NOTES ON MATHEMATICAL ANALYSIS

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CONTENTS

THE CALCULUS OF VARIATIONS

1.1 LINEAR FORMS

Let F be a field, and let V be a vector space over F. A linear map from V into F is referred to as a *linear form on V*. Equivalently, a function $f\colon V \to F$ is a linear form if $f(\lambda \boldsymbol{a} + \boldsymbol{b}) = \lambda f(\boldsymbol{a}) + f(\boldsymbol{b})$ for any $\lambda \in F$ and any $\boldsymbol{a}, \boldsymbol{b} \in V$. Linear forms are also known as *linear functionals*.

Let $[x_0, x_1]$ be a closed interval on $\mathbb{R}$, and let $C^0\big([x_0, x_1]\big)$ be the vector space of continuous real functions on $[x_0, x_1]$. Then $J\colon C^0\big([x_0, x_1]\big) \to \mathbb{R}$ defined by

$$J(f) = \int_{x_0}^{x_1} f(x)\,\mathrm{d}x$$

is a linear form on $C^0\big([x_0, x_1]\big)$.

1.2 FUNCTIONALS AND THEIR EXTREMA

Let $[x_0, x_1]$ be a closed interval on $\mathbb{R}$, and let $C^2\big([x_0, x_1]\big)$ be the set of twice continuously differentiable real functions on $[x_0, x_1]$. We refer to linear forms on $C^2\big([x_0, x_1]\big)$ as *functionals*. We denote a functional by enclosing its variable in square brackets.

Let $\Omega \subseteq C^2\big([x_0, x_1]\big)$ be a set of functions. A functional $J\colon \Omega \to \mathbb{R}$ is said to obtain an *extremum at function f* if there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that $J[g] - J[f]$ has the same sign for any $g \in \Omega$ which satisfies $\forall x \in [x_0, x_1]\,\big(|g(x) - f(x)| < \varepsilon\big)$.

1.3 VANISHING OF THE FIRST VARIATION

Let $\Omega \subset C^2\big([x_0, x_1]\big)$ be given by $\Omega = \big\{f \in C^2\big([x_0, x_1]\big) : y_0 = f(x_0) \wedge y_1 = f(x_1)\big\}$ wherein $y_0, y_1 \in \mathbb{R}$ are prescribed. Consider a functional of the form

$$J[f] = \int_{x_0}^{x_1} L\big(x, f'(x), f(x)\big)\,\mathrm{d}x \tag{1}$$

wherein L is a twice continuously differentiable function with respect to x, f', and f.

Suppose $f \in \Omega$ is a function whereat the functional J obtains an extremum. Take another function $\eta \in C^2([x_0, x_1])$ which vanishes at x_0 and x_1. We then form the family of functions

$$\varphi(x, \varepsilon) = f(x) + \varepsilon\eta(x)$$

with $\varepsilon \in \mathbb{R}$. Note that with any given ε we have $\varphi \in \Omega$.

We see that

$$\eta(x) = \frac{\partial\varphi}{\partial\varepsilon},$$

and so we refer to $\varepsilon\eta(x)$ as a *variation of* f and denote it by δf.

Let $\psi(\varepsilon) = J[f + \varepsilon\eta]$ be a real function. The postulate that f shall give an extremum of J implies that ψ shall possess an extremum for $\varepsilon = 0$, and so it is necessary that

$$\psi'(0) = 0.$$

If f satisfies $\psi'(0) = 0$ for any η, we then say that J is *stationary at* f.

In general, we refer to $\varepsilon\psi'(0)$ as the *first variation of* J and denote it by δJ. Thus, the stationary character of J at f is equivalent to the vanishing of the first variation.

1.4 THE FUNDAMENTAL LEMMA OF THE CALCULUS OF VARIATIONS

Lemma 1 (fundamental lemma of the calculus of variations). *If a function* $f \in C^0([x_0, x_1])$ *satisfies*

$$\int_{x_0}^{x_1} \eta(x) f(x)\,\mathrm{d}x = 0$$

for any $\eta \in C^2([x_0, x_1])$ *such that* $\eta(x_0) = \eta(x_1) = 0$, *then* $f(x) = 0$ *for any* $x \in [x_0, x_1]$.

Proof. We assume, for the sake of contradiction, that there exists a $\xi \in [x_0, x_1]$ such that $f(\xi) > 0$ and which satisfies all the prescribed conditions. Then, as f is continuous on $[x_0, x_1]$, there exists an $\alpha \in \mathbb{R}_{>0}$ such that $f(x) > 0$ for any $x \in [\xi - \alpha, \xi + \alpha] \subseteq [x_0, x_1]$.

Let $\eta \in C^2([x_0, x_1])$ be defined by

$$\eta(x) = \begin{cases} \left((x-\xi)^2 - \alpha^2\right)^4 & \text{if } x \in [\xi - \alpha, \xi + \alpha], \\ 0 & \text{otherwise.} \end{cases}$$

We see that $\eta(x)f(x) > 0$ for any $x \in [\xi - \alpha, \xi + \alpha]$ and that $\eta(x)f(x) = 0$ for any $x \in [x_0, \xi - \alpha) \cup (\xi + \alpha, x_1]$. It follows that

$$\int_{x_0}^{x_1} \eta(x) f(x) \,\mathrm{d}x > 0,$$

which is a contradiction. Therefore, $f(\xi)$ cannot be positive. For the same reasons, $f(\xi)$ cannot be negative. Hence, $f(x)$ must vanish for any $x \in [x_0, x_1]$.

Thus, the lemma holds. □

1.5 THE EULER–LAGRANGE EQUATION

Theorem 1 (Euler–Lagrange)**.** *The functional J defined in (1) is stationary at function f if and only if*

$$\frac{\partial L}{\partial f}\left(x, f(x), f'(x)\right) - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial f'}\left(x, f(x), f'(x)\right) = 0$$

for any $x \in [x_0, x_1]$.

Proof. As shown in subsection 1.3, that J is stationary at f is equivalent to the vanishing of

$$\begin{aligned} \frac{\mathrm{d}\psi}{\mathrm{d}\varepsilon}(\varepsilon)\bigg|_{\varepsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{x_0}^{x_1} J\big[f(x) + \varepsilon\eta(x)\big]\,\mathrm{d}x\bigg|_{\varepsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{x_0}^{x_1} L\big(x, f(x) + \varepsilon\eta(x), f'(x) + \varepsilon\eta'(x)\big)\,\mathrm{d}x\bigg|_{\varepsilon=0} \\ &= \int_{x_0}^{x_1} \frac{\partial}{\partial\varepsilon} L\big(x, f(x) + \varepsilon\eta(x), f'(x) + \varepsilon\eta'(x)\big)\bigg|_{\varepsilon=0}\,\mathrm{d}x \\ &= \int_{x_0}^{x_1} \left(\frac{\partial}{\partial\varepsilon}\big(f(x) + \varepsilon\eta(x)\big)\bigg|_{\varepsilon=0} \frac{\partial L}{\partial(f + \varepsilon\eta)}\big(x, f(x) + \varepsilon\eta(x), f'(x) + \varepsilon\eta'(x)\big)\bigg|_{\varepsilon=0} \right. \end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{\partial}{\partial \varepsilon}\big(f'(x)+\varepsilon\eta'(x)\big)\Big|_{\varepsilon=0} \frac{\partial L}{\partial (f'+\varepsilon\eta')}\big(x, f(x)+\varepsilon\eta(x), f'(x)+\varepsilon\eta'(x)\big)\Big|_{\varepsilon=0}\Bigg)\,\mathrm{d}x \\
&= \int_{x_0}^{x_1}\left(\eta(x)\frac{\partial L}{\partial f}\big(x, f(x), f'(x)\big) + \eta'(x)\frac{\partial L}{\partial f'}\big(x, f(x), f'(x)\big)\right)\mathrm{d}x.
\end{aligned}
$$

We further see that

$$
\begin{aligned}
\int_{x_0}^{x_1} \eta'(x)\frac{\partial L}{\partial f'}\big(x, f(x), f'(x)\big)\,\mathrm{d}x &= \eta(x)\frac{\partial L}{\partial f'}\big(x, f(x), f'(x)\big)\Big|_{x_0}^{x_1} \\
&\quad - \int_{x_0}^{x_1} \eta(x)\,\mathrm{d}\frac{\partial L}{\partial f'}\big(x, f(x), f'(x)\big) \\
&= -\int_{x_0}^{x_1} \eta(x)\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial f'}\big(x, f(x), f'(x)\big)\,\mathrm{d}x.
\end{aligned}
$$

By integration by parts, the equivalent condition is then

$$
\int_{x_0}^{x_1} \eta(x)\left(\frac{\partial L}{\partial f}\big(x, f(x), f'(x)\big) - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial f'}\big(x, f(x), f'(x)\big)\right)\mathrm{d}x = 0.
$$

By lemma 1, we then conclude that

$$
\frac{\partial L}{\partial f}\big(x, f(x), f'(x)\big) - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial f'}\big(x, f(x), f'(x)\big) = 0
$$

for any $x \in [x_0, x_1]$.

Hence, the theorem holds. $\square$