第五章 机器人速度与雅可比

Chapter 5 Instantaneous Kinematics

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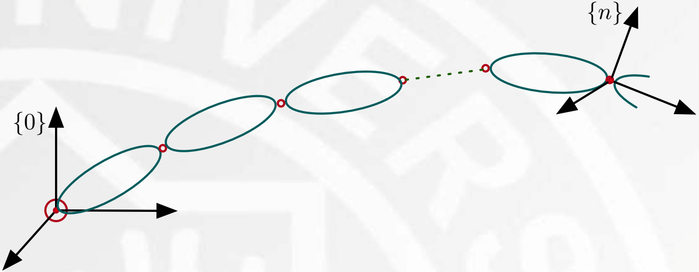
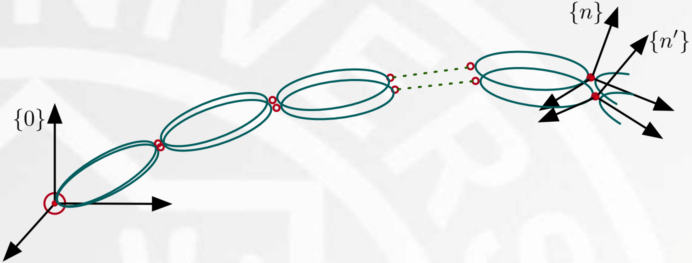
Differential Motion



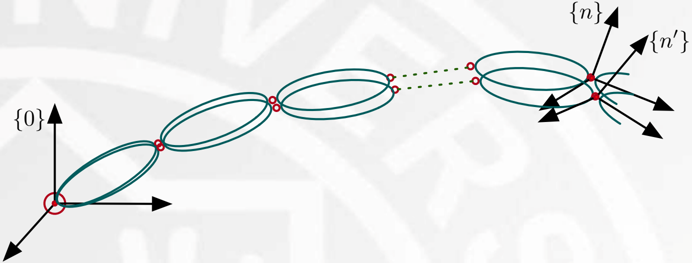
- Forward Kinematics: $\theta \rightarrow x$

Differential Motion



- Forward Kinematics: $\theta \to x$
- Instantaneous Kinematics:

Differential Motion



- Forward Kinematics: $\theta \rightarrow x$
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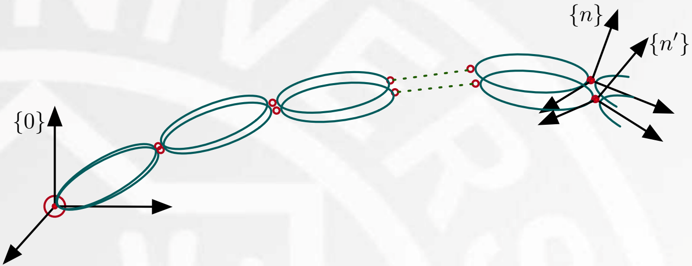
Differential Motion



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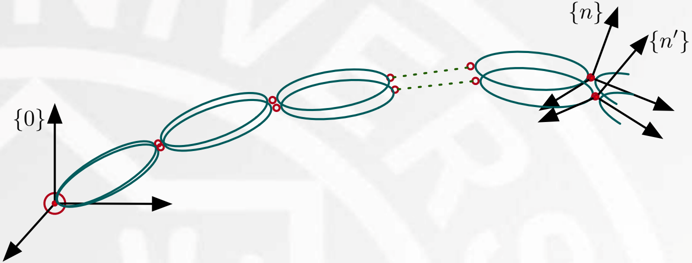
- Relationship: $\boxed{\delta\theta \leftrightarrow \delta x}$

Differential Motion



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- Relationship: $\boxed{\delta\theta \leftrightarrow \delta x}$

$$\boxed{\dot{\theta} \leftrightarrow \dot{x}} \begin{cases} \text{Linear Velocity} \\ \text{Angular Velocity} \end{cases}$$

Jacobian

- Differential Motion
- Linear & Angular Motion
- Velocity Propagation
- Explicit Form
- Static Forces

Joint Coordinates

$$\text{coordinate-}i : \begin{cases} \theta_i & \text{Revolute joint} \\ d_i & \text{Prismatic joint} \end{cases}$$

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$$\text{with} \quad \varepsilon_i = \begin{cases} 0 & \text{Revolute joint} \\ 1 & \text{Prismatic joint} \end{cases}$$

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Joint Coordinate Vector: $$\boxed{q = (q_1, \; q_2, \; \cdots, \; q_n)}$$

Jacobians: Direct Differentiation

$$x = f(q);$$

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$$\begin{aligned} \delta x_1 &= \frac{\partial f_1}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_1}{\partial q_n}\delta q_n \\ &\vdots \\ \delta x_m &= \frac{\partial f_m}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_m}{\partial q_n}\delta q_n \end{aligned}$$

