$h(n) = \{1, 0, -1\}$

a) $H(\omega) = \sum_n h(n) e^{-j\omega n}$

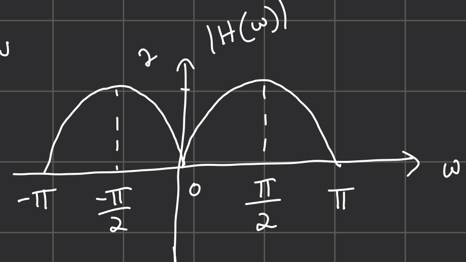
$= 1 + 0 - e^{-j\omega}$

$= 1 - e^{-2j\omega}$ $\frac{1}{2}$

$= 1 - \cos 2\omega + j \sin 2\omega$

$|H(\omega)| = \sqrt{2(1-\cos 2\omega)}$

$= 2|\sin\omega|$



1

b) $H(\omega) = 1 - e^{-2j\omega}$

$= 2j\, e^{-j\omega} \frac{(e^{j\omega} - e^{-j\omega})}{2j} = 2j \sin\omega \cdot e^{-j\omega} = 2|\sin\omega| \cdot j e^{-j\omega} \cdot \text{sign}(\sin\omega)$

(or)

tan⁻¹ with conditions for each quadrant

↓

full marks given even if tan⁻¹ is taken without piecewise

$\angle H(\omega) = \begin{cases} \angle e^{-j(\omega - \frac{\pi}{2})} & \omega > 0 \\ \angle e^{-j(\omega + \frac{\pi}{2})} & \omega < 0 \end{cases}$

linear phase for $\omega > 0$ 1.5

plot 0.5

$\phi(\omega) = \boxed{\angle H(\omega) = -\omega + \frac{\pi}{2} \quad \forall\, \omega > 0}$

c) $x(n) = 1 + \sin\left(\frac{\pi n}{6}\right) + \sin\left(\frac{\pi n}{2}\right) + \cos(\pi n)$

$\omega = 0$ $\quad \omega = \frac{\pi}{6}$ $\quad \omega = \frac{\pi}{2}$ $\quad \omega = \pi$

$\frac{d\phi(\omega)}{d\omega} = -1 \Rightarrow$ group delay of -1

→ calculating exact fourier and concluding

$x(n) - x(n-2)$ — 1

$Y(\omega) = H(\omega) \cdot X(\omega)$ $\quad |H(\omega)| = 1$ $\quad |H(\omega)| = 2$

↳ passes $\omega = \frac{\pi}{6}$ & $\omega = \frac{\pi}{2}$

$y(n) = \sin\left(\frac{\pi n}{6} - \frac{\pi}{3}\right) + 2\sin\left(\frac{\pi}{2}(n-1)\right)$

final ans 3

↳ wrong coefficients or phase 1.5m – 2.5m

d) $h_1(n) = h(n-1)$

$X(\omega) = 2\pi\delta(\omega) + \frac{\pi}{j}\left(\delta(\omega - \frac{\pi}{6}) + \delta(\omega + \frac{\pi}{6})\right) + \frac{\pi}{j}\left(\delta(\omega - \frac{\pi}{2}) + \delta(\omega + \frac{\pi}{2})\right) + \pi\left(\delta(\omega - \pi) + \delta(\omega + \pi_0)\right)$

$H_1(\omega) = H(\omega)\, e^{-j\omega}$

$|H_1(\omega)| = |H(\omega)|$ $\quad \angle H_1(\omega) = \angle H(\omega) - \omega \Rightarrow$ linear phase

$Y_1(\omega) = H(\omega) X(\omega) e^{-j\omega}$

$y_1(n) = y(n-1) = \sin\left(\frac{\pi}{6}(n-2)\right) + \sin\left(\frac{\pi}{2}(n-2)\right)$

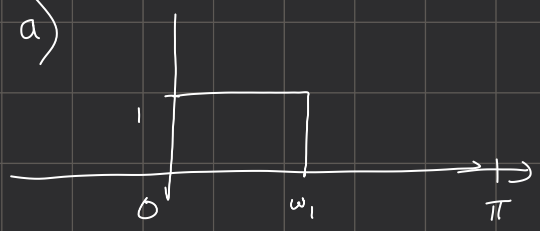
↳ or phase as $\frac{\pi}{2} - 2\omega$

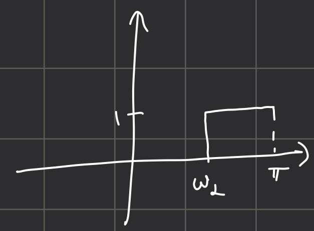
Linear phase – 1

mag & phase relation – 1

$y(n)$ – 2

2) a)





