NOTES ON THE THEORY OF COMPUTATION

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Contents

AUTOMATA AND FORMAL LANGUAGES

An **alphabet** is a finite set Σ, and a **word over the alphabet** Σ is a finite sequence of the elements of Σ. If a word w is the sequence $(w_0, \dots, w_n)$ for some $n \in \mathbb{N}$, we may write w as the concatenation $w_0 \cdots w_n$. If $w = a \cdots a$ wherein a is repeated n times for some $n \in \mathbb{Z}_{>0}$, we may write w as a^n. The empty word is denoted by ϵ, and for any element a of an alphabet a^0 is the empty word. The set of all words over Σ is Σ^*[1]. A **formal language over the alphabet** Σ is a subset of Σ^*. The attributive "formal" connotes that such languages lack semantics.

An **automaton** is an ordered sequence that **accepts** some words over an alphabet. The set of words an automaton accepts forms a language, which is unique, in which case we say the automaton **recognises** the language. Given an automaton M, we may speak of the unique language recognised by M as the **language of the automaton** M. An automaton may accept no word, in which case the language thereof is $\varnothing$. Two automata are equivalent if they recognise the same language.

1.1 FINITE-STATE AUTOMATA AND REGULAR LANGUAGES

1.1.1 DETERMINISTIC FINITE-STATE AUTOMATA

DEFINITION 1. A **deterministic finite-state automaton** is an ordered quintuple $(\Sigma, Q, \delta, q_0, F)$ wherein

(a) Σ is an alphabet,

(b) Q is a finite set of **states**,

(c) $\delta : Q \times \Sigma \to Q$ is the **transition function**,

(d) $q_0 \in Q$ is the **initial state**, and

(e) $F \subseteq Q$ is the set of **accepting states**.

Let $M = (\Sigma, Q, \delta, q_0, F)$ be a deterministic finite-state automaton and let $w = w_0 \cdots w_n$ wherein $n \in \mathbb{N}$ be a word over Σ. Then M accepts w if there exists a sequence of states $(r_0, \dots, r_{n+1})$ in Q such that

[1] * denotes the unary operator of Kleene star, defined as $A^* = \{a_0 \cdots a_n : n \in \mathbb{N} \wedge \forall\, i \in \mathbb{N}_{<n+1}\, (a_i \in A)\} \cup \{\epsilon\}$ for a subset A of an alphabet, and $a^* = \{a^n : n \in \mathbb{N}\}$ for an element a of an alphabet.

(a) $r_0 = q_0$,

(b) $\delta(r_i, w_i) = r_{i+1}$ for $i \in \mathbb{N}_{<n+1}$, and

(c) $r_{n+1} \in F$.

Furthermore, M accepts ϵ if $q_0 \in F$.

1.1.2 Nondeterministic Finite-State Automata

Definition 2. A **nondeterministic finite-state automaton** is an ordered quintuple $(\Sigma, Q, \delta, q_0, F)$ wherein

(a) Σ is an alphabet,

(b) Q is a finite set of states,

(c) $\delta : Q \times (\Sigma \cup \{\epsilon\}) \to \mathscr{P}(Q)$ is the transition function,

(d) $q_0 \in Q$ is the initial state, and

(e) $F \subseteq Q$ is the set of accepting states.

Let $N = (\Sigma, Q, \delta, q_0, F)$ be a nondeterministic finite-state automaton and let w be a word over Σ. Then N accepts w if $w = w_0 \cdots w_n$ wherein $n \in \mathbb{N}$ such that each $w_i \in \Sigma \cup \{\epsilon\}$ for some $i \in \mathbb{N}_{<n+1}$ and that there exists a sequence of states $(r_0, \dots, r_{n+1})$ in Q such that

(a) $r_0 = q_0$,

(b) $r_{i+1} \in \delta(r_i, w_i)$ for $i \in \mathbb{N}_{<n+1}$, and

(c) $r_{n+1} \in F$.