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$$\vdots$$

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$$\delta x = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix} \delta q$$

$$\boxed{\delta x_{(m\times 1)} = J_{(m\times n)}(q)\delta q_{(n\times 1)}}$$

Jacobian

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Jacobian

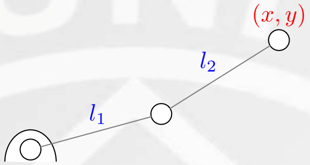
$$\delta x_{(m\times 1)} = J_{(m\times n)}(q)\delta q_{(n\times 1)}$$

$$\dot{x}_{(m\times 1)} = J_{(m\times n)}(q)\dot{q}_{(n\times 1)}$$

where

$$\boxed{J_{ij}(q) = \frac{\partial}{\partial q_j} f_i(q)}$$

Example



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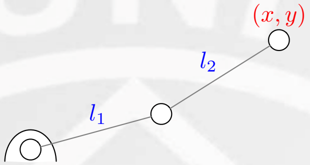
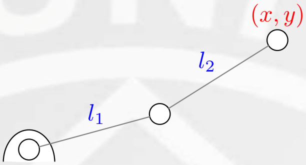


$$x = l_1 c_1 + l_2 c_{12}$$
$$y = l_1 s_1 + l_2 s_{12}$$

Example



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$$\delta x = -(l_1 s_1 + l_2 s_{12})\delta\theta_1 - l_2 s_{12}\delta\theta_2$$
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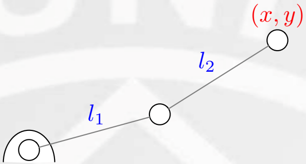
Example



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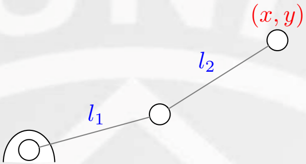
$$\boxed{\delta X = J(\theta)\delta\theta}$$

Example



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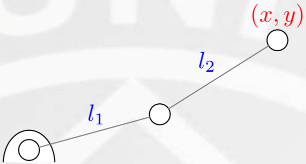

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$$\boxed{\delta X = J(\theta)\delta\theta}$$

$$\dot{X} = J(\theta)\dot{\theta}$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial\theta_1} & \frac{\partial x}{\partial\theta_2} \\ \frac{\partial y}{\partial\theta_1} & \frac{\partial y}{\partial\theta_2} \end{bmatrix} = \begin{bmatrix} -y & -l_2 s_{12} \\ x & l_2 c_{12} \end{bmatrix}$$

Stanford Scheinman Arm

Stanford Scheinman Arm

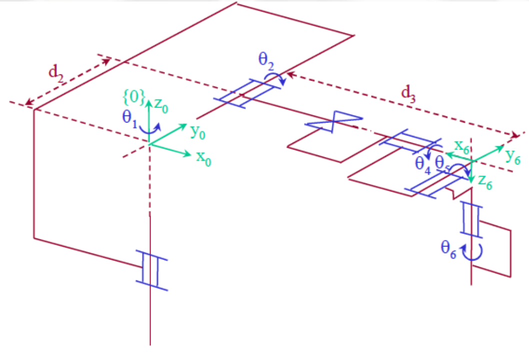

Table: DH 参数

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	d_2	θ_2
3	90	0	d_3	0
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6

Stanford Scheinman Arm

$$X = \begin{bmatrix} X_P \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} d_3C_1S_2 - d_2S_1 \\ d_3S_1S_2 + d_2C_1 \\ d_3C_2 \\ C_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] - S_1(S_4C_5C_6 + C_4S_6) \\ S_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] + C_1(S_4C_5C_6 + C_4S_6) \\ -S_2(C_4C_5C_6 - S_4S_6) - C_2S_5C_6 \\ C_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] - S_1(-S_4C_5S_6 + C_4C_6) \\ S_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6) + C_1(-S_4C_5S_6 + C_4C_6) \\ S_2(C_4C_5S_6 + S_4C_6) + C_2S_5S_6 \\ C_1(C_2C_4S_5 + S_2C_5) - S_1S_4S_5 \\ S_1(C_2C_4S_5 + S_2C_5) + C_1S_4S_5 \\ -S_2C_4S_5 + C_2C_5 \end{bmatrix}$$

Stanford Scheinman Arm

Poistion:

$$X_P = \begin{bmatrix} d_3C_1S_2 - d_2S_1 \\ d_3S_1S_2 + d_2C_1 \\ d_3C_2 \end{bmatrix}$$

$$\dot{X}_P = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$

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Stanford Scheinman Arm

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Stanford Scheinman Arm

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Stanford Scheinman Arm

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$$\dot{X}_{P(3\times1)} = J_{X_{P(3\times6)}}(q)\dot{q}_{(6\times1)}$$

Stanford Scheinman Arm

Poistion:

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$$\boxed{\dot{X}_{P(3\times1)}} = J_{X_{P(3\times6)}}(q)\dot{q}_{(6\times1)}$$

Linear Velocity V

Orientation: Direction Cosines

$$X_R = \begin{bmatrix} r_1(q) \\ r_2(q) \\ r_3(q) \end{bmatrix}$$

$$\dot{X}_R = J_{X_R}(q)\dot{q}$$

Orientation: Direction Cosines

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$$\dot{X}_R = \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{bmatrix}_{(9\times1)} = \begin{bmatrix} \frac{\partial r_1}{\partial q_1} & \frac{\partial r_1}{\partial q_2} & \cdots & \frac{\partial r_1}{\partial q_6} \\ \frac{\partial r_2}{\partial q_1} & \frac{\partial r_2}{\partial q_2} & \cdots & \frac{\partial r_2}{\partial q_6} \\ \frac{\partial r_3}{\partial q_1} & \frac{\partial r_3}{\partial q_2} & \cdots & \frac{\partial r_3}{\partial q_6} \end{bmatrix}_{(9\times6)} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}_{(6\times1)}$$

Stanford Scheinman Arm

$$X_R = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} C_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] - S_1(S_4C_5C_6 + C_4S_6) \\ S_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] + C_1(S_4C_5C_6 + C_4S_6) \\ -S_2(C_4C_5C_6 - S_4S_6) - C_2S_5C_6 \\ C_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] - S_1(-S_4C_5S_6 + C_4C_6) \\ S_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6) + C_1(-S_4C_5S_6 + C_4C_6) \\ S_2(C_4C_5S_6 + S_4C_6) + C_2S_5S_6 \\ C_1(C_2C_4S_5 + S_2C_5) - S_1S_4S_5 \\ S_1(C_2C_4S_5 + S_2C_5) + C_1S_4S_5 \\ -S_2C_4S_5 + C_2C_5 \end{bmatrix}$$

$$\dot{X}_R = \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{bmatrix}_{(9\times1)} = \begin{bmatrix} \frac{\partial r_1}{\partial q_1} & \frac{\partial r_1}{\partial q_2} & \dots & \frac{\partial r_1}{\partial q_6} \\ \frac{\partial r_2}{\partial q_1} & \frac{\partial r_2}{\partial q_2} & \dots & \frac{\partial r_2}{\partial q_6} \\ \frac{\partial r_3}{\partial q_1} & \frac{\partial r_3}{\partial q_2} & \dots & \frac{\partial r_3}{\partial q_6} \end{bmatrix}_{(9\times6)} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}_{(6\times1)}$$

Representations

$$X = \begin{bmatrix} X_P \\ X_R \end{bmatrix}$$

Representations

- Cartesian
- Spherical
- Cylindrical
- ...

$$X = \begin{bmatrix} \boxed{X_P} \\ X_R \end{bmatrix}$$

Representations

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- Cartesian
- Spherical
- Cylindrical
- ...

- Euler Angles
- Quaternion
- Direction Cosines
- Euler Parameters

Jacobian for X

$$\dot{X}_P = J_{X_P}(q)\dot{q}$$
$$\dot{X}_R = J_{X_R}(q)\dot{q}$$

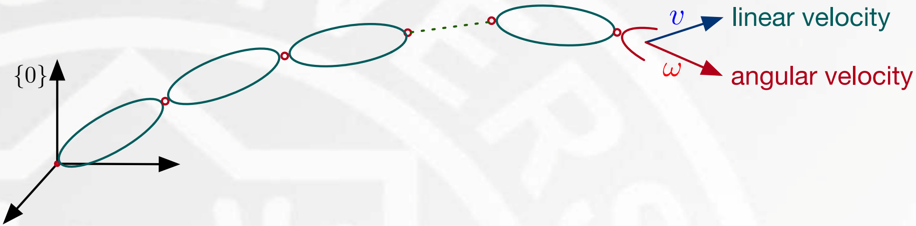
$$\begin{bmatrix} \dot{X}_P \\ \dot{X}_R \end{bmatrix} = \begin{bmatrix} J_{X_P}(q) \\ J_{X_R}(q) \end{bmatrix} \dot{q}$$

Cartesian & Direction Cosines：

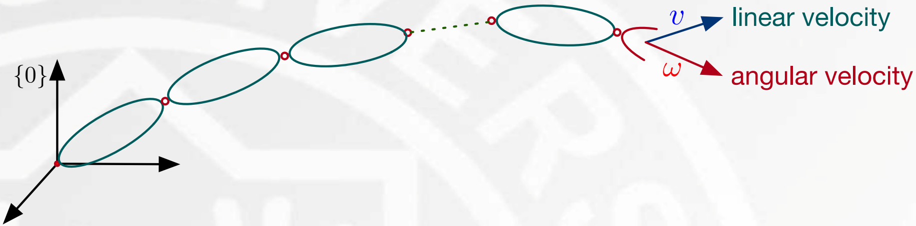
$$\dot{X}_{(12\times1)} = J_X(q)_{(12\times6)}\dot{q}_{(6\times1)}$$

The Jacobian is dependent on the representation

Basic Jacobian

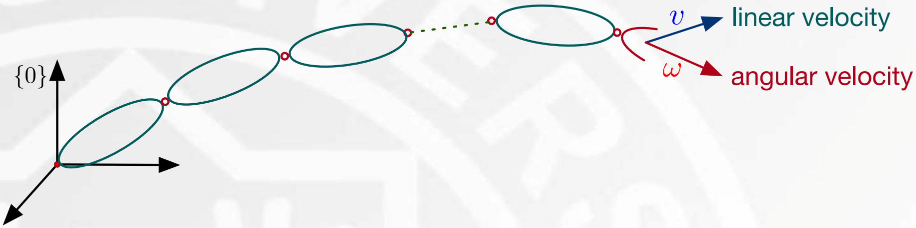


Basic Jacobian



$$\begin{bmatrix} v \\ \omega \end{bmatrix}_{(6\times1)} = J_0(q)_{(6\times n)}\dot{q}_{(n\times1)}$$

Basic Jacobian



$$\begin{bmatrix} v \\ \omega \end{bmatrix}_{(6\times 1)} = J_0(q)_{(6\times n)} \dot{q}_{(n\times 1)}$$

$$\dot{X}_P = E_P(X_P) v$$
$$\dot{X}_R = E_R(X_R) \omega$$

Examples

- $X_R = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}; \quad E_R(X_R) = \begin{bmatrix} -\dfrac{c\alpha c\beta}{s\beta} & -\dfrac{s\alpha c\beta}{s\beta} & 1 \\ -s\alpha & c\alpha & 0 \\ \dfrac{c\alpha}{s\beta} & \dfrac{s\alpha}{s\beta} & 0 \end{bmatrix}$

- $X_P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad E_P(X_P) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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Given a representation $X = \begin{bmatrix} X_P & X_R \end{bmatrix}^T$

$$\dot{X} = J_X(q)\dot{q}$$

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$$\boxed{J_{X_P} = E_P \cdot J_v, \quad J_{X_R} = E_R \cdot J_\omega}$$

