NOTES ON PROBABILITY THEORY

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Contents

MEASURES

Suppose that $(A_n)_{n\in\mathbb{N}}$ is a countably infinite sequence of sets. The sequence is ***increasing*** if $A_n \subseteq A_{n+1}$ for each $n \in \mathbb{N}$, and ***decreasing*** if $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$. The sequence is ***monotone*** if it is increasing or decreasing.

If $(A_n)_{n\in\mathbb{N}}$ is increasing, the ***increasing union thereof*** is $\bigcup_{n\in\mathbb{N}} A_n$. If $(A_n)_{n\in\mathbb{N}}$ is decreasing, the ***decreasing intersection thereof*** is $\bigcap_{n\in\mathbb{N}} A_n$.

The ***limit inferior of the countably infinite sequence*** $(A_n)_{n\in\mathbb{N}}$ is

$$\liminf_{n\to\infty} A_n = \bigcup_{n=0}^{\infty} \bigcap_{i=n}^{\infty} A_i,$$

and the ***limit superior thereof*** is

$$\limsup_{n\to\infty} A_n = \bigcap_{n=0}^{\infty} \bigcup_{i=n}^{\infty} A_i.$$

1.1 TOPOLOGICAL SPACES AND σ-ALGEBRAS

DEFINITION 1. Let S be a set. Then $\mathcal{T} \subseteq \mathcal{P}(S)$ is a ***topology on*** S if and only if

(a) $\varnothing \in \mathcal{T}$;

(b) $S \in \mathcal{T}$;

(c) $\mathcal{T}$ is closed under finite intersections; and

(d) $\mathcal{T}$ is closed under unions.

A ***topological space*** is an ordered pair $(S, \mathcal{T})$ wherein S is a set and $\mathcal{T}$ is a topology thereon, and a subset of S is an ***open set*** if it is in $\mathcal{T}$.

For each $\mathcal{S} \subseteq \mathcal{P}(S)$, there exists a smallest topology on S which includes $\mathcal{S}$, referred by as the ***topology generated by*** $\mathcal{S}$.

DEFINITION 2. Let S be a set. Then $\Sigma \subseteq \mathcal{P}(S)$ is a σ-***algebra on*** S if and only if

(a) $\Sigma \neq \varnothing$;

(b) Σ is closed under complementation; and

(c) Σ is closed under countable unions.

A ***measurable space*** is an ordered pair (S, Σ) wherein S is a set and Σ is a σ-algebra thereon, and a subset of S is a ***measurable set*** if it is in Σ.

It follows that $S \in \Sigma$ and $\varnothing \in \Sigma$ for each σ-algebra on S and that each σ-algebra is closed under countable intersections. Hence, the smallest σ-algebra on S is $\{S, \varnothing\}$ and the largest thereon is $\mathcal{P}(S)$.

A σ-algebra is closed under both increasing unions and decreasing intersections.

For each $P \in \mathcal{P}(S)$, there exists a smallest σ-algebra on S which includes P, referred to as the ***σ-algebra generated by*** P and denoted by $\sigma(P)$.

A ***π-system on*** S is a set of subsets of S which is closed under finite intersections. A ***Dynkin system on*** S is a set of subsets of S which contains S and is closed under both proper differences and increasing unions.

Theorem 1 (Sierpiński–Dynkin's π–λ theorem)**.** *Let $\mathcal{C}$ be a π-system on S, and let $\mathcal{D}$ be a Dynkin system thereon.*

If $\mathcal{C} \subseteq \mathcal{D}$, then $\sigma(\mathcal{C}) \subseteq \mathcal{D}$.

Proof. First, $\mathcal{C}$ is a σ-algebra if and only if it is both a π-system and a Dynkin system.

Assume that $\mathcal{D}$ is the smallest Dynkin system which includes $\mathcal{C}$. It then suffices to show that $\mathcal{D}$ is also a π-system. Hence, we need to show that $A \cap B \in \mathcal{D}$ for any $A, B \in \mathcal{D}$.

To demonstrate the above proposition, first define $\mathcal{S}_B = \{A \subseteq S : A \cap B \in \mathcal{D}\}$ for each $B \in \mathcal{C}$. Then each $\mathcal{S}_B$ is a Dynkin system including $\mathcal{C}$, and so it includes $\mathcal{D}$. Thus, $A \cap B \in \mathcal{D}$ for each $A \in \mathcal{D}$ and $B \in \mathcal{C}$. Next, define $\mathcal{S}'_A = \{B \subseteq S : A \cap B \in \mathcal{D}\}$ for each $A \in \mathcal{D}$. Similarly, we note that each $\mathcal{S}'_A$ includes $\mathcal{D}$.

Therefore, the theorem holds. □

DEFINITION 3. Let $(S, \mathcal{T})$ be a topological space. The ***Borel σ-algebra of*** $(S, \mathcal{T})$, denoted by $\mathcal{B}(S, \mathcal{T})$, is the σ-algebra generated by $\mathcal{T}$.

A ***Borel set of*** $(S, \mathcal{T})$ is an element of $\mathcal{B}(S, \mathcal{T})$.

Let $(S_i, \mathcal{T}_i)_{i \in I}$ be topological spaces with I as an index set. Let $\mathcal{X}$ be the cartesian product of $(S_i)_{i \in I}$; i.e.,

$$\mathcal{X} = \prod_{i \in I} S_i.$$

For each $i \in I$, let $\mathrm{pr}_i : \mathcal{X} \to S_i$ denote the i-th projection on $\mathcal{X}$; i.e.,

$$\mathrm{pr}_i(x) = s_i$$

wherein $x = (s_i)_{i \in I}$. Then, the ***product topology on*** $\mathcal{X}$ is the topology generated by

$$\{\mathrm{pr}_i^{-1}[U] : U \in \mathcal{T}_i \wedge i \in I\}.$$

Let $(S_i, \Sigma_i)_{i \in I}$ be measurable spaces with I as an index set. The ***product σ-algebra of*** $(\Sigma_i)_{i \in I}$ is

$$\bigotimes_{i \in I} \Sigma_i = \sigma\left(\left\{A_i \times \prod_{j \in I \wedge j \neq i} S_j : A_i \in \Sigma_i \wedge i \in I\right\}\right).$$