y$ is continuous at $a \in X$ if $a$ is an isolated point, or $\lim_{x\to a} f(x) = f(a)$.
   - Every function is continuous at isolated points.
   - Intuitively, a continuous function is a function for which sufficiently small changes in the input result in arbitrarily small changes in the output.
3. Quantitative Meansures of Continuity
   - $f : X \to Y$ is Lipschitz continuous if there exists a constant $M$ such that $d_Y(f(x_1), f(x_2)) \le M d_X(x_1, x_2)$. $M$ is called the Lipschitz constant.
     - If $f : (a, b) \to \mathbb{R}$ is differentiable and $f$ is bounded then $f$ is Lipschitz continuous. This follows from the mean value theorem: if $|f(\xi)| \le L$ for all $\xi \in (a, b)$ then $|f(x)-f(y)| \le |f(\xi)||x-y| \le L|x-y|$ for all $x, y \in (a, b)$.
     - If $f$ is Lipschitz continuous and differentiable at $x$ then $f(x)$ is bounded by the Lipschitz constant.
   - $f : X \to Y$ is Hölder continuous with exponent $\alpha \in (0, 1]$ if there exists a constant $M$ such that $d_Y(f(x_1), f(x_2)) \le M\,[d_X(x_1, x_2)]^\alpha$.
4. Continuity and Compactness: $f : X \to Y$ is continuous, if $K \subset X$ is compact, then $f(K)$ is compact in Y.

   *Proof.* $\forall(y_n) \subset f(K)$, $\exists(x_n) : f(x_n) = y_n$. $K$ compact $\Rightarrow \exists(x_{n_j}) \to x$; $f$ continuous $\Rightarrow f(x_{n_j}) \to f(x) \Rightarrow y_{n_j} \to f(x)$.

   *Cor.* Continuous function on a compact set is bounded.
5. Extreme Value Theorem: Each continuous function on a compact set attains its maximum and minimum.

   *Proof.* $K$ compact $\Rightarrow f(K)$ compact $\Rightarrow f(K)$ closed and bounded $\Rightarrow$ exist least upper bound $\gamma$ and $\gamma \in f(K)$ (take a sequence approaching $\gamma$ and extract its convergent subsequence). Therefore, $\exists x_0 \in K$ s.t. $\gamma = f(x_0)$.
6. Continuity and Open Sets: The following statements are equivalent:
   (a) $f : X \to Y$ is continuous on $X$;
   (b) $f^{-1}(E)$ is open whenever $E$ is an open set in $Y$;
   (c) $f^{-1}(E)$ is closed whenever $E$ is a closed set in $Y$.

   *Proof.* $f$ continuous $\Rightarrow \forall\epsilon\ \exists\delta\ f(B_\delta(x)) \subset B_\epsilon(f(x)) \subset E \Rightarrow B_\delta(x) \subset f^{-1}(E)$. On the other hand, $f^{-1}(B_\epsilon(f(x)))$ is an open set $\Rightarrow \exists B_\delta(x) \subset f^{-1}(B_\epsilon(f(x))) \Rightarrow f(B_\delta(x)) \subset B_\epsilon(f(x)) \Rightarrow f$ is continuous.

   *Note.* Continuous functions do *not* necessarily map open sets to open sets, or closed sets to closed sets.

   *Cor.* If $f : X \to \mathbb{R}$ is continuous, $\{x : f(x) < 0\}$ is open.
7. The function $f : X \to Y$ is uniformly continuous if for each $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \epsilon$, forall $x, x' \in X$.
   - In general continuity, $\delta$ depends on both $\epsilon$ and $x$.
   - In uniform continuity, $\delta$ depends only on $\epsilon$, not on $x$.
   - If $f$ is continuous and $X$ is *compact*, then $f$ is uniformly continuous.
   - A continuous real-valued functions defined on closed and bounded subset of $\mathbb{R}^n$ is uniformly continuous.
   - If $f$ is Lipschitz continuous then it is uniformly continuous. The converse is not true, for example, $f(x) = \sqrt{|x|}$ is uniformly continuous but not Lipschitz.
8. Uniform continuity and Cauchy sequences: If $f : X \to Y$ is uniformly continuous, then $(x_n)$ is Cauchy in $X \Rightarrow (f(x_n))$ is Cauchy in $Y$. If $f$ is only continuous, $(f(x_n))$ may not converge (consider $f(x) = \frac{1}{x}$ on $(0, \infty)$).
9. Continuous Extension Theorem: If $f : A \to Y$ is uniformly continuous, then $f$ can be uniquely extended to $\overline{A}$ maintaining the uniform continuity. $\bar{f}(x)$ on the boundary of $A$ can be unambiguously defined by $\lim_{n\to\infty} f(x_n)$ for $x \in \partial A$ with $(x_n) \subset A$ and $x_n \to x$.