Theorem 1. *Every nondeterministic finite-state automaton has an equivalent deterministic finite-state automaton.*

Proof. Let Σ be an alphabet, let A be a language over Σ, and let $N = (\Sigma, Q, \delta, q_0, F)$ be a nondeterministic finite-state automaton recognising A. We construct a deterministic finite-state automaton $M = (\Sigma, Q', \delta', q_0', F')$ which also recognises A.

We first see that $Q' = \mathscr{P}(Q)$ and that $F' = \{R \in Q' : R \cap F \neq \varnothing\}$.

Let $\delta_0 : Q \times \{\epsilon\} \to \mathscr{P}(Q)$ be defined as $\delta_0(q, \epsilon) = \delta(q, \epsilon)$ for each $q \in Q$. Assume first that, thus induced, $\delta_0 = \varnothing$ for N. For each $R \in Q'$ and each $a \in \Sigma$, let $\delta'(R, a) = \big\{q \in Q : \exists r \in R\big(q \in \delta(r, a)\big)\big\}$.

Equivalently,

$$\delta'(R,a) = \bigcup_{r \in R} \delta(r,a).$$

Also let $q_0' = \{q_0'\}$. We then see that $M = (\Sigma, Q', \delta', q_0', F')$ recognises A.

Assume then that $\delta_0 \neq \emptyset$ for N. For each $R \subseteq Q$, let

$$E(R) = \big\{q \in Q \,:\, \exists n \in \mathbb{N}\, \exists r \in R \,\big(q = \delta^n(r,\epsilon)\big)\big\}.$$

We then let

$$\delta'(R,a) = \big\{q \in Q \,:\, \exists r \in R\, s \in E\big(\delta(r,a)\big)\big\}$$

and let $q_0' = E(\{q_0\})$. We similarly see that $M = (\Sigma, Q', \delta', q_0', F')$ recognises A.

Therefore, the theorem holds. $\square$

1.1.3 Regular Expressions and Regular Languages

Definition 3. Let Σ be an alphabet. Then R is a **regular expression over Σ** if

(a) $R = \emptyset$,

(b) $R = \epsilon$,

(c) $R = a$ for some $a \in \Sigma$,

(d) $R = R_1 \cup R_2$ wherein R_1 and R_2 are regular expressions over Σ,

(e) $R = R_1R_2$[2] wherein R_1 and R_2 are regular expressions over Σ, or

(f) $R = R_1^*$ wherein R_1 is a regular expression over Σ.

The language described by a regular expression is a **regular language**, which is unique. If R is a regular expression, we denote the regular language it describes by $L(R)$.

Let Σ be an alphabet, let $a \in \Sigma$, and let R, R_1, and R_2 be regular expressions over Σ. If $R = \emptyset$, then $L(R) = \emptyset$. If $R = \epsilon$, then $L(R) = \{\epsilon\}$. If $R = a$, then $L(R) = \{a\}$. If $R = R_1 \cup R_2$, then $L(R) = L(R_1) \cup L(R_2)$. If $R = R_1R_2$, then $L(R) = L(R_1)L(R_2)$[3]. If $R = R_1^*$, then $L(R) = L(R_1)^*$.

[2] R_1R_2 denotes the concatenation of R_1 and R_2.

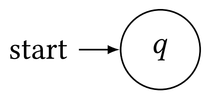
[3] If A and B are languages, AB denotes the concatenation of A and B, defined as $AB = \{ab \,:\, a \in A \wedge b \in B\}$.

1.1.4 EQUIVALENCE BETWEEN FINITE-STATE AUTOMATA AND REGULAR LANGUAGES

Lemma 1. *If a language is regular, then some nondeterministic finite-state automaton recognises it.*

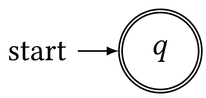
Proof. Let Σ be an alphabet and let R be a regular expression over Σ.

If $R = \emptyset$, then the nondeterministic finite-state automaton N characterised by the following diagram recognises $L(R)$.



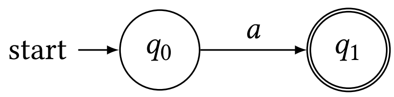
Equivalently, $N = (\Sigma, \{q\}, \delta, q, \emptyset)$ wherein $\delta(r, b) = \emptyset$ for any r and b.

