$Y^2 + (E[X])^2 - 2XE[X]$

$+ 2sX - 4sE[X]$

End-Semester Examination

(MA6.102) Probability and Random Processes, Monsoon 2024

29 November, 2024

Max. Duration: 3 Hours

Question 1. (a) (2 Marks) Suppose the events A and B are independent, and the events A and C are also independent. Are the events A and $B \cup C$ independent? Justify your answer.
(b) (2 Marks) Suppose that the events A, B, and C are independent. Are the events A and $B \cup C$ independent? Justify your answer.

Question 2 (3 Marks). Let X be a geometric random variable with parameter p, where $0 < p < 1$. The probability mass function (PMF) of X is given by $P_X(k) = (1-p)^{k-1}p$, $k \in \mathbb{N}$. Find $\mathbb{E}[X]$ and $\text{var}(X)$.

Question 3 (3 Marks). Let X and Y be jointly continuous random variables with joint probability density function (PDF) $f_{X,Y}(x,y) = \mathrm{e}^{-x-y}$, for $x, y \geq 0$. Find the conditional joint cumulative distribution function (CDF) $F_{(X,Y)|X<Y}(x,y)$ defined as

$$F_{(X,Y)|X<Y}(x,y) = P\left((X \leq x, Y \leq y) \mid X < Y\right).$$

Hint: You may need to consider two cases separately, one for $x \leq y$ and the other for $x > y$.

Question 4. (a) (2 Marks) Prove that $P(X - \mathbb{E}[X] \geq t) \leq \frac{\text{var}(X)}{\text{var}(X)+t^2}$, for $t > 0$.
Hint: Obtain a bound on $P((X - \mathbb{E}[X] + s)^2 \geq (t+s)^2)$, for $s > 0$.
(b) (1 Mark) Given $a > 0$, construct a non-constant random variable X such that $P(X \geq a) = \frac{\mathbb{E}[X]}{a}$.
(c) (2 marks) Let $X_1, X_2, \ldots$ be independent random variables uniformly distributed over $[-1, 1]$. Show that the sequence $Y_n = \prod_{i=1}^{n} X_i$, $n \in \mathbb{N}$ converges in probability to some limit.
Hint: Use Chebyshev's inequality. $P(|X_n - M| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$

Question 5 (5 Marks). Let $X_t = A\cos(\omega_c t + \Theta)$, where ω_c is a non-zero constant, A and Θ are independent random variables with $P(A > 0) = 1$ and $\mathbb{E}[A^2] < \infty$. If Θ is uniformly distributed over $[0, 2\pi]$, show that X_t is wide-sense stationary (WSS). Is X_t strict-sense stationary also?

Question 6 (10 Marks). For each of the statements below, write True or False.
Note: Explanations are not required. Ensure that your answer to this question fits within a single page of the answer sheet. Answers spread across more than one page will NOT be evaluated.

(a) A random variable defined on a countable sample space is always a discrete random variable.
(b) A random variable defined on an uncountable sample space is always a continuous random variable.
(c) Let X and Y be independent random variables, each taking the values -1 or 1 with equal probability $\frac{1}{2}$, and let $Z = XY$. Then Z and X are not independent because Z is a function of X and Y.
(d) If $Y(\omega_1) = y_1 \neq y_2 = Y(\omega_2)$, then $\mathbb{E}[X|Y](\omega_1) \neq \mathbb{E}[X|Y](\omega_2)$.
(e) If $P_{X,Y,Z}(x,y,z) = P_X(x)P_Y(y)P_Z(z)$, for all x, y, z, then $P_{X,Y}(x,y) = P_X(x)P_Y(y)$, for all x, y. Similarly, if $P(A \cap B \cap C) = P(A)P(B)P(C)$, then $P(A \cap B) = P(A)P(B)$.
(f) If X is a Gaussian random variable with mean 0 and variance 1, and $\sqrt{2\pi}\Phi(x) = \int_{-\infty}^{x} \mathrm{e}^{-\frac{t^2}{2}}\,dt$, then $\Phi(x) + \Phi(1-x) = 1$.
(g) For any random variable X and $a \in \mathbb{R}$, we have $P(X \leq a) \leq \inf_{s<0} \frac{\mathbb{E}[\mathrm{e}^{sX}]}{e^{as}}$.
(h) A sequence of random variables $X_1, X_2, \ldots$ is said to converge almost surely to X if $P(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1$.
(i) Suppose $(N_t, t \in [0, \infty))$ is a Poisson process with rate λ. The probability of exactly one arrival in the interval $[2024, 2026]$ is given by $\mathrm{e}^{-\lambda}\lambda$.
(j) If $(X_t, t \in \mathbb{R})$ is a strict-sense stationary (SSS) process, then $F_{X_{t_1}}(x_1) = F_{X_{t_2}}(x_2)$, $\forall\, t_1, t_2, x_1, x_2 \in \mathbb{R}$.