NOTES ON PROBABILITY THEORY

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Contents

MEASURES

Suppose that $(A_n)_{n\in\mathbb{N}}$ is a countably infinite sequence of sets. The sequence is ***increasing*** if $A_n \subseteq A_{n+1}$ for each $n \in \mathbb{N}$, and ***decreasing*** if $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$. The sequence is ***monotone*** if it is increasing or decreasing.

If $(A_n)_{n\in\mathbb{N}}$ is increasing, the ***increasing union thereof*** is

$$\bigcup_{n\in\mathbb{N}} A_n;$$

if $(A_n)_{n\in\mathbb{N}}$ is decreasing, the ***decreasing intersection thereof*** is

$$\bigcap_{n\in\mathbb{N}} A_n.$$

The ***limit inferior of the countably infinite sequence*** $(A_n)_{n\in\mathbb{N}}$ is

$$\liminf_{n\to\infty} A_n = \bigcup_{n=0}^{\infty}\bigcap_{i=n}^{\infty} A_i,$$

and the ***limit superior thereof*** is

$$\limsup_{n\to\infty} A_n = \bigcap_{n=0}^{\infty}\bigcup_{i=n}^{\infty} A_i.$$

1.1 TOPOLOGICAL SPACES AND METRIC SPACES

DEFINITION 1. Let X be a set. Then $\mathcal{T} \subseteq \mathcal{P}(X)$ is a ***topology on*** X if and only if

(a) $\varnothing \in \mathcal{T}$;

(b) $X \in \mathcal{T}$;

(c) $\mathcal{T}$ is closed under finite intersections; and

(d) $\mathcal{T}$ is closed under unions.

A ***topological space*** is an ordered pair $(X, \mathcal{T})$ wherein X is a set and $\mathcal{T}$ is a topology thereon, and a subset of X is ***open*** if it is in $\mathcal{T}$.

A subset of X is ***closed*** if and only if the relative complement thereof is open. For each $A \subseteq X$, the ***closure of*** A is the intersection of all closed sets which include A. A subset of X is ***dense*** if and only if the closure thereof is X. The topological space is ***separable*** if and only if there exists a countable dense subset of X.

For each $\mathcal{X} \subseteq \mathcal{P}(X)$, there exists a smallest topology on X which includes $\mathcal{X}$, referred by as the ***topology generated by*** $\mathcal{X}$.

Let $(X_i, \mathcal{T}_i)_{i\in I}$ be topological spaces with I as an index set. Let $\mathcal{X}$ be the cartesian product of $(X_i)_{i\in I}$; i.e.,

$$\mathcal{X} = \prod_{i\in I} X_i.$$

For each $i \in I$, let $\mathrm{pr}_i : \mathcal{X} \to X_i$ denote the i-th projection on $\mathcal{X}$; i.e.,

$$\mathrm{pr}_i(x) = x_i$$

wherein $x = (x_i)_{i\in I}$. The ***product topology on*** $\mathcal{X}$ is the topology generated by

$$\{\mathrm{pr}_i^{-1}[U] : U \in \mathcal{T}_i \wedge i \in I\}.$$

DEFINITION 2. Let X be a set. Then $d : X^2 \to \mathbb{R}$ is a ***metric on*** X if and only if

(a) $d(x, x) = 0$ for each $x \in X$;

(b) $d(x, y) > 0$ if $x \neq y$ for each $x, y \in X$;

(c) $d(x, y) = d(y, x)$ for each $x, y \in X$; and

(d) (***triangle inequality***) $d(x, y) + d(y, z) \geq d(x, z)$ for each $x, y, z \in X$.

A ***metric space*** is an ordered pair (X, d) wherein X is a set and d is a metric thereon.

Given $\varepsilon \in \mathbb{R}_{>0}$ and $x \in X$, the ***open*** ε***-ball of*** x ***in*** X is

$$B_d(x, \varepsilon) = \{y : d(x, y) < \varepsilon\}.$$

The ***metric topology on*** X ***induced by*** d is

$$\{U \subseteq X : \forall x \in U \big(\exists \varepsilon \in \mathbb{R}_{>0}\ (B_d(x,\varepsilon) \subseteq U)\big)\}.$$

A topological space $(X, \mathcal{T})$ is ***metrisable*** if and only if there exists a metric on X whereby $\mathcal{T}$ is the induced metric topology.

