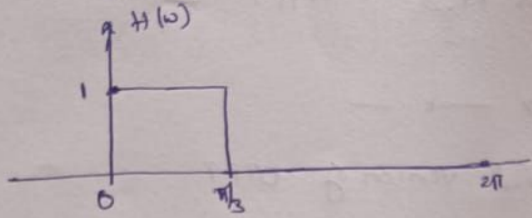
Q1.a. $DTFT\left(\cos(\pi/9\, n) + \cos(2\pi/3\, n)\right)$

$\downarrow \omega_1 \qquad \downarrow \omega_2$

$= 2\pi\,\delta(\omega - \pi/9) + 2\pi\,\delta(\omega + \pi/9) + 2\pi\,\delta(\omega + 2\pi/3) + 2\pi\,\delta(\omega - 2\pi/3)$

——— x ——— x ———

1.b.



phase = 0.

$h[n] = \frac{1}{2\pi}\int_0^{2\pi} H(\omega)\, e^{j\omega n}\, d\omega$

$= \frac{1}{2\pi}\int_0^{\pi/3} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left.\frac{e^{j\omega n}}{jn}\right]_0^{\pi/3}$

$= j\frac{1}{2\pi n}\left(e^{j\pi/3\, n} - 1\right)$

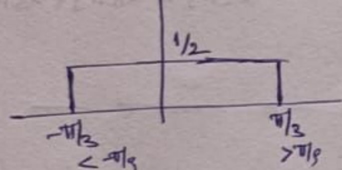
——— x ——— x ———

1.c

$h[n] \xrightarrow{DTFT} H(\omega)$

$h_R[n] \xrightarrow{DTFT} H_e(\omega) = \frac{H(\omega) + H^*(-\omega)}{2}$

$H_e(\omega) =$



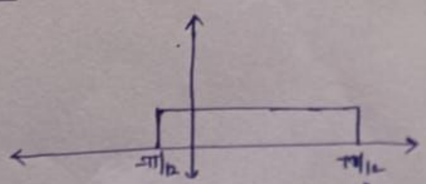
$y[n] = \frac{1}{2}\, x_1[n]$ ∵ since this spectrum can't pass the $\cos(2\pi/3\, n)$ component

——— x ——— x ———

1.d. $h_{Am}[n] \xrightarrow{DTFT} H_{Am}(\omega)$

$\|$

$e^{\pi/4\, n}\, h_R[n] \longrightarrow H_e(\omega + \pi/4)$



hence not equal ⇐ $> \pi/9$ $< 2\pi/3$

Q2a.

required for resolution, $GCD(\pi/3, 2\pi/3)$

$= \frac{\pi}{9}$

$\Rightarrow N_x = 2\pi / \pi/9 = \underline{18}$

—x —x —

b yes: DFT is sampled version of DTFT

—x —x —

c. NO; because

$h_R[n]$ is an infinite length signal. thus

IDFT $(H_1(k))$ is an time aliased version of $h_R[n]$.

—x —x —

d. $X[k-N_x] = X[k]$

$\Downarrow$ IDFT

$x[n] = x_1[n] + x_2[n]$

—x —x —

Q3.a. $X_1(\omega) = \frac{X(\frac{\omega}{2}) + X(\frac{\omega}{2}+\pi)}{2}$. proof: $X_1(\omega) = \sum_{n=-\infty}^{\infty} x[2n] \cdot e^{-j\omega n}$

$n_1 = 2n$

$= \sum_{\substack{n_1=-\infty \\ n_1 \text{ even}}}^{\infty} x[n_1] \cdot e^{-j\omega \frac{n_1}{2}}$

$= \sum_{n_1=-\infty}^{\infty} \frac{1+(-1)^n}{2} x[n_1] \cdot e^{-j\omega/2 \cdot n_1}$

$= \frac{X(\omega/2) + X(\omega/2+\pi)}{2}$

b. $\{1,1,1,0,0\} \circledast \{1,1,1,0,0\}$

$= \{\underset{\uparrow}{1}, 2, 3, 2, 1\} = y[n]$

—x—x—

c. $x_3[n] =$ ~~[illegible]~~ $y[(n+2)_6]$

—x—x—

Q4 (a) $h[n] = \alpha\,\delta[n] + \beta\,\delta[n-1]$

[4+1+3]

$$\Rightarrow H(z) = \alpha + \beta z^{-1}$$ 1

Input : $u[n]$ $\longrightarrow$ Output : $\delta[n]$

$$X(z) = \frac{1}{1-z^{-1}} \longrightarrow Y(z) = 1$$ 1

we know $H(z) = \dfrac{Y(z)}{X(z)}$ 1

$$\Rightarrow \alpha + \beta z^{-1} = \frac{1}{\frac{1}{1-z^{-1}}} = 1 - z^{-1}$$

$$\Rightarrow \alpha = 1 \text{ and } \beta = -1.$$ 1

(b) we have, $h[n] = \delta[n] - \delta[n-1]$

This is a difference system

$$y[n] = x[n] * h[n] = x[n] - x[n-1]$$

Constant input $\Rightarrow x[n] = c \quad \forall n$

$$\therefore y[n] = c - c = 0 \quad \forall n$$

$\Rightarrow$ constant input $\longrightarrow$ zero output 1

(c) If S_1 & S_2 are inverse systems,

$$H_1(z)\, H_2(z) = 1$$

$$H_1(z) = 1 - z^{-1}$$

$$\Rightarrow \quad H_2(z) = \frac{1}{1-z^{-1}} \qquad (1)$$

* There are two time domain signals associated with the above z transform.

