南开大学
Nankai University

第四章 机器人逆运动学

《机器人学导论》

韩建达 教授

Email: hanjianda@nankai.edu.cn

课程内容

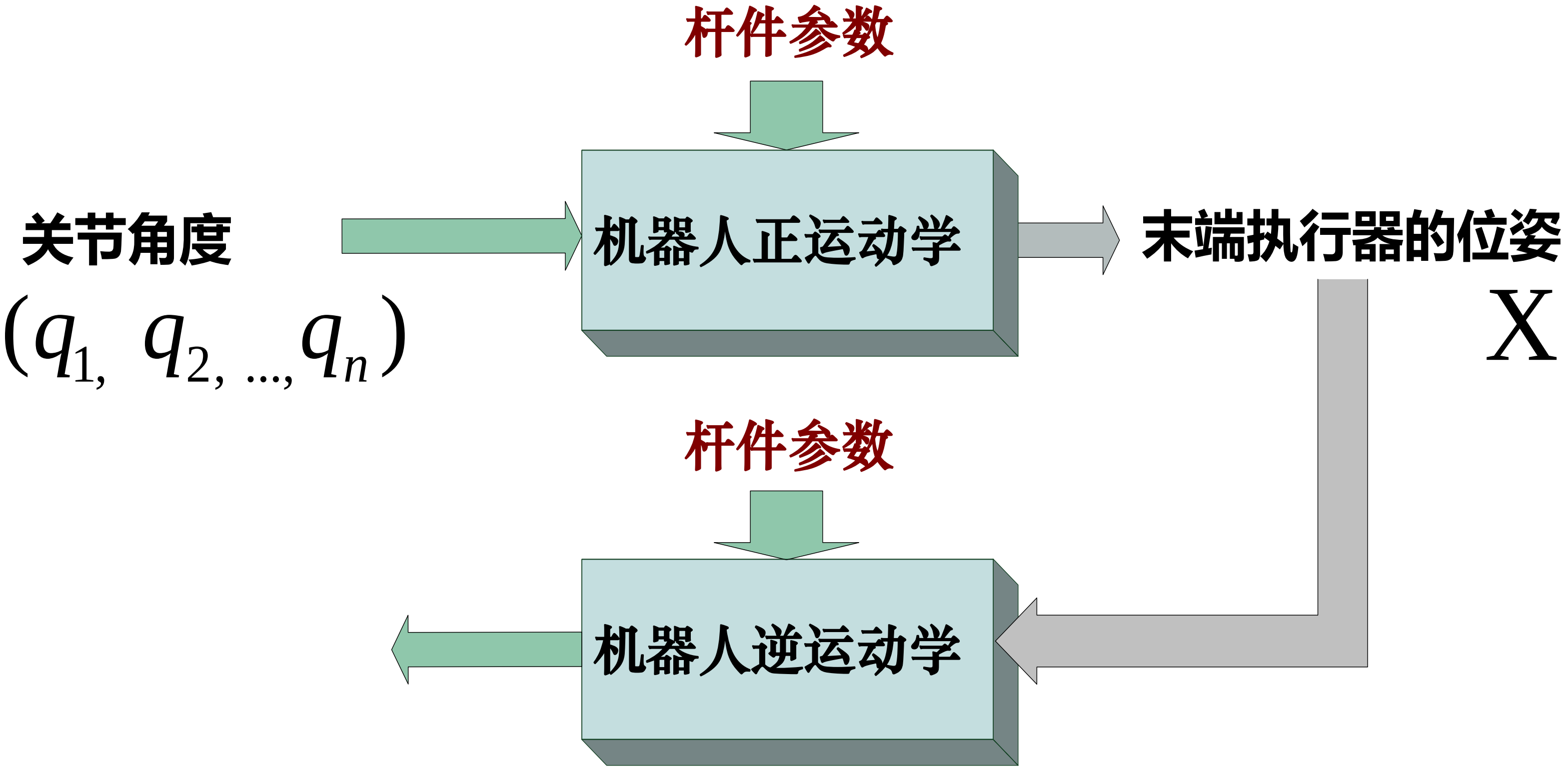
1. 机器人逆运动学基本概念

2. 机器人逆运动学求取方法

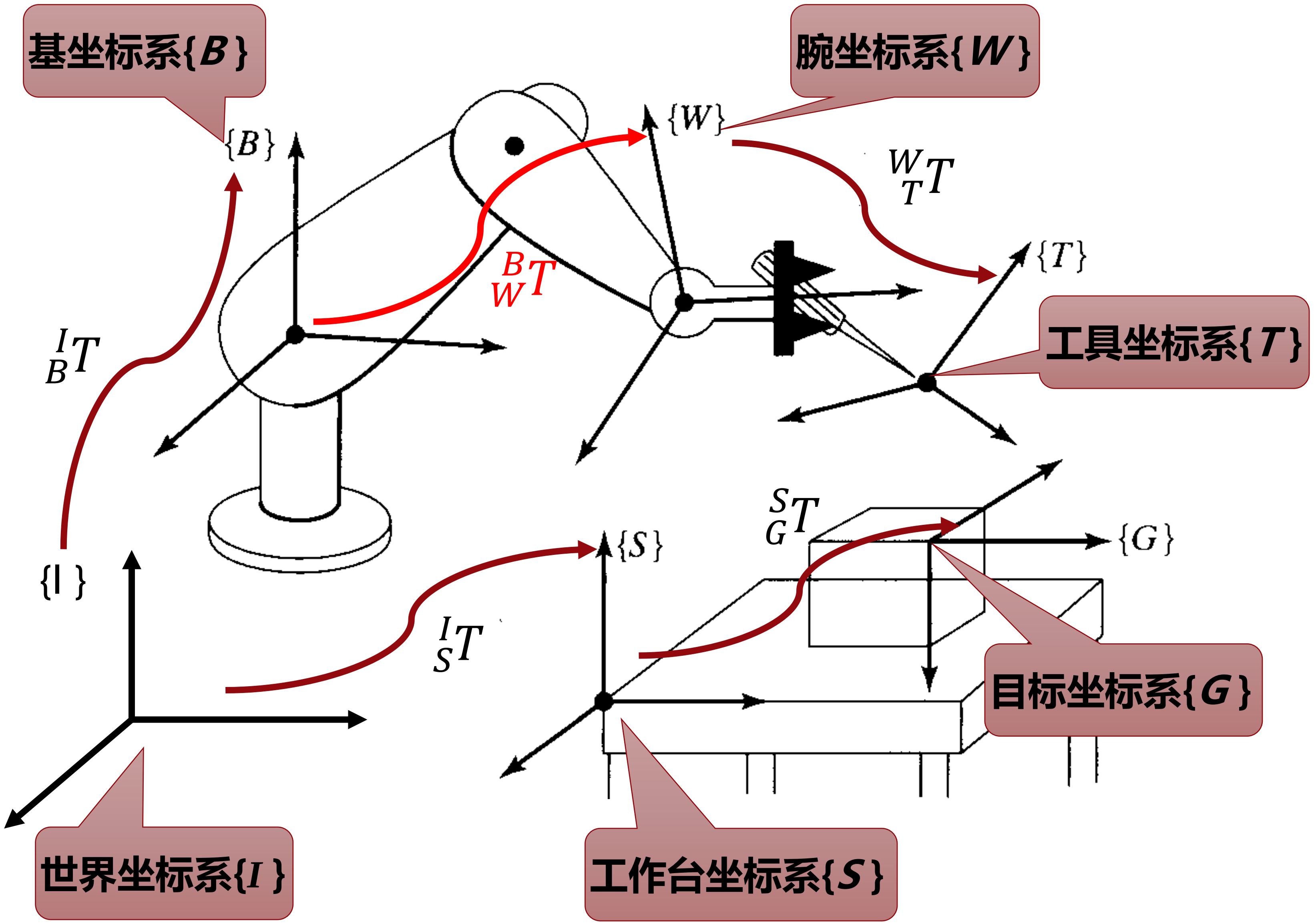
3. 机器人工作空间分析

1.1 机器人逆运动学基本概念

- **机器人逆运动学：**
  - **已知机器人末端执行器的（期望）位姿 X，求解机器人关节角度 q**



1.2 机器人相关坐标系



作业目标： $^{I}_{G}T = {}^{I}_{S}T\,{}^{S}_{G}T$

机械臂把持的工具

$$^{I}_{T}T = {}^{I}_{B}T \cdot {}^{B}_{W}T(q) \cdot {}^{W}_{T}T$$

工具到达指定目标

$$^{I}_{T}T = {}^{I}_{G}T$$

求解出腕部相对于基座坐标系的位姿

$$\begin{aligned} {}^{B}_{W}T(q) &= {}^{I}_{B}T^{-1}\,{}^{I}_{T}T \cdot {}^{W}_{T}T^{-1} \\ &= {}^{I}_{B}T^{-1}\,{}^{I}_{G}T \cdot {}^{W}_{T}T^{-1} \end{aligned}$$

根据运动学方程

$$^{0}_{6}T(q) = {}^{B}_{W}T(q)$$

求解出对应的关节角

已知$^{0}_{6}T(q)$，求解q

■ **机器人逆运动学求取方法：**

- □ **代数法：** 直接从正运动学推导出的齐次变换阵出发进行求解
- □ **几何法：** 先不关注机器人变换矩阵，从机器人空间构型出发，利用空间解析几何求解，将未知数维度进行降维，之后利用降维后的变换阵求解剩余关节角。
- □ **数值法：** 利用迭代搜索方法求取近似解—如何优化迭代方向和步长

课程内容

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2.1 机器人逆运动学求取

The general problem of inverse kinematics can be stated as follows. Given a 4×4 homogeneous transformation

已知期望位姿 $$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3)$$

with $R \in SO(3)$, find (one or all) solutions of the equation

$$T_n^0(q_1, \ldots, q_n) = H$$

求各关节角

where

已知机器人正运动学 $$T_n^0(q_1, \ldots, q_n) = A_1(q_1) \cdots A_n(q_n).$$

Here, H represents the desired position and orientation of the end-effector, and our task is to find the values for the joint variables $q_1, \ldots, q_n$ so that $T_n^0(q_1, \ldots, q_n) = H$.

由 $$T_n^0(q_1,\ldots,q_n) = H$$

可得12个方程

$$T_{ij}(q_1,\ldots,q_n) = h_{ij}, \qquad i=1,2,3, \quad j=1,\ldots,4$$

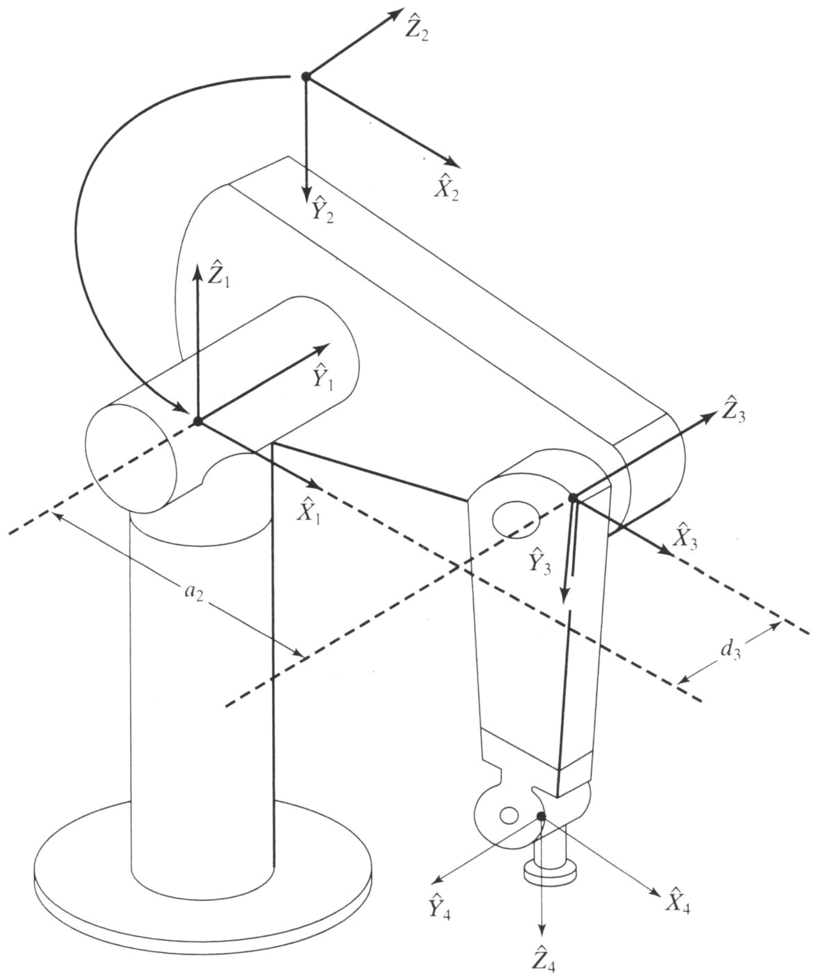
where T_{ij}, h_{ij} refer to the twelve nontrivial entries of T_n^0 and H.

