NOTES ON THE THEORY OF COMPUTATION

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Contents

Automata and Formal Languages

An **alphabet** is a finite set Σ, and a **word over the alphabet** Σ is a finite sequence of the elements of Σ. If a word w is the sequence $(w_0, \dots, w_n)$ for some $n \in \mathbb{N}$, we may write the word as the concatenation $w_0 \cdots w_n$. The empty word is denoted by ϵ. The set of all words over Σ is Σ^*[1]. A **formal language over the alphabet** Σ is a subset of Σ^*.

An **automaton** is an ordered sequence that **accepts** some words over an alphabet. The set of words an automaton accepts forms a language, which is unique, in which case we say the automaton **recognises** the language. Given an automaton M, we may speak of the unique language recognised by M as the **language of the automaton** M. An automaton may accept no word, in which case the language thereof is $\emptyset$.

1.1 Finite-State Automata and Regular Languages

1.1.1 Deterministic Finite-State Automata

Definition 1. A **deterministic finite-state automaton** is an ordered quintuple $(\Sigma, S, \delta, s_0, F)$ wherein

(a) Σ is an alphabet,

(b) S is a finite set of **states**,

(c) $\delta : S \times \Sigma \to S$ is the **transition function**,

(d) $s_0 \in S$ is the **initial state**, and

(e) $F \subseteq S$ is the set of **accepting states**.

Let $M = (\Sigma, S, \delta, s_0, F)$ be a deterministic finite-state automaton and let $w = w_0 \cdots w_n$ wherein $n \in \mathbb{N}$ be a word over Σ. Then M accepts w if there exists a sequence of states $(r_0, \dots, r_{n+1})$ in S such that

(a) $r_0 = s_0$,

(b) $\delta(r_i, w_i) = r_{i+1}$ for $i \in \mathbb{N}_{<n+1}$, and

(c) $r_{n+1} \in F$.

[1] * denotes the unary operator of Kleene star, defined as $A^* = \{a_0 \cdots a_n : n \in \mathbb{N} \wedge \forall i \in \mathbb{N}_{<n+1} (a_i \in A)\} \cup \{\epsilon\}$.

Furthermore, M accepts ϵ if $s_0 \in F$.

1.1.2 Nondeterministic Finite-State Automata

Definition 2. A **nondeterministic finite-state automaton** is an ordered quintuple $(\Sigma, S, \delta, s_0, F)$ wherein

(a) Σ is an alphabet,

(b) S is a finite set of states,

(c) $\delta : S \times (\Sigma \cup \{\epsilon\}) \to \mathscr{P}(S)$ is the transition function,

(d) $s_0 \in S$ is the initial state, and

(e) $F \subseteq S$ is the set of accepting states.

Let $M = (\Sigma, S, \delta, s_0, F)$ be a nondeterministic finite-state automaton and let w be a word over Σ. Then M accepts w if $w = w_0 \cdots w_n$ wherein $n \in \mathbb{N}$ such that each $w_i \in \Sigma \cup \{\epsilon\}$ for $i \in \mathbb{N}_{<n+1}$ and that there exists a sequence of states $(r_0, \dots, r_{n+1})$ in S such that

(a) $r_0 = s_0$,

(b) $r_{i+1} \in \delta(r_i, w_i)$ for $i \in \mathbb{N}_{<n+1}$, and

(c) $r_{n+1} \in F$

We say that two automata are equivalent if they recognise the same language.

Theorem 1. *Every nondeterministic finite-state automaton has an equivalent deterministic finite-state automaton.*

Proof. Let $N = (\Sigma, S, \delta, s_0, F)$ be the nondeterministic finite-state automaton recognising some language A over Σ. We construct a deterministic finite-state automaton $M = (\Sigma, S', \delta', s_0', F')$ recognising A.

We first see that $S' = \mathscr{P}(S)$ and that $F' = \{R \in S' : R \cap F \neq \emptyset\}$.

