# Science-1, Monsoon 2024

## QUIZ-2, 17-Oct-2024, 45 min

ROLL NUMBER: ____________________ SEAT: ____________________

1 4

| Question | Points | Graded by: |
|---|---|---|
| 1 | . | . |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |

Instructions:

- Calculator is not required and not allowed in this exam.
- Write your roll number and seat number on top-page.
- Question paper consists of 5 questions of equal weightage.
- Answer in the space given for the question.

1. In an inertial frame, A and B are undergoing uniform motion; A is moving along x-axis with speed $u$, and B is moving along y-axis with speed $v$. A has a light source whose frequency is $\nu_0$ in rest-frame. What is the frequency of this light source as seen by B?

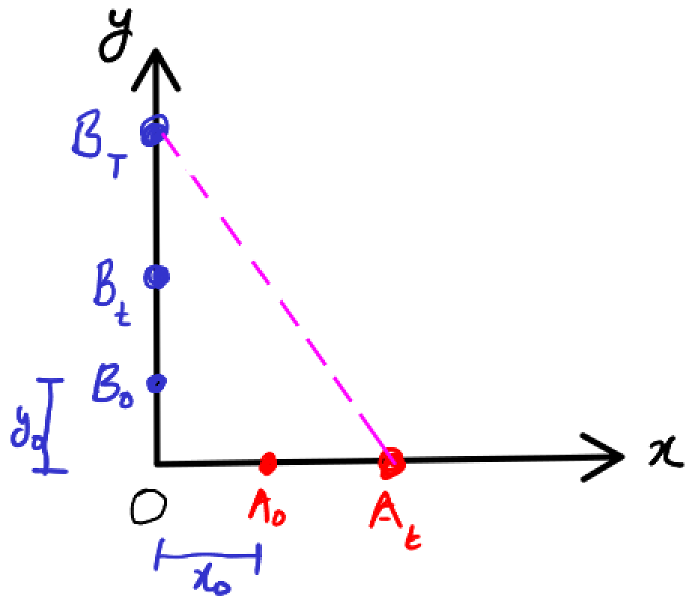


Figure 1: x-y plot indicating positions of A and B

(a) At time $t$, position vectors are $A(t) = (x_0 + ut, 0, 0)$ and $B(t) = (0, y_0 + vt, 0)$

(b) A light flash at $A(t)$ reaches $B$ at time $T$. Clearly $T - t$ will depend on $x_0$ and $y_0$ (see figure). And can be calculated by:

i. for x-distance: $x_0 + ut = (T - t)c\cos\theta$

ii. for y-distance: $y_0 + vT = (T - t)c\sin\theta$

(c) If flashes at A happen at $t_1, t_2, \cdots$, and these reach at $T_1, T_2, \cdots$. We assume that $\theta$ does not change much during this time, we get using

i. for x-axis: $u(t_2 - t_1) = ((T_2 - T_1) - (t_2 - t_1))\, c\cos\theta \implies (u + c\cos\theta)\Delta t = \Delta T \cdot c\cos\theta$

ii. for y-axis: $v(T_2 - T_1) = ((T_2 - T_1) - (t_2 - t_1))\, c\sin\theta \implies c\sin\theta \cdot \Delta t = (c\sin\theta - v)\Delta T$

These relations connect time differences, in the given inertial frame, of the light flashes at starting at A ($\Delta t$) and reaching B ($\Delta T$)

(d) Clearly $\Delta t = \gamma(v)\tau_0$ (with $\tau_0 = 1/\nu_0$), giving us $\Delta T = \frac{u + c\cos\theta}{c\cos\theta}\gamma(v)\tau_0$, or alternatively (from y-disance formula above) $\Delta T = \frac{c\sin\theta}{c\sin\theta - v}\gamma(v)\tau_0$

(e) In rest frame of B, $\Delta T' = \Delta T/\gamma(u)$ and hence frequency $\nu' = \nu_0 \frac{\gamma(u)}{\gamma(v)} \frac{c\sin\theta - v}{c\sin\theta}$ or an equivalent formula is $\nu' = \nu_0 \frac{\gamma(u)}{\gamma(v)} \frac{c\cos\theta}{u + c\cos\theta}$

2. A bob (mass $M$) is suspended from a point support by a 'ideal' string (length $l$). Let the vertical axis be $z-$axis; for a simple pendulum the bob will oscillate in $x-z$ plane. However in the present case, when the string lies in the x-z plane, bob is pushed in $y-$direction so that it is no longer the simple pendulum case. Your task is to find the equation of motion for this case by (a) determine the appropriate generalised coordinates (b) setup the Lagrangian and finally (c) find the equation of motion. Finally, from the equation of motion (with out solving it), (d) comment on the path of the bob.

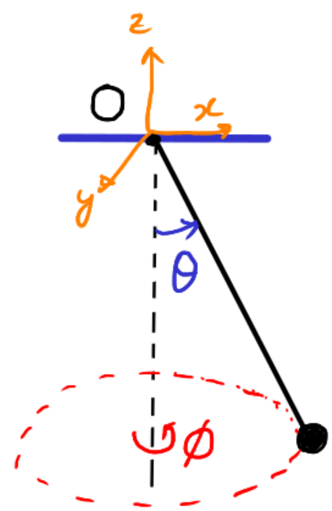


Figure 2: Spherical coordinate system

(a) Spherical coordinates are coordinates, with $r = b$ (a constant), leaving only $\theta$ and $\phi$ as generalised coordinates. [Cylindrical coordinates are also a good set of generalised coordinates for this problem, but a constraint has to bee used]

(b) Lagrangian $L =$ Kinetic energy - potential energy

  i. Kinetic energy $K = \frac{1}{2}m(b^2\dot{\theta}^2 + b^2 \sin^2\theta\dot{\phi}^2)$

  ii. Potential energy: $U = -mgb\cos\theta$

  Giving us the Lagrangian $L = \frac{1}{2}m(b^2\dot{\theta}^2 + b^2\sin^2\theta\dot{\phi}^2) + mgb\cos\theta = L(\theta, \dot{\theta}, \dot{\phi})$

(c) Euler-Lagrange equations:

  i. $\frac{\partial L}{\partial \theta} = mb^2 \sin\theta\ \cos\theta\ \dot{\phi}^2 - mgb\sin\theta$ and $\frac{\partial L}{\partial \dot{\theta}} = mb^2\dot{\theta}$, which gives $mb^2\ddot{\theta} = mb^2\sin\theta\ \cos\theta\ \dot{\phi}^2 - mgb\sin\theta$

  ii. $\frac{\partial L}{\partial \phi} = 0$! giving us $\frac{\partial L}{\partial \dot{\phi}} = mb^2\sin^2\theta\ \dot{\phi} =$constant

(d) Plugging in (ii) in (i) above, we get

$$mb^2\ddot{\theta} = \frac{C}{mb^2}\frac{\cos\theta}{\sin^3\theta} - mgb\sin\theta$$

which for small angles is:

$$mb^2\ddot{\theta} = \frac{C_2}{\theta^3} - mgb\theta$$

which means: $\theta$ oscillates about a non-zero value of $\theta_0$

Additionally $\dot{\phi} =$constant (approximately)

3. A charged particle is moving with velocity $\vec{v} = (v_x, v_y, v_z)$ at some time $t$, in a magnetic field given by $\vec{B} = (0, 0, B_0)$. Determine the trajectory of the particle. Describe the path.

(a) Force on particle is the Lorentz force $\vec{F} = q\vec{v} \times \vec{B} = q(v_x, v_y, v_z) \times (0, 0, B_0) = q((v_y B_0 - 0), -(v_x B_0 - 0), (0 - 0) = qB_0(v_y, -v_x, 0)$

(b) Using Newtons IInd Law: $F = m(\dot{v}_x, \dot{v}_y, \dot{v}_z)$, and thus $m\dot{v}_x = qB_0 v_y$, $m\dot{v}_y = -qB_0 v_x$ and $m\dot{v}_z = 0$

(c) For the z-component: $v_z$ =constant is a solution and hence $z = v_z^0\ t$

(d) Differentiating x-component once with respect to time: $m\ddot{v}_x = qB_0\dot{v}_y = qB_0(-qB_0 v_x/m)$ i.e. $\ddot{v}_x = -(qB_0/m)^2\ v_x$. Define $\alpha = qB_0/m$
and thus solutions $v_x(t) = A\cos\alpha t + B\sin\alpha t$

(e) Differentiating y-component similarly gives $\ddot{v}_y = -\alpha^2 v_y$, and
thus the solution $v_y(t) = C\cos\alpha t + D\sin\alpha t$

(f) To calculate the constants A,B,C and D:

   i. $\dot{v}_x = \alpha v_y \implies -A\alpha\sin\alpha t + B\alpha\cos\alpha t = \alpha(C\cos\alpha t + D\sin\alpha t)$ which gives:
   $B = C$ and $-A = D$

   ii. Thus $v_x = A\cos\alpha t + B\sin\alpha t$ and $v_y = B\cos\alpha t - A\sin\alpha t$

   iii. At $t = 0$, $v = (v_x^0, v_y^0, v_z^0)$ gives us $A = v_x^0$ and $B = v_y^0$

   iv. $\dot{x} = v_x \implies x(t) = \frac{1}{\alpha}(-A\sin\alpha t + B\cos\alpha t)$

   v. $\dot{y} = v_y \implies y(t) = \frac{1}{\alpha}(B\sin\alpha t + A\cos\alpha t)$

   vi. Clearly $x^2(t) + y^2(t) = \frac{1}{\alpha^2}(A^2 + B^2) = \frac{m^2(v_x^{0\,2} + v_y^{0\,2})}{q^2 B_0^2}$

   vii. Helix with radius $mv/qB_0$, and pitch $v_z^0 \times 2\pi/\alpha$

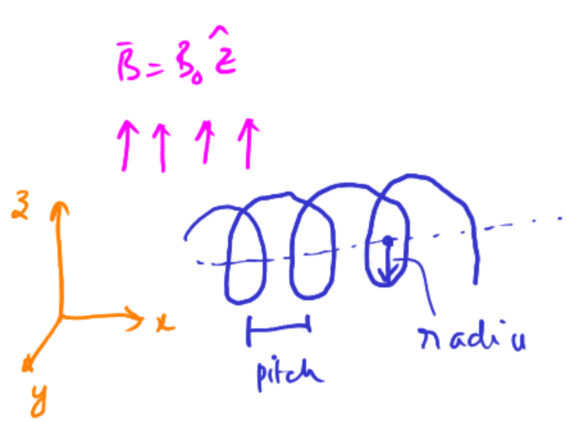


Figure 3: With a constant magnetic field along $z-$axis ($\vec{B} = B_0\hat{B}$), particle's trajectory is a helix, with radius and pitch calculated in main text.

4. Consider a particle with potential energy given by $U(\vec{r}) = -\frac{C}{r}$, where $r = ||\vec{r}||$. If the particle has velocity $\vec{v} = (v_x, v_y, v_z)$ at some time, what can you infer about the path of the particle at any later time? Mathematically derive your inference.

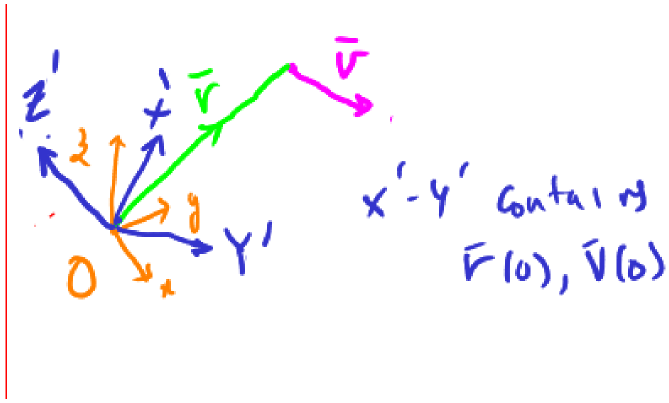


Figure 4: Orange colored $x-, y-, z-$ axis. New orthogonal frame $X'-, Y'-, Z'-$axis are blue in color, with $Z'-$ axis perpendicular to $\vec{r}(0)$ and $\vec{v}(0)$.

(a) Consider the plane containing $\vec{r}$ and $\vec{v}$, the line perpendicular to this plane passing though origin will be the new z-axis, called $Z'$-axis. Since force $\vec{F} = -\nabla U = +\frac{C}{r^2}\hat{r}$, is in this plane, as is the velocity vector $\vec{v}$. Since there is no force that is perpendicular to this plane, the particle stays in this plane.

(b) $\hat{F} = \hat{r}$, and hence Torque about the $Z'-$axis is $\vec{N} = \vec{r} \times \vec{F} = 0$. Using relation between torque and angular momentum, $\frac{d}{dt}\vec{L} = \vec{N}$, so that in this case we have $\vec{L}$ =constant. In this new orthogonal frame,

$$(r \times v) \cdot \hat{x}' = 0 \; and \; (r \times v) \cdot \hat{y}' = 0$$

This means that $\vec{r}$ and $\vec{v}$ remains in $X' - Y'$ plane.