pole of $H_2(z)$: $z = 1$

* For non-causal system, select the ROC : $|z| < 1$.

$$\Rightarrow \quad h_2[n] = -u[-n-1] \qquad (1)$$

verification :

$$h_1[n] * h_2[n] = \delta[n]$$

$$LHS = \left(\delta[n] - \delta[n-1]\right) * \left(-u[-n-1]\right)$$

$$= -u[-n-1] + u[-n]$$

$$= \delta[n] = RHS \qquad (1)$$

Q5 [2+3] $\quad y[n] = \alpha y[n-1] + \alpha^2 y[n-2] + x[n]$

$\alpha \in \mathbb{R}$ (real number)

(a) system transfer function : $H(z) = \dfrac{Y(z)}{X(z)}$

Taking z - Transform on both sides

$$Y(z) = \alpha z^{-1} Y(z) + \alpha^2 z^{-2} Y(z) + X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = H(z) = \frac{1}{1 - \alpha z^{-1} - \alpha^2 z^{-2}}$$

$$\therefore \quad H(z) = \frac{z^2}{z^2 - \alpha z - \alpha^2}$$

poles : $\quad z^2 - \alpha z - \alpha^2 = 0$

$\Rightarrow$ roots : $\dfrac{\alpha \pm \sqrt{\alpha^2 + 4\alpha^2}}{2}$

$$z_1 = \alpha\left(\frac{1+\sqrt{5}}{2}\right) \quad \& \quad z_2 = \alpha\left(\frac{1-\sqrt{5}}{2}\right) \qquad 1$$

A system is stable if the unit circle ($|z| = 1$) is part of the ROC.

For the system to be unstable, unit circle contains poles.

$\therefore$ The system is not stable if

$$|z_1| = 1 \quad \text{or} \quad |z_2| = 1$$

$$\Rightarrow \alpha = \pm \frac{2}{1+\sqrt{5}} \quad \underline{\text{OR}} \quad \alpha = \pm \frac{2}{1-\sqrt{5}}$$ 1

b For causal & stable system, all poles must be inside the unit circle $(|z| = 1)$ 1

$$\Rightarrow |z_1| < 1 \quad \text{and} \quad |z_2| < 1$$

$$\therefore \left|\alpha \left(\frac{1+\sqrt{5}}{2}\right)\right| < 1 \quad \text{and} \quad \left|\alpha \left(\frac{1-\sqrt{5}}{2}\right)\right| < 1$$

$$|\alpha| < \frac{2}{\sqrt{5}+1} \quad \underline{\text{AND}} \quad |\alpha| < \frac{2}{\sqrt{5}-1}$$ 1

$$\Rightarrow |\alpha| < \frac{2}{\sqrt{5}+1}$$

$$\Rightarrow \alpha \in \left(\frac{-2}{\sqrt{5}+1}, \frac{2}{\sqrt{5}+1}\right)$$ 1

Q6. [3]

$$y[n] = \alpha y[n-1] + x[n]$$

$$\alpha \in \mathbb{R}$$

eigen signals : $e^{j\omega n} \xrightarrow{LTI} H(e^{j\omega})\, e^{j\omega n}$

reduction in amplitude : $|H(e^{j\omega})| < 1 \quad \forall\, \omega$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$$

$$H(z = e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega}-\alpha}$$ 1

$$|H(e^{j\omega})| = \left| \frac{e^{j\omega}}{e^{j\omega}-\alpha} \right| = \frac{1}{|e^{j\omega}-\alpha|}$$

$$|H(e^{j\omega})| < 1 \Rightarrow$$

$$|e^{j\omega} - \alpha| > 1 \quad \forall\, \omega$$ 1

$$(e^{j\omega}-\alpha)(e^{j\omega}-\alpha)^* > 1$$

$$(e^{j\omega}-\alpha)(e^{-j\omega}-\alpha) > 1$$

$$1 - \alpha\left(e^{j\omega}+e^{-j\omega}\right) + \alpha^2 > 1$$

$$\alpha^2 - 2\alpha\cos(\omega) + 1 > 1$$

$$\alpha^2 - 2\alpha\cos(\omega) > 0$$

$$\alpha\left(\alpha - 2\cos(\omega)\right) > 0 \quad \forall\ \omega$$ ①

$$\Rightarrow \left\{\alpha > 0 \quad \underline{\text{AND}} \quad \alpha - 2\cos(\omega) > 0 \quad \forall\ \omega\right\}$$

$$\underline{\underline{\text{OR}}}$$

$$\left\{\alpha < 0 \quad \underline{\underline{\text{AND}}} \quad \alpha - 2\cos(\omega) < 0 \quad \forall\ \omega\right\}$$

$$\Rightarrow \left\{\alpha > 2\right\} \quad \underline{\underline{\text{OR}}} \quad \left\{\alpha < -2\right\}$$

Since $\cos(\omega) \in [-1, 1]$.

$$\Rightarrow \alpha \in (-\infty, -2) \cup (2, \infty)$$