Jacobian and Basic Jacobian

$$J=\left[\begin{array}{c} J_{X_P} \\ \hdashline J_{X_R} \end{array}\right]=\left[\begin{array}{c:c} E_P & 0 \\ \hdashline 0 & E_R \end{array}\right]\left[\begin{array}{c} J_v \\ \hdashline J_\omega \end{array}\right]$$

Jacobian and Basic Jacobian

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With Cartesian Coordinates:

$$J_{X_P} = J_v \Rightarrow E_P = I_3$$

and

$$E = \left[\begin{array}{c:c} I & 0 \\ \hdashline 0 & E_R \end{array}\right]$$

Position Representations

- Cartesian Coordinates $(x,\ y,\ z)$

$$E_P(X) = I_3$$

- Cylindrical Coordinates $(\rho,\ \theta,\ z)$ Using

$$\begin{pmatrix} x & y & z \end{pmatrix}^T = \begin{pmatrix} \rho\cos\theta & \rho\sin\theta & z \end{pmatrix}^T$$

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Position Representations

- Spherical Coordinates $(\rho,\ \theta,\ \phi)$ Using

$$\begin{pmatrix} x & y & z \end{pmatrix}^T = \begin{pmatrix} \rho\cos\theta\sin\phi & \rho\sin\theta\sin\phi & \rho\cos\phi \end{pmatrix}^T$$

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$$\Rightarrow \quad E_P(X) = \begin{pmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -\sin\theta/(\rho\sin\phi) & \cos\theta/(\rho\sin\phi) & 0 \\ \cos\theta\cos\phi/\rho & \sin\theta\cos\phi/\rho & -\sin\phi/\rho \end{pmatrix}$$

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When $\Delta\theta \to 0$, then $c\Delta\theta \to 1$, $s\Delta\theta \to \Delta\theta$, $v\Delta\theta = 1 - c\Delta\theta \to 0$, the following equation could be obtained:

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Orientation Representations

- Rigid angular velocity and the direction cosines:

$$\dot{R} = \lim_{\Delta t \to 0} \left(\frac{R_K(\Delta\theta) - I_3}{\Delta t} R(t) \right) = \lim_{\Delta t \to 0} \left(\frac{R_K(\Delta\theta) - I_3}{\Delta t} \right) R(t)$$

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$$^A_BR_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

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Orientation Representations

- 欧拉角速度 $\dot{\alpha}, \dot{\beta}, \dot{\gamma}$, 与笛卡尔空间角速度 Ω

Z-Y-Z 欧拉角

首先将坐标系 $\{B\}$ 和一个已知参考坐标系 $\{A\}$ 重合。先将 $\{B\}$ 绕 $\hat{Z}_B$ 旋转 α 角，绕 $\hat{Y}_B$ 旋转 β 角，最后绕 $\hat{Z}_B$ 旋转 γ 角。

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$$R_Z(\alpha)R_Y(\beta)R_Z(\gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

Orientation Representations

- 欧拉角速度 $\dot{\alpha}, \dot{\beta}, \dot{\gamma}$, 与笛卡尔空间角速度 Ω

Z-Y-Z 欧拉角

首先将坐标系 $\{B\}$ 和一个已知参考坐标系 $\{A\}$ 重合。先将 $\{B\}$ 绕 $\hat{Z}_B$ 旋转 α 角，绕 $\hat{Y}_B$ 旋转 β 角，最后绕 $\hat{Z}_B$ 旋转 γ 角。

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$$\begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\alpha} + \begin{bmatrix} -s\alpha \\ c\alpha \\ 0 \end{bmatrix} \dot{\beta} + \begin{bmatrix} c\alpha s\beta \\ s\alpha s\beta \\ c\beta \end{bmatrix} \dot{\gamma} = \begin{bmatrix} 0 & -s\alpha & c\alpha s\beta \\ 0 & c\alpha & s\alpha s\beta \\ 1 & 0 & c\beta \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

Orientation Representations

- XYZ 固定角速度 $\dot{\alpha}, \dot{\beta}, \dot{\gamma}$, 与笛卡尔空间角速度 Ω

XYZ 固定角

首先将坐标系 $\{B\}$ 和一个已知参考坐标系 $\{A\}$ 重合。先将 $\{B\}$ 绕 $\hat{X}_A$ 旋转 γ 角，绕 $\hat{Y}_A$ 旋转 β 角，最后绕 $\hat{Z}_A$ 旋转 α 角。

Orientation Representations

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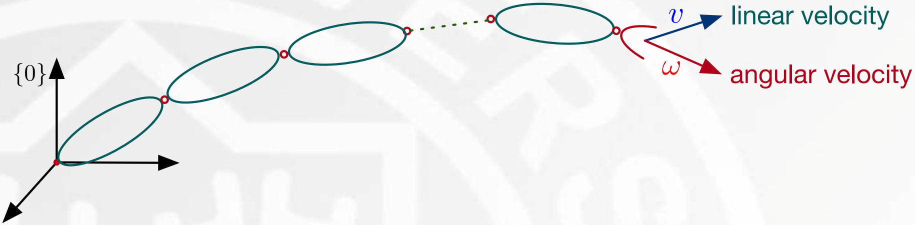
Jacobian

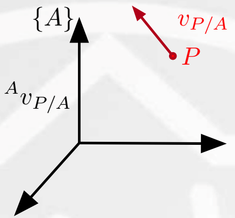
- Differential Motion
- Linear & Angular Motion
- Velocity Propagation
- Explicit Form
- Static Forces

