NOTES ON MATHEMATICAL ANALYSIS

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MEASURES

1.1 TOPOLOGICAL SPACES

DEFINITION 1. Let S be a set. Then $\mathcal{T} \subseteq \mathcal{P}(S)$ is a ***topology on*** S if and only if

(a) $\varnothing \in \mathcal{T}$;

(b) $S \in \mathcal{T}$;

(c) $\mathcal{T}$ is closed under finite intersections; and

(d) $\mathcal{T}$ is closed under unions.

A ***topological space*** is an ordered pair $(S, \mathcal{T})$ wherein S is a set and $\mathcal{T}$ is a topology thereon, and a subset of S is an ***open set*** if it is in $\mathcal{T}$.

1.2 σ-ALGEBRAS

DEFINITION 2. Let S be a set. Then $\Sigma \subseteq \mathcal{P}(S)$ is a σ***-algebra on*** S if and only if

(a) $\Sigma \neq \varnothing$;

(b) Σ is closed under complementation; and

(c) Σ is closed under countable unions.

A ***measurable space*** is an ordered pair (S, Σ) wherein S is a set and Σ is a σ-algebra thereon, and a subset of S is a ***measurable set*** if it is in Σ.

It follows that $S \in \Sigma$ and $\varnothing \in \Sigma$ for each σ-algebra on S and that each σ-algebra is closed under countable intersections. Hence, the smallest σ-algebra on S is $\{S, \varnothing\}$ and the largest thereon is $\mathcal{P}(S)$.

Each σ-algebra on S is also closed under monotone limits.

The intersection of any set of σ-algebras is also a σ-algebra. For each $P \in \mathcal{P}(S)$, there exists a smallest σ-algebra on S containing P, referred to as the σ***-algebra generated by*** P and denoted by $\sigma(P)$, which is obtained as the intersection of all σ-algebras containing P.

DEFINITION 3. Let $(S, \mathcal{T})$ be a topological space. The ***Borel σ-algebra of* $(S, \mathcal{T})$**, denoted by $\mathcal{B}(S, \mathcal{T})$, is the σ-algebra generated by $\mathcal{T}$.

A ***Borel set of*** $(S, \mathcal{T})$ is an element of $\mathcal{B}(S, \mathcal{T})$.

A ***π-system on*** S is a set of subsets of S which is closed under finite intersections. A ***Dynkin system on*** S is a set of subsets of S which contains S and is closed under both proper differences and increasing limits.