NOTES ON MATHEMATICAL ANALYSIS

YANNAN MAO

7th March 2024

Contents

CALCULUS OF VARIATIONS

1.1 LINEAR FORMS

Let F be a field, and let V be a vector space over F. A linear map from V into F is referred to as a *linear form on V*. Equivalently, a function $f\colon V \to F$ is a linear form if $f(\lambda\boldsymbol{a}+\boldsymbol{b}) = \lambda f(\boldsymbol{a}) + f(\boldsymbol{b})$ for any $\lambda \in F$ and any $\boldsymbol{a}, \boldsymbol{b} \in V$. Linear forms are also known as *linear functionals*.

Let $[x_0, x_1]$ be a closed interval on $\mathbb{R}$, and let $C^0\big([x_0, x_1]\big)$ be the vector space of continuous real functions on $[x_0, x_1]$. Then $J\colon C^0\big([x_0, x_1]\big) \to \mathbb{R}$ defined by

$$J(f) = \int_{x_0}^{x_1} f(x)\,\mathrm{d}x$$

is a linear form on $C^0\big([x_0, x_1]\big)$.

1.2 FUNCTIONALS AND THEIR EXTREMA

Let $[x_0, x_1]$ be a closed interval on $\mathbb{R}$, and let $C^2\big([x_0, x_1]\big)$ be the set of twice continuously differentiable real functions on $[x_0, x_1]$. We refer to linear forms on $C^2\big([x_0, x_1]\big)$ as *functionals*. We denote a functional by enclosing its variable in square brackets.

Let $\Omega \subseteq C^2\big([x_0, x_1]\big)$ be a set of functions. A functional $J\colon \Omega \to \mathbb{R}$ is said to obtain an *extremum at function f* if there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that $J[g] - J[f]$ has the same sign for any $g \in \Omega$ which satisfies $\forall x \in [x_0, x_1]\,\big(|g(x) - f(x)| < \varepsilon\big)$.

1.3 VARIATIONS

Let $\Omega \subset C^2\big([x_0, x_1]\big)$ be given by $\Omega = \big\{f \in C^2\big([x_0, x_1]\big) : y_0 = f(x_0) \wedge y_1 = f(x_1)\big\}$ wherein $y_0, y_1 \in \mathbb{R}$ are prescribed. Consider a functional of the form

$$J(f) = \int_{x_0}^{x_1} L\big(x, f'(x), f(x)\big)\,\mathrm{d}x \tag{1}$$

wherein L is a twice continuously differentiable function with respect to x, f', and f.

Suppose $g \in \Omega$ is a function whereat the functional J obtains an extremum. Take another function $\eta \colon [x_0, x_1] \to \mathbb{R}$ which vanishes at x_0 and x_1. We then form the family of functions

$$\varphi(x, \varepsilon) = g(x) + \varepsilon\eta(x)$$

with $\varepsilon \in \mathbb{R}$. Note that with any given ε we have $\varphi \in \Omega$.

We see that

$$\eta(x) = \frac{\partial\varphi}{\partial\varepsilon}$$

and so we refer to $\varepsilon\eta(x)$ as a *variation of* g and denote it by δg.

Let $\psi(\varepsilon) = J[g + \varepsilon\eta]$ be a function of ε. The postulate that g shall give an extremum of {J} implies that ψ shall possess a minimum for $\varepsilon = 0$, so as a necessary condition we have the equation

$$\psi'(0) = 0.$$

1.4 Euler–Lagrange Equation

Theorem 1 (Euler–Lagrange equation)**.** *The functional J defined in* (1) *obtains an extremum at function f if and only if*

$$\frac{\partial L}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial f'}.$$