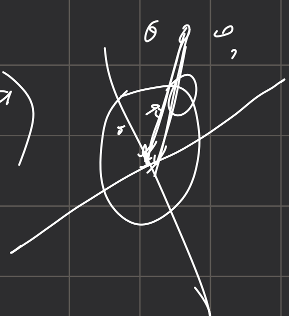
$\omega_1 > \omega_2$

$\boxed{\omega_2 < \omega\_range < \omega_1}$ 2

b) in regions of $\pi > \omega > \omega_1$ & $0 < \omega < \omega_2 \Rightarrow \omega = 0$ 2

3) $H(z) \rightarrow$ stable, causal & linear phase & FIR

a)



IIR $\rightarrow$ no linear phase

IIR $\rightarrow$ noncausal

without reasoning expect a 0

stable, non causal 1.5

↳ cases of zeros inside & outside for stability

b) stable, not causal, FIR 1.5

c) FIR, case $\begin{cases} \rightarrow \text{odd} \\ \rightarrow \text{even} \end{cases}$ linear with reason $-$ 1 2

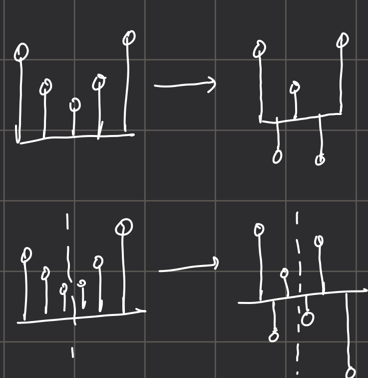
causal stable $-$ 1

d) causal stable, linear 1

$\sum h(n) z^{-n}$

$\sum (-1)^n h(n) z^{-n}$

$h'(n) = (-1)^n h(n)$



SP final

Tuesday, 3 December 2024 6:48 AM

Q4 Solution

Shiva : $s = 1 - z^{-1} = \frac{z-1}{z}$ (derivative approximation of differential eqn.)

Madhuri : $s = \frac{z}{z+1}$ (modified Bilinear)

(a) [3 marks] How does $j\omega$-axis map to z-plane ?

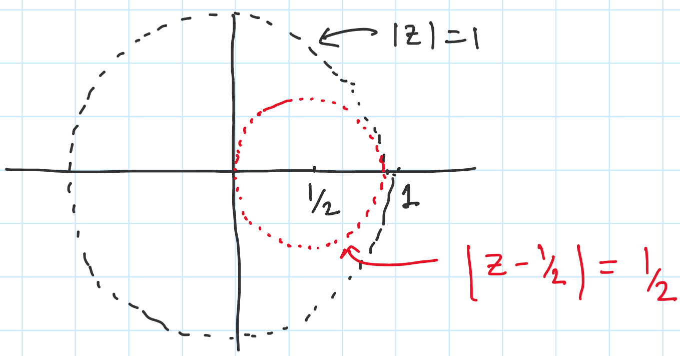
Shiva: $s = 1 - z^{-1} \Rightarrow z^{-1} = 1 - s$

$$\Rightarrow z = \frac{1}{1-s} \qquad \left(\tfrac{1}{2}\right)$$

$s = j\omega$ gives $\rightarrow$ $z = \frac{1}{1-j\omega}$

$$z - \frac{1}{2} = \frac{1}{1-j\omega} - \frac{1}{2} = \frac{1}{2}\,\frac{1+j\omega}{1-j\omega} \qquad \left(\tfrac{1}{2}\right)$$

$\left| z - \frac{1}{2} \right| = \frac{1}{2}$ $\Rightarrow$ Circle of radius $\frac{1}{2}$ with center at $\frac{1}{2}$. $\left(\tfrac{1}{2}\right)$