Linear & Angular Velocities



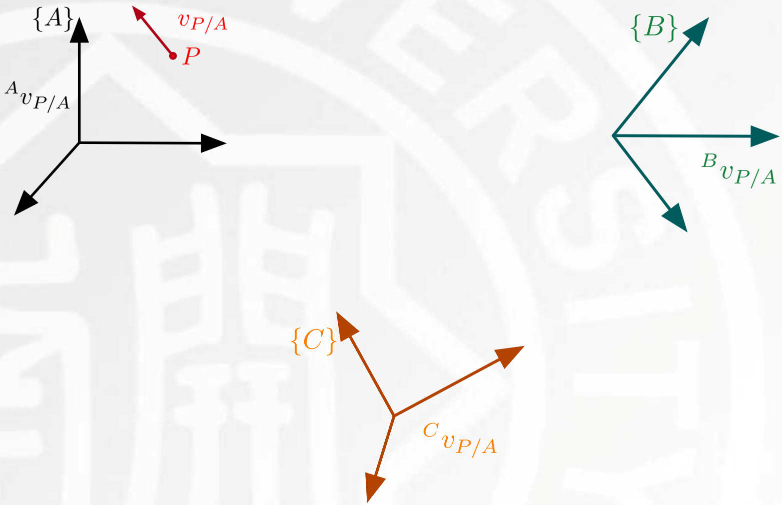
$$\begin{bmatrix} v \\ \omega \end{bmatrix}_{(6\times 1)} = J_0(q)_{(6\times n)}\dot{q}_{(n\times 1)} \qquad \begin{cases} \dot{X}_P = E_P(X_P)v \\ \dot{X}_R = E_R(X_R)\omega \end{cases}$$