求解 q：机器人逆运动学

2.1 机器人逆运动学求取

已知期望位姿

$$H=\begin{pmatrix} r11 & r12 & r13 & px \\ r21 & r22 & r23 & py \\ r31 & r32 & r33 & pz \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



已知正运动学方程

$$r_{11}=C_1[C_{23}(C_4C_5C_6-S_4S_6)-S_{23}S_5C_6]+S_1(S_4C_5C_6+C_4S_6)$$
$$r_{21}=S_1[C_{23}(C_4C_5C_6-S_4S_6)-S_{23}S_5C_6]-C_1(S_4C_5C_6+C_4S_6)$$
$$r_{31}=-S_{23}(C_4C_5C_6-S_4S_6)-C_{23}S_5C_6$$
$$r_{12}=C_1[C_{23}(-C_4C_5S_6-S_4C_6)+S_{23}S_5S_6]+S_1(C_4C_6-S_4C_5S_6)$$
$$r_{22}=S_1[C_{23}(-C_4C_5S_6-S_4C_6)+S_{23}S_5S_6]-C_1(C_4C_6-S_4C_5S_6)$$
$$r_{32}=-S_{23}(-C_4C_5S_6-S_4C_6)+C_{23}S_5S_6$$
$$r_{13}=-C_1(C_{23}C_4S_5+S_{23}C_5)-S_1S_4S_5$$
$$r_{23}=-S_1(C_{23}C_4S_5+S_{23}C_5)+C_1S_4S_5$$
$$r_{33}=S_{23}C_4S_5-C_{23}C_5$$
$$p_x=C_1[a_2C_2+a_3C_{23}-d_4S_{23}]-d_3S_1$$
$$p_y=S_1[a_2C_2+a_3C_{23}-d_4S_{23}]+d_3C_1$$
$$p_z=-a_3S_{23}-a_2S_2-d_4C_{23}$$

求关节角

- **解的存在性：是否有解，为什么没有解—工作空间**
- **多解问题：是否是唯一解，多组解，无穷多组解，如何优化**

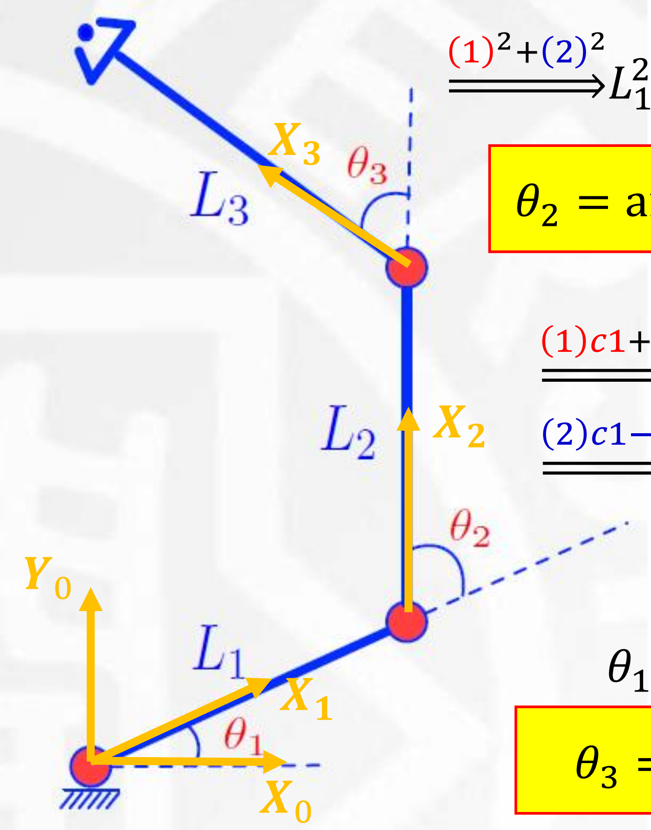
2.1 机器人逆运动学求取：代数法--平面RRR机器人举例《机器人学导论》

DH参数表

关节	α_{i-1}	$\vec{a}_{i-1}$	d_i	θ_i
1	0	0	0	$\boldsymbol{\theta_1}$
2	0	$\boldsymbol{L_1}$	0	$\boldsymbol{\theta_2}$
3	0	$\boldsymbol{L_2}$	0	$\boldsymbol{\theta_3}$

$$ {}^0_3T = \begin{pmatrix} r11 & r12 & r13 & px \\ r21 & r22 & r23 & py \\ r31 & r32 & r33 & pz \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^0_3T = \begin{pmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} (1) \\ (2) \\ \\ \\ \end{matrix} $$



$$ \xRightarrow{(1)^2+(2)^2} L_1^2 + L_2^2 + 2L_1L_2(\cos\theta_1\cos(\theta_1+\theta_2) + \sin\theta_1\sin(\theta_1+\theta_2)) = L_1^2 + L_2^2 + 2L_1L_2\cos\theta_2 = p_x^2 + p_y^2 $$

$$ \theta_2 = \arccos\left(\left(p_x^2 + p_y^2 - L_1^2 - L_2^2\right)/2L_1L_2\right) \qquad \theta_2 = -\arccos\left(\left(p_x^2 + p_y^2 - L_1^2 - L_2^2\right)/2L_1L_2\right) $$

$$ \xRightarrow{(1)c1+(2)s1} p_x c1 + p_y s1 = L_1 + L_2 c2 $$

$$ \xRightarrow{(2)c1-(1)s1} p_y c1 - p_x s1 = L_2 s2 $$

$$ \Rightarrow s1 = \frac{(L_1+L_2c2)p_y - p_x L_2 s2}{p_x^2 + p_y^2}, c1 = \frac{(L_1+L_2c2)p_x + p_y L_2 s2}{p_x^2 + p_y^2} $$

$$ \theta_1 = \arctan2(s1, c1) $$

$$ \theta_1 + \theta_2 + \theta_3 = \arctan2(r_{11}, r_{21}) $$

$$ \theta_3 = \arctan2(r_{21}, r_{11}) - \theta_1 - \theta_2 $$

$$ p_x^2 + p_y^2 = 0 \Rightarrow L_1^2 + L_2^2 + 2L_1L_2\cos\theta_2 = 0 $$

$$ L_1^2 + L_2^2 \geq 2L_1L_2, \quad \text{and } |\cos\theta_2| \leq 1 $$

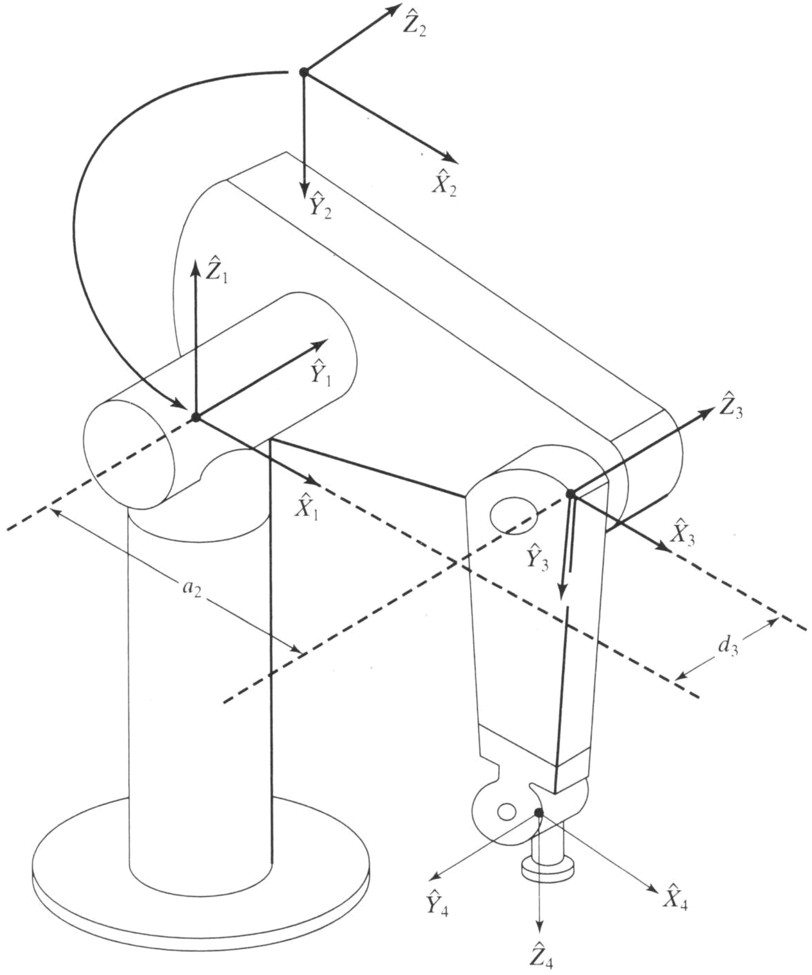
$$ \Rightarrow (L_1 - L_2)^2 = 0, and \cos\theta_2 = -1 $$

$$ \Rightarrow L_1 = L_2, and\ \ \theta_2 = \pi $$

1、3关节完全重合

2.1 机器人逆运动学求取：代数法—PUMA机器人举例

$$
{}^{0}_{6}T = \begin{pmatrix} r11 & r12 & r13 & px \\ r21 & r22 & r23 & py \\ r31 & r32 & r33 & pz \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

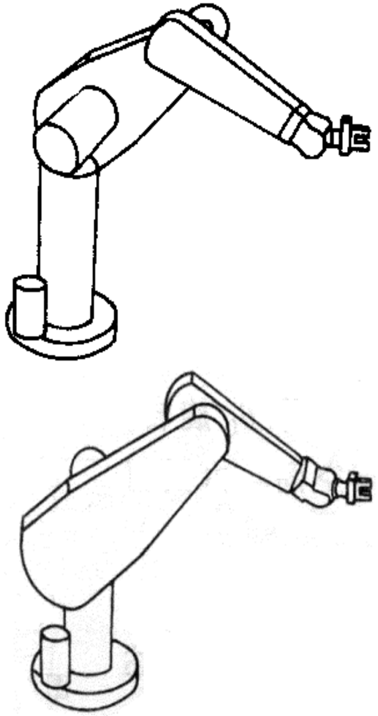


$$
\begin{aligned}
r_{11} &= C_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] + S_1(S_4C_5C_6 + C_4S_6) \\
r_{21} &= S_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] - C_1(S_4C_5C_6 + C_4S_6) \\
r_{31} &= -S_{23}(C_4C_5C_6 - S_4S_6) - C_{23}S_5C_6 \\
r_{12} &= C_1[C_{23}(-C_4C_5S_6 - S_4C_6) + S_{23}S_5S_6] + S_1(C_4C_6 - S_4C_5S_6) \\
r_{22} &= S_1[C_{23}(-C_4C_5S_6 - S_4C_6) + S_{23}S_5S_6] - C_1(C_4C_6 - S_4C_5S_6) \\
r_{32} &= -S_{23}(-C_4C_5S_6 - S_4C_6) + C_{23}S_5S_6 \\
r_{13} &= -C_1(C_{23}C_4S_5 + S_{23}C_5) - S_1S_4S_5 \\
r_{23} &= -S_1(C_{23}C_4S_5 + S_{23}C_5) + C_1S_4S_5 \\
r_{33} &= S_{23}C_4S_5 - C_{23}C_5
\end{aligned}
$$

$$
\begin{aligned}
p_x &= C_1[a_2C_2 + a_3C_{23} - d_4S_{23}] - d_3S_1 \\
p_y &= S_1[a_2C_2 + a_3C_{23} - d_4S_{23}] + d_3C_1 \\
p_z &= -a_3S_{23} - a_2S_2 - d_4C_{23}
\end{aligned}
$$

$$ {}^0_6T = \begin{pmatrix} r11 & r12 & r13 & px \\ r21 & r22 & r23 & py \\ r31 & r32 & r33 & pz \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^0_1T \cdot {}^1_2T \cdot {}^2_3T \cdot {}^3_4T \cdot {}^4_5T \cdot {}^5_6T = \begin{pmatrix} * & * & * & -\mathrm{Sin}[\theta_1]\boxed{L_3} + \mathrm{Cos}[\theta_1]\boxed{(\mathrm{Cos}[\theta_2]L_2 + \mathrm{Cos}[\theta_2+\theta_3]L_4 - \mathrm{Sin}[\theta_2+\theta_3]L_5)} \\ * & * & * & \mathrm{Cos}[\theta_1]\boxed{L_3} + \mathrm{Sin}[\theta_1]\boxed{(\mathrm{Cos}[\theta_2]L_2 + \mathrm{Cos}[\theta_2+\theta_3]L_4 - \mathrm{Sin}[\theta_2+\theta_3]L_5)} \\ * & * & * & -\mathrm{Sin}[\theta_2]L_2 - \mathrm{Sin}[\theta_2+\theta_3]L_4 - \mathrm{Cos}[\theta_2+\theta_3]L_5 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$



$$ {}^0_1T = \begin{pmatrix} \mathrm{Cos}[\theta_1] & -\mathrm{Sin}[\theta_1] & 0 & 0 \\ \mathrm{Sin}[\theta_1] & \mathrm{Cos}[\theta_1] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ \theta_1 = \phi - \arcsin\frac{L_3}{\rho} \text{ 或 } \theta_1 = \phi + \arcsin\frac{L_3}{\rho} - \pi $$