1.2 σ-ALGEBRAS

DEFINITION 3. Let X be a set. Then $\Sigma \subseteq \mathcal{P}(X)$ is a ***σ-algebra on*** X if and only if

(a) $\Sigma \neq \varnothing$;

(b) Σ is closed under complementation; and

(c) Σ is closed under countable unions.

A ***measurable space*** is an ordered pair (X, Σ) wherein X is a set and Σ is a σ-algebra thereon, and a subset of X is ***measurable*** if it is in Σ.

It follows that $X \in \Sigma$ and $\varnothing \in \Sigma$ for each σ-algebra on X and that each σ-algebra is closed under countable intersections. Hence, the smallest σ-algebra on X is $\{X, \varnothing\}$ and the largest thereon is $\mathcal{P}(X)$.

A σ-algebra is closed under both increasing unions and decreasing intersections.

For each $P \in \mathcal{P}(X)$, there exists a smallest σ-algebra on X which includes P, referred to as the ***σ-algebra generated by*** P and denoted by $\sigma(P)$.

A ***π-system on*** X is a set of subsets of X which is closed under finite intersections. A ***Dynkin system on*** X is a set of subsets of X which contains X and is closed under both proper differences and increasing unions.

Theorem 1 (Sierpiński–Dynkin's π–λ theorem). *Let $\mathcal{C}$ be a π-system on X, and let $\mathcal{D}$ be a Dynkin system thereon.*

If $\mathcal{C} \subseteq \mathcal{D}$, then $\sigma(\mathcal{C}) \subseteq \mathcal{D}$.

Proof. First, $\mathcal{C}$ is a σ-algebra if and only if it is also a Dynkin system.

Assume that $\mathcal{D}$ is the smallest Dynkin system which includes $\mathcal{C}$. It then suffices to show that $\mathcal{D}$ is also a π-system. Hence, we need to show that $A \cap B \in \mathcal{D}$ for any $A, B \in \mathcal{D}$.

To demonstrate the above proposition, first define $\mathcal{X}_B = \{A \subseteq X : A \cap B \in \mathcal{D}\}$ for each $B \in \mathcal{C}$. Then each $\mathcal{X}_B$ is a Dynkin system including $\mathcal{C}$, and so it includes $\mathcal{D}$. Thus, $A \cap B \in \mathcal{D}$ for each $A \in \mathcal{D}$ and $B \in \mathcal{C}$. Next, define $\mathcal{X}'_A = \{B \subseteq X : A \cap B \in \mathcal{D}\}$ for each $A \in \mathcal{D}$. Similarly, we note that each $\mathcal{X}'_A$ includes $\mathcal{D}$.

Therefore, the theorem holds. □

DEFINITION 4. Let $(X, \mathcal{T})$ be a topological space. The ***Borel σ-algebra of*** $(X, \mathcal{T})$, denoted by $\mathcal{B}(X, \mathcal{T})$, is the σ-algebra generated by $\mathcal{T}$.

A ***Borel set of*** $(X, \mathcal{T})$ is an element of $\mathcal{B}(X, \mathcal{T})$.

Let $(X_i, \Sigma_i)_{i \in I}$ be measurable spaces with I as an index set. The ***product σ-algebra of*** $(\Sigma_i)_{i \in I}$ is

$$\bigotimes_{i \in I} \Sigma_i = \sigma\left(\left\{A_i \times \prod_{j \in I \wedge j \neq i} X_j : A_i \in \Sigma_i \wedge i \in I\right\}\right).$$

Lemma 1. *Let $(X_i)_{i \in I}$ be separable metric spaces with I as an index set. Then*

$$\mathcal{B}\left(\prod_{i \in I} X_i\right) = \bigotimes_{i \in I} \mathcal{B}(X_i).$$