Root Locus for Systems with Delays

Objectives:

To be able to design a feedback compensator for a system with a delay.

Delays:

A delay can model fast dynamics which you ignore to simplify the model. It can also be part of the actual system.

A delay has a LaPlace transform of

$$\text{delay(T seconds)} \Leftrightarrow e^{-sT}$$

This is a problem for root locus analysis since you need to know the poles and zeros to sketch the root loucs. e^{-sT} doesn't have poles or zeros.

One way around this is to the a Pade approximation for a delay:

Pade Approximation:

$$e^{-sT} = \left(\frac{e^{-\frac{sT}{2}}}{e^{\frac{sT}{2}}} \right)$$

Using the Taylors series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

a delay of T seconds has a LaPlace transform of

$$e^{-sT} = \left(\frac{1+\left(\frac{-sT}{2}\right)+\left(\frac{-sT}{2}\right)^2\left(\frac{1}{2!}\right)+\left(\frac{-sT}{2}\right)^3\left(\frac{1}{3!}\right)+\left(\frac{-sT}{2}\right)^4\left(\frac{1}{4!}\right)+\ldots}{1+\left(\frac{sT}{2}\right)+\left(\frac{sT}{2}\right)^2\left(\frac{1}{2!}\right)+\left(\frac{sT}{2}\right)^3\left(\frac{1}{3!}\right)+\left(\frac{sT}{2}\right)^4\left(\frac{1}{4!}\right)+\ldots} \right)$$

or

$$e^{-sT} = \left(\frac{1-\left(\frac{T}{2}\right)s+\left(\frac{T^2}{8}\right)s^2-\left(\frac{T^3}{48}\right)s^3+\left(\frac{T^4}{384}\right)s^4+\ldots}{1+\left(\frac{T}{2}\right)s+\left(\frac{T^2}{8}\right)s^2+\left(\frac{T^3}{48}\right)s^3+\left(\frac{T^4}{384}\right)s^4+\ldots} \right)$$

The more terms you add, the better the approximation.

Example: Find 'k' so that the following system has 10% overshoot for its step response: (A DC servo motor with a 1/2 second delay)

$$Y = \left(\frac{1000 \cdot e^{-0.5s}}{s(s+5)(s+20)} \right) U$$

Solution #1: Let's use a Pade approximation:

$$e^{-sT} = \left(\frac{1 - \left(\frac{T}{2}\right)s + \left(\frac{T^2}{8}\right)s^2 - \left(\frac{T^3}{48}\right)s^3 + \left(\frac{T^4}{384}\right)s^4 + \ldots}{1 + \left(\frac{T}{2}\right)s + \left(\frac{T^2}{8}\right)s^2 + \left(\frac{T^3}{48}\right)s^3 + \left(\frac{T^4}{384}\right)s^4 + \ldots} \right)$$

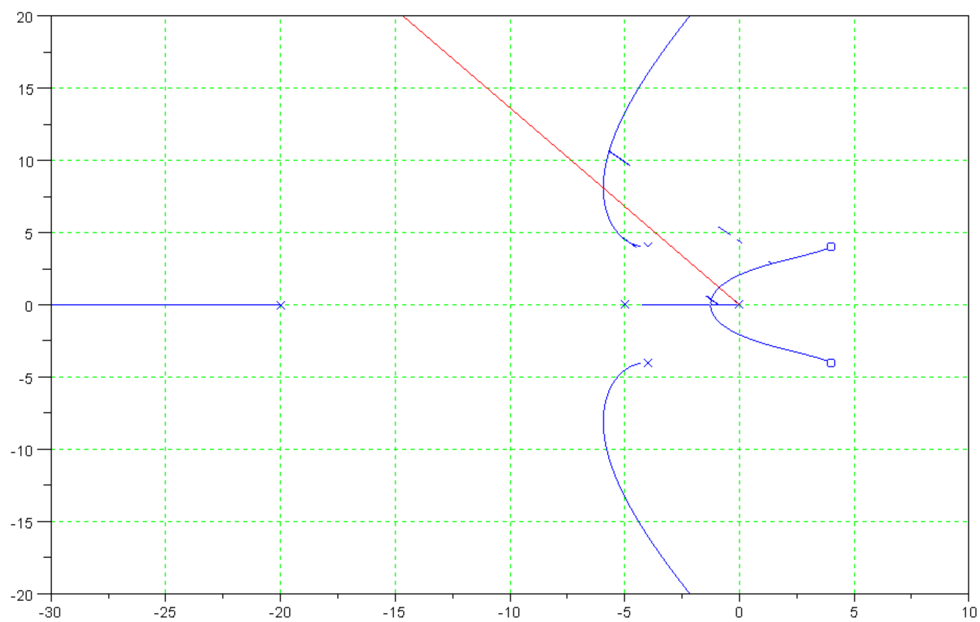
Pluggint in T = 0.5 and using the first two terms results in