对应的空间关系是什么?

$$ {}^0_1T^{-1} \cdot {}^0_6T = {}^1_2T \cdot {}^2_3T \cdot {}^3_4T \cdot {}^4_5T \cdot {}^5_6T $$

$$ \begin{pmatrix} * & * & * & px\mathrm{Cos}[\theta_1] + py\mathrm{Sin}[\theta_1] \\ * & * & * & \boxed{py\mathrm{Cos}[\theta_1] - px\mathrm{Sin}[\theta_1]} \\ * & * & * & pz \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} * & * & * & \mathrm{Cos}[\theta_2]L_2 + \mathrm{Cos}[\theta_2+\theta_3]L_4 - \mathrm{Sin}[\theta_2+\theta_3]L_5 \\ * & * & * & \boxed{L_3} \\ * & * & * & -\mathrm{Sin}[\theta_2]L_2 - \mathrm{Sin}[\theta_2+\theta_3]L_4 - \mathrm{Cos}[\theta_2+\theta_3]L_5 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ py\mathrm{Cos}[\theta_1] - px\mathrm{Sin}[\theta_1] = L_3 \xRightarrow{\rho=\sqrt{p_x^2+p_y^2},\phi=\mathrm{Atan2}(p_y,p_x)} s_\phi c_1 - c_\phi s_1 = \boxed{s(\phi-\theta_1) = \frac{L_3}{\rho} \Rightarrow \phi - \theta_1 = \arcsin\frac{L_3}{\rho}} \Rightarrow \boxed{\theta_1 = \phi - \arcsin\frac{L_3}{\rho}} $$

$$\begin{pmatrix} * & * & * & \boxed{\mathrm{px}\mathrm{Cos}[\theta_1]+\mathrm{py}\mathrm{Sin}[\theta_1]} \\ * & * & * & \mathrm{py}\mathrm{Cos}[\theta_1]-\mathrm{px}\mathrm{Sin}[\theta_1] \\ * & * & * & \boxed{\mathrm{pz}} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} * & * & * & \boxed{\mathrm{Cos}[\theta_2]L_2+\mathrm{Cos}[\theta_2+\theta_3]L_4-\mathrm{Sin}[\theta_2+\theta_3]L_5} \\ * & * & * & L_3 \\ * & * & * & \boxed{-\mathrm{Sin}[\theta_2]L_2-\mathrm{Sin}[\theta_2+\theta_3]L_4-\mathrm{Cos}[\theta_2+\theta_3]L_5} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix}(1)\\(3)\\(2)\\ \end{matrix}$$

$$\xRightarrow{\rho=\sqrt{L_4^2+L_5^2},\phi=\mathrm{Atan2}(L_5,L_4)} \begin{cases} \mathrm{Cos}[\theta_2+\theta_3]L_4-\mathrm{Sin}[\theta_2+\theta_3]L_5=\rho(\mathrm{Cos}[\theta_2+\theta_3]c_\phi-\mathrm{Sin}[\theta_2+\theta_3]s_\phi)=\rho\cdot c(\theta_2+\theta_3+\phi) \\ -\mathrm{Sin}[\theta_2+\theta_3]L_4-\mathrm{Cos}[\theta_2+\theta_3]L_5=-\rho\big(\mathrm{Sin}[\theta_2+\theta_3]c_\phi+\mathrm{Cos}[\theta_2+\theta_3]s_\phi\big)=-\rho\cdot s(\theta_2+\theta_3+\phi) \end{cases}$$

$$\mathrm{px}\mathrm{Cos}[\theta_1]+\mathrm{py}\mathrm{Sin}[\theta_1]=\mathrm{Cos}[\theta_2]L_2+\rho\cdot c(\theta_2+\theta_3+\phi) \qquad (1)$$

$$-\mathrm{pz}=\mathrm{Sin}[\theta_2]L_2+\rho\cdot s(\theta_2+\theta_3+\phi) \qquad (2)$$

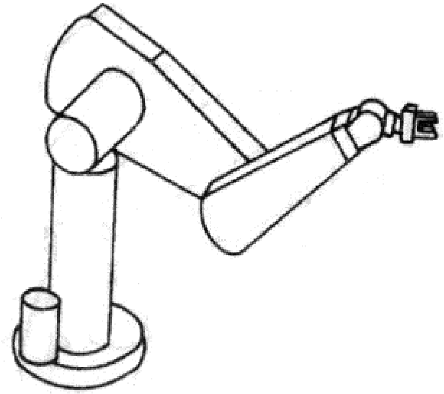
$$\mathrm{py}\mathrm{Cos}[\theta_1]-\mathrm{px}\mathrm{Sin}[\theta_1]=L_3 \qquad (3)$$

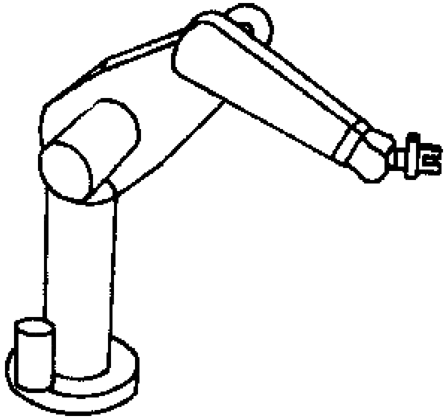
$$\xRightarrow{(1)^2+(2)^2+(3)^2} p_x^2+p_y^2+p_z^2=L_2^2+L_3^2+L_4^2+L_5^2+2\rho L_2(\mathrm{Cos}[\theta_2]c(\theta_2+\theta_3+\phi)+\mathrm{Sin}[\theta_2]s(\theta_2+\theta_3+\phi))$$

$$\Rightarrow c(\theta_3+\phi)=\frac{\left(p_x^2+p_y^2+p_z^2-(L_2^2+L_3^2+L_4^2+L_5^2)\right)}{2\rho L_2} \xRightarrow{k=\frac{\left(p_x^2+p_y^2+p_z^2-(L_2^2+L_3^2+L_4^2+L_5^2)\right)}{2\rho L_2}} \boxed{\theta_3=\arccos k-\phi}$$

$$\boxed{\theta_3=\arccos k-\phi \text{ 或 } \theta_3=-\arccos k-\phi}$$

对应的空间关系是什么?





$$
{}^3_2T^{-1}\cdot{}^2_1T^{-1}\cdot{}^0_1T^{-1}\cdot{}^0_6T=\begin{pmatrix} * & * & \text{r13Cos}[\theta_1]\text{Cos}[\theta_2+\theta_3]+\text{r23Cos}[\theta_2+\theta_3]\text{Sin}[\theta_1]-\text{r33Sin}[\theta_2+\theta_3] & \boxed{\text{pxCos}[\theta_1]\text{Cos}[\theta_2+\theta_3]+\text{pyCos}[\theta_2+\theta_3]\text{Sin}[\theta_1]-\text{pzSin}[\theta_2+\theta_3]-\text{Cos}[\theta_3]L_2} \\ * & * & \boxed{-\text{r33Cos}[\theta_2+\theta_3]-\text{r13Cos}[\theta_1]\text{Sin}[\theta_2+\theta_3]-\text{r23Sin}[\theta_1]\text{Sin}[\theta_2+\theta_3]} & \boxed{-\text{pzCos}[\theta_2+\theta_3]-\text{pxCos}[\theta_1]\text{Sin}[\theta_2+\theta_3]-\text{pySin}[\theta_1]\text{Sin}[\theta_2+\theta_3]+\text{Sin}[\theta_3]L_2} \\ * & * & \text{r23Cos}[\theta_1]-\text{r13Sin}[\theta_1] & \text{pyCos}[\theta_1]-\text{pxSin}[\theta_1]-L_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^3_4T\cdot{}^4_5T\cdot{}^5_6T=\begin{pmatrix} \text{Cos}[\theta_4]\text{Cos}[\theta_5]\text{Cos}[\theta_6]-\text{Sin}[\theta_4]\text{Sin}[\theta_6] & -\text{Cos}[\theta_6]\text{Sin}[\theta_4]-\text{Cos}[\theta_4]\text{Cos}[\theta_5]\text{Sin}[\theta_6] & \boxed{-\text{Cos}[\theta_4]\text{Sin}[\theta_5]} & \boxed{L_4} \\ \text{Cos}[\theta_6]\text{Sin}[\theta_5] & -\text{Sin}[\theta_5]\text{Sin}[\theta_6] & \boxed{\text{Cos}[\theta_5]} & \boxed{L_5} \\ -\text{Cos}[\theta_5]\text{Cos}[\theta_6]\text{Sin}[\theta_4]-\text{Cos}[\theta_4]\text{Sin}[\theta_6] & -\text{Cos}[\theta_4]\text{Cos}[\theta_6]+\text{Cos}[\theta_5]\text{Sin}[\theta_4]\text{Sin}[\theta_6] & \boxed{\text{Sin}[\theta_4]\text{Sin}[\theta_5]} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{matrix}(1)\\(2)\\ \\ \\ \end{matrix}
$$

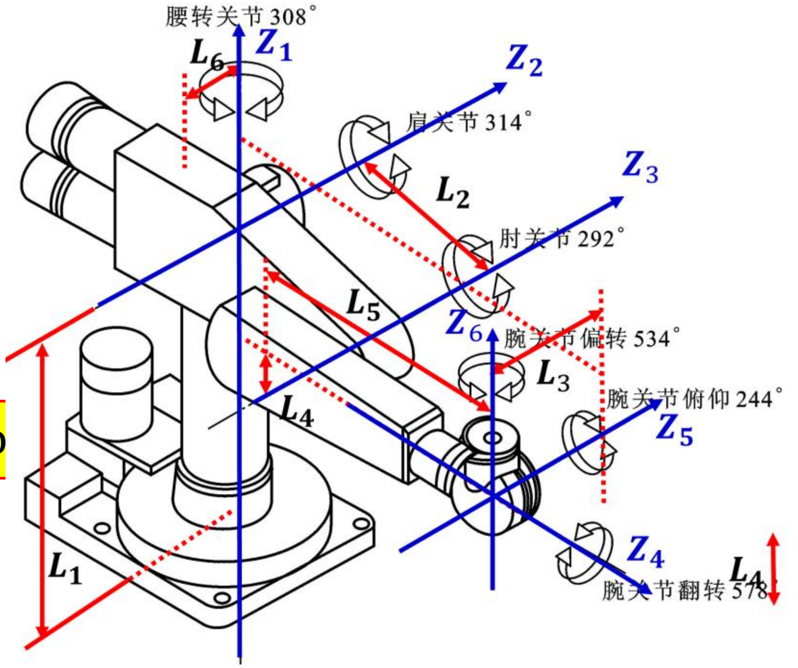
$$\overset{(1)}{\Rightarrow} c_1c_{23}p_x+s_1c_{23}p_y-s_{23}p_z-c_3L_2=L_4$$

$$\overset{(2)}{\Rightarrow} -c_1s_{23}p_x+s_1s_{23}p_y-c_{23}p_z+s_3L_2=L_5$$

$$\Longrightarrow s_{23}\ ,\ c_{23}\Longrightarrow \boxed{\theta_{23}=\text{arctan2}(s_{23},c_{23})}$$ **唯一解**

$$\boxed{\theta_2=\theta_{23}-\theta_3}$$ **唯一解**

$$\text{Cos}[\theta_5]=-\text{r33Cos}[\theta_2+\theta_3]-\text{r13Cos}[\theta_1]\text{Sin}[\theta_2+\theta_3]-\text{r23Sin}[\theta_1]\text{Sin}[\theta_2+\theta_3]=k'\Longrightarrow \boxed{\theta_5=\text{arcco}}$$



$$\boxed{\theta_5=\text{arccos}k' \text{ 或 } \theta_5=-\text{arccos}k'}$$

对应的空间关系是什么?