## VII. CONVERGENCE OF FUNCTIONS

1. Let $S$ be a set and $(Y, \rho)$ a metric space. A sequence of functions $f_n : S \to Y$ converges to a function $f : S \to Y$ pointwise if for each $x \in S$ and any $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $n > N \implies \rho(f_n(x), f(x)) < \epsilon$. (Note: $N$ depends on both $x$ and $\epsilon$)
2. A sequence $f_n : S \to Y$ converge uniformly to a function $f : S \to Y$ if $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $n > N \implies \rho(f_n(x), f(x)) < \epsilon, \forall x \in S$. (Note: convergence is uniform with respect to $x$. $N$ depends only on $\epsilon$, not on $x$)
   - Uniform convergence $\not\Leftarrow\!\!\Rightarrow$ Pointwise convergence
   - $(f_n)$ converging to $f$ uniformly is equivalent to $f_n \to f$ in uniform metric: $d_u(f_n, f) = \sup_{x\in S} \rho(f_n(x), f(x))$.
3. A sequence $(f_n)$ is uniformly Cauchy if $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$, such that $m, n > N \implies \rho(f_m(x), f_n(x)) < \epsilon, \forall x \in S$.
   - $(f_n)$ is uniformly Cauchy if and only if it is Cauchy in uniform metric.
4. Theorem: Let $\mathcal{F} = \{f | f : S \to Y\}$. If $(Y, \rho)$ is complete, then every Cauchy sequence $(f_n) \subset \mathcal{F}$ converges uniformly to some $f \in \mathcal{F}$, i.e. $\mathcal{F}$ is complete.

   *Proof.* Let $(f_n)$ be Cauchy $\Rightarrow \sup|f_m(x) - f_n(x)| < \epsilon \Rightarrow |f_m(x) - f_n(x)| < \epsilon, \forall x \Rightarrow \forall x, (f_n(x))$ is Cauchy in $Y$. $Y$ is complete $\Rightarrow f_n(x) \to f(x) \in Y$. $\rho(f_m(x), f(x)) \le \rho(f_m(x), f_n(x)) + \rho(f_n(x), f(x))$. Let $n \to \infty$, we have $\rho(f_m(x), f(x)) \le \epsilon, \forall x$. i.e. $f_m \to f$ uniformly.
5. Theorem: Suppose $(f_n)$ is a sequence of continuous functions from $X$ to $Y$. Suppose $f_n \to f$ uniformly. Then $f$ is also continuous (uniform limits of continuous functions are continuous).

   *Proof.* $f_n \to f$ uniformly $\Rightarrow \rho(f_n(x), f(x)) < \epsilon$. $f_n$ continuous $\Rightarrow d(x, x_0) < \delta \Rightarrow \rho(f_n(x), f_n(x_0)) < \epsilon$. $\rho(f(x), f(x_0)) \le \rho(f(x), f_n(x)) + \rho(f_n(x), f_n(x_0)) + \rho(f_n(x_0), f(x_0)) < 3\epsilon$. Therefore $f$ is continuous at $x_0$.
6. Corollary: Let $C_b(X, Y)$ be the set of continuous and bounded functions from $X$ to $Y$. If $(Y, \rho)$ is complete, then $C_b(X, Y)$ is a complete metric space when equipped with the uniform metric.
7. Suppose $f_n \to f$ uniformly, then
   - $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$ uniformly;
   - $\int_a^x f_n(t)dt \to \int_a^x f(t)dt$ uniformly;

· $\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx$;

· The limit of differentiable may not be differentiable.

## VIII. COMPACTNESS

1. A metric space $(X,d)$ is <u>sequentially compact</u> if every sequence in $X$ has a convergent subsequence.
2. A space $K$ is <u>compact</u> if every open cover of $K$ ($K \subset \cup_{i\in I} U_i$) has a finite subcover ($K \subset \cup_{i=1}^N U_i$).
3. If $(X,d)$ is a metric space, $K \subset X$ is compact iff $K$ is sequentially compact. This is not true in general topological space.
4. Consequences of compactness:

   · $K$ compact $\Rightarrow$ $K$ is closed and bounded;

   · In $\mathbb{R}^n$, $K$ is compact $\Leftrightarrow$ $K$ is closed and bounded;

   · A closed subset of a compact space is compact;
   *Proof.* Let $A \subset X$ be closed. Let $\{U_i\}$ cover $A$. Then $(\cup_i U_i) \cup (X \setminus A)$ is an open cover for $X$. There exists finite subcover $(\cup_{i=1}^N U_i) \cup (X \setminus A)$. Then $\cup_{i=1}^N U_i$ is a finite subcover for $A$.