Madhuri: $s = \frac{z}{1+z} \Rightarrow \frac{1}{s} = \frac{z+1}{z} = 1 + \frac{1}{z}$

$$\Rightarrow \frac{1}{z} = \frac{1}{s} - 1 = \frac{1-s}{s}$$

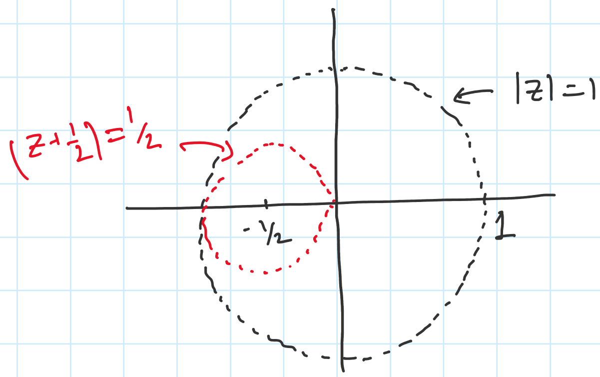
$$\Rightarrow \quad \frac{1}{z} = \frac{1}{s} - 1 = \frac{1-s}{s}$$

$$\Rightarrow \quad z = \frac{s}{1-s} \qquad \left(\frac{1}{2}\right)$$

for $s = j\omega$, $\quad z = \dfrac{j\omega}{1-j\omega}$

$$z + \frac{1}{2} = \frac{j\omega}{1-j\omega} + \frac{1}{2} = \frac{1}{2}\,\frac{1+j\omega}{1-j\omega} \qquad \left(\frac{1}{2}\right)$$

$\therefore \left|z + \frac{1}{2}\right| = \frac{1}{2} \quad \Rightarrow$ circle of radius $1/2$ with centre at $-1/2$ $\quad \left(\frac{1}{2}\right)$



(b) [3 Marks] Stable & Causal

$s = \sigma + j\omega$

Shiva: $\quad z = \dfrac{1}{1-s} = \dfrac{1}{1-\sigma-j\omega}$

$$z - \frac{1}{2} = \frac{1}{1-\sigma-j\omega} - \frac{1}{2} = \frac{1}{2}\,\frac{1+\sigma+j\omega}{1-\sigma-j\omega}$$

$$\left|z - \frac{1}{2}\right| = \frac{1}{2}\sqrt{\frac{(1+\sigma)^2+\omega^2}{(1-\sigma)^2+\omega^2}}$$

For $\sigma < 0$, $\left|z - \frac{1}{2}\right| < \frac{1}{2}$ as $(1-\sigma)^2 > (1+\sigma)^2$

$\Rightarrow \quad Re(s) = \sigma < 0 \quad \Rightarrow \quad \left|z - \frac{1}{2}\right| < \frac{1}{2}$ (1)

$\therefore$ Left half s-plane (i.e. $\sigma < 0$) maps to inside the circle $\left|z - \frac{1}{2}\right| = \frac{1}{2}$, which is inside the unit circle.

$\therefore$ If the continuous-time filter is causal & stable i.e. poles are in the left half s-plane, they map to poles inside unit circle in z-plane

$\Rightarrow$ the discrete-time filter is also causal & stable. ($\frac{1}{2}$)

Madhuri: Similarly show that,

$$z + \frac{1}{2} = \frac{1}{2}\,\frac{1+\sigma+j\omega}{1-\sigma-j\omega}$$

$\therefore \quad Re(s) = \sigma < 0 \quad \Rightarrow \quad \left|z + \frac{1}{2}\right| < \frac{1}{2}$ (1)

Using same argument as above,

Stable & Causal in C-T $\Rightarrow$ Stable & causal in D-T ($\frac{1}{2}$)

(c) [4 Marks]

Shiva: Analog filter $H_a(s) = \frac{1}{s+2}$

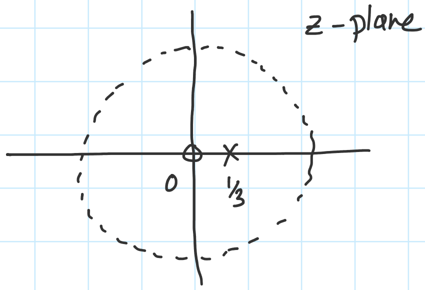
Digital filter $H(z) = H_a(s)\Big|_{s = 1 - z^{-1}}$

$$H(z) = \frac{1}{1 - z^{-1} + 2} = \frac{1}{3 - z^{-1}} = \frac{z}{3z - 1}$$

$$H(z) = \frac{1}{3} \frac{z}{z - 1/3} \qquad (1)$$

zero : $z = 0$

pole : $z = 1/3$



(1/2)