If $R = \epsilon$, then the nondeterministic finite-state automaton N characterised by the following diagram recognises $L(R)$.



Equivalently, $N = (\Sigma, \{q\}, \delta, q, \{q\})$ wherein $\delta(r, b) = \emptyset$ for any r and b.

If $R = a$ for some $a \in \Sigma$, then the nondeterministic finite-state automaton N characterised by the following diagram recognises $L(R)$.



Equivalently, $N = (\Sigma, \{q_0, q_1\}, \delta, q_0, \{q_1\})$ wherein $\delta(q_0, a) = \{q_1\}$ and $\delta(r, b) = \emptyset$ if $r \neq q_0$ or $b \neq a$.

Assume that R_1 and R_2 are regular expressions over Σ, that $N_1 = (\Sigma, Q_1, \delta_1, q_1, F_1)$ is a nondeterministic finite-state automaton recognising $L(R_1)$, and that $N_2 = (\Sigma, Q_2, \delta_2, q_2, F_2)$ is a nondeterministic finite-state automaton recognising $L(R_2)$.

If $R = R_1 \cup R_2$, let $\{q_0\}$ be disjoint from Q_1 and Q_2, let $Q = Q_1 \cup Q_2 \cup \{q_0\}$, and let $F = F_1 \cup F_2$.

Define $\delta : Q \times (\Sigma \cup \{\epsilon\}) \to \mathscr{P}(Q)$ so that for each $r \in Q$ and each $b \in \Sigma \cup \{\epsilon\}$ we have

$$\delta(r,b) = \begin{cases} \delta_1(r,b) & \text{if } r \in Q_1, \\ \delta_2(r,b) & \text{if } r \in Q_2, \\ \{q_1, q_2\} & \text{if } r = q_0 \wedge b = \epsilon, \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$

We see that $N = (\Sigma, Q, \delta, q_0, F)$ is a nondeterministic finite-state automaton recognising $L(R)$.

If $R = R_1R_2$, let $Q = Q_1 \cup Q_2$. Define $\delta : Q \times (\Sigma \cup \{\epsilon\}) \to \mathscr{P}(Q)$ so that for each $r \in Q$ and each $b \in \Sigma \cup \{\epsilon\}$ we have

$$\delta(r,b) = \begin{cases} \delta_1(r,b) & \text{if } (r \in Q_1 \wedge r \notin F_1) \vee (r \in F_1 \wedge b \neq \epsilon), \\ \delta_1(r,b) \cup \{q_2\} & \text{if } r \in F_1 \wedge b = \epsilon, \text{ and} \\ \delta_2(r,b) & \text{otherwise.} \end{cases}$$

We see that $N = (\Sigma, Q, \delta, q_1, F_2)$ is a nondeterministic finite-state automaton recognising $L(R)$.

If $R = R_1^*$, let $\{q_0\}$ be disjoint from Q_1, let $Q = Q_1 \cup \{q_0\}$, and let $F = F_1 \cup \{q_0\}$. Define $\delta : Q \times (\Sigma \cup \{\epsilon\}) \to \mathscr{P}(Q)$ so that for each $r \in Q$ and each $b \in \Sigma \cup \{\epsilon\}$ we have

$$\delta(r,b) = \begin{cases} \delta_1(r,b) & \text{if } r \in Q_1 \setminus F_1 \vee (r \in F_1 \wedge b \neq \epsilon), \\ \delta_1(r,b) \cup \{q_1\} & \text{if } r \in F_1 \wedge b = \epsilon, \\ \{q_1\} & \text{if } r = q_0 \wedge b = \epsilon, \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$

We see that $N = (\Sigma, Q, \delta, q_0, F)$ is a nondeterministic finite-state automaton recognising $L(R)$.

Therefore, the lemma holds by the principle of induction. $\square$

DEFINITION 4. A **generalised nondeterministic finite-state automaton** is an ordered quintuple $(\Sigma, Q, \delta, q_0, q_1)$ wherein

(a) Σ is an alphabet,

(b) Q is a finite set of states,

(c) $\delta : (Q \setminus \{q_1\}) \times (Q \setminus \{q_0\}) \to \mathscr{R}$ wherein $\mathscr{R}$ is the set of all regular expressions over Σ is the transition function,

(d) $q_0 \in Q$ is the initial state, and

(e) $q_1 \neq q_0 \in Q$ is the accepting state.

Let $G = (\Sigma, Q, \delta, q_0, q_1)$ be a generalised nondeterministic finite-state automaton and let w be a word over Σ. Then M accepts w if $w = w_0 \cdots w_n$ wherein $n \in \mathbb{N}$ such that each $w_i \in \Sigma^*$ for some $i \in \mathbb{N}_{<n+1}$ and that there exists a sequence of states $(r_0, \dots, r_{n+1})$ in Q such that

(a) $r_0 = q_0$,

(b) $r_{n+1} = q_1$, and

(c) $w_i \in L\big(\delta(r_i, r_{i+1})\big)$ for $i \in \mathbb{N}_{<n+1}$.

Lemma 2. *If a nondeterministic finite-state automaton recognises a language, then it is regular.*

Proof. Let Σ be an alphabet, let A be a language over Σ, and let $N = (\Sigma, Q, \delta, q_0, F)$ be a nondeterministic finite-state automaton recognising A. We argue that A is described by some regular expression R over Σ.

Let $G = (\Sigma, Q', \delta', q_0', q_1')$ be a generalised nondeterministic finite-state automaton such that

(a) $\{q_0', q_0'\} \cap Q = \varnothing$,

(b) $Q' = Q \cup \{q_0', q_1'\}$, and

(c) for each $r_0 \in Q' \setminus \{q_1'\}$ and each $r_1 \in Q' \setminus \{q_0'\}$ we have

$$\delta'(r_0, r_1) = \begin{cases} \epsilon & \text{if } (r_0 = q_0' \wedge r_1 = q_0) \vee (r_0 \in F \wedge r_1 = q_1'), \\ R' & \text{if } r_0 \in Q \wedge r_1 \in Q \wedge \forall\, r \in L(R')\,\big(r_1 \in \delta(r_0, r)\big), \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$

We see that G also recognises A. We shall then convert G into regular expression R.

Let $k = |Q'|$.

If $k = 2$, then $Q' = \{q'_0, q'_1\}$, and so $R = \delta'(q'_0, q'_1)$ is the regular expression.

If $k > 2$, let $q \in Q'$ be distinct from q'_0 and q'_1, and let $G' = (\Sigma, Q'', \delta'', q'_0, q'_1)$ be a generalised nondeterministic finite-state automaton such that

(a) $Q'' = Q' \setminus \{q\}$,

(b) for each $r_0 \in Q'' \setminus \{q'_0\}$ and each $r_1 \in Q'' \setminus \{q'_1\}$ we have

$$\delta''(r_0, r_1) = R_0 R_1^* R_2 \cup R_3$$

wherein $R_0 = \delta'(r_0, q)$, $R_1 = \delta'(q, q)$, $R_2 = \delta'(q, r_1)$, and $R_3 = \delta'(r_0, r_1)$.

We see that G' is equivalent to G.

Because G' has one fewer state than G, by the principle of induction, there exists regular expression R converted from G for any generalised nondeterministic finite-state automaton.

Therefore, the lemma holds. □

Theorem 2. *A language is regular if and only if some nondeterministic finite-state automaton recognises it.*

Proof. The theorem holds by Lemma 1 and Lemma 2. □

Corollary 1. *A language is regular if and only if some deterministic finite-state automaton recognises it.*

Proof. The corollary holds by Theorem 1 and Theorem 2. □

1.1.5 NONREGULAR LANGUAGES

Theorem 3 (pumping lemma). *Let Σ be an alphabet. If A is a regular language over Σ, then there is some $p \in \mathbb{Z}_{>0}$, the* ***pumping length****, such that if $w \in A$ satisfies $|w| \geq p$, then there exist x, y, and $z \in \Sigma^*$ which satisfy*

(a) $w = xyz$,

(b) $xy^iz \in A$ *for each* $i \in \mathbb{N}$,

(c) $|y| > 0$, *and*

(d) $|xy| \leq p$.