$$\boxed{\theta_4=\text{arctan2}(\text{Sin}[\theta_4]\text{Sin}[\theta_5],\text{Cos}[\theta_4]\text{Sin}[\theta_5])}$$ **唯一解**

$$\boxed{\theta_6=\text{arctan2}(\text{Sin}[\theta_4]\text{Sin}[\theta_5],\text{Cos}[\theta_4]\text{Sin}[\theta_5])}$$ **唯一解**

$$\mathbf{Sin}[\theta_5]=\mathbf{0}$$

$$
{}^3_6T=\begin{pmatrix} c_{46} & -s_{46} & 0 & L_4 \\ 0 & 0 & 1 & L_5 \\ -s_{46} & -c_{46} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ 或 } \begin{pmatrix} -c_{46} & -s_{46} & 0 & L_4 \\ 0 & 0 & 1 & L_5 \\ s_{46} & -c_{46} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

当$\mathbf{Sin}[\theta_5]=\mathbf{0}$时，机器人处于奇异构型

奇异构型：机器人自由度丢失

2.1 机器人逆运动学求取：代数法—PUMA机器人举例

1 $\theta_1 = \phi - \arcsin\frac{L_3}{\rho}$ 或 $\theta_1 = \phi + \arcsin\frac{L_3}{\rho} - \pi$

2 $\theta_3 = \arccos k - \phi$ 或 $\theta_3 = -\arccos k - \phi$

3 $\theta_{23} = \mathrm{arctan2}(s_{23}, c_{23})$

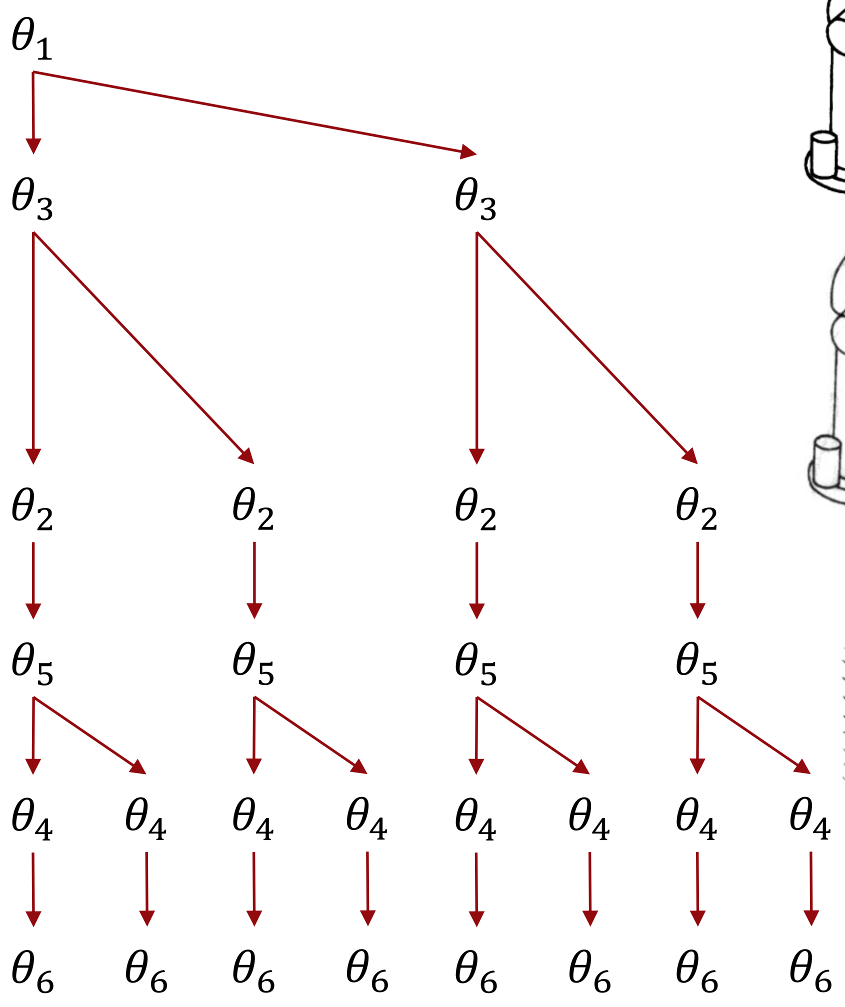
4 $\theta_2 = \theta_{23} - \theta_3$

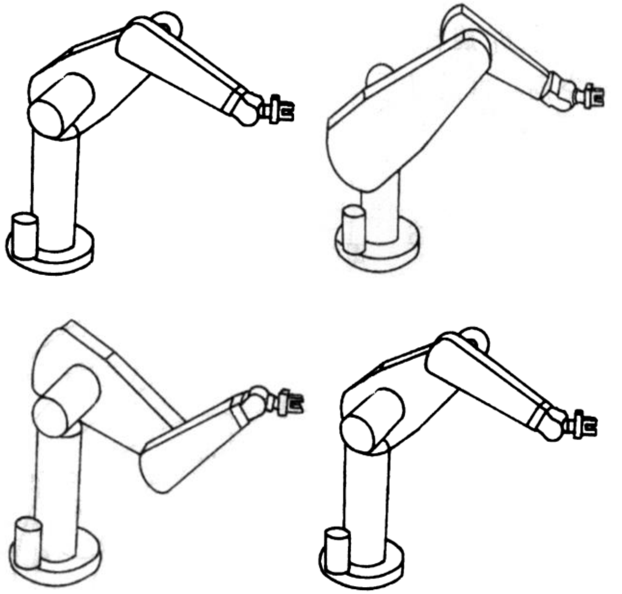
5 $\theta_5 = \arccos k'$ 或 $\theta_5 = -\arccos k'$

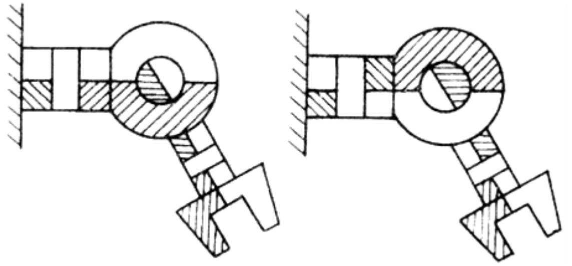
6 $\theta_4 = \mathrm{arctan2}(\mathrm{Sin}[\theta_4]\mathrm{Sin}[\theta_5], \mathrm{Cos}[\theta_4]\mathrm{Sin}[\theta_5])$

7 $\theta_6 = \mathrm{arctan2}(\mathrm{Sin}[\theta_4]\mathrm{Sin}[\theta_5], \mathrm{Cos}[\theta_4]\mathrm{Sin}[\theta_5])$

PUMA机器人逆运动学的8个解







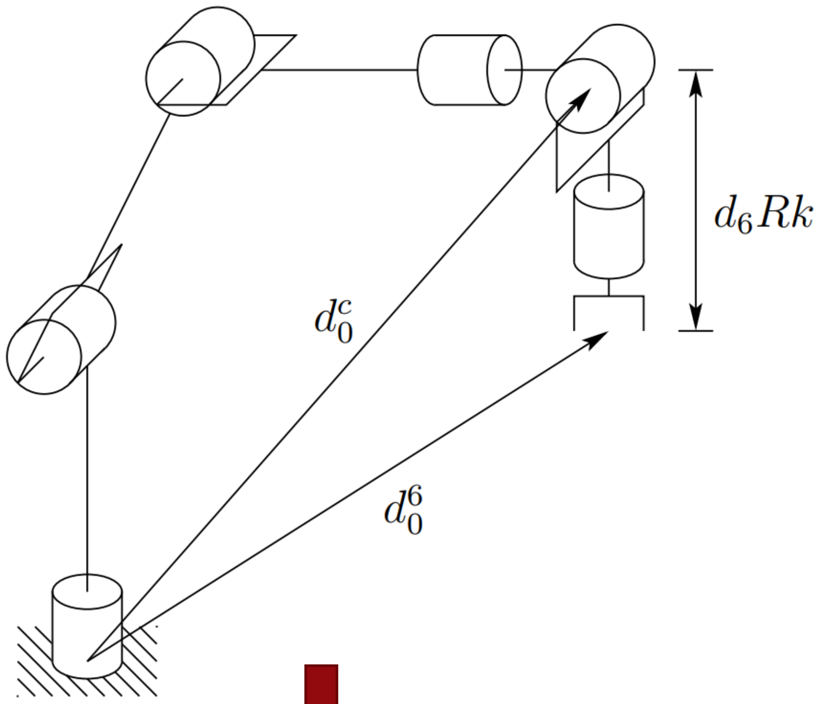
若$\rho < L_3$或$k > 1$或$k' > 1$，无解

目标点超出了机器人工作空间

- PUMA机器人逆运动学求解总结：
  - 求解目标为腕部在基座坐标系(Z_0处)的位置姿态与关节空间的映射
  - PUMA机器人末端4，5，6关节相交于一点——封闭解的充分条件
    - ✓ 机器人**腕部坐标系**的**位置仅由**1，2，3关节有关：逆运动学的起点
    - ✓ 末端4，5，6关节相交于一点，可以形成任意姿态
  - PUMA机器人逆运动学存在8个解，如何取舍
    - ✓ 能量最小：加权，最后一个关节连接的质量越小
    - ✓ 规避障碍物
  - PUMA机器人逆运动学解的奇异性
    - ✓ 退化奇异：当 $\mathrm{Sin}[\theta_5] = 0$时，机器人丢失自由度
  - PUMA机器人逆运动学求解的存在性
    - ✓ 工作空间：若$\sqrt{p_x^2 + p_y^2} > L_3$且$k < 1$且$k' < 1$

2.1 机器人逆运动学求取：代数法

- **对于末端为3自由度球形关节的6自由度机械臂**
- **Decouple the inverse kinematics into two simpler problems: inverse position kinematics, and inverse orientation kinematics**



已知 $H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix}$

$$R_6^0(q_1, \ldots, q_6) = R$$
$$o_6^0(q_1, \ldots, q_6) = o$$

$$o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2.1 机器人逆运动学求取：代数法

由 $\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$ 求解前3个关节变量

期望手腕位置

球形手腕坐标系原点只与前3各关节有关

期望工具位置

Step 1: Find q_1, q_2, q_3 such that the wrist center o_c has coordinates given by

$$o_c^0 = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

由手腕关节运动学，求解后3个关节角

$$R = R_3^0 R_6^3$$

$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R.$$

Step 2: Using the joint variables determined in Step 1, evaluate R_3^0.

Step 3: Find a set of Euler angles corresponding to the rotation matrix

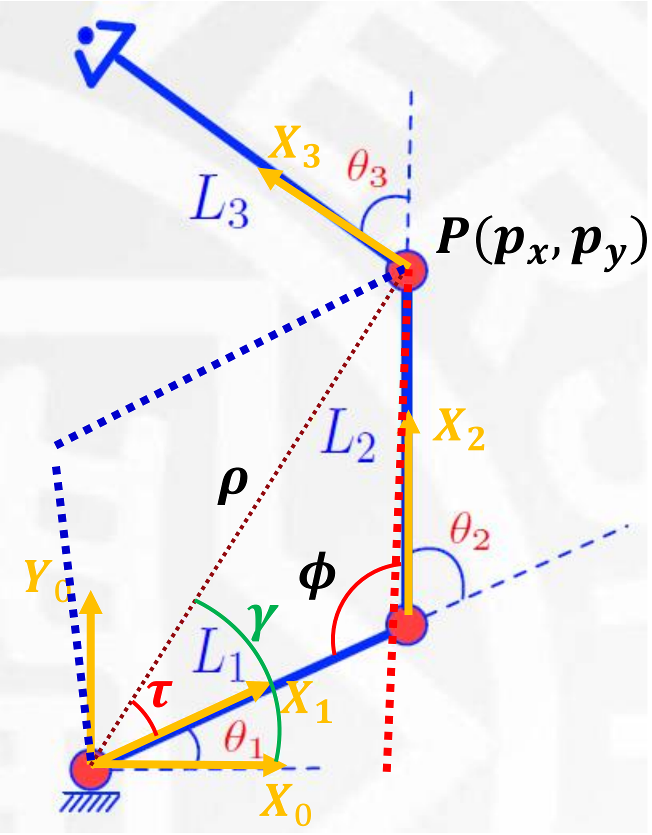
$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R.$$