   · $f : X \to Y$ being continuous, if $K \subset X$ is compact, then $f(K)$ is compact in $Y$;
   *Proof.* Let $\{U_i\}$ cover $f(K)$. Then $f^{-1}(\cup U_i) = \cup f^{-1}(U_i)$ cover $K$. $K$ is compact $\Rightarrow \exists$ finite subcover $\cup_{i=1}^N f^{-1}(U_i)$. Then $\cup_{i=1}^N U_i$ is a finite subsover for $f(K)$.

   · $f : X \to Y$ being continuous, if $K \subset X$ is compact, then $f$ is uniformly continuous on $K$;
   *Proof.* $f$ is continuous $\Rightarrow \forall\epsilon > 0$, $\forall x \in X$, $\exists\delta_x > 0$ such that $|x - x'| < \delta_x \Rightarrow |f(x) - f(x')| < \epsilon$. $\cup_{x\in X} B(x, \delta_x)$ cover $X$, $X$ is compact, there exists finite subcover. Let $\delta = \min\{\delta_{x_1}, \dots, \delta_{x_n}\}$. Then there exists a $\delta$ independent of $x$.

   · $f : X \to Y$ being continuous, if $K \subset X$ is compact, then $f$ achieves its maximum and minimum on $K$.
5. A set $X$ is <u>totally bounded</u> if for any $\delta$, there exist finitely many points $x_1, \dots x_N \in X$ such that $X \subset \cup_{i=1}^N B_\delta(x_i)$.

   · Totally boundedness $\not\Longleftarrow\!\Rightarrow$ boundedness.

   · A subset $E$ in $\mathbb{R}^n$ is totally bounded iff $E$ is bounded.

   · A subset of a metric space is totally bounded iff its closure is totally bounded.
6. <u>Theorem</u>: A space $X$ is compact if and only if it is complete and totally bounded.
   *Proof.* ($\Rightarrow$) $\{B_\delta(x)\}_{x\in X}$ cover $X \Rightarrow$ finite subcover $\Rightarrow$ totally bounded. ($\Leftarrow$) Let $(x_n) \subset X$. Cover $X$ with finite many balls of radius 1. There must be a subsequence $(x_n^1) \subset (x_n)$ trapped in of one these balls, i.e. $(x_n^1) \subset B_1$. Cover $X$ with finite many balls of radius $\frac{1}{2}$, then $\exists (x_n^2) \subset B_{\frac{1}{2}}$. Keep going, $(x_n^3) \subset B_{\frac{1}{3}}, \dots$ Let $y_k = x_k^k$. Then $(y_n)$ is Cauchy, by completeness, $(y_n)$ is a convergent subsequence of $(x_n)$.
7. Let $\mathcal{F} \subset C(X,Y)$ (continuous functions from $X$ to $Y$) $\mathcal{F}$ is <u>uniformly equicontinuous</u> if $\forall\epsilon > 0$, $\exists\delta > 0$, such that $d(x,x') < \delta \Rightarrow \rho(f(x), f(x')) < \epsilon$, $\forall x, x' \in X$, $\forall f \in \mathcal{F}$. (The same $\delta$ holds for all $x \in X$ and all $f \in \mathcal{F}$)

   · Uniformly continuity is usually shown by showing the family of functions $\mathcal{F}$ is Lipschitz with the same constant for all $f \in \mathcal{F}$; or every $f$ is Holder continuous with the same $\alpha$ and the same Holder constant.
8. <u>Arzela-Ascoli theorem</u>: $\mathcal{F} \subset C(X, \mathbb{R}^n)$ is *compact* iff $X$ is compact and $\mathcal{F}$ is closed,bounded and uniformly equicontinuous.
   *Proof.*

   · Uniform equicontinuous $\Rightarrow d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon$;

   · $X$ totally bounded $\Rightarrow \exists x_1, \dots x_n$, $|x - x_i| < \delta$, and this implies $|f(x) - f(x_i)| < \epsilon$;

   · $|f(X)| \le K$ totally bounded $\Rightarrow \exists y_1 \dots y_m$, $|f(x) - y_i| < \epsilon$;

   · Construct discrete function $\alpha : \{x_1 \dots x_n\} \to \{y_1 \dots y_m\}$ (there are at most $m^n$ such $\alpha$);

