NOTES ON CLASSICAL MECHANICS

YANNAN MAO

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CONTENTS

THE EQUATIONS OF MOTION

The position of a particle in three-dimensional Euclidean space is defined by a vector $\boldsymbol{r} \in \mathbb{R}^3$. Its derivative $\boldsymbol{v} = \dfrac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$ with respect to time $t \in \mathbb{R}$ is the ***velocity of the particle***, denoted by a dot above: $\boldsymbol{v} = \dot{\boldsymbol{r}}$. The second derivative $\ddot{\boldsymbol{r}} = \dfrac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}$ is the ***acceleration of the particle***.

1.1 GENERALISED COORDINATES

In general, the number of independent quantities which are necessary to uniquely define the position of a system is the number of ***degrees of freedom of the system***. Any $n \in \mathbb{Z}_{>0}$ quantities $q_0, \dots, q_{n-1}$, each of which is in some subset of $\mathbb{R}$, which completely define the position of a system with n degrees of freedom are referred to as ***generalised coordinates of the system***, and the derivatives $\dot{q}_0, \dots, \dot{q}_{n-1}$ are its ***generalised velocities***. Generalised coordinates span the ***configuration space of the system***. We denote generalised coordinates by an n-dimensional vector $\boldsymbol{q}$.

In principle, if all the coordinates $\boldsymbol{q}$ and velocities $\dot{\boldsymbol{q}}$ of a system are simultaneously specified for some instant, then accelerations $\ddot{\boldsymbol{q}}$ for that instant are uniquely determined. The relations between the coordinates, velocities, and accelerations are the ***equations of motions of the system***, which are second-order differential equations for the function $\boldsymbol{q}(t)$. Solving for $\boldsymbol{q}(t)$ makes possible the determination of the motion of the system.

1.2 THE STATIONARY-ACTION PRINCIPLE

The most general formulation of the law governing the motion of mechanical systems is the ***stationary-action principle*** or the ***principle of least action***, according to which every mechanical system is characterised by a definite function $L\big(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t\big)$, referred to as the ***Lagrangian***, and the motion of the system is such that a certain condition is satisfied.

Let the system occupy, at the instants t_0 and t_1, positions defined by two sets of values of the coordinates, $\boldsymbol{q}_0$ and $\boldsymbol{q}_1$. Then the condition is that the system moves between these positions in such a way that

$$S[\boldsymbol{q}] = \int_{t_0}^{t_1} L\big(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t\big)\, \mathrm{d}t$$

is stationary. The functional $S[\boldsymbol{q}]$ is referred to as the ***action***. The fact that the Lagrangian contains only $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ expresses the aforementioned result that the mechanical state of the system is determined when the coordinates and velocities are given.

From the calculus of variations, we know that the functional $S[\boldsymbol{q}]$ is stationary if and only if its first variation vanishes. Thus, the stationary-action principle may also be written

$$\delta S[\boldsymbol{q}] = \delta \int_{t_0}^{t_1} L\big(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t\big)\, \mathrm{d}t = 0.$$

And per Euler–Lagrange, the above condition is equivalent to n differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i \in \mathbb{N}_{<n}.$$

The general solution contains $2n$ arbitrary constants, which are determined by the initial conditions which specify the state of the system at some given instant, for example the initial values of all the coordinates and velocities.

The Lagrangian is additive among systems between which interactions may be neglected as the distance between which tends to infinity, which expresses the fact that the equations of motion of either of the two non-interacting parts cannot involve quantities pertaining to the other part.

The multiplication of the Lagrangian of a mechanical system by an arbitrary constant also has no effect on the equations of motion. This corresponds to the natural arbitrariness in the choice of the unit of measurement of the Lagrangian.

As the stationary-action principle is equivalent to the vanishing of the first variation, the Lagrangian is defined only to within an additive total time derivative of any function or coordinates and time; i.e., two Lagrangians differing by $\dfrac{\mathrm{d}}{\mathrm{d}t}f(\boldsymbol{q}, t)$ wherein f is some function of coordinates and time give rise to the same equations of motion.

1.3 GALILEAN INVARIANCE

It is necessary to choose a ***frame of reference*** when considering mechanical phenomena, and the equations of motion are in general different for different frames of reference.

It is found that a frame of reference can always be chosen in which space is homogeneous and isotropic and time is homogeneous, referred to as an ***inertial frame of reference***.

We may draw some immediate inferences concerning the form of the Lagrangian of a particle moving freely in an inertial frame of reference. The homogeneity of space and time implies that the Lagrangian cannot contain explicitly either $\boldsymbol{r}$ or t. Thus, L must be a function of $\boldsymbol{v}$ only. Since space is isotropic, the Lagrangian must be independent of the direction of $\boldsymbol{v}$ and is therefore a function only of its norm $\|\boldsymbol{v}\|$. As such, we see that the Euler–Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{v}} = 0,$$

which implies that $\boldsymbol{v}$ is constant.

Therefore, we conclude that, in an inertial frame of reference, any free motion takes place with a velocity constant in both magnitude and direction. This is ***Newton's first law of motion*** or the ***principle of inertia***.

Experiments show that another frame moving uniformly in a straight line relative to an inertial frame of reference is entirely equivalent in all mechanical respects to the inertial frame. Thus, there is an infinity of inertial frames moving, relative to one another, uniformly in a straight line, and in all these frames the properties of space and time are identical, and the laws of motion are the same. This is referred to as ***Galilean invariance***.

In what follows, unless the contrary is specifically stated, we shall consider only inertial frames of reference.

The coordinates $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$ of a given point in two different frames of reference, of which the latter moves relative to the former with velocity $\boldsymbol{v}$, are related by

$$\boldsymbol{r}_0 = \boldsymbol{r}_1 + \boldsymbol{v}t.$$

Here, it is assumed that time is the same in the two frames, $t = t_0 = t_1$, which is one of the foundations of classical mechanics. These formulae are referred to as a ***Galilean transformation***.