Based on location of pole & zero, this is a low pass filter $\left(\frac{1}{2}\right)$

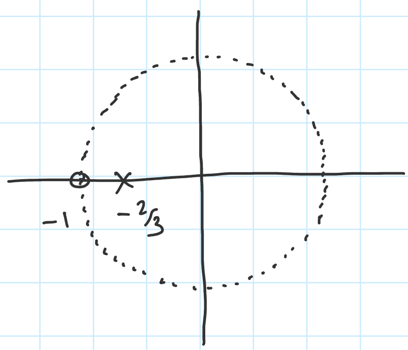
Madhuri :

digital filter $H(z) = H_a(s) \Big|_{s = \frac{z}{z+1}}$

$$H(z) = \frac{1}{\frac{z}{z+1} + 2} = \frac{z+1}{3z+2} = \frac{1}{3} \frac{z+1}{z + 2/3} \qquad (1)$$

zero : $z = -1$

pole : $z = -2/3$



(1/2)

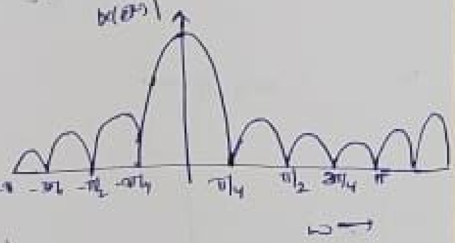
From above pole-zero plot, this is also a low pass filter. $\left(\frac{1}{2}\right)$

IIITH Roll No:

1. $X(e^{j\omega}) = \sum_{n=0}^{7} e^{-j\omega n} = \underbrace{\sum_{n=0}^{7} e^{-j\omega n} + \sum_{n=-\infty}^{\infty} e^{-j\omega n}}_{} - \sum_{n=5}^{\infty} e^{-j\omega n}$

$= \frac{1}{1-e^{-j\omega}} - e^{-j\omega 8} \frac{1}{1-e^{-j\omega}} \Rightarrow |X(e^{j\omega})| = \frac{\sin 4\omega}{\sin \omega/2}$

$X(e^{j\omega}) = \frac{1-e^{-j\omega 8}}{1-e^{-j\omega}}$



$\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 = \sum_{n=0}^{7} |x[n]|^2$ using parseval's theorem

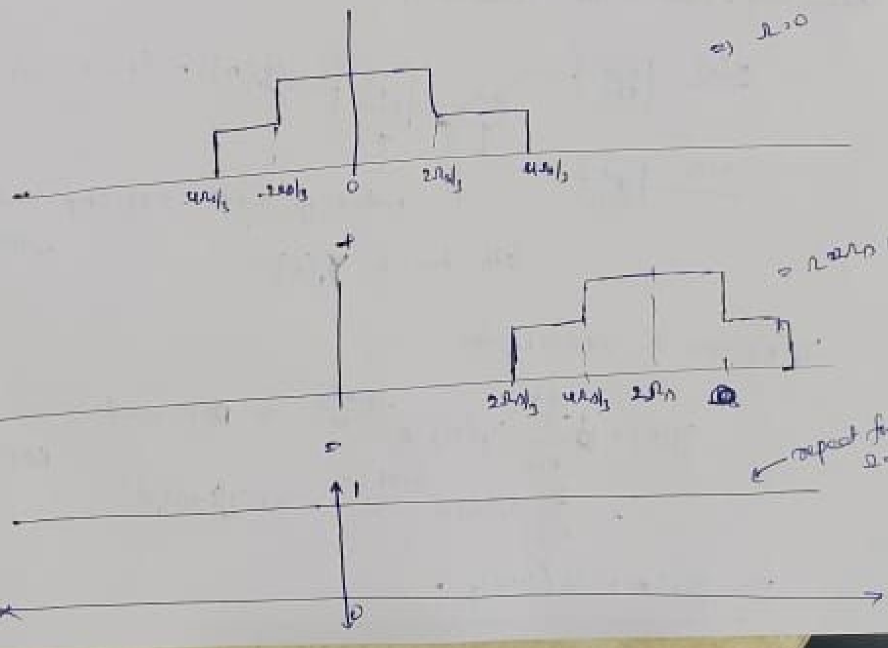
$= 8$

$X_R(e^{j\omega}) \cdot e^{-j2\omega} \xRightarrow{DTFT} x_e[n+2] = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$