2.1 机器人逆运动学求取：几何法--平面RRR机器人举例《机器人学导论》

DH参数表

关节	α_{i-1}	$\vec{a}_{i-1}$	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

已知 $${}^0_3T = \begin{pmatrix} r11 & r12 & r13 & px \\ r21 & r22 & r23 & py \\ r31 & r32 & r33 & pz \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



$$\rho = \sqrt{p_x^2 + p_y^2}$$

$$\phi = \arccos\frac{(L_1^2 + L_2^2) - \rho^2}{2L_1L_2}$$

$$\boxed{\theta_2 = \pi - \phi}$$

$$\boxed{-\theta_2 = \pi - \phi \Longrightarrow \theta_2 = \phi - \pi}$$

$$\tau = \arccos\frac{(L_1^2 + \rho^2) - L_2^2}{2L_1\rho}$$

$$\gamma = \arctan 2(p_y, p_x)$$

$$\boxed{\theta_1 = \gamma - \tau}$$

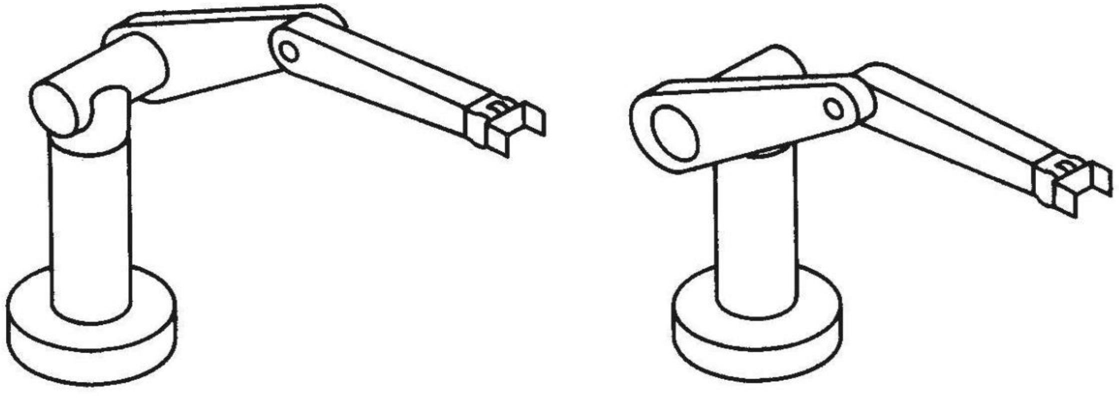
$$\boxed{\theta_1 = \gamma + \tau}$$

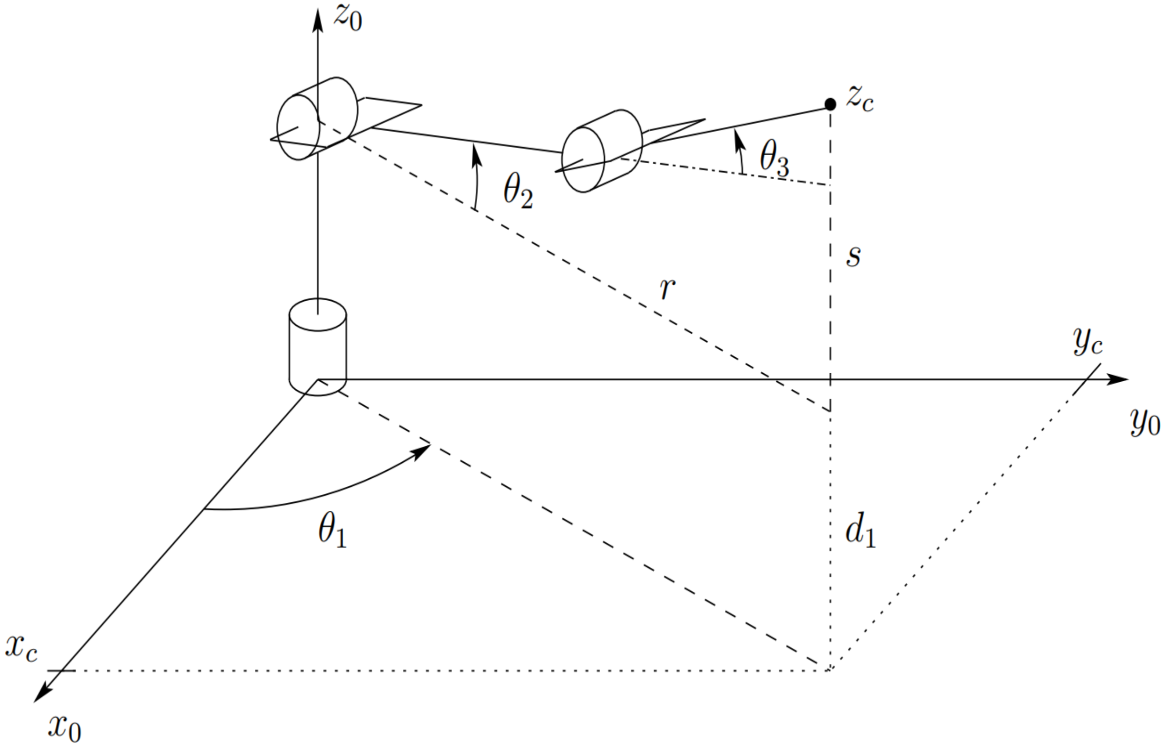
$$\boxed{\theta_3 = \arctan2(r_{21}, r_{11}) - \theta_1 - \theta_2}$$

2.1 机器人逆运动学求取：几何法

$$\theta_1 = A\tan(x_c, y_c),$$

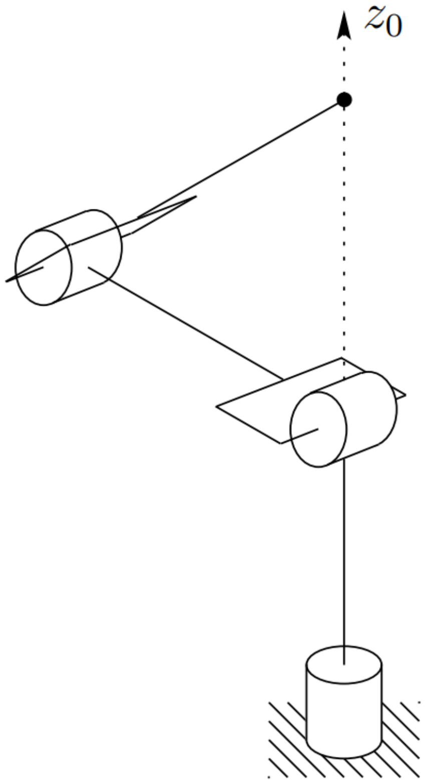
$$\theta_1 = \pi + A\tan(x_c, y_c)$$



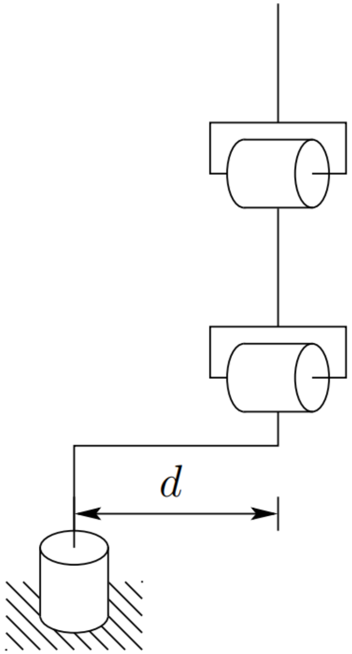


奇异点：$x_c=y_c=0$

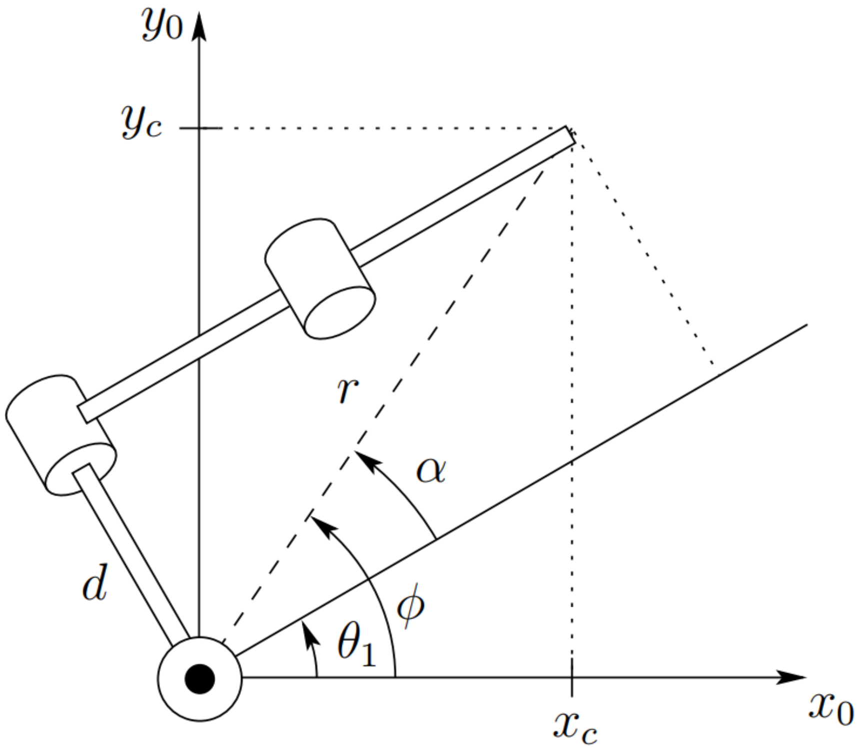
wrist center o_c intersects z_0; hence any value of ϑ_1 leaves o_c fixed. There are thus infinitely many solutions for ϑ_1 when o_c intersects z_0.

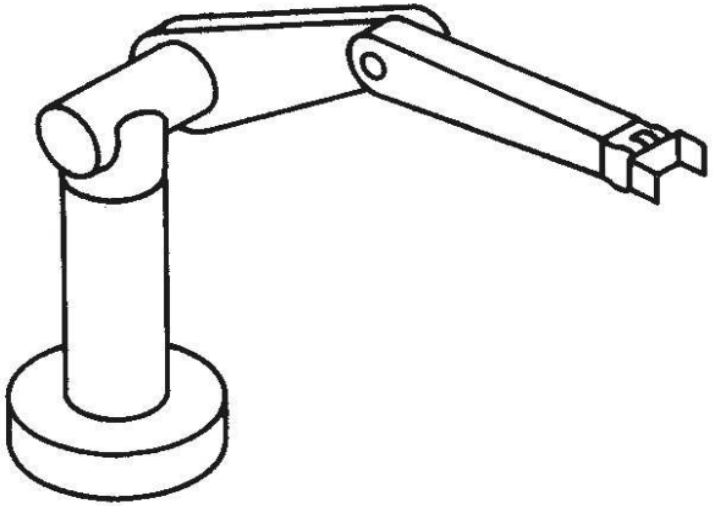


2.1 机器人逆运动学求取：几何法



Elbow manipulator with shoulder offset





LEFT and ABOVE Arm

$$\theta_1 = \phi - \alpha$$

$$\begin{aligned}\phi &= A\tan(x_c, y_c) \\ \alpha &= A\tan\left(\sqrt{r^2 - d^2}, d\right) \\ &= A\tan\left(\sqrt{x_c^2 + y_c^2 - d^2}, d\right)\end{aligned}$$

2.1 机器人逆运动学求取：几何法

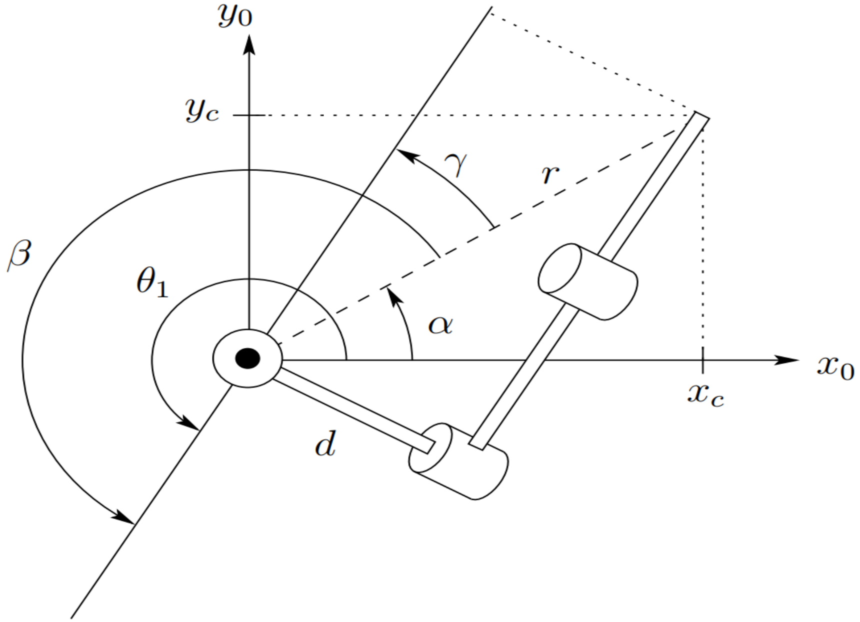
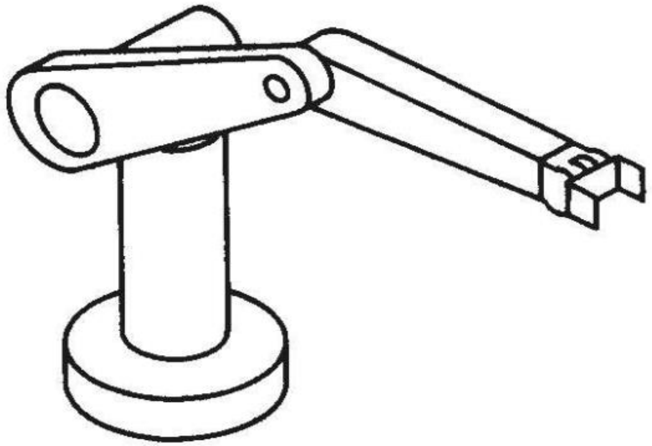


Figure 4.7: Right arm configuration.