Proof. Let $M = (\Sigma, Q, \delta, q_0, F)$ be a deterministic finite-state automaton recognising A and let $p = |Q|$.

Let $w = w_0 \cdots w_n$ wherein $n \in \mathbb{N}$ be a word in R of length $n + 1$ which satisfies $n + 1 \geq p$. Let $(r_0, \dots, r_{n+1})$ be the sequence of states that M enters when accepting w. This sequence has length $n + 2$, which must be at least $p + 1$. Among the first $p + 1$ elements in the sequence, two must be the same state by the pigeonhole principle. Let the first of these be r_i and the second r_j. We note that $i \leq j - 1$ and that $j \leq p$. Now let $x = w_0 \cdots w_{i-1}$, $y = w_i \cdots w_{j-1}$, and $z = w_j \cdots w_n$.

Thus induced, $w = xyz$ satisfies the pumping lemma. □

Exercise 1. Let $\Sigma = \{0, 1\}$ be an alphabet. Prove that the language $A = \{0^n 1^n : n \in \mathbb{N}\}$ is not regular.

Solution. Assume for the sake of contradiction that A is regular. Let p be the pumping length thereof, and let $w = 0^p 1^p$. Then there exist x, y, and $z \in \Sigma^*$ such that $w = xyz$, that $xy^i z \in A$ for $i \in \mathbb{N}$, that $|y| > 0$, and that $|xy| \leq p$ by the pumping lemma. We argue that it is impossible that there exist such words.

We first see that $y = 0^j$ wherein $j \in \mathbb{Z}_{>0}$, for $|y| > 0$ and $|xy| \leq p$. Thus, $xyyz = 0^{p+j} 1^p \notin A$, which is a contradiction of $xy^i z \in A$ for $i \in \mathbb{N}$.

By the contradiction obtained above, the original proposition holds. ◊

1.2 Pushdown Automata and Context-Free Languages

1.2.1 Pushdown Automata

Definition 5. A **pushdown automaton** is an ordered sextuple $(\Sigma, \Gamma, Q, \delta, q_0, F)$ wherein

(a) Σ is an alphabet for the input,

(b) Γ is another alphabet for the **stack**,

(c) Q is a finite set of states,

(d) $\delta : Q \times (\Sigma \cup \{\epsilon\}) \times (\Gamma \cup \{\epsilon\}) \to \mathscr{P}\big(Q \times (\Gamma \cup \{\epsilon\})\big)$ is the transition function,

(e) $q_0 \in Q$ is the initial state, and

(f) $F \subseteq Q$ is the set of accepting states.

Let $P = (\Sigma, \Gamma, Q, \delta, q_0, F)$ be a pushdown automaton and let w be a word over Σ. Then M accepts $w = w_0 \cdots w_n$ wherein $n \in \mathbb{N}$ such that $w_i \in \Sigma \cup \{\epsilon\}$ for some $i \in \mathbb{N}_{<n+1}$ and that there exist a sequence of states $(r_0, \dots, r_{n+1})$ in Q and a sequence of words $(s_0, \dots, s_{n+1})$ in Γ^* such that

(a) $r_0 = q_0$,

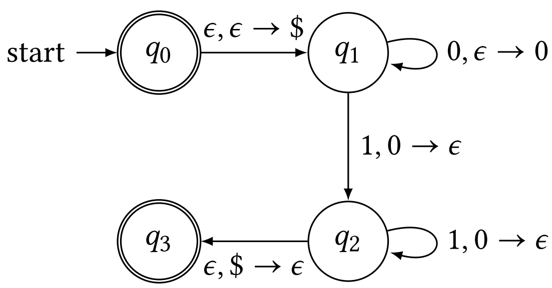
(b) $s_0 = \epsilon$,

(c) for each $i \in \mathbb{N}_{<n+1}$ there exist some a and $b \in \Gamma \cup \{\epsilon\}$ and some $t \in \Gamma^*$ such that $(r_{i+1}, b) \in \delta(r_i, w_i, a)$, that $s_i = at$, and that$s_{i+1} = bt$, and

(d) $r_{n+1} \in F$.