Linear Velocity

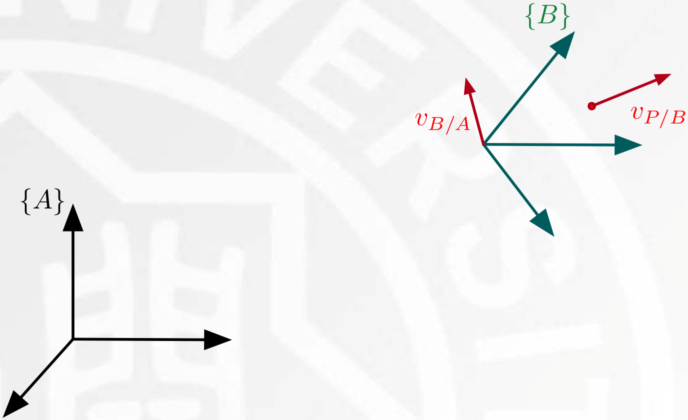


Linear Velocity

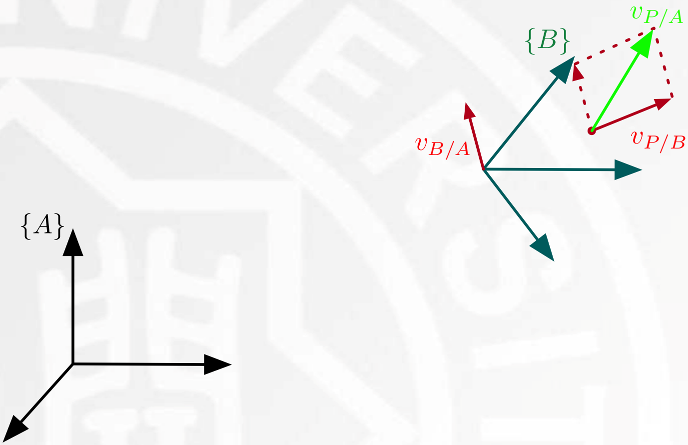
Linear Velocity



Pure Translation



Pure Translation



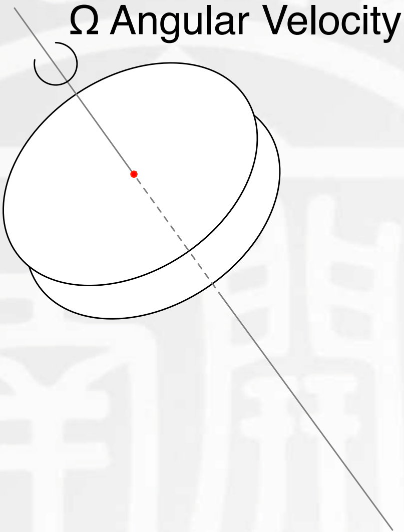
$$v_{P/A} = v_{B/A} + v_{P/B}$$

Rotational Motion

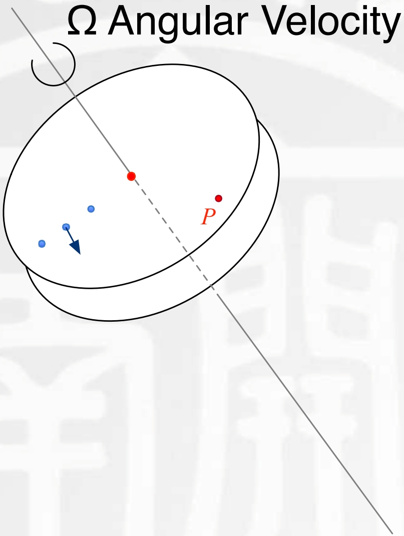
Axis of rotation

fixed points on the rigid body

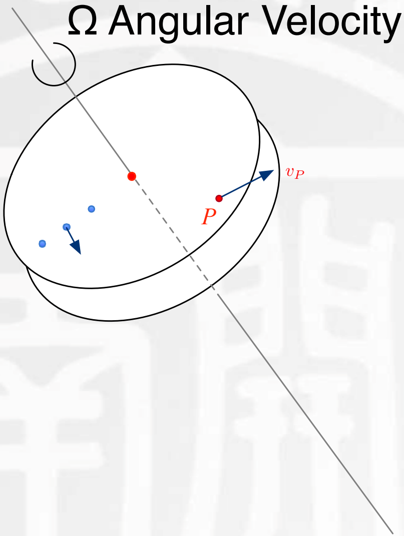
Rotational Motion



Rotational Motion

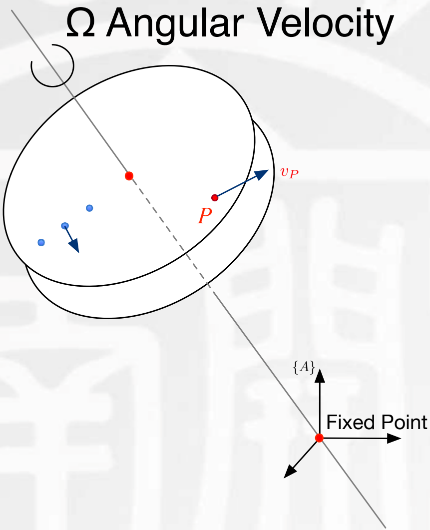


Rotational Motion

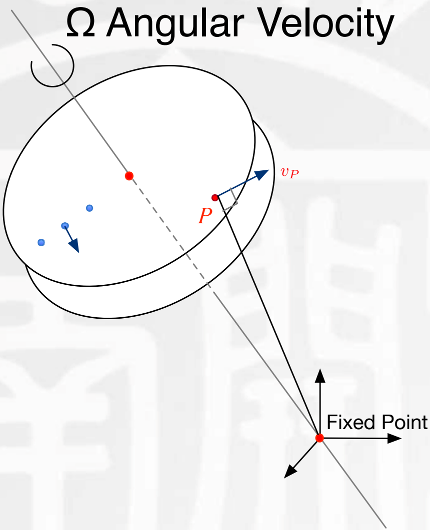


$$\boxed{v_p = \ ?}$$

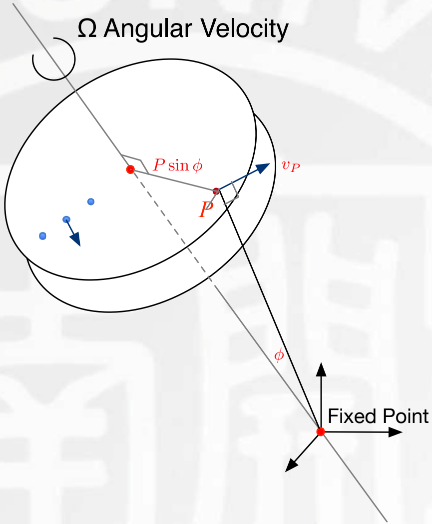
Rotational Motion



Rotational Motion



Rotational Motion



v_p is proportional to

- $\|\Omega\|$
- $\|P\sin\phi\|$

and

- $v_p \perp \Omega$
- $v_p \perp P$

$$\boxed{v_p = \Omega \times P}$$

Cross Product Operator

Let $a = [a_x,\ a_y,\ a_z]^T$ and $b = [b_x,\ b_y,\ b_z]^T$

$$c = a \times b \quad \Rightarrow \quad c = \hat{a}b$$

$$\text{vectors} \quad \Rightarrow \quad \text{matrics}$$

$a\times \ \Rightarrow\ \hat{a}$: a skew-symmetric matrix

$$c = a \times b = \hat{a}b = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$\boxed{c = \hat{a}b}$$

Cross Product Operator

$$\boxed{v_p = \Omega \times P}$$

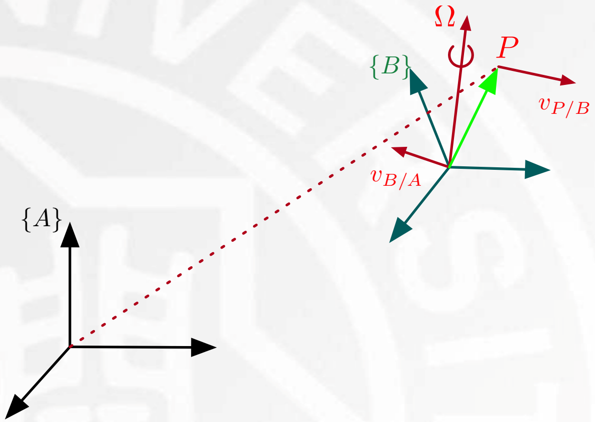
$\Omega\times \Rightarrow \hat{\Omega}$: a skew-symmetric matrix

Let $\Omega = [\Omega_x, \ \Omega_y, \ \Omega_z]^T$ and $P = [P_x, \ P_y, \ P_z]^T$

