MA2.101: Linear Algebra (Spring 2022)

Exam

Wednesday, 28 March 2024

Course outcomes: CO1, CO3, CO6.

1. ([4 marks]) Solve one of the following.

   - The system of equations

$$x + y + z = 6$$
$$x + 4y + 6z = 20$$
$$x + 4y + \lambda z = \phi.$$

   Find the values of λ and ϕ for which this system of equations has no solutions.

   - If $A\mathbf{x} = \mathbf{b}$ always has at least one solution, show that the only solution to $A^T\mathbf{y} = 0$ is $\mathbf{y} = 0$. Here A^T denotes the transposition of matrix A.

2. ([3 marks]) V is a finite-dimensional vector space and let $T : V \to V$ be a linear operator on V. Suppose that $\text{rank}(T^2) = \text{rank}(T)$. Prove that the range and nullspace of T have only the zero vector $\mathbf{0}$ in common.

3. ([4 marks]) Two vector spaces are called *isomprphic* if there exists an invertible linear transformation from one vector space onto the other one. Prove that two finite-dimensional vector spaces over $\mathbf{F}$ are isomorphic if and only if they have the same dimension.

4. ([4 marks]) Solve one of the following.

   (a) Prove both of the following statements.

   - The image or the range of a linear transformation $T : V \to W$ is a subspace of W.
   - A linear transformation $T : V \to W$ is one-to-one if and only if the nullspace of T only contains $0 \in V$.

$T(\alpha_2) = \beta$

$T(\alpha_1) = \beta$

$T(\alpha_1 + \alpha_2) = \beta + \beta$

(b) Consider the ordered bases $\mathcal{A} = \{(1,2), (-2,-3)\}$ and $\mathcal{B} = \{(2,1), (1,3)\}$ for a vector space $\mathbf{V}$. Then find the following

- Matrix P that changes coordinates of any vector $\vec{\alpha} \in \mathbf{V}$ w.r.t. the ordered basis $\mathcal{A}$ to coordinates w.r.t. the ordered basis $\mathcal{B}$.
- Matrix Q that changes coordinates of any vector $\vec{\alpha} \in \mathbf{V}$ w.r.t. the ordered basis $\mathcal{B}$ to coordinates w.r.t. the ordered basis $\mathcal{A}$.