(c) Now, for this 2-D problem, et us use new generalised coordiantes: (a) distance from origin $r$ and (b) angle $\theta$ that the position vector makes with $X'-$ axis. [POLAR COORDINATES.] Lagrangian is:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{C}{r}$$

which give the Euler-Lagrange equations for $L(r, \dot{r}, \dot{\theta})$ as:

$$\text{EL for } \theta : \quad \frac{d}{dt}(mr^2\dot{\theta}) = 0 \implies \dot{\theta} = \frac{C}{mr^2}$$

and

$$\text{EL for } r : \quad \frac{d}{dt}(m\dot{r}) = mr\dot{\theta}^2 - \frac{C}{r^2} \implies m\ddot{r} = \frac{C^2/m}{r^3} - \frac{C}{r^2}$$

Here, EL for $\theta$, gives the fact that AREAL VELOCITY $\frac{1}{2}r\ r\dot{\theta}$ =constant, which is Kepler's Second Law.

5. Let the force $\vec{F} = (ayz + bx + c, axz + bz, axy + by)$. Calculate the work done by the force to go from $(0, 0, 0)$ to $(1, 1, 1)$.

   (a) There are infinitely many paths that connect points $P_1 = (0, 0, 0)$ and $P_2 = (1, 1, 1)$

   (b) We notice that points are so chosen to have all coordinates the same value, so one possible path is $x = y = z$

   (c) On $x = y = z$ path, $\vec{F} = (ax^2 + bx + c, ax^2 + bx, ax^2 + bx)$ and $d\vec{r} = (dx, dy, dz)$ and hence $W = \int_{x=0}^{x=1} dx\ \vec{F} \cdot (dx, dx, dx) = \int_0^1 dx((ax^2 + bx + c) + (ax^2 + bx) + (ax^2 + bx))dx$
   $= \int_0^1 dx(3ax^2 + 3bx + c) = 3ax^3/3 + 3bx^2/2 + cx|_{x=0}^{x=1}$
   $= a + 3b/2 + c$

---

It is interesting to check for another path $(0, 0, 0) \to (1, 0, 0) \to (1, 1, 0) \to (1, 1, 1)$.

$$W = \int_{(0,0,0)}^{(1,0,0)} F_x(x, y = 0, z = 0)dx + \int_{(1,0,0}^{(1,1,0)} F_y(x = 1, y, z = 0)dy + \int_{(1,1,0)}^{(1,1,1)} F_z(x = 1, y = 1, z)dx$$

$$= \int_0^1 (bx + c)\ dx + \int_0^1 (0)\ dy + \int_0^1 (a + b)\ dz$$

$$= (b/2 + c) + (0) + (a + b)$$

$$W = a + 3b/2 + c$$

Two different paths having same work is indication that work is independent of path, and hence this being a force from a potential is possibility. To confirm, check

- $\frac{\partial F_x}{\partial y} \stackrel{?}{=} \frac{\partial F_y}{\partial x}$ meaning $az \stackrel{?}{=} az$.... TRUE
- $\frac{\partial F_x}{\partial z} \stackrel{?}{=} \frac{\partial F_z}{\partial x}$ meaning $ay \stackrel{?}{=} ay$.... TRUE
- $\frac{\partial F_y}{\partial z} \stackrel{?}{=} \frac{\partial F_z}{\partial y}$ meaning $ax + b \stackrel{?}{=} ax + b$.... TRUE

These are equivalent to Maxwell Relations in Thermodynamics. All three conditions being true means that force $\vec{F} = -\nabla U(x, y, z)$

- $F_x = ayz + bx + c = -\frac{\partial U}{\partial x} \implies -U = axyz + bx^2/2 + cx + f_1(y, z)$
- $F_y = axz + bz = -\frac{\partial U}{\partial y} \implies -U = axyz + byz + f_2(x, z)$
- $F_y = axy + by = -\frac{\partial U}{\partial z} \implies -U = axyz + byz + f_3(x, z)$

Function that satisfies all three above conditions is $-U = axyz + byz + bx^2/2 + cx + D$ where $D$ =constant. This means

$$-U(0, 0, 0) = D, \quad -U(1, 1, 1) = a + b + b/2 + c + D = a + 3b/2 + c + D$$

this gives $W = -(U(1, 1, 1) - U(0, 0, 0)) = a + 3b/2 + c$