NOTES ON THE THEORY OF COMPUTATION

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13th September 2022

Contents

Automata and Formal Languages

An **alphabet** is a finite set Σ, and a **word over the alphabet** Σ is a finite sequence or **string** of the elements of Σ. If a word w is the sequence $(w_0, \dots, w_n)$ for some $n \in \mathbb{N}$, we may write the word as $w_0 \cdots w_n$. The empty word is denoted by ε. The set of all words over Σ is Σ^*[1]. A **formal language over the alphabet** Σ is a subset of Σ^*.

An **automaton** is an ordered sequence that **accepts** some words over an alphabet. The set of words an automaton accepts forms a language, which is unique, in which case we say the automaton **recognises** the language. Given an automaton M, we may speak of the unique language recognised by M as the **language of the automaton** M and denote it by $L(M)$. An automaton may accept no string, in which case the language thereof is $\varnothing$.

1.1 Finite-State Automata and Regular Languages

Definition 1. A **deterministic finite-state automaton** is an ordered quintuple $(\Sigma, S, \delta, s_0, F)$ wherein

(a) Σ is an alphabet,

(b) S is a finite set of **states**,

(c) $\delta : S \times \Sigma \to S$ is the **transition function**,

(d) $s_0 \in S$ is the **initial state**, and

(e) $F \subseteq S$ is the set of **final states** or **accepting states**.

Let $M = (\Sigma, S, \delta, s_0, F)$ be a deterministic finite-state automaton and let $w = w_0 \cdots w_n$ wherein $n \in \mathbb{N}$ be a word over Σ. Then M accepts w if there exists a finite sequence of states $(r_0, \dots, r_{n+1})$ in S such that

(a) $r_0 = s_0$,

(b) $\delta(r_i, w_i) = r_{i+1}$ for $i = 0, \dots, n$, and

(c) $r_{n+1} \in F$.

[1] $*$ is the unary operator of Kleene star, defined as $A^* = \{a_0 \cdots a_n : n \in \mathbb{N} \wedge \forall\, i \in \mathbb{N}_{<n+1}\, (a_i \in A)\}$.

DEFINITION 2. A **nondeterministic finite-state automaton** is an ordered quintuple $(\Sigma, S, \delta, s_0, F)$ wherein

(a) Σ is an alphabet,

(b) S is a finite set of states,

(c) $\delta : S \times \Sigma_\varepsilon \to \mathscr{P}(S)$[2] is the transition function,

(d) $s_0 \in S$ is the initial state, and

(e) $F \subseteq S$ is the set of final or accepting states.

Let $M = (\Sigma, S, \delta, s_0, F)$ be a nondeterministic finite-state automaton and let w be a word over Σ. Then M accepts w if $w = w_0 \cdots w_n$ wherein $n \in \mathbb{N}$ such that each $w_i \in \Sigma_\varepsilon$, $i \in \mathbb{N}_{<n+1}$, and that there exists a finite sequence of states $(r_0, \dots, r_{n+1})$ in S such that

(a) $r_0 = s_0$,

(b) $r_{i+1} \in \delta(r_i, w_i)$ for $i = 0, \dots, n$, and

(c) $r_{n+1} \in F$

We say that two automata are equivalent if they recognise the same language.

Theorem 1. *Every nondeterministic finite-state automaton has an equivalent deterministic finite-state automaton.*

Proof. Let $N = (\Sigma, S, \delta, s_0, F)$ be the nondeterministic finite-state automaton recognising some language A. We construct a deterministic finite-state automaton $M = (\Sigma, S', \delta', s_0', F)$ recognising A.

We first see that $S' = \mathscr{P}(S)$ and that $F' = \{R \in S' : R \cap F \neq \varnothing\}$.