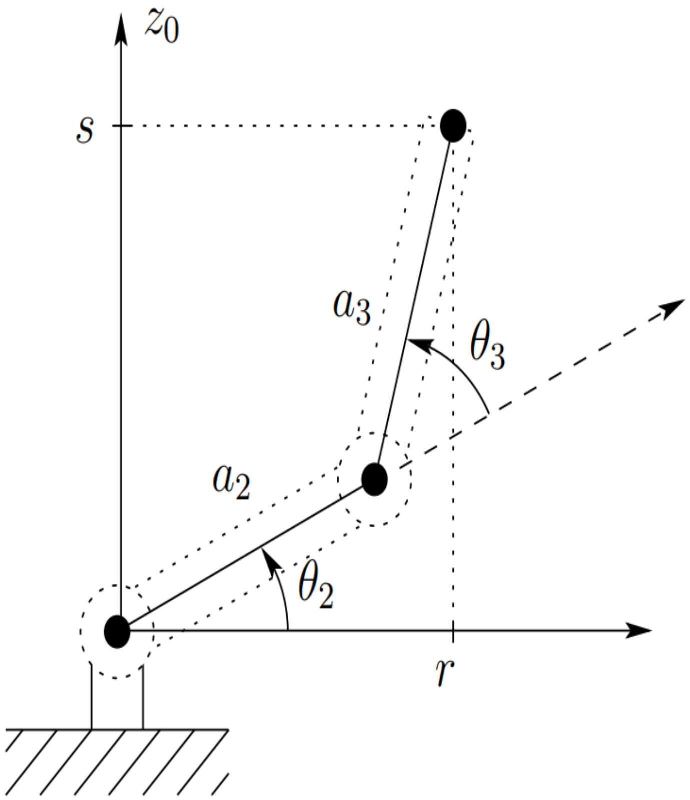
RIGHT and ABOVE Arm

$$\theta_1 = A\tan(x_c, y_c) + A\tan\left(-\sqrt{r^2 - d^2}, -d\right)$$

$$\begin{aligned} \theta_1 &= \alpha + \beta \\ \alpha &= A\tan(x_c, y_c) \\ \beta &= \gamma + \pi \\ \gamma &= A\tan(\sqrt{r^2 - d^2}, d) \end{aligned}$$

$$\beta = A\tan\left(-\sqrt{r^2 - d^2}, -d\right)$$

2.1 机器人逆运动学求取：几何法



$$
\begin{aligned}
\cos\theta_3 &= \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3} \\
&= \frac{x_c^2 + y_c^2 - d^2 + z_c^2 - a_2^2 - a_3^2}{2a_2a_3} := D,
\end{aligned}
$$

since $r^2 = x_c^2 + y_c^2 - d^2$ and $s = z_c$. Hence, θ_3 is given by

$$
\theta_3 = A\tan\left(D, \pm\sqrt{1 - D^2}\right).
$$

Similarly θ_2 is given as

$$
\begin{aligned}
\theta_2 &= A\tan(r, s) - A\tan(a_2 + a_3c_3, a_3s_3) \\
&= A\tan\left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c\right) - A\tan(a_2 + a_3c_3, a_3s_3)
\end{aligned}
$$

2.1 机器人逆运动学求取：几何法

对于球形手腕的三个角度，采用欧拉角旋转变换公式

期望姿态变换到手腕坐标系

The matrix $R_6^3 = A_4 A_5 A_6$ *is given as*

$$R_6^3 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix}. \quad (4.37)$$

The equation to be solved now for the final three variables is therefore

$$R_6^3 = (R_3^0)^T R \quad (4.38)$$

and the Euler angle solution can be applied to this equation. For example, the three equations given by the third column in the above matrix equation are given by

$$c_4s_5 = c_1c_{23}r_{13} + s_1c_{23}r_{23} + s_{23}r_{33} \quad (4.39)$$

$$s_4s_5 = -c_1s_{23}r_{13} - s_1s_{23}r_{23} + c_{23}r_{33} \quad (4.40)$$

$$c_5 = s_1r_{13} - c_1r_{23}. \quad (4.41)$$

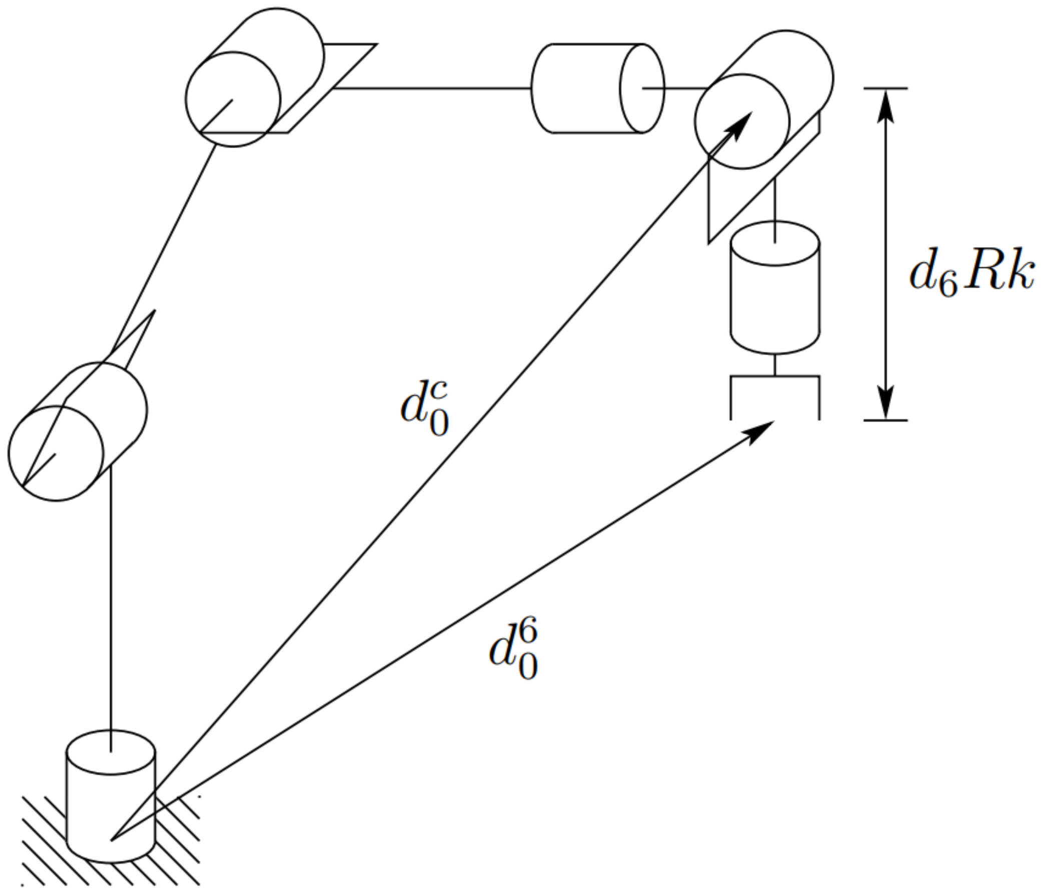
Hence, if not both of the expressions ($\boxed{4.39}$*), (*$\boxed{4.40}$*) are zero, then we obtain* θ_5

$$\theta_5 = A\tan\left(s_1r_{13} - c_1r_{23}, \pm\sqrt{1 - (s_1r_{13} - c_1r_{23})^2}\right). \quad (4.42)$$

If the positive square root is chosen in ($\boxed{4.42}$*), then* θ_4 *and* θ_6 *are given by*

$$\theta_4 = A\tan(c_1c_{23}r_{13} + s_1c_{23}r_{23} + s_{23}r_{33}, -c_1s_{23}r_{13} - s_1s_{23}r_{23} + c_{23}r_{33}) \quad (4.43)$$

$$\theta_6 = A\tan(-s_1r_{11} + c_1r_{21}, s_1r_{12} - c_1r_{22}). \quad (4.44)$$



If $s_5 = 0$, *this is a singular configuration and only the sum* $\theta_4 + \theta_6$ *can be determined. One solution is to choose* θ_4 *arbitrarily and then determine* θ_6.

2.2 PUMA机器人逆运动学：几何法

- PUMA机器人逆运动学几何法求解总结：
  - 求解目标为腕部在基座坐标系(Z_0处)的位置姿态与关节空间的映射
  - PUMA机器人末端4，5，6关节相互垂直且交于一点
    - ✓ 机器人腕部坐标系的位置仅由1，2，3关节有关
    - ✓ 末端4，5，6关节相交于一点，可以形成任意姿态
  - 关节3大小决定腕部到基坐标原点距离——余弦定理
  - 关节1决定机械臂在XOY平面投影的方向
  - 关节2、关节3与腕部Z方向高度和腕部到基坐标原点距离相关
  - 几何法就是通过几何关系不考虑正运动学，解出对应的三个关节角，进而将原来6维的未知量降维至3维未知量

课程内容

1. 机器人逆运动学基本概念

2. 机器人逆运动学求取方法

3. 机器人工作空间分析

3.1 机器人逆运动学解的存在性

- 工作空间W：Workspace is that volume of space that the end-effector of the manipulator can reach.
- 可达空间RW：The reachable workspace is that volume of space that the robot can reach in at least one orientation.
- 灵巧空间DW：Dextrous workspace is that volume of space that the robot end-effector can reach with all orientations.
- 注意：

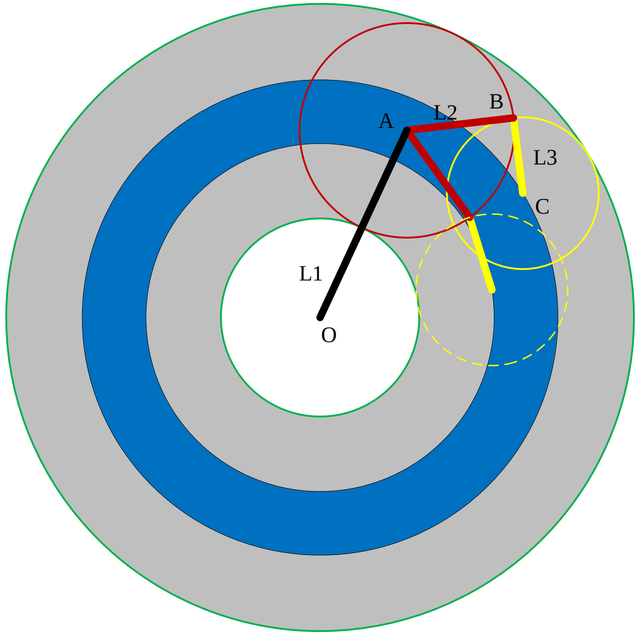
$$\boxed{DW \subset RW = W}$$

灵巧空间指的是末端执行器的位置，不是机械臂最后关节坐标系的原点

3.2 机器人灵巧空间

■ **灵巧空间**：**Dextrous workspace** is that volume of space that the robot end-effector can reach with all orientations.