EXERCISE 2. Let $\Sigma = \{0, 1\}$ be an alphabet. Construct a pushdown automaton which recognises the language $A = \{0^n 1^n : n \in \mathbb{N}\}$.

Solution. The pushdown automaton P characterised by the following diagram recognises A.



Equivalently, $P = (\Sigma, \Gamma, Q, \delta, q_0, F)$ wherein

(a) $\Gamma = \{0, \$\}$,

(b) $Q = \{q_0, q_1, q_2, q_3\}$,

(c) $F = \{q_0, q_3\}$, and

(d) for each $q \in Q$, each $b \in \Sigma \cup \{\epsilon\}$, and each $s \in \Gamma \cup \{\epsilon\}$ we have

$$\delta(q,b,s) = \begin{cases} \{(q_1,\$)\} & \text{if } q = q_0 \wedge b = \epsilon \wedge s = \epsilon, \\ \{(q_1,0)\} & \text{if } q = q_1 \wedge b = 0 \wedge s = \epsilon, \\ \{(q_2,\epsilon)\} & \text{if } (q = q_1 \vee q = q_2) \wedge b = 1 \wedge s = 0, \\ \{(q_3,\epsilon)\} & \text{if } q = q_2 \wedge b = \epsilon \wedge s = \$, \text{ and} \\ \varnothing & \text{otherwise} \end{cases}$$

is a pushdown automaton which recognises A. ◊

1.2.2 Context-Free Grammars and Context-Free Languagse

DEFINITION 6. A **context-free grammar** is an ordered quadruple (Σ, V, R, S) wherein

(a) Σ is an alphabet of **terminals**,

(b) V is another alphabet of **variables**, which is disjoint from Σ,

(c) $R : V \to (\Sigma \cup V)^*$ is a finite set of **production rules**, and

(d) $S \in V$ is the **start variable**.

Let (Σ, V, R, S) be a context-free grammar. If $R(A) = w$ wherein $A \in V$ and $w \in (\Sigma \cup V)^*$ is a production rule, we write $A \to w$. Let u, v, and $w \in (\Sigma \cup V)^*$. If $A \to w$ is a production rule, we say that uAv **yields** uwv and write $uAv \Rightarrow uwv$. We say that u **derives** v and write $u \Rightarrow^* v$ if $u = v$, $u \Rightarrow v$, or there exists a sequence $(u_0, \dots, u_n)$ in $(\Sigma \cup V)^*$ for some $n \in \mathbb{N}$ such that

$$u \Rightarrow u_0 \Rightarrow \cdots \Rightarrow u_n \Rightarrow v.$$

If $A \to u$ and $A \to v$ are production rules of the grammar, we may denote them by $A \to u \,|\, v$. The **language generated by the grammar** is $\{w \in \Sigma^* : S \Rightarrow^* w\}$.

The language generated by a context-free grammar is a **context-free language**.

EXERCISE 3. Let $\Sigma = \{0, 1\}$ be an alphabet. Construct a context-free grammar which generates the language $A = \{0^n 1^n : n \in \mathbb{N}\}$.

Solution. Let (Σ, V, R, S) be the context-free grammar wherein $V = \{S\}$ and R consists of the following production rule

$$S \to 0S1 \mid \epsilon.$$

The language generated by the above context-free grammar is A. ◇

A derivation of a word in a context-free grammar is a **leftmost derivation** if at every step of production the leftmost remaining variable is the one substituted according to a production rule.

Definition 7. A word is derived **ambiguously** in a context-free grammar if there exist two or more distinct leftmost derivations for it.

A context-free grammar is **ambiguous** is it generates some words ambiguously.

Some context-free languages can only be generated by ambiguous context-free grammars. Such languages are **inherently ambiguous**.

1.2.3 Chomsky Normal Form

Definition 8. A context-free grammar is **in Chomsky normal form** if every production rule thereof is

(a) $S \to \epsilon$ wherein S is the start variable,

(b) $A \to BC$ wherein A, B, and C are variables and B and C are not the start variable, or

(c) $A \to a$ wherein A is a variable and a is a terminal.