Let $\delta_0 : S \times \{\epsilon\} \to \mathscr{P}(S)$ be defined as $\delta_0(s, \epsilon) = \delta(s, \epsilon)$ for each $s \in S$. Assume first that, thus induced, $\delta_0 = \emptyset$ for N. For each $R \in S'$ and for each $a \in \Sigma$, let $\delta'(R, a) = \{s \in S : \exists r \in R\,(s \in \delta(r, a))\}$. Equivalently,

$$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a).$$

Also let $s_0' = \{s_0'\}$. We then see that $M = (\Sigma, S', \delta', s_0', F')$ recognises A.

Assume then that $\delta_0 \neq \emptyset$ for N. For each $Q \subseteq S$, let

$$E(Q) = \{s \in S : \exists n \in \mathbb{N} \exists r \in Q \left(s = \delta^n(r, \epsilon)\right)\}.$$

We then let

$$\delta'(Q, a) = \{s \in S : \exists r \in Q \, s \in E\big(\delta(r, a)\big)\}$$

and let $s_0' = E(\{s_0\})$. We similarly see that $M = (\Sigma, S', \delta', s_0', F')$ recognises A.

Therefore, the theorem holds. □

1.1.3 Regular Expressions and Regular Languages

Definition 3. Let Σ be an alphabet, and let $a \in \Sigma$. Then R is a **regular expression over** Σ if

(a) $R = \emptyset$,

(b) $R = \epsilon$,

(c) $R = a$,

(d) $R = R_1 \cup R_2$ wherein R_1 and R_2 are regular expressions over Σ,

(e) $R = R_1 R_2$[2] wherein R_1 and R_2 are regular expressions over Σ, or

(f) $R = R_1^*$ wherein R_1 is a regular expression over Σ.

The language described by a regular expression is a **regular language**. Each regular expression describes a unique regular language, while a regular language may have multiple distinct regular expressions describing it. If R is a regular expression, we denote the regular language it describes by $L(R)$.

Let Σ be an alphabet, let $a \in \Sigma$, and let R, R_1, and R_2 be regular expressions over Σ. If $R = \emptyset$, then $L(R) = \emptyset$. If $R = \epsilon$, then $L(R) = \{\epsilon\}$. If $R = a$, then $L(R) = \{a\}$. If $R = R_1 \cup R_2$, then $L(R) = L(R_1) \cup L(R_2)$. If $R = R_1 R_2$, then $L(R) = L(R_1)L(R_2)$[3]. If $R = R_1^*$, then $L(R) = L(R_1)^*$.

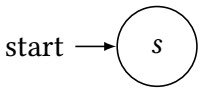
[2] $R_1 R_2$ denotes the concatenation of R_1 and R_2.

[3] If A and B are languages, AB denotes the concatenation of A and B, defined as $AB = \{ab : a \in A \wedge b \in B\}$.

Lemma 1. *If a language is regular, then some nondeterministic finite-state automaton recognises it.*

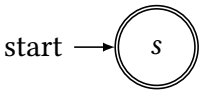
Proof. Let Σ be an alphabet, let A be a regular language over Σ, and let R be a regular expression which describes A.

If $R = \emptyset$, then the nondeterministic finite-state automaton N characterised by the following diagram recognises A.



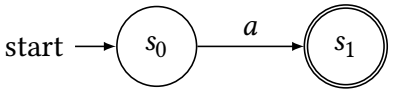
Equivalently, $N = (\Sigma, \{s\}, \delta, s, \emptyset)$ wherein $\delta(r, b) = \emptyset$ for any r and b.

If $R = \epsilon$, then the nondeterministic finite-state automaton N characterised by the following diagram recognises A.



Equivalently, $N = (\Sigma, \{s\}, \delta, s, \{s\})$ wherein $\delta(r, b) = \emptyset$ for any r and b.

If $R = a$ for some $a \in \Sigma$, then the nondeterministic finite-state automaton N characterised by the following diagram recognises it.



Equivalently, $N = (\Sigma, \{s_0, s_1\}, \delta, s_0, \{s_1\})$ wherein $\delta(s_0, a) = \{s_1\}$ and $\delta(r, b) = \emptyset$ if $r \neq s_0$ or $b \neq a$.

Assume that R_1 and R_2 are regular expressions over Σ, that $N_1 = (\Sigma, S_1, \delta_1, s_1, F_1)$ is a nondeterministic finite-state automaton recognising $L(R_1)$, and that $N_2 = (\Sigma, S_2, \delta_2, s_2, F_2)$ is a nondeterministic finite-state automaton recognising $L(R_2)$.

If $R = R_1 \cup R_2$, let s_0 be a state not in S_1 or S_2, let $S = S_1 \cup s_2 \cup \{s_0\}$, and let $F = F_1 \cup F_2$.

Define $\delta : S \times (\Sigma \cup \{\epsilon\}) \to \mathscr{P}(S)$ so that for each $r \in S$ and each $b \in \Sigma \cup \{\epsilon\}$ we have

$$\delta(r,b) = \begin{cases} \delta_1(r,b) & \text{if } r \in S_1, \\ \delta_2(r,b) & \text{if } r \in S_2, \\ \{s_1, s_2\} & \text{if } r = s_0 \wedge b = \epsilon, \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$

We see that $N = (\Sigma, S, \delta, s_0, F)$ is a nondeterministic finite-state automaton recognising A.

If $R = R_1 R_2$, let $S = S_1 \cup S_2$. Define $\delta : S \times (\Sigma \cup \{\epsilon\}) \to \mathscr{P}(S)$ so that for each $r \in S$ and each $b \in \Sigma \cup \{\epsilon\}$ we have

$$\delta(r,b) = \begin{cases} \delta_1(r,b) & \text{if } (r \in S_1 \wedge r \notin F_1) \vee (r \in F_1 \wedge b \neq \epsilon), \\ \delta_1(r,b) \cup \{s_2\} & \text{if } r \in F_1 \wedge b = \epsilon, \text{ and} \\ \delta_2(r,b) & \text{otherwise.} \end{cases}$$

We see that $N = (\Sigma, S, \delta, s_1, F_2)$ is a nondeterministic finite-state automaton recognising A.

If $R = R_1^*$, let s_0 be a state not in S_1, let $S = S_1 \cup \{s_0\}$, and let $F = F_1 \cup \{s_0\}$. Define $\delta : S \times (\Sigma \cup \{\epsilon\}) \to \mathscr{P}(S)$ so that for each $r \in S$ and each $b \in \Sigma \cup \{\epsilon\}$ we have

$$\delta(r,b) = \begin{cases} \delta_1(r,b) & \text{if } (r \in S_1 \wedge r \notin F_1) \vee (r \in F_1 \wedge b \neq \epsilon), \\ \delta_1(r,b) \cup \{s_1\} & \text{if } r \in F_1 \wedge b = \epsilon, \\ \{s_1\} & \text{if } r = s_0 \wedge b = \epsilon, \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$

We see that $N = (\Sigma, S, \delta, s_0, F)$ is a nondeterministic finite-state automaton recognising A.

Therefore, the lemma holds by the principle of induction. □

Definition 4. A **generalised nondeterministic finite-state automaton** is an ordered quintuple $(\Sigma, S, \delta, s_0, s_1)$ wherein

(a) Σ is an alphabet,

(b) S is a finite set of states,

(c) $\delta : (S \setminus \{s_1\}) \times (S \setminus \{s_0\}) \to \mathscr{R}$ wherein $\mathscr{R}$ is the set of all regular expressions over Σ is the transition function,

(d) $s_0 \in S$ is the initial state, and

(e) $s_1 \neq s_0 \in S$ is the accepting state.

Let $M = (\Sigma, S, \delta, s_0, s_1)$ be a generalised nondeterministic finite-state automaton and let w be a word over Σ. Then M accepts w if $w = w_0 \cdots w_n$ wherein $n \in \mathbb{N}$ such that each $w_i \in \Sigma^*$ for $i \in \mathbb{N}_{<n+1}$ and that there exists a sequence of states $(r_0, \ldots, r_{n+1})$ in S such that

(a) $r_0 = s_0$,

(b) $r_{n+1} = s_1$, and

(c) $w_i \in L(R_i)$ wherein $R_i = \delta(r_i, r_{i+1})$ for $i \in \mathbb{N}_{<n+1}$.

Lemma 2. *If a nondeterministic finite-state automaton recognises a language, then it is regular.*

Proof. Let $N = (\Sigma, S, \delta, s_0, F)$ be a nondeterministic finite-state automaton recognising the language A over Σ. We argue that A is described by some regular expression R over Σ.

Let $G = (\Sigma, S', \delta', s_0', s_1')$ be a generalised nondeterministic finite-state automaton such that

(a) $s_0' \notin S$,

(b) $s_1' \notin S$,

(c) $S' = S \cup \{s_0', s_1'\}$, and

(d) for each $r_0 \in S \cup \{s_0'\}$ and each $r_1 \in S \cup \{s_1'\}$ we have

$$\delta'(r_0, r_1) = \begin{cases} \epsilon & \text{if } (r_0 = s_0' \wedge r_1 = s_0) \vee (r_0 \in F \wedge r_1 = s_1'), \\ R' & \text{if } r_0 \in S \wedge r_1 \in S \wedge \forall r \in L(R') \left(r_1 \in \delta(r_0, r)\right), \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$

We see that G also recognises A. We shall then convert G into regular expression R.

Let $k = |S'|$.

If $k = 2$, then $S' = \{s_0', s_1'\}$, and so $R = \delta'(s_0', s_1')$ is the regular expression.

If $k > 2$, let $s \in S'$ be distinct from s_0' and s_1', and let $G' = (\Sigma, S'', \delta'', s_0', s_1')$ be a generalised nondeterministic finite-state automaton such that

(a) $S'' = S' \setminus \{s\}$,

(b) for each $r_0 \in S'' \setminus \{s_0'\}$ and each $r_1 \in S'' \setminus \{s_1'\}$ we have

$$\delta''(r_0, r_1) = R_0 R_1^* R_2 \cup R_3$$

wherein $R_0 = \delta'(r_0, s)$, $R_1 = \delta'(s, s)$, $R_2 = \delta'(s, r_1)$, and $R_3 = \delta'(r_0, r_1)$.

We see that G' is equivalent to G.

Because G' has one fewer state than G, by the principle of induction, there exists regular expression R converted from G for any generalised nondeterministic finite-state automaton.

Therefore, the lemma holds. □

Theorem 2. *A language is regular if and only if some nondeterministic finite-state automaton recognises it.*

Proof. The theorem holds by Lemma 1 and Lemma 2. □

Corollary 1. *A language is regular if and only if some deterministic finite-state automaton recognises it.*

Proof. The corollary holds by Theorem 1 and Theorem 2 □

1.1.4 NONREGULAR LANGUAGES

Theorem 3 (Pumping lemma). *If A is a regular language over Σ, then there is a number $p \in \mathbb{Z}_{>0}$, the **pumping length**, such that if $w \in A$ is of length at least p, then there exist x, y, and $z \in \Sigma^*$ which satisfies*

(a) $w = xyz$,

(b) $xy^iz \in A$ for each $i \in \mathbb{N}$,

(c) $|y| > 0$, and

(d) $|xy| \leq p$.

Proof. Let $M = (\Sigma, S, \delta, s_0, F)$ be a deterministic finite-state automaton recognising A and let $p = |S|$.

Let $w = w_0 \cdots w_n$ wherein $n \in \mathbb{N}$ be a word in R of length $n + 1$ which satisfies $n + 1 \geq p$. Let $r_0, \ldots, r_{n+1}$ be the sequence of states that M enters when accepting r. This sequence has length $n + 2$, which must be at least $p + 1$. Among the first $p + 1$ elements in the sequence, two must be the same state by the pigeonhole principle. Let the first of these be r_i and the second r_j. We note that $i \leq j - 1$ and that $j \leq p$. Now let $x = w_0 \cdots w_{i-1}$, $y = w_i \cdots w_{j-1}$, and $z = w_j \cdots w_n$.

Thus induced, $r = xyz$ satisfies the pumping lemma. □

We may use the pumping lemma to prove that a language is not regular. To do so, first assume that the language is regular, and then use the pumping lemma to guarantee the existence of a pumping length p such that all words of length p or greater in the language can be pumped. Next, find a word in the language with length p or greater but cannot be pumped. Finally, demonstrate that this word cannot be pumped by considering all ways of diving it into three such substrings described by the pumping lemma. The existence of this word contradicts the pumping lemme assuming the language is regular. Hence, the language is not regular.

1.2 Pushdown Automata and Context-Free Languages

1.2.1 Pushdown Automata

Definition 5. A **pushdown automaton** is an ordered sextuple $(\Sigma, \Gamma, S, \delta, s_0, F)$ wherein

(a) Σ is an alphabet, for the input,

(b) Γ is another alphabet, for the **stack**,

(c) S is a finite set of states,

(d) $\delta : S \times (\Sigma \cup \{\epsilon\}) \times (\Gamma \cup \{\epsilon\}) \to \mathscr{P}\big(S \times (\Gamma \cup \{\epsilon\})\big)$ is the transition function,

(e) $s_0 \in S$ is the initial state, and

(f) $F \subseteq S$ is the set of final or accepting states.

Let $M = (\Sigma, \Gamma, S, \delta, s_0, F)$ be a pushdown automaton and let w be a word over Σ. Then M accepts $w = w_0 \cdots w_n$ wherein $n \in \mathbb{N}$ such that $w_i \in \Sigma \cup \{\epsilon\}$ for $i \in \mathbb{N}_{<n+1}$ and that there exist a sequence of states $(r_0, \ldots, r_{n+1})$ in S and a sequence of words $(q_0, \ldots, q_{n+1})$ in Γ^* such that

(a) $r_0 = s_0$,

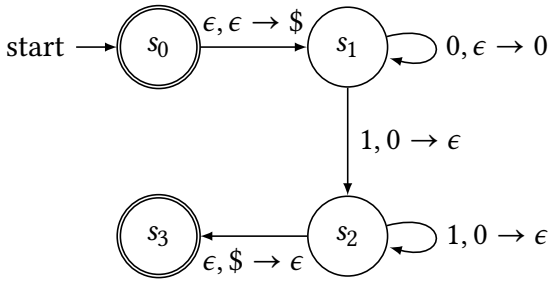
(b) $q_0 = \epsilon$,

(c) $(r_{i+1}, b) \in \delta(r_i, w_i, a)$, $q_i = at$, and $q_{i+1} = bt$ for some a and $b \in \Gamma \cup \{\epsilon\}$ and some $t \in \Gamma^*$ for $i \in \mathbb{N}_{<n+1}$, and

(d) $r_{n+1} \in F$.

Exercise 1. Let $\Sigma = \{0, 1\}$ be an alphabet. Construct a pushdown automaton which recognises the language $\{0^n 1^n : n \in \mathbb{N}\}$.

Solution. The pushdown automaton M characterised by the following diagram recognises the given language.



Equivalently, $M = (\Sigma, \Gamma, S, \delta, s_0, F)$ wherein

(a) $\Gamma = \{0, \$\}$,

(b) $S = \{s_0, s_1, s_2, s_3\}$,

(c) $F = \{s_0, s_3\}$, and

(d) for each $s \in S$, each $a \in \Sigma \cup \{\epsilon\}$, and each $b \in \Gamma \cup \{\epsilon\}$ we have

$$\delta(s,a,b) = \begin{cases} \{(s_1, \$)\} & \text{if } s = s_0 \wedge a = \epsilon \wedge b = \epsilon, \\ \{(s_1, 0)\} & \text{if } s = s_1 \wedge a = 0 \wedge b = \epsilon, \\ \{(s_2, \epsilon)\} & \text{if } (s = s_1 \vee s = s_2) \wedge a = 1 \wedge b = 0, \\ \{(s_3, \epsilon)\} & \text{if } s = s_2 \wedge a = \epsilon \wedge b = \$, \text{ and} \\ \varnothing & \text{otherwise} \end{cases}$$

is a pushdown automaton which recognises the given language. ◊

1.2.2 Context-Free Grammars and Context-Free Languagse

Definition 6. A **context-free grammar** is an ordered quadruple (Σ, V, R, S) wherein

(a) Σ is an alphabet, the elements whereof are **terminals**,
(b) V is another alphabet, the elements whereof are **variables**, which is disjoint from Σ,
(c) $R : V \to (\Sigma \cup V)^*$ is a finite set of **production rules**, and
(d) $S \in V$ is the **start variable**.

Let (Σ, V, R, S) be a context-free grammar. If (A, w) wherein $A \in V$ and $w \in (\Sigma \cup V)^*$ is a production rule, we write $A \to w$. If u, v, and $w \in (\Sigma \cup V)^*$, and $A \to w$ is a production rule of the grammar, we say that uAv **yields** uwv, written $uAv \Rightarrow uwv$. We say that u **derives** v, written $u \Rightarrow^* v$, if $u = v$, $u \Rightarrow v$, or there exists a sequence $(u_0, \dots, u_n)$ wherein $n \in \mathbb{N}$ such that

$$u \Rightarrow u_0 \Rightarrow \cdots \Rightarrow u_n \Rightarrow v.$$

If $A \to u$ and $A \to v$ are production rules of the grammar, we may denote them by $A \to u \,|\, v$. The **language of the grammar** is $\{w \in \Sigma^* : S \Rightarrow^* w\}$.

The language of a context-free grammar is a **context-free language**.

Exercise 2. Let $\Sigma = \{0, 1\}$ be an alphabet. Construct a context-free grammar whose language is $\{0^n 1^n : n \in \mathbb{N}\}$.

Solution. Let (Σ, V, R, S) be the context-free grammar wherein $V = \{S\}$ and R consists of the following production rule

$$S \to 0S1 \mid \epsilon.$$

The language of the above context-free grammar is the given language. ◊

A derivation of a word in a context-free grammar is a **leftmost derivation** if at every step of production the leftmost remaining variable is the one substituted according to a production rule.

Definition 7. A word is derived **ambiguously** in a context-free grammar if there exist two or more distinct leftmost derivations for it.

A context-free grammar is **ambiguous** is it generates some words ambiguously.

Some context-free languages can only be generated by ambiguous context-free grammars. Such languages are **inherently ambiguous**.

1.2.3 Chomsky Normal Form

Definition 8. A context-free grammar is **in Chomsky normal form** if every production rule thereof is

(a) $S \to \epsilon$ wherein S is the start variable,

(b) $A \to BC$ wherein A, B, and C are variables and B and C are not the start variable, or

(c) $A \to a$ wherein A is a variable and a is a terminal.

Theorem 4. *Any context-free language is generated by a context-free grammar in Chomsky normal form.*

Proof. Let (Σ, V, R, S) be a context-free grammar. We demonstrate a procedure to convert it into another context-free grammar in Chomsky normal form (Σ, V', R', S').

We first add $S' \to S$ as a production rule.

Second, if there exist rules of the form $A \to \epsilon$ wherein $A \neq S'$, we remove them and repeatedly replace any rule of the form $B \to uAv$ wherein $B \in V'$ and u and $v \in (\Sigma \cup V')^*$ with $B \to uv$ for each occurrence of A.

Third, if there exist rules of the form $A \to B$ wherein A and $B \in V'$, we remove them and replace any rule of the form $B \to u$ wherein $u \in (\Sigma \cup V')^*$ with $A \to u$.

Lastly, we replace each rule of the form $A \to u_0 \cdots u_n$ wherein $n \in \mathbb{N}$ and $u_i \in \Sigma \cup V'$ for $i \in \mathbb{N}_{<n+1}$ such that $n > 1$ with the rules $A \to u_0 A_0$, $A_0 \to u_1 A_1$, ..., $A_{n-2} \to u_{n-1} u_n$ and add A_i for $i \in \mathbb{N}_{<n-1}$ as variables. We then replace any terminal u_i for $i \in \mathbb{N}_{<n+1}$ with the new variable U_i while adding the rule $U_i \to u_i$.

The resultant context-free grammar is in Chomsky normal form, and therefore the theorem holds. □