灵巧空间特指的是**末端执行器**的位置，不是DH法中的最后关节坐标系的原点



- 以平面3R机器人的灵巧空间分析为例，其3个连杆的端点为OABC。
- 平面3R机器人在平面上的灵巧空间就是末端执行器能以360°姿态逼近C点的区域。
- 而这样的C点等同于第二连杆末端B的可达空间（图中灰色区域）包围以C点为圆心以第三连杆长度L3为半径的圆周即可。根据连杆长度的不同，分别讨论如下：

情况1：只有圆环状灵巧空间

条件：$2L_3 \le L_1 + L_2 - |L_1 - L_2|$且$L_3 \le |L_1 - L_2|$; 等号成立时圆环退化为圆周

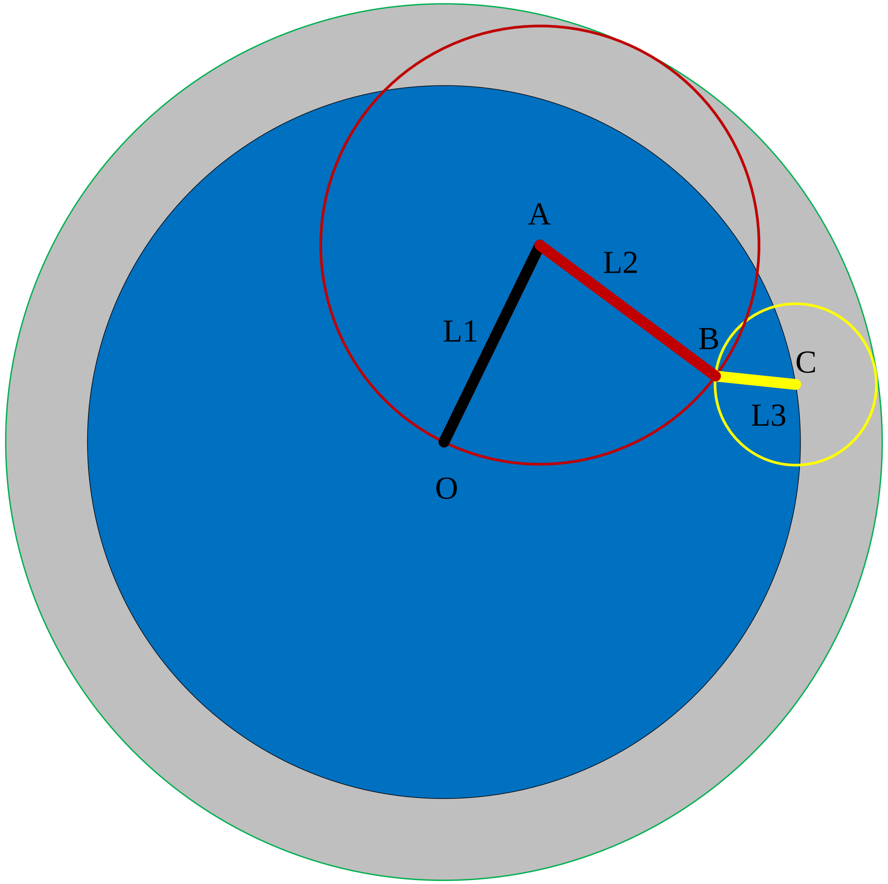
即：图中黄色圆比中央绿色圆小但能完全落到灰色环状带内的状态

3.3 3R操作臂灵巧空间的详细分析

情况2：当$L_1 = L_2$时，中心绿圆消失，出现圆形灵巧空间

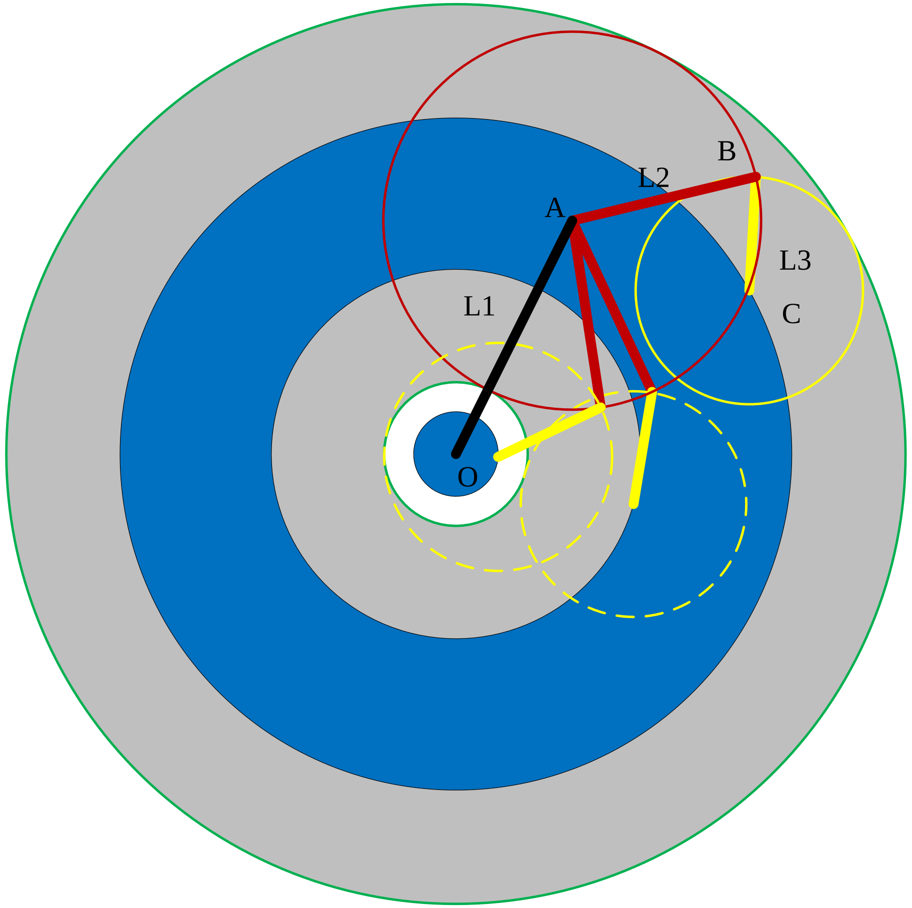
条件：$L_3 \leq L_1 + L_2$

等号成立时圆形退化为一点



注意黄圈变大的过程，变大过程中出现了各种情况，最后大到灵巧空间消失。

3.3 3R操作臂灵巧空间的详细分析



情况3：同时有圆环状和中心圆状灵巧空间

条件：$2L_3 \le L_1 + L_2 - |L_1 - L_2|$且$L_3 \ge |L_1 - L_2|$；等号成立时圆环退化为圆周，中心圆退化为一点

即：图中黄色圆比中央绿色圆大同时能完全落到灰色环状带内的状态

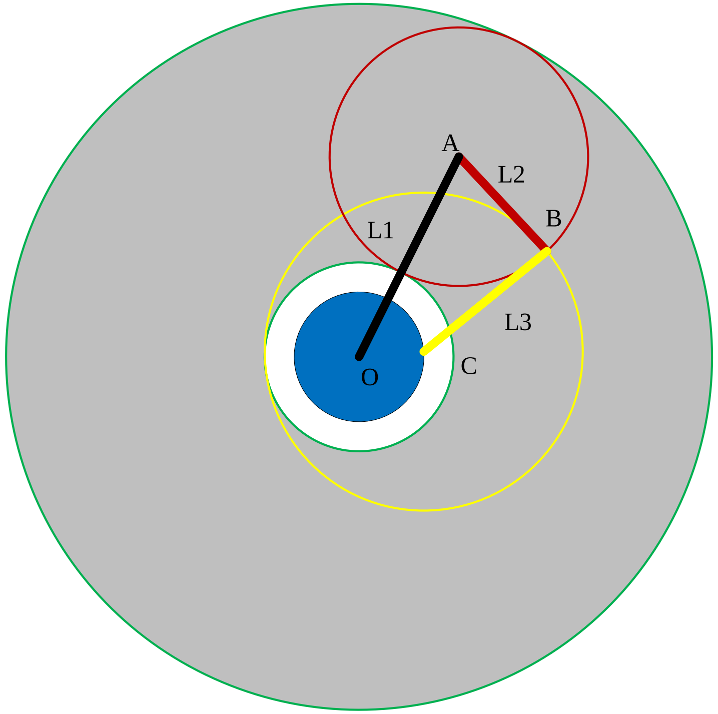
3.3 3R操作臂灵巧空间的详细分析

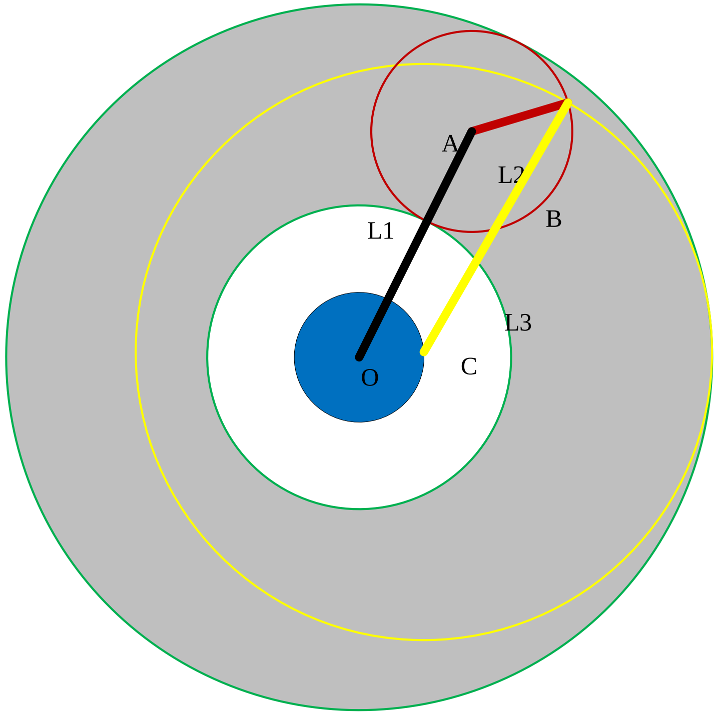
情况4：灵巧空间只有中心圆，此时又分两种状态，即黄色圆与内侧绿色圆外切和与外侧绿色圆内切两种，如左右两图

条件：$|L_1 - L_2| \leq L_3 \leq L_1 + L_2$且$2L_3 \geq L_1 + L_2 - |L_1 - L_2|$

等号成立时中心圆退化为一点

即：图中黄色圆比小绿色圆大，且直径超过灰色环状带宽度的状态



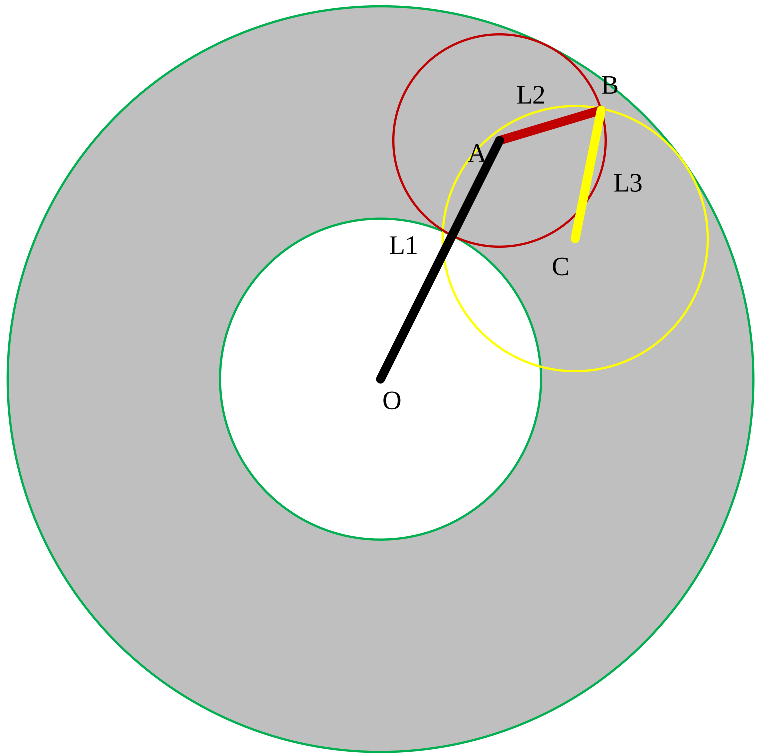


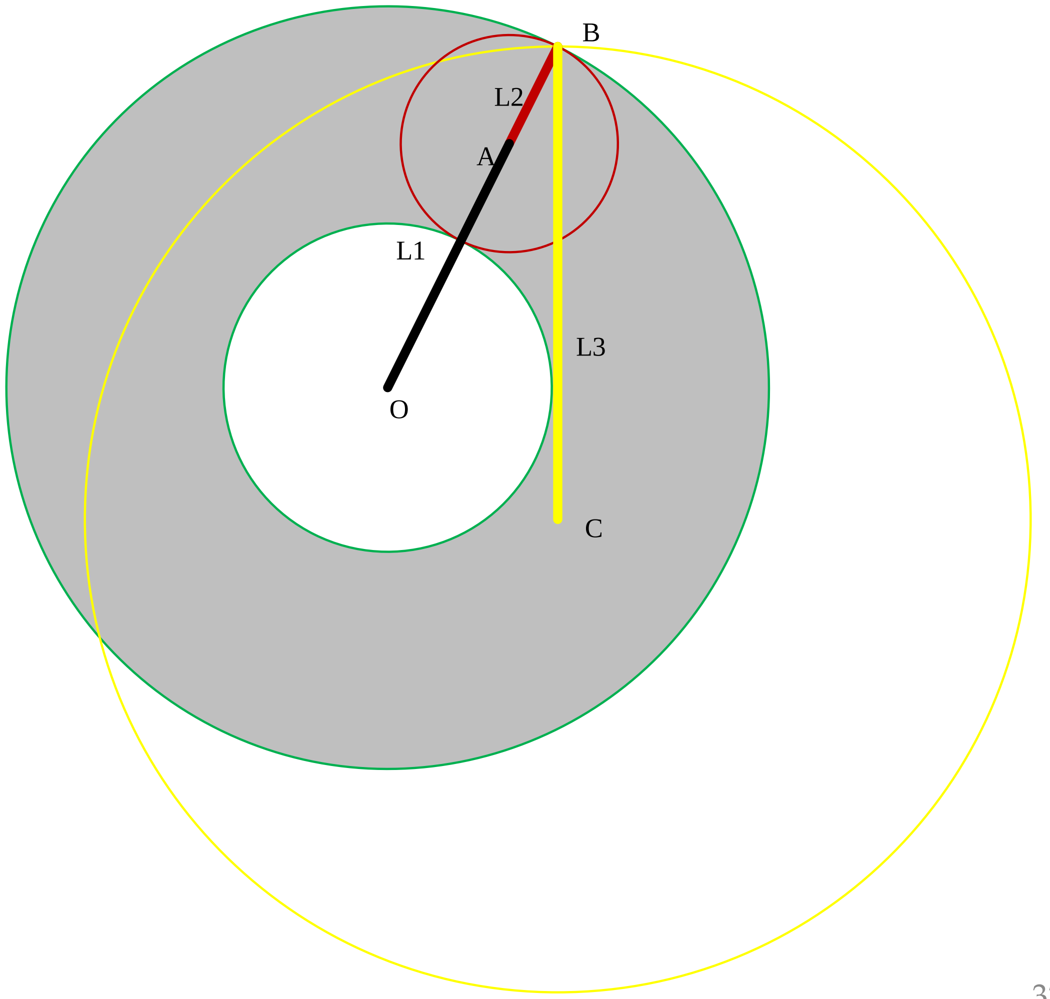
3.3 3R操作臂灵巧空间的详细分析

情况5：没有灵巧空间，有两种状态；

左图是由于黄色的圆比中心绿色圆小又不能完全落在灰色环状带内。

右图是由于黄色圆太大，无法完整落在在灰色环内



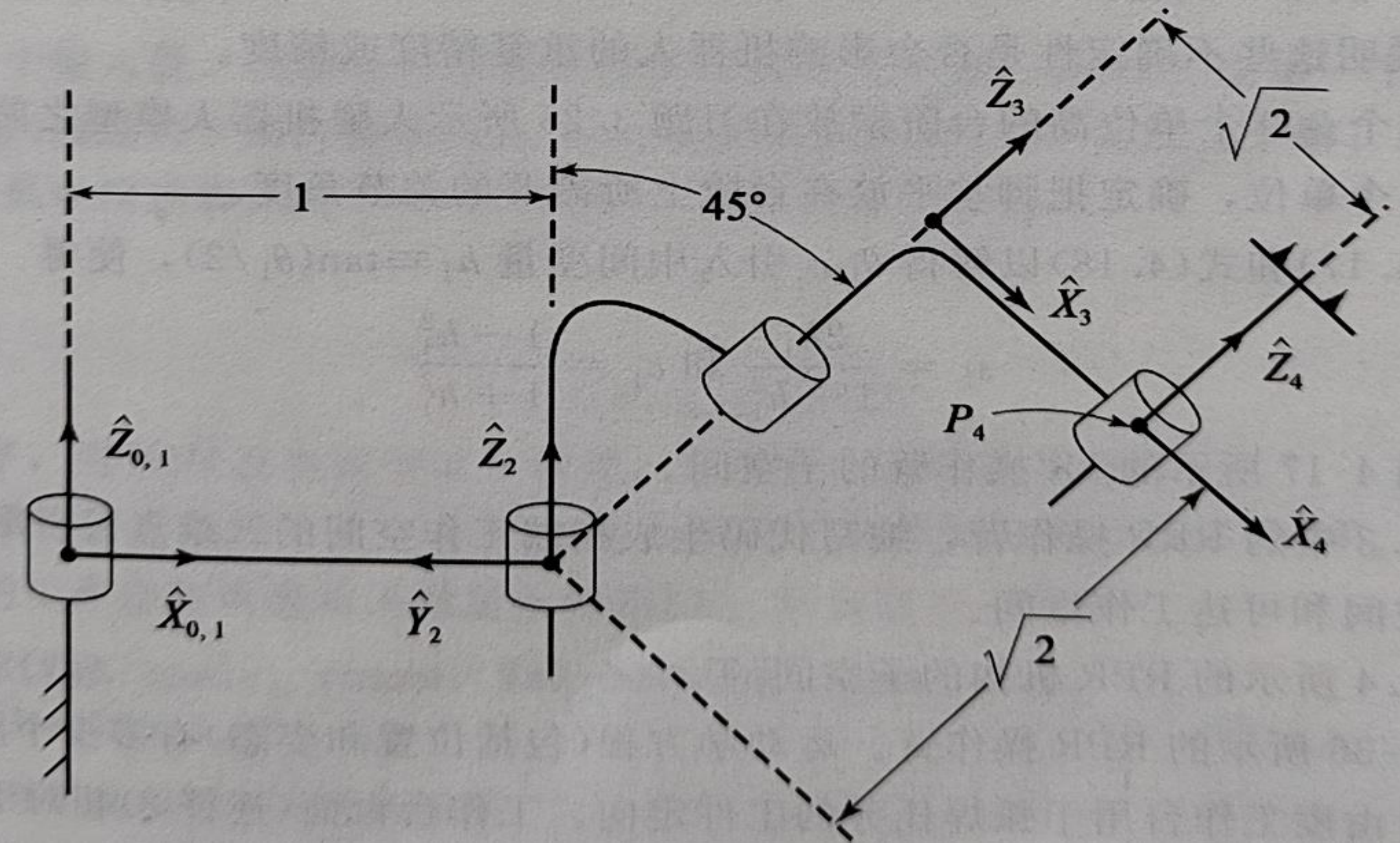


作 业

4.16 [25]图 4-15 所示为一个 4R 操作臂，非零连杆参数为 $a_1=1$，$\alpha_2=45°$，$d_3=\sqrt{2}$ 和 $a_3=\sqrt{2}$，这个机构的位形为 $\Theta=(0, 90°, -90°, 0)^T$，每个关节的运动范围为 $\pm180°$，对于

$$ {}^0P_{4ORG}=(1.1, 1.5, 1.707)^T $$

~~求所有 θ_3 的值。~~求正、逆运动学，分析多解和奇异情况



作 业

■ 求解如图所示4R机器人的正运动学模型

关节	α_{i-1}	$\vec{a}_{i-1}$	d_i	θ_i
1	0	0	0	θ_1
2	0	1	0	θ_2
3	45^0	0	$\sqrt{2}$	θ_3
4	0	$\sqrt{2}$	0	θ_4

分别求出 ${}^0_1T = [\quad]$, ${}^1_2T = [\quad]$, ${}^2_3T = [\quad]$, ${}^3_4T = [\quad]$

求 ${}^0_4T = [\quad]$

然后依次求 θ_3、 θ_4、 θ_1、 θ_2

End of Chapter-4

作业

■ 求解如图所示4R机器人的正运动学模型

关节	α_{i-1}	$\vec{a}_{i-1}$	d_i	θ_i
1	0	0	0	θ_1
2	0	1	0	θ_2
3	45^0	0	$\sqrt{2}$	θ_3
4	0	$\sqrt{2}$	0	θ_4

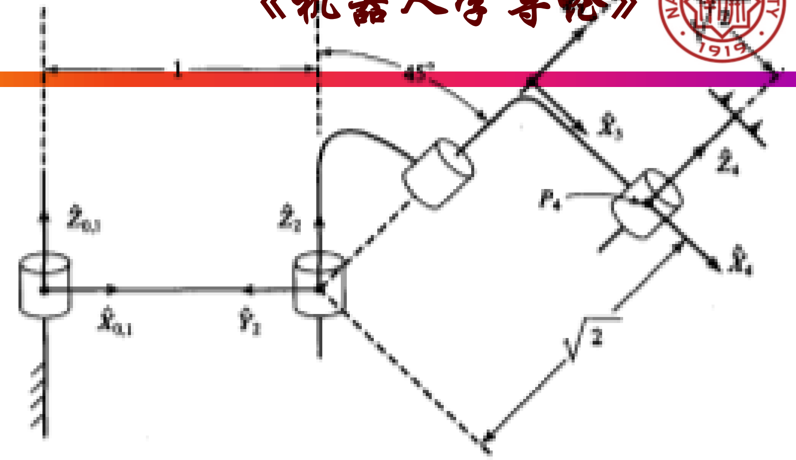


图4-15 4R操作臂，图示位置Θ=[0, 90°, −90°, 0]ᵀ（习题4.16）

$$
{}^0_1T=\begin{bmatrix} c_1 & -s_1 & 0 & 0\\ s_1 & c_1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad
{}^1_2T=\begin{bmatrix} c_2 & -s_2 & 0 & 1\\ s_2 & c_2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad
{}^2_3T=\begin{bmatrix} c_3 & -s_3 & 0 & 0\\ \frac{\sqrt{2}}{2}s_3 & \frac{\sqrt{2}}{2}c_3 & \frac{\sqrt{2}}{2} & 1\\ -\frac{\sqrt{2}}{2}s_3 & -\frac{\sqrt{2}}{2}c_3 & \frac{\sqrt{2}}{2} & 1\\ 0 & 0 & 0 & 1\end{bmatrix},\quad
{}^3_4T=\begin{bmatrix} c_4 & -s_4 & 0 & \sqrt{2}\\ s_4 & c_4 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

$$
{}^0_6T=\begin{bmatrix}
c_{12}c_{34}-\frac{\sqrt{2}}{2}s_{12}s_{34} & -c_{12}s_{34}-\frac{\sqrt{2}}{2}s_{12}c_{34} & \frac{\sqrt{2}}{2}s_{12} & c_1+s_{12}+\sqrt{2}c_{12}c_3-s_{12}s_3\\
s_{12}c_{34}+\frac{\sqrt{2}}{2}c_{12}s_{34} & -s_{12}s_{34}+\frac{\sqrt{2}}{2}c_{12}c_{34} & -\frac{\sqrt{2}}{2}c_{12} & s_1-c_{12}+\sqrt{2}s_{12}c_3+c_{12}s_3\\
\frac{\sqrt{2}}{2}s_{34} & \frac{\sqrt{2}}{2}c_{34} & \frac{\sqrt{2}}{2} & s_3+1\\
0 & 0 & 0 & 1
\end{bmatrix}
$$

作业

- 求解如图所示4R机器人逆运动学模型，注意多解和奇异

$$
{}^0_6T=\begin{bmatrix}
c_{12}c_{34}-\frac{\sqrt{2}}{2}s_{12}s_{34} & -c_{12}s_{34}-\frac{\sqrt{2}}{2}s_{12}c_{34} & \frac{\sqrt{2}}{2}s_{12} & c_1+s_{12}+\sqrt{2}c_{12}c_3-s_{12}s_3\\
s_{12}c_{34}+\frac{\sqrt{2}}{2}c_{12}s_{34} & -s_{12}s_{34}+\frac{\sqrt{2}}{2}c_{12}c_{34} & -\frac{\sqrt{2}}{2}c_{12} & s_1-c_{12}+\sqrt{2}s_{12}c_3+c_{12}s_3\\
\frac{\sqrt{2}}{2}s_{34} & \frac{\sqrt{2}}{2}c_{34} & \frac{\sqrt{2}}{2} & s_3+1\\
0 & 0 & 0 & 1
\end{bmatrix}=\begin{bmatrix}
r_{11} & r_{12} & r_{13} & p_x\\
r_{21} & r_{22} & r_{23} & p_y\\
r_{31} & r_{32} & r_{33} & p_z\\
0 & 0 & 0 & 1
\end{bmatrix}
$$

由 $\boxed{\quad\quad\quad}$ 所示的对应位置可得：

$$\theta_3=\arcsin(p_z-1)\ \text{或}\ \theta_3=\pi-\arcsin(p_z-1)$$

由 $\boxed{\quad\quad\quad}$ 所示的对应位置可得：

$$\theta_4=\arctan2(r_{31},r_{32})-\theta_3$$

由 $\boxed{\quad\quad\quad}$ 所示的对应位置可得：

$$\theta_1=\arctan2(p_y-\sqrt{2}r_{23}-2r_{13}c_3+\sqrt{2}r_{23}(p_z-1),p_x-\sqrt{2}r_{13}+2r_{23}c_3+\sqrt{2}r_{13}(p_z-1))$$

由 $\boxed{\quad\quad\quad}$ 所示的对应位置可得：

$$\theta_2=\arctan2(r_{13},-r_{23})-\theta_1$$