

NOTES ON MATHEMATICAL ANALYSIS

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THE CALCULUS OF VARIATIONS

1.1 LINEAR FORMS

Let F be a field, and let V be a vector space over F . A linear map from V into F is referred to as a **linear form on V** . Equivalently, a function $f: V \rightarrow F$ is a linear form if $f(\lambda \mathbf{a} + \mathbf{b}) = \lambda f(\mathbf{a}) + f(\mathbf{b})$ for any $\lambda \in F$ and any $\mathbf{a}, \mathbf{b} \in V$. Linear forms are also known as **linear functionals**.

Let $[x_0, x_1]$ be a closed interval on \mathbb{R} , and let $C^0([x_0, x_1])$ be the vector space of continuous real functions on $[x_0, x_1]$. Then $J: C^0([x_0, x_1]) \rightarrow \mathbb{R}$ defined by

$$J(f) = \int_{x_0}^{x_1} f(x) dx$$

is a linear form on $C^0([x_0, x_1])$.

1.2 FUNCTIONALS AND THEIR EXTREMA

Let $[x_0, x_1]$ be a closed interval on \mathbb{R} , and let $C^2([x_0, x_1])$ be the set of twice continuously differentiable real functions on $[x_0, x_1]$. We refer to linear forms on $C^2([x_0, x_1])$ as **functionals**. We denote a functional by enclosing its variable in square brackets.

Let $\Omega \subseteq C^2([x_0, x_1])$ be a set of functions. A functional $J: \Omega \rightarrow \mathbb{R}$ is said to obtain an **extremum at function f** if there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that $J[g] - J[f]$ has the same sign for any $g \in \Omega$ which satisfies $\forall x \in [x_0, x_1] (|g(x) - f(x)| < \varepsilon)$.

1.3 VANISHING OF THE FIRST VARIATION

Let $\Omega \subset C^2([x_0, x_1])$ be given by $\Omega = \{f \in C^2([x_0, x_1]) : y_0 = f(x_0) \wedge y_1 = f(x_1)\}$ wherein $y_0, y_1 \in \mathbb{R}$ are prescribed. Consider a functional of the form

$$J[f] = \int_{x_0}^{x_1} L(x, f'(x), f(x)) dx \tag{1}$$

wherein L is a twice continuously differentiable function with respect to x , f' , and f .

Suppose $f \in \Omega$ is a function whereat the functional J obtains an extremum. Take another function $\eta \in C^2([x_0, x_1])$ which vanishes at x_0 and x_1 . We then form the family of functions

$$\varphi(x, \varepsilon) = f(x) + \varepsilon\eta(x)$$

with $\varepsilon \in \mathbb{R}$. Note that with any given ε we have $\varphi \in \Omega$.

We see that

$$\eta(x) = \frac{\partial \varphi}{\partial \varepsilon},$$

and so we refer to $\varepsilon\eta(x)$ as a **variation of f** and denote it by δf .

Let $\psi(\varepsilon) = J[f + \varepsilon\eta]$ be a real function. The postulate that f shall give an extremum of J implies that ψ shall possess an extremum for $\varepsilon = 0$, and so it is necessary that

$$\psi'(0) = 0.$$

If f satisfies $\psi'(0) = 0$ for any η , we then say that J is **stationary at f** .

In general, we refer to $\varepsilon\psi'(0)$ as the **first variation of J** and denote it by δJ . Thus, the stationary character of J at f is equivalent to the vanishing of the first variation.

1.4 THE FUNDAMENTAL LEMMA OF THE CALCULUS OF VARIATIONS

Lemma 1 (fundamental lemma of the calculus of variations). *If a function*

$f \in C^0([x_0, x_1])$ satisfies

$$\int_{x_0}^{x_1} \eta(x)f(x) dx = 0$$

for any $\eta \in C^2([x_0, x_1])$ such that $\eta(x_0) = \eta(x_1) = 0$, then $f(x) = 0$ for any $x \in [x_0, x_1]$.

Proof. We assume, for the sake of contradiction, that there exists a $\xi \in [x_0, x_1]$ such that $f(\xi) > 0$ and which satisfies all the prescribed conditions. Then, as f is continuous on

$[x_0, x_1]$, there exists an $\alpha \in \mathbb{R}_{>0}$ such that $f(x) > 0$ for any $x \in [\xi - \alpha, \xi + \alpha] \subseteq [x_0, x_1]$.