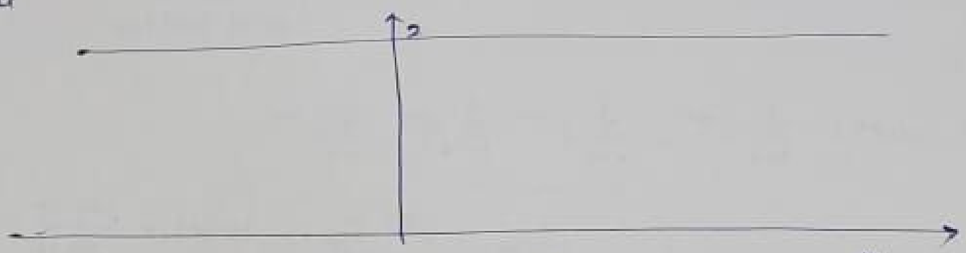
$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{8} = 2\pi/8 \cdot k}$

$X[k] = \frac{1-e^{-j\frac{2\pi}{8} \cdot 8k}}{1-e^{-j\frac{2\pi k}{8}}} = \frac{1-e^{-j2\pi k}}{1-e^{-j\frac{2\pi k}{8}}} \Rightarrow X[0] = \{1, 0, 0, 0, 0, 0, 0\}$

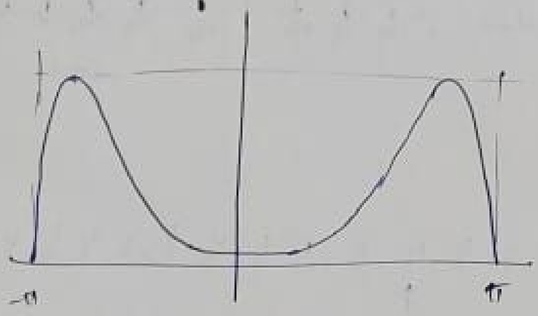
2.

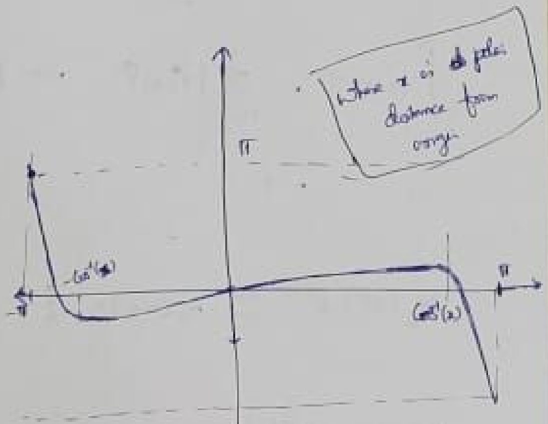


result



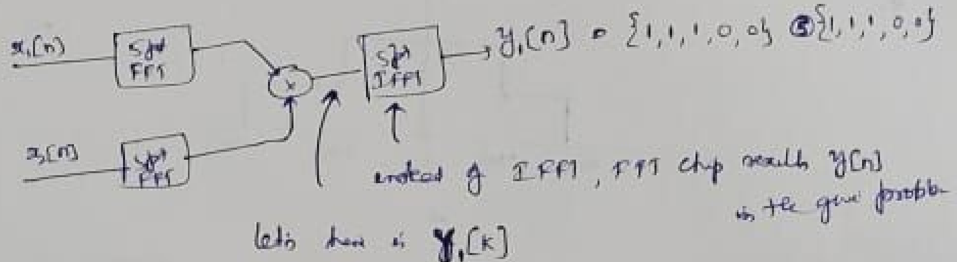
3. mag spectrum phase spectra





4. Based on Quiz-2 solution only

5. ~~Quiz~~ mid-exam solution available for the below



$Y_1[K]$ goes to FFT chip the

$$y[n] = \sum_{k=0}^{N-1} Y_1[K]\, e^{\frac{-j2\pi kn}{N}}$$ ← DFT equation ∵ FFT chip computes DFT on the input

$$= \sum_{k=0}^{N-1} Y_1[K]\, e^{\frac{j2\pi k(-n)}{N}} = N \cdot y_1((-n))_N$$

$\Rightarrow$ $y[n] = 5\, y_1((-n))_5$.