   · Find $g_\alpha$ for each $\alpha$ such that $|g_\alpha(x_i) - \alpha(x_i)| < \epsilon$;

   · For any $f \in \mathcal{F}$, choose the $\alpha$ such that $|f(x_i) - \alpha(x_i)| < \epsilon$ and the corresponding $g_\alpha$, then choose $x_i$ such that $|x - x_i| < \delta$, then we have $|f(x) - g_\alpha(x)| \le |f(x) - f(x_i)| + |f(x_i) - \alpha(x_i)| + |\alpha(x_i) - g_\alpha(x_i)| + |g_\alpha(x_i) - g_\alpha(x)|$;

   · So all $f \in \mathcal{F}$ are within the balls of finitely many $g_\alpha$.

## IX. CONNECTEDNESS

1. A space $X$ is <u>connected</u> if there do not exist nonempty open sets $U, V$ such that $U \cap V = \emptyset$ and $X = U \cup V$.

   · Equivalently, $X$ is connected if there do not exist nonempty closed sets $U, V$ such that $U \cap V = \emptyset$ and $X = U \cup V$.

   · $X$ is connected if and only if the only subset of $X$ that are both open and closed are $\emptyset$ and $X$.

   · If $A$ is connected, then $\overline{A}$ is connected.

   · Let $A_1, A_2, \dots$ be connected, if $A_i \cap A_{i+1} \neq \emptyset$, then $\cup_{i=1}^\infty A_i$ is connected.
2. A space $X$ is <u>path connected</u> if $\forall a, b \in X$, there exists a continuous function $f : [0,1] \to X$ such that $f(0) = a$, $f(1) = b$.

   · Path connected $\Rightarrow$ connected. The converse if not true, counter-example: $X = \{(x,y) : y = \sin\frac{1}{x}\} \cup \{(0,y) : -1 \le y \le 1\}$ is connected but not path-connected.
3. Connectedness in $\mathbb{R}^n$:

   · $S \subset \mathbb{R}$ is connected iff it is an interval.

   · $U$(open)$\subset \mathbb{R}^n$ is connected iff it is path-connected.
4. <u>Theorem</u>: The continuous image of a connected set is connected.
   *Proof.* Let $f : X \to Y$ be continuous. Suppose $f(X)$ is not connected. Then $\exists E \subset f(X)$ that is both open and closed. $\exists E', E'' \subset Y$ s.t. $E = E' \cap f(X)$, $E = E'' \cap f(X)$. Then we have $f^{-1}(E) = f^{-1}(E') = f^{-1}(E'') \neq \emptyset$ which is also both open and closed $\Rightarrow X$ is not connected — contradiction.
5. Corollary (Intermediate value theorem) Suppose $X$ is connected, $f : X \to \mathbb{R}$ is continuous. Let $x, y \in X$, $f(x) = a$, $f(y) = b$, $a < b$. Then for any $c \in (a,b)$, $\exists z \in X$, such that $f(z) = c$.
   *Proof.* $f(X)$ is connected because $f$ is continuous. $f(X) \subset \mathbb{R}$, so it is an interval. It then follows $c \in f(X)$ for any $c \in (a,b)$.

## X. APPLICATION: DIFFERENTIAL EQUATIONS

1. Assume $U$ (open) $\subset \mathbb{R} \times \mathbb{R}^n$, and $f : U \to \mathbb{R}^n$ is continuous. The problem of the form

$$\begin{cases} \frac{d\boldsymbol{x}}{dt} = f(t, \boldsymbol{x}) \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \end{cases}$$

is called an initial value problem (IVP). A solution to an IVP is a function $\boldsymbol{x} : I \to \mathbb{R}^n$ for some open interval $I$ containing $(t_0, \boldsymbol{x}_0)$ such that $\boldsymbol{x}$ solves the differential equation and the initial condition.

- Solving the IVP is equivalent to solving the integral equation:

$$\boldsymbol{x}(t) = t_0 + \int_{t_0}^{t} f(s, \boldsymbol{x}(s))ds$$

2. Peano's existence theorem: Assume $f$ is continuous on $U \subset \mathbb{R} \times \mathbb{R}$. Let $(t_0, x_0) \subset U$. Then there exists $h > 0$ such that the IVP has a $C^1$ solution for $t \in [t_0 - h, t_0 + h]$.