Let $\delta_0 : S \times \{\varepsilon\} \to \mathscr{P}(S)$ be defined as $\delta_0(s, \varepsilon) = \delta(s, \varepsilon)$ for each $s \in S$. Assume first that, thus induced, $\delta_0 = \varnothing$ for N. For each $R \in S'$ and for each $a \in \Sigma$, let $\delta'(R, a) = \{s \in S : \exists r \in R\, (s \in \delta(r, a))\}$. Equivalently,

$$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a).$$

Also let $s_0' = \{s_0'\}$. We then see that $M = (\Sigma, S', \delta', s_0', F')$ recognises A.

[2] Σ_ε denotes the union of Σ and $\{\varepsilon\}$.

Assume then that $\delta_0 \neq \varnothing$ for N. For each $R \subseteq S$, let

$$E(R) = \big\{s \in S \,:\, \exists n \in \mathbb{N}\, \exists r \in R \big(s = \delta^n(r, \varepsilon)\big)\big\}.$$

We then let

$$\delta'(R, a) = \big\{s \in S \,:\, \exists r \in R\, s \in E\big(\delta(r, a)\big)\big\}$$

and let $s_0' = E(\{s_0\})$. We similarly see that $M = (\Sigma, S', \delta', s_0' F')$ recognises A.

Therefore, the theorem holds. $\square$

DEFINITION 3. Let Σ be an alphabet, and let $a \in \Sigma$. Then some $R \subseteq \Sigma^*$ is a **regular language** if

(a) $R = \varnothing$,

(b) $R = \{\varepsilon\}$,

(c) $R = \{a\}$,

(d) $R = R_1 \cup R_2$ wherein R_1 and R_2 are regular languages over Σ,

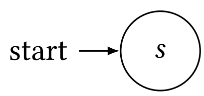
(e) $R = R_1 R_2$[3] wherein R_1 and R_2 are regular languages over Σ, or

(f) $R = R_0^*$ wherein R_0 is a regular language over Σ.

An expression used to describe a regular language is a **regular expression**. If R is a regular expression, we denote the regular language it describes by $L(R)$.

Lemma 1. *If a language is regular, then some nondeterministic finite-state automaton recognises it.*

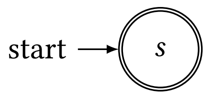
Proof. Let Σ be an alphabet, and let R be a regular language over Σ. If $R = \varnothing$, then the nondeterministic finite-state automaton characterised by the following diagram recognises it.



Equivalently, $N = (\Sigma, \{s\}, \delta, s, \varnothing)$ wherein $\delta(r, b) = \varnothing$ for any $r \in \{s\}$ and for any $b \in \Sigma$ recognises R.

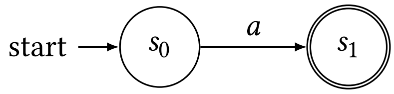
[3] $R_1 R_2$ is the concatenation of R_1 and R_2, defined as $R_1 R_2 = \{xy \,:\, x \in R_1 \wedge y \in R_2\}$

If $R = \{\varepsilon\}$, then the nondeterministic finite-state automaton characterised by the following diagram recognises it.



Equivalently, $N = (\Sigma, \{s\}, \delta, s, \{s\})$ wherein $\delta(r, b) = \emptyset$ for any $r \in \{s\}$ and for any $b \in \Sigma$ recognises R.

If $R = \{a\}$ for some $a \in \Sigma$, then the nondeterministic finite-state automaton characterised by the following diagram recognises it.



Equivalently, $N = (\Sigma, \{s_0, s_1\}, \delta, s_0, \{s_1\})$ wherein $\delta(s_0, a) = \{s_1\}$ and $\delta(r, b) = \emptyset$ if $r \neq s_0$ or $b \neq a$.

Assume that R_1 and R_2 are regular languages over Σ, that $N_1 = (\Sigma, S_1, \delta_1, s_1, F_1)$ is a nondeterministic finite-state automaton recognising R_1, and that $N_2 = (\Sigma, S_2, \delta_2, s_2, F_2)$ is a nondeterministic finite-state automaton recognising R_2.

If $R = R_1 \cup R_2$, let s_0 be a state not in S_1 or S_2, let $S = \{s_0\} \cup S_1 \cup S_2$, and let $F = F_1 \cup F_2$. Define $\delta : S \times \Sigma \to \mathscr{P}(S)$ so that for each $r \in S$ and for each $b \in \Sigma$ we have

$$\delta(r, b) = \begin{cases} \delta_1(r, b) & \text{if } r \in S_1, \\ \delta_2(r, b) & \text{if } r \in S_2, \\ \{s_1, s_2\} & \text{if } r = s_0 \wedge b = \varepsilon, \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

We see that $N = (\Sigma, S, \delta, s_0, F)$ is a nondeterministic finite-state automaton recognising R.

If $R = R_1 R_2$, let $S = S_1 \cup S_2$. Define $\delta : S \times \Sigma \to S$ so that for each $r \in S$ and for each $b \in \Sigma$

we have

$$\delta(r,b) = \begin{cases} \delta_1(r,b) & \text{if } (r \in S_1 \wedge r \notin F_1) \vee (r \in F_1 \wedge b \neq \varepsilon), \\ \delta_1(r,b) \cup \{s_2\} & \text{if } r \in F_1 \wedge b = \varepsilon, \text{ and} \\ \delta_2(r,b) & \text{otherwise.} \end{cases}$$

We see that $N = (\Sigma, S, \delta, s_1, F_2)$ is a nondeterministic finite-state automaton recognising R.

If $R = R_1^*$, let s_0 be a state not in S_1, let $S = \{s_0\} cup S_1$, and let $F = \{s_0\} \cup F_1$. Define $\delta : S \times \Sigma \to S$ so that for each $r \in S$ and for each $b \in \Sigma$ we have

$$\delta(r,b) = \begin{cases} \delta_1(r,b) & \text{if } (r \in S_1 \wedge r \notin F_1) \vee (r \in F_1 \wedge b \neq \varepsilon), \\ \delta_1(r,b) \cup \{s_1\} & \text{if } r \in F_1 \wedge b = \varepsilon, \\ \{s_1\} & \text{if } r = s_0 \wedge b = \varepsilon, \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$

We see that $N = (\Sigma, S, \delta, s_0, F)$ is a nondeterministic finite-state automaton recognising R.

Therefore, the lemma holds by the principle of induction. □

DEFINITION 4. A **generalised nondeterministic finite-state automaton** is an ordered quintuple $(\Sigma, S, \delta, s_0, s_1)$ wherein

(a) Σ is an alphabet,

(b) S is a finite set of states,

(c) $\delta : (S \setminus \{s_1\}) \times (S \setminus \{s_0\}) \to \mathscr{R}$ wherein $\mathscr{R}$ is the set of all regular expressions over Σ is the transition function,

(d) $s_0 \in S$ is the initial state, and

(e) $s_1 \in S$ is the final or accepting state.

Let $M = (\Sigma, S, \delta, s_0, s_1)$ be a generalised nondeterministic finite-state automaton and let w be a word over Σ. Then M accepts w if $w = w_0 \cdots w_n$ wherein $n \in \mathbb{N}$ such that each $w_i \in \Sigma^*$, $i \in \mathbb{N}_{<n+1}$, and that there exists a finite sequence of states $(r_0, \ldots, r_{n+1})$ such that

(a) $r_0 = s_0$,

(b) $r_{n+1} = s_1$, and

(c) for each i we have $w_i \in L(R_i)$ wherein $R_i = \delta(r_i, r_{i+1})$.

Lemma 2. *If a nondeterministic finite-state automaton recognises a language, then it is regular.*

Proof. Let N be a nondeterministic finite-state automaton. We prove that $L(N)$ is described by some regular expression R over Σ.

TODO □

Theorem 2. *A language is regular if and only if some nondeterministic finite-state automaton recognises it.*

Proof. The theorem holds by Lemma 1 and Lemma 2. □