$$e^{-0.5s} \approx \left(\frac{1 - 0.25s + 0.0313s^2}{1 + 0.25s + 0.0313s^2} \right)$$

$$e^{-0.5T} \approx \left(\frac{(s-4+j4)(s-4-j4)}{(s+4+j4)(s+4-j4)} \right)$$

In SciLab:

```
-->G = zp2ss([4+j*4,4-j*4],[0,-5,-20,-4+j*4,-4-j*4],1000);
-->k = logspace(-2,2,1000)';
-->R = rlocus(G,k,0.5910);
```



Root locus for G(s) with a 0.5 second delay

Note that there are poles at -4 +/- j4 and zeros at +4 +/- j4. These model the delay.

There are two solutions for a damping ratio of 0.5910 (for 10% overshoot):

s = -0.8767 + j1.1966 k = 0.0775

s = -5.9293 + j8.0930 k = 0.4370

The former is the dominant pole (the one we care about). Hence,

$$\boxed{K(s) = 0.0775}$$

Problem: Repeat for a 4th-order Pade Approximation

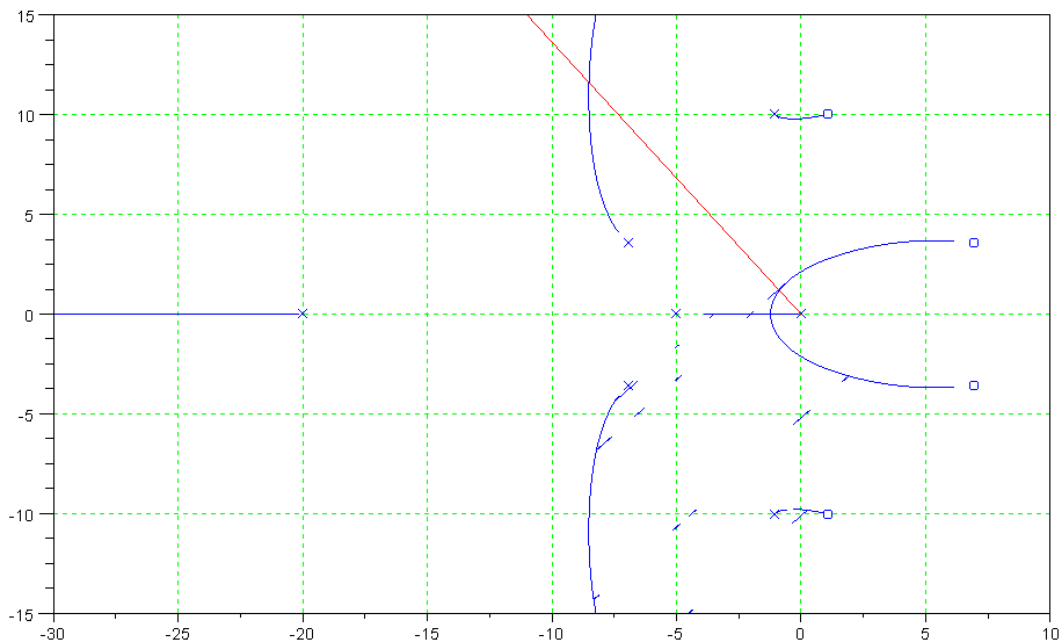
$$e^{-sT} = \left(\frac{1-\left(\frac{T}{2}\right)s+\left(\frac{T^2}{8}\right)s^2-\left(\frac{T^3}{48}\right)s^3+\left(\frac{T^4}{384}\right)s^4+...}{1+\left(\frac{T}{2}\right)s+\left(\frac{T^2}{8}\right)s^2+\left(\frac{T^3}{48}\right)s^3+\left(\frac{T^4}{384}\right)s^4+...} \right)$$

$$e^{-0.5s} \approx \left(\frac{1-0.25s+0.0313s^2-0.0026s^3+0.002s^4}{1+0.25s+0.0313s^2+0.0026s^3+0.002s^4} \right)$$