*Proof.*

- Let $A_{h,k}(t_0, x_0) = \{(t, x) : |t - t_0| \leq h, |x - x_0| \leq k\}$.
- $A_{h,k}$ is a compact $\Rightarrow f$ is bounded, $|f(t, x)| \leq M$.
- Let $h \leq \frac{k}{M}$.
- Approximate $x(t)$ at $t_0$ with

$$x_n(t) = \begin{cases} x_0 + (t - t_0)f(t_0, x_0) \\ \qquad \text{for } t \in [t_0, t_0 + \frac{h}{n}] \\ x_n(t_0 + \frac{h}{n}) + (t - (t_0 + \frac{h}{n}))f(t_0 + \frac{h}{n}, x_n(t_0 + \frac{h}{n})) \\ \qquad \text{for } t \in [t_0 + \frac{h}{n}, t_0 + \frac{2h}{n}] \\ \qquad \vdots \\ x_n(t_0 + i\frac{h}{n}) + (t - (t_0 + i\frac{h}{n}))f(t_0 + i\frac{h}{n}, x_n(t_0 + i\frac{h}{n})) \\ \qquad \text{for } t \in [t_0 + i\frac{h}{n}, t_0 + (i+1)\frac{h}{n}] \\ \qquad \vdots \end{cases}$$

- $|\frac{d}{dt}x_n(t)| = |f(\cdot)| \leq M$, so $x_n$ is Lipschitz on $A_{h,k}$ with constant $M$.
- $(x_n)$ is a sequence of closed, bounded and uniformly equicontinuous functions. Therefore $(x_n)$ is a compact subset of $C[t_0 - h, t_0 + h]$. There exists $(x_{n_j})$ such that $x_{n_j} \to x$.
- $x$ is the solution to the IVP problem.

3. Assume $f : U(\subset \mathbb{R} \times \mathbb{R}) \to \mathbb{R}$ is continuous. $f$ is locally Lipschitz with respect to $x$ if for every $A_{h,k}(t_0, x_0) \subset U$, there exists $K$ such that

$$|f(t, x_1) - f(t, x_2)| \leq K|x_1 - x_2|$$

for all $(t, x_1), (t, x_2) \in A_{h,k}(t_0, x_0)$.

4. Local uniqueness: Assume $f$ is continuous on $U \subset \mathbb{R}\times\mathbb{R}$ and *locally Lipschitz* with respect to $x$. Let $(t_0, x_0) \subset U$. Then there exists $h > 0$ such that the IVP has a *unique* $C^1$ solution for $t \in [t_0 - h, t_0 + h]$.

*Proof.*

- Define $A_{h,k}(t_0, x_0) = \{(t, x) : |t - t_0| \leq h, |x - x_0| \leq k\}$.
- $A_{h,k}$ is a compact $\Rightarrow f$ is bounded, $|f(t, x)| \leq M$.
- $f$ is locally Lipschitz on $A_{h,k}(t_0, x_0)$, so $\exists K$ s.t. $|f(t, x_1) - f(t, x_2)| \leq K|x_1 - x_2|$, for all $x_1, x_2 \in A_{h,k}$.
- Assume $h \leq \min\left\{\frac{k}{M}, \frac{1}{2K}\right\}$.
- Define $C^*[t_0 - h, t_0 + h] = C[t_0 - h, t_0 + h] \cap \{x : |x(t) - x_0| \leq k\}$. $C^*[t_0 - h, t_0 + h]$ can be shown to be a complete space.
- Define $T : C^*[t_0 - h, t_0 + h] \to C^*[t_0 - h, t_0 + h]$

$$(Tx)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

- Show $T$ is a contraction: $|(Tx_1)(t) - (Tx_2)(t)| \leq \int_{t_0}^{t} |f(s, x_1(s)) - f(s, x_2(s))|ds \leq K \int_{t_0}^{t} |x_1(s) - x_2(s)|ds \leq Kd(x_1, x_2)h \leq \frac{1}{2}d(x_1, x_2)$.
- $T$ has a fixed point, which is the solution to the IVP for $t \in [t_0 - h, t_0 + h]$.

Usually we show the solution exists on $[t_0 - h, t_0 + h]$ and then show it can be extended beyond this scope.

*Proof.*

- Go back by $\frac{h}{2}$. Consider the IVP:

$$\begin{cases} y' = f(t, y) \\ y(t_0 + \frac{h}{2}) = x(t_0 + \frac{h}{2}) \end{cases}$$

- Solution exists on $[t_0 - \frac{h}{2}, t_0 + \frac{3}{2}h]$.
- $x$ and $y$ overlap on $[t_0 - \frac{h}{2}, t_0 + h]$. By uniqueness of the solution, $x$ and $y$ must coincide.
- Solution $x$ can be extended to $[t_0 - h, t_0 + \frac{3}{2}h]$.
- Iterate the argument until reach the boundary of $U$.