Let $\eta \in C^2([x_0, x_1])$ be defined by

$$\eta(x) = \begin{cases} ((x - \xi)^2 - \alpha^2)^4 & \text{if } x \in [\xi - \alpha, \xi + \alpha], \\ 0 & \text{otherwise.} \end{cases}$$

We see that $\eta(x)f(x) > 0$ for any $x \in [\xi - \alpha, \xi + \alpha]$ and that $\eta(x)f(x) = 0$ for any $x \in [x_0, \xi - \alpha) \cup (\xi + \alpha, x_1]$. It follows that

$$\int_{x_0}^{x_1} \eta(x)f(x) dx > 0,$$

which is a contradiction. Therefore, $f(\xi)$ cannot be positive. For the same reasons, $f(\xi)$ cannot be negative. Hence, $f(x)$ must vanish for any $x \in [x_0, x_1]$.

Thus, the lemma holds. □

1.5 THE EULER-LAGRANGE EQUATION

Theorem 1 (Euler-Lagrange). *The functional J defined in (1) is stationary at function f if and only if*

$$\frac{\partial L}{\partial f}(x, f(x), f'(x)) - \frac{d}{dx} \frac{\partial L}{\partial f'}(x, f(x), f'(x)) = 0$$

for any $x \in [x_0, x_1]$.

Proof. As shown in subsection 1.3, that J is stationary at f is equivalent to the vanishing of

$$\begin{aligned} \left. \frac{d\psi}{d\varepsilon}(\varepsilon) \right|_{\varepsilon=0} &= \left. \frac{d}{d\varepsilon} \int_{x_0}^{x_1} J[f(x) + \varepsilon\eta(x)] dx \right|_{\varepsilon=0} \\ &= \left. \frac{d}{d\varepsilon} \int_{x_0}^{x_1} L(x, f(x) + \varepsilon\eta(x), f'(x) + \varepsilon\eta'(x)) dx \right|_{\varepsilon=0} \\ &= \int_{x_0}^{x_1} \left. \frac{\partial}{\partial \varepsilon} L(x, f(x) + \varepsilon\eta(x), f'(x) + \varepsilon\eta'(x)) \right|_{\varepsilon=0} dx \\ &= \int_{x_0}^{x_1} \left(\left. \frac{\partial}{\partial \varepsilon} (f(x) + \varepsilon\eta(x)) \right|_{\varepsilon=0} \frac{\partial L}{\partial (f + \varepsilon\eta)}(x, f(x) + \varepsilon\eta(x), f'(x) + \varepsilon\eta'(x)) \right) \Big|_{\varepsilon=0} \end{aligned}$$

$$\begin{aligned}
& + \left. \frac{\partial}{\partial \varepsilon} (f'(x) + \varepsilon \eta'(x)) \right|_{\varepsilon=0} \frac{\partial L}{\partial (f' + \varepsilon \eta')} (x, f(x) + \varepsilon \eta(x), f'(x) + \varepsilon \eta'(x)) \Big|_{\varepsilon=0} \Big) dx \\
& = \int_{x_0}^{x_1} \left(\eta(x) \frac{\partial L}{\partial f} (x, f(x), f'(x)) + \eta'(x) \frac{\partial L}{\partial f'} (x, f(x), f'(x)) \right) dx.
\end{aligned}$$

We further see that

$$\begin{aligned}
\int_{x_0}^{x_1} \eta'(x) \frac{\partial L}{\partial f'} (x, f(x), f'(x)) dx & = \eta(x) \frac{\partial L}{\partial f'} (x, f(x), f'(x)) \Big|_{x_0}^{x_1} \\
& \quad - \int_{x_0}^{x_1} \eta(x) d \frac{\partial L}{\partial f'} (x, f(x), f'(x)) \\
& = - \int_{x_0}^{x_1} \eta(x) \frac{d}{dx} \frac{\partial L}{\partial f'} (x, f(x), f'(x)) dx.
\end{aligned}$$

By integration by parts, the equivalent condition is then

$$\int_{x_0}^{x_1} \eta(x) \left(\frac{\partial L}{\partial f} (x, f(x), f'(x)) - \frac{d}{dx} \frac{\partial L}{\partial f'} (x, f(x), f'(x)) \right) dx = 0.$$

By **lemma 1**, we then conclude that

$$\frac{\partial L}{\partial f} (x, f(x), f'(x)) - \frac{d}{dx} \frac{\partial L}{\partial f'} (x, f(x), f'(x)) = 0$$

for any $x \in [x_0, x_1]$.

Hence, the theorem holds. □