

NOTES ON MATHEMATICAL ANALYSIS

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THE CALCULUS OF VARIATIONS

1.1 LINEAR FORMS

Let F be a field, and let V be a vector space over F . A linear map from V into F is referred to as a *linear form on V* . Equivalently, a function $f : V \rightarrow F$ is a linear form if $f(\lambda \mathbf{a} + \mathbf{b}) = \lambda f(\mathbf{a}) + f(\mathbf{b})$ for any $\lambda \in F$ and any $\mathbf{a}, \mathbf{b} \in V$. Linear forms are also known as *linear functionals*.

Let $[x_0, x_1]$ be a closed interval on \mathbb{R} , and let $C^0([x_0, x_1])$ be the vector space of continuous real functions on $[x_0, x_1]$. Then $J : C^0([x_0, x_1]) \rightarrow \mathbb{R}$ defined by

$$J(f) = \int_{x_0}^{x_1} f(x) dx$$

is a linear form on $C^0([x_0, x_1])$.

1.2 FUNCTIONALS AND THEIR EXTREMA

Let $[x_0, x_1]$ be a closed interval on \mathbb{R} , and let $C^2([x_0, x_1])$ be the set of twice continuously differentiable real functions on $[x_0, x_1]$. We refer to linear forms on $C^2([x_0, x_1])$ as *functionals*. We denote a functional by enclosing its variable in square brackets.

Let $\Omega \subseteq C^2([x_0, x_1])$ be a set of functions. A functional $J : \Omega \rightarrow \mathbb{R}$ is said to obtain an *extremum at function f* if there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that $J[g] - J[f]$ has the same sign for any $g \in \Omega$ which satisfies $\forall x \in [x_0, x_1] (|g(x) - f(x)| < \varepsilon)$.

1.3 VANISHING OF THE FIRST VARIATION

Let $\Omega \subset C^2([x_0, x_1])$ be given by $\Omega = \{f \in C^2([x_0, x_1]) : y_0 = f(x_0) \wedge y_1 = f(x_1)\}$ wherein $y_0, y_1 \in \mathbb{R}$ are prescribed. Consider a functional of the form

$$J[f] = \int_{x_0}^{x_1} L(x, f'(x), f(x)) dx \quad (1)$$

wherein L is a twice continuously differentiable function with respect to x , f' , and f .

Suppose $f \in \Omega$ is a function whereat the functional J obtains an extremum. Take another function $\eta \in C^2([x_0, x_1])$ which vanishes at x_0 and x_1 . We then form the family of functions

$$\varphi(x, \varepsilon) = f(x) + \varepsilon\eta(x)$$

with $\varepsilon \in \mathbb{R}$. Note that with any given ε we have $\varphi \in \Omega$.

We see that

$$\eta(x) = \frac{\partial \varphi}{\partial \varepsilon}$$

and so we refer to $\varepsilon\eta(x)$ as a *variation of f* and denote it by δf .

Let $\psi(\varepsilon) = J[f + \varepsilon\eta]$ be a real function. The postulate that f shall give an extremum of J implies that ψ shall possess an extremum for $\varepsilon = 0$, and so it is necessary that

$$\psi'(0) = 0.$$

If f satisfies $\psi'(0) = 0$ for any η , we then say that J is *stationary at f* .

In general, we refer to $\varepsilon\psi'(0)$ as the *first variation of J* and denote it by δJ . Thus, the stationary character of J at f is equivalent to the vanishing of the first variation.

1.4 THE FUNDAMENTAL LEMMA OF THE CALCULUS OF VARIATIONS

Lemma 1 (fundamental lemma of the calculus of variations). *If a function $f \in C^0([x_0, x_1])$ satisfies*

$$\int_{x_0}^{x_1} \eta(x)f(x) dx = 0$$

for any $\eta \in C^2([x_0, x_1])$ such that $\eta(x_0) = \eta(x_1) = 0$, then $f(x) = 0$ for any $x \in [x_0, x_1]$.

Proof. We assume, for the sake of contradiction, that there exists a $\xi \in [x_0, x_1]$ such that $f(\xi) > 0$ and which satisfies all the prescribed conditions. Then, as f is continuous on $[x_0, x_1]$, there exists an $\alpha \in \mathbb{R}_{>0}$ such that $f(x) > 0$ for any $x \in [\xi - \alpha, \xi + \alpha] \subseteq [x_0, x_1]$.

Let $\eta \in C^2([x_0, x_1])$ be defined by

$$\eta(x) = \begin{cases} ((x - \xi)^2 - \alpha^2)^4 & \text{if } x \in [\xi - \alpha, \xi + \alpha], \\ 0 & \text{otherwise.} \end{cases}$$

We see that $\eta(x)f(x) > 0$ for any $x \in [\xi - \alpha, \xi + \alpha]$ and that $\eta(x)f(x) = 0$ for any $x \in [x_0, \xi - \alpha) \cup (\xi + \alpha, x_1]$. It follows that

$$\int_{x_0}^{x_1} \eta(x)f(x) dx > 0,$$

which is a contradiction. Therefore, $f(\xi)$ cannot be positive. For the same reasons, $f(\xi)$ cannot be negative. Hence, $f(x)$ must vanish for any $x \in [x_0, x_1]$.

Thus, our original proposition holds. □

1.5 THE EULER-LAGRANGE EQUATION

Theorem 1 (Euler-Lagrange). *The functional J defined in (1) is stationary at function f if and only if*

$$\frac{\partial L}{\partial f} - \frac{d}{dx} \frac{\partial L}{\partial f'} = 0.$$