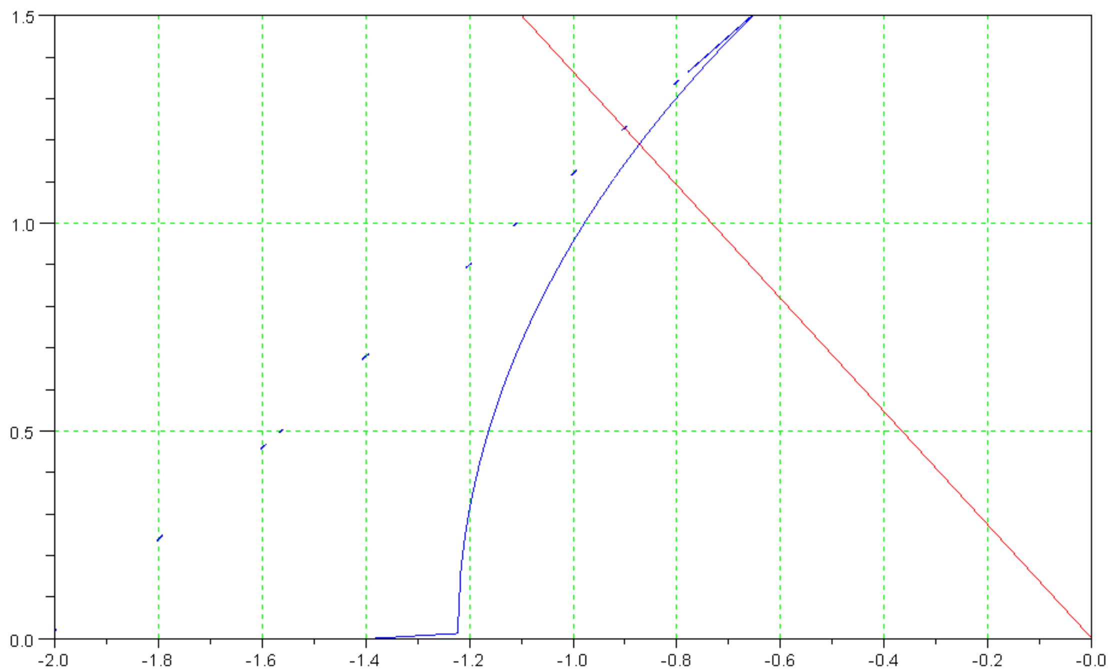
$$e^{-0.5s} \approx \left(\frac{(s-1.08222\pm j10.019)(s-6.9177\pm j3.5559)}{(s+1.08222\pm j10.019)(s+6.9177\pm j3.5559)} \right)$$

The root locus of G + delay is a bit more complicated:

```
-->P = roots({0.5^4/384,0.5^3/48,0.5^2/8,0.25,1]);
-->Z = roots({0.5^4/384,-0.5^3/48,0.5^2/8,-0.25,1]);
-->G = zp2ss(Z,[P;0;-5;-20],1000);
-->R = rlocus(G,k,0.5910);
```



Zooming in on the dominant pole:



4th order Pade Model:

s = -0.8724 + j1.1907 k = 0.0786

Option #2:

This option doesn't really have a name. It's based upon root locus techniques. The idea is, for a damping ratio of 0.5910, you're searching along the line

$$s = \alpha\angle 126.228^0$$

until the angle of G(s) is 180 degrees. You don't really need to use a Pade approximation to do this. Simply find the point where

$$angle\left(\frac{1000\cdot e^{-0.5s}}{s(s+5)(s+20)}\right)_{s=\alpha\angle 126.228^0} = 180^0$$

This method doesn't have a name and might not be taught anywhere else other than NDSU.

- It isn't really root locus design, since you're not drawing the root locus
- It sort of is root locus design, since you're finding the point on the root locus which intersects the desired damping line. That's the only point you care about anyway, so you don't need (and won't use) the rest of the root loucs.
- It's a lot easier and more accurate since you using e^{-sT} to model a delay, not an approximation. (Typing in four poles and four zeros for the 4th-order model was a bit of a pain. Typing in e^{-sT} was easy.)

Anyway, if you use this method, the result is

exact solution:

$s = -0.8725 + j1.1909$ $k = 0.0786$

4th order Pade:

$s = -0.8724 + j1.1907$ $k = 0.0786$

2nd Order Pade Model:

$s = -0.8767 + j1.1966$ $k = 0.0775$

The real test of a design is if it works. Using VisSim with an actual 1/2 second delay, we do get 10% overshoot: