

MATHEMATICAL ANALYSIS

YANNAN MAO

6TH MARCH 2024

CONTENTS

1	Calculus of Variations	1
1.1	Linear Forms	1
1.2	Functionals and Their Extrema	1
1.3	Variations	1
1.4	Euler–Lagrange Equation	2

CALCULUS OF VARIATIONS

1.1 LINEAR FORMS

Let F be a field, and let V be a vector space over F . A linear map from V into F is referred to as a *linear form on V* . Equivalently, a function $f : V \rightarrow F$ is a linear form if $f(\lambda \mathbf{a} + \mathbf{b}) = \lambda f(\mathbf{a}) + f(\mathbf{b})$ for any $\lambda \in F$ and any $\mathbf{a}, \mathbf{b} \in V$. Linear forms are also known as *linear functionals*.

Let $[x_0, x_1]$ be a closed interval on \mathbb{R} , and let $C^0([x_0, x_1])$ be the vector space of continuous real functions on $[x_0, x_1]$. Then $J : C^0([x_0, x_1]) \rightarrow \mathbb{R}$ defined by

$$J(f) = \int_{x_0}^{x_1} f(x) dx$$

is a linear form on $C^0([x_0, x_1])$.

1.2 FUNCTIONALS AND THEIR EXTREMA

Let $[x_0, x_1]$ be a closed interval on \mathbb{R} , and let $C^2([x_0, x_1])$ be the set of twice continuously differentiable real functions on $[x_0, x_1]$. We refer to linear forms on $C^2([x_0, x_1])$ as *functionals*. We denote a functional by enclosing its variable in square brackets.

Let $\Omega \subseteq C^2([x_0, x_1])$ be a set of functions. A functional $J : \Omega \rightarrow \mathbb{R}$ is said to obtain an *extremum at function f* if there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that $J[g] - J[f]$ has the same sign for any $g \in \Omega$ which satisfies $\forall x \in [x_0, x_1] (|g(x) - f(x)| < \varepsilon)$.

1.3 VARIATIONS

Suppose $g \in \Omega$ is a function whereat the functional J obtains an extremum. Take another function $\eta : [x_0, x_1] \rightarrow \mathbb{R}$ which vanishes at x_0 and x_1 . We then form the family of functions

$$\varphi(x, \varepsilon) = g(x) + \varepsilon \eta(x)$$

with $\varepsilon \in \mathbb{R}$. Note that with any given ε we have $\varphi \in \Omega$.

We see that

$$\eta(x) = \frac{\partial \varphi}{\partial \varepsilon}$$

and so we refer to $\varepsilon \eta(x)$ as a *variation of g* and denote it by δg .

Let $\psi(\varepsilon) = J[g + \varepsilon \eta]$ be a function of ε . The postulate that g shall give an extremum of $\{J\}$ implies that ψ shall possess a minimum for $\varepsilon = 0$, so as a necessary condition we have the equation

$$\psi'(0) = 0.$$

1.4 EULER-LAGRANGE EQUATION

Let $\Omega \subset C^2([x_0, x_1])$ be given by $\Omega = \{f \in C^2([x_0, x_1]) : y_0 = f(x_0) \wedge y_1 = f(x_1)\}$ wherein $y_0, y_1 \in \mathbb{R}$ are prescribed. Consider a functional of the form

$$J(f) = \int_{x_0}^{x_1} L(x, f'(x), f(x)) dx \quad (1)$$

wherein L is a twice continuously differentiable function with respect to x, f' , and f .

Theorem 1 (Euler–Lagrange equation). *The functional J defined in (1) obtains an extremum at function f if and only if*

$$\frac{\partial L}{\partial f} - \frac{d}{dx} \frac{\partial L}{\partial f'} = 0.$$