Theorem 4. *Any context-free language is generated by a context-free grammar in Chomsky normal form.*

Proof. Let (Σ, V, R, S) be a context-free grammar. We demonstrate a procedure to convert it into another context-free grammar in Chomsky normal form (Σ, V', R', S').

We first add $S' \to S$ as a production rule.

Second, if there exist rules of the form $A \to \epsilon$ wherein $A \neq S'$, we remove them and repeatedly replace any rule of the form $B \to uAv$ wherein $B \in V'$ and u and $v \in (\Sigma \cup V')^*$ with $B \to uv$ for each occurrence of A.

Third, if there exist rules of the form $A \to B$ wherein A and $B \in V'$, we remove them and replace any rule of the form $B \to u$ wherein $u \in (\Sigma \cup V')^*$ with $A \to u$.

Lastly, we replace each rule of the form $A \to u_0 \cdots u_n$ wherein $n \in \mathbb{N}$ and $u_i \in \Sigma \cup V'$ for $i \in \mathbb{N}_{<n+1}$ such that $n > 1$ with the rules $A \to u_0 A_0$, $A_0 \to u_1 A_1$, ..., $A_{n-2} \to u_{n-1} u_n$ and add A_i for $i \in \mathbb{N}_{<n-1}$ as variables. We then replace any terminal u_i for $i \in \mathbb{N}_{<n+1}$ with the new variable U_i while adding the rule $U_i \to u_i$.

The resultant context-free grammar is in Chomsky normal form, and thus the theorem holds. □

1.2.4 Equivalence Between Pushdown Automata and Context-Free Languages

Lemma 3. *If a language is context-free, then some pushdown automaton recognises it.*

Proof. Let Σ be an alphabet, let A be a context-free language over Σ, and let $G = (\Sigma, V, R, S)$ be a context-free grammar which generates A. We construct a pushdown automaton $P = (\Sigma, \Gamma, Q, \delta, q_0, F)$ which recognises A.

Let $b \in \Sigma \cup \{\epsilon\}$, let $s \in \Gamma \cup \{\epsilon\}$, and let q and $r \in Q$. Let $u = u_0 \cdots u_i$ wherein $i \in \mathbb{N}$ be a word over Γ. We denote by $(r, u) \in \delta(q, b, s)$ that there exist a sequence $(q_0, \dots, q_{i-1})$ in Q such that

(a) $(q_0, u_i) \in \delta(q, b, s)$,

(b) $\{(q_{j+1}, u_{i-j-1})\} = \delta(q_j, \epsilon, \epsilon)$ for $j \in \mathbb{N}_{<i-1}$, and

(c) $\{(r, u_0)\} = \delta(q_{i-1}, \epsilon, \epsilon)$.

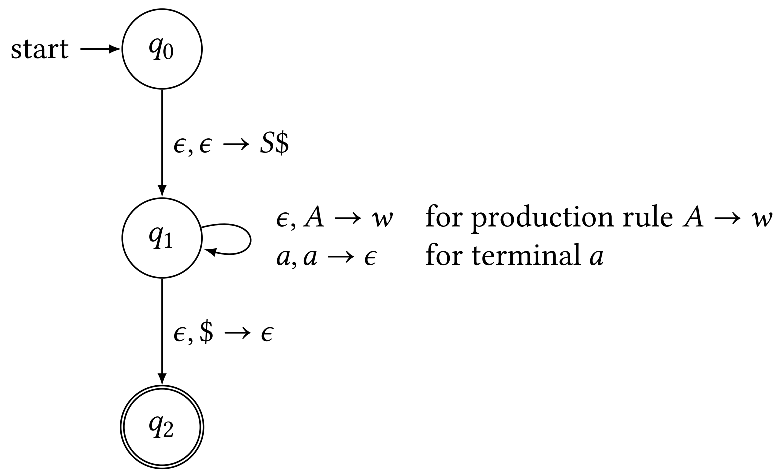
Let $Q = E \cup \{q_0, q_1, q_2\}$ and let $F = \{q_2\}$. Let $\{\$\}$ be disjoint from Σ and V, and let $\Gamma =$

$\Sigma \cup V \cup \{\$\}$. Let δ be defined as

$$\delta(q, b, s) = \begin{cases} \{(q_1, S\$)\} & \text{if } q = q_0 \wedge b = \epsilon \wedge s = \epsilon, \\ \{(q_1, w)\} & \text{if } q = q_1 \wedge b = \epsilon \wedge s = A \wedge (A \to w) \in R, \\ \{(q_1, \epsilon)\} & \text{if } q = q_1 \wedge b = a \wedge s = a \in \Sigma, \\ \{(q_2, \epsilon)\} & \text{if } q = q_1 \wedge b = \epsilon \wedge s = \$, \text{ and}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $E \subseteq Q$ consist of those states necessary to make the δ as described above well-defined per the notation given in the previous paragraph.

The following diagram illustrates the constructed P.



Thus defined, the pushdown automaton P recognises A. Therefore, the lemma holds. □

Lemma 4. *If a pushdown automaton recognises a language, then it is context-free.*

Proof. Let $P = (\Sigma, \Gamma, Q, \delta, q_0, F)$ be a pushdown automaton. We construct a context-free grammar $G = (\Sigma, V, R, S)$ which generates all words over Σ accepted by P.

We first let $P' = (\Sigma, \Gamma, Q, \delta', q_0, F')$ be a pushdown automaton equivalent to P such that

(a) $F' = \{q_1\}$,

(b) there exist $q \in Q$, $b \in \Sigma \cup \{\epsilon\}$, and $s \in \Gamma \cup \{\epsilon\}$ which satisfy $\{q_1, \epsilon\} \in \delta'(q, b, s)$, and

(c) if $\{r_1, s_1\} \in \delta(r_0, b, s_0)$ for some r_0 and $r_1 \in Q$, some $b \in \Sigma \cup \{\epsilon\}$, and some s_0 and $s_1 \in \Gamma \cup \{\epsilon\}$, then $s_0 = \epsilon$ or $s_1 = \epsilon$.

TODO □

Theorem 5. *A language is context-free if and only if some pushdown automaton recognises it.*

Proof. The theorem holds by Lemma 3 and Lemma 4. □

Corollary 2. *Every regular language is context-free.*

Proof. Let Σ be an alphabet and let A be a regular language over Σ. Let $(\Sigma, Q, \delta, q_0, F)$ be a nondeterministic finite-state automaton recognising A. Then the pushdown automaton $(\Sigma, \emptyset, Q, \delta', q_0, F)$ wherein $\delta'(q, b, \epsilon) = \delta(q, b)$ for each $q \in Q$ and each $b \in \Sigma \cup \{\epsilon\}$ also recognises A. Thus, A is context-free. □

EXERCISE 4. Let $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be an alphabet. Then $R = (1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9)(0 \cup 1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9)^*$ is a regular expression over Σ, and $L(R)$ is the set of positive integers in base 10 written in the Indo–Arabic numeral system.

Construct a context-free grammar which generates $L(R)$.

Solution. The context-free grammar (Σ, V, R, S) wherein $V = \{S, A, B\}$ and R consists of the production rules

$$S \to AB^*$$

$$A \to 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$$

$$B \to A \mid 0$$

generates $L(R)$. ◇

1.2.5 NON-CONTEXT-FREE LANGUAGES

Theorem 6 (pumping lemma for context-free languages). *Let Σ be an alphabet. If A is a context-free language over Σ, then there is some $p \in \mathbb{Z}_{>0}$, the pumping* *length, such that if* $w \in A$ *satisfies* $|w| \geq p$*, then there exist* u, v, x, y*, and* $z \in \Sigma^*$ *which satisfy*

(a) $w = uvxyz$,

(b) $uv^ixy^iz \in A$ *for each* $i \in \mathbb{N}$,

(c) $|vy| > 0$*, and*

(d) $|vxy| \leq p$.

Proof. TODO $\square$

1.2.6 Deterministic Pushdown Automata and Deterministic Context-Free Languages