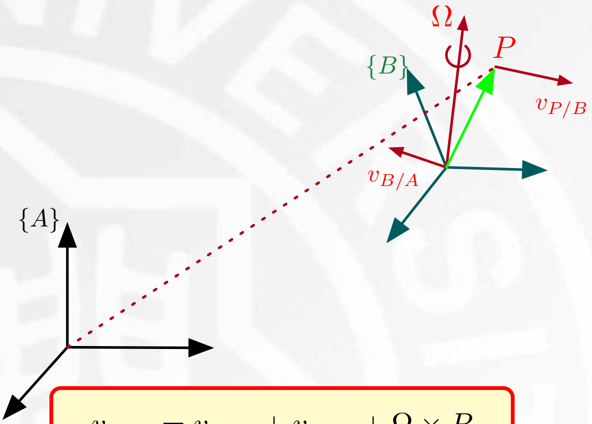
$$v_p = \hat{\Omega} P = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

$$\boxed{v_P = \hat{\Omega} P}$$

Simultaneous linear and angular motion

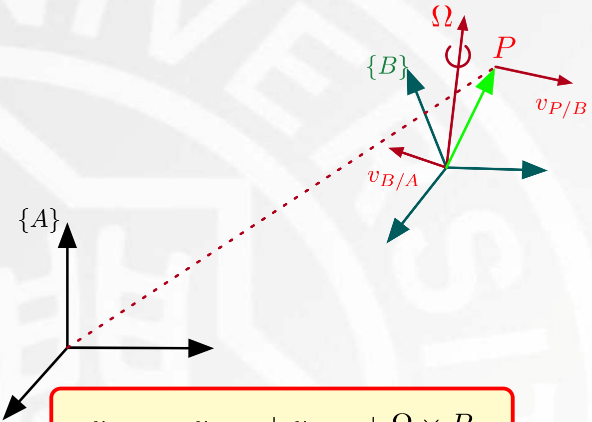


Simultaneous linear and angular motion



$$v_{P/A} = v_{B/A} + v_{P/B} + \Omega \times P$$

Simultaneous linear and angular motion



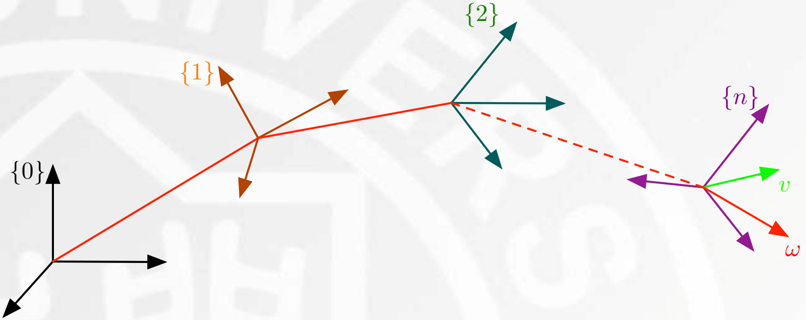
$$\boxed{v_{P/A} = v_{B/A} + v_{P/B} + \Omega \times P}$$

$$^{A}v_{P/A} = {}^{A}v_{B/A} + {}^{A}_{B}R \cdot {}^{B}v_{P/B} + {}^{A}\Omega \times {}^{A}_{B}R \cdot {}^{B}P$$

Jacobian

- Differential Motion
- Linear & Angular Motion
- Velocity Propagation
- Explicit Form
- Static Forces

Spatial Mechanisms

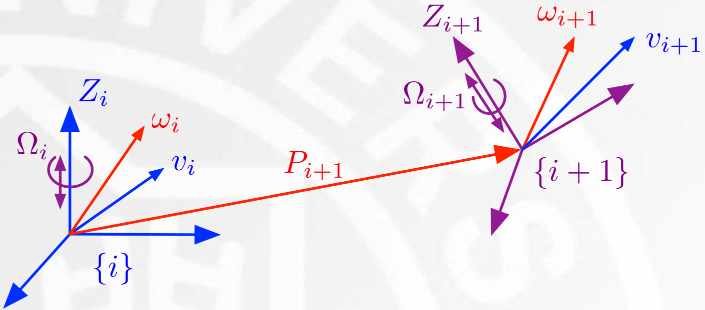


Propagation fo velocities

$$\dot{x} : \begin{cases} v : \text{ linear velocity} \\ \omega : \text{ angular velocity} \end{cases}$$

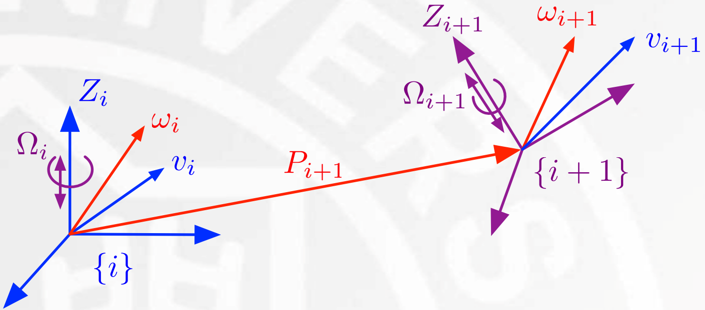
$$\dot{x} = J(\theta) \cdot \dot{\theta}$$

Velocity propagation



- Linear velocity:

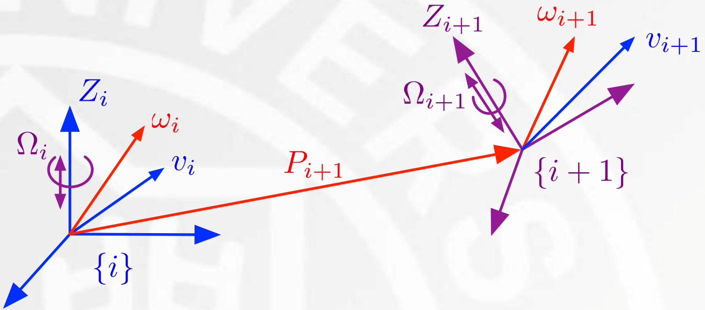
Velocity propagation



- Linear velocity:

$$v_{i+1} = v_i + \omega_i \times P_{i+1} + \dot{d}_{i+1} \cdot Z_{i+1}$$

Velocity propagation

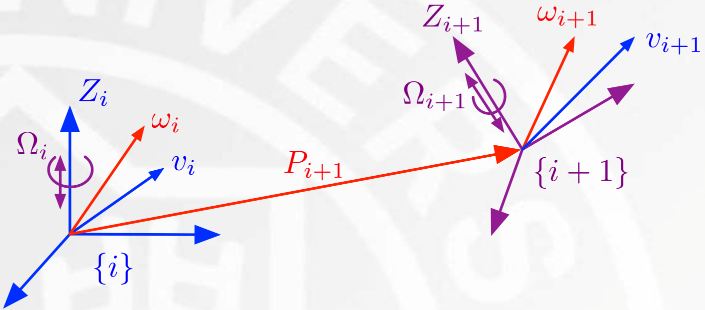


- Linear velocity:

$$v_{i+1} = v_i + \omega_i \times P_{i+1} + \dot{d}_{i+1} \cdot Z_{i+1}$$

- Angular velocity:

Velocity propagation



- Linear velocity:

$$v_{i+1} = v_i + \omega_i \times P_{i+1} + \dot{d}_{i+1} \cdot Z_{i+1}$$

- Angular velocity:

$$\begin{aligned} \omega_{i+1} &= \omega_i + \Omega_{i+1} \\ \Omega_{i+1} &= \dot{\theta}_{i+1} \cdot Z_{i+1} \end{aligned}$$

Velocity propagation

- Joint 1
  v_1 amd ω_1 in frame $\{1\}$
- Joint $i+1$

Velocity propagation

- Joint 1
  v_1 amd ω_1 in frame $\{1\}$
- Joint $i+1$

$$\begin{aligned} {}^{i+1}\omega_{i+1} &= {}^{i+1}_{i}R \cdot {}^{i}\omega_i + \dot{\theta}_{i+1} \cdot {}^{i+1}Z_{i+1} \\ {}^{i+1}v_{i+1} &= {}^{i+1}_{i}R \cdot ({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) + \dot{d}_{i+1} \cdot {}^{i+1}Z_{i+1} \end{aligned}$$

- $\Rightarrow {}^{n}\omega_n$ and ${}^{n}v_n$

Velocity propagation

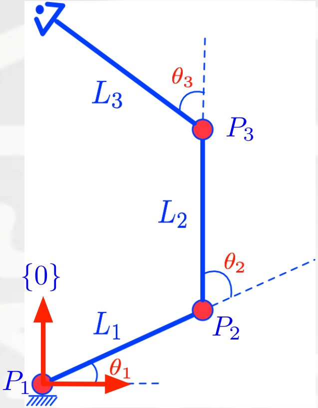
- Joint 1
  v_1 amd ω_1 in frame $\{1\}$
- Joint $i+1$

$$\begin{aligned} {}^{i+1}\omega_{i+1} &= {}^{i+1}_{i}R \cdot {}^{i}\omega_i + \dot{\theta}_{i+1} \cdot {}^{i+1}Z_{i+1} \\ {}^{i+1}v_{i+1} &= {}^{i+1}_{i}R \cdot ({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) + \dot{d}_{i+1} \cdot {}^{i+1}Z_{i+1} \end{aligned}$$

- $\Rightarrow {}^{n}\omega_n$ and ${}^{n}v_n$

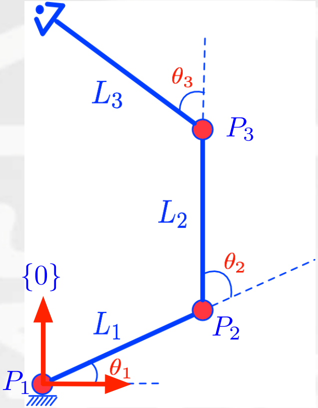
$$\begin{bmatrix} {}^{0}v_n \\ {}^{0}\omega_n \end{bmatrix} = \begin{bmatrix} {}^{0}_{n}R & 0 \\ 0 & {}^{0}_{n}R \end{bmatrix} \begin{bmatrix} {}^{n}v_n \\ {}^{n}\omega_n \end{bmatrix}$$

Example



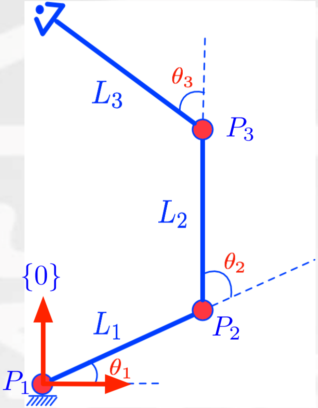
- $v_{i+1} = v_i + \omega_i \times P_{i+1}$

Example



- $v_{i+1} = v_i + \omega_i \times P_{i+1}$
- $v_{P_1} = 0$

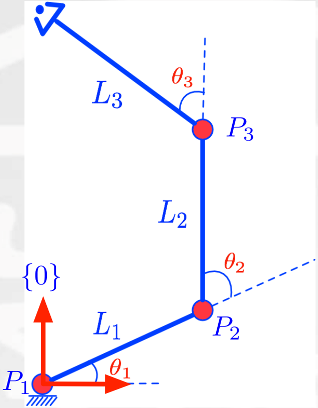
Example



- $v_{i+1} = v_i + \omega_i \times P_{i+1}$
- $v_{P_1} = 0$
- $v_{P_2} = v_{P_1} + \omega_1 \times P_2$

$$^0v_{P_2} = 0 + \begin{bmatrix} 0 & -\dot{\theta}_1 & 0 \\ \dot{\theta}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} l_1 \cdot c_1 \\ l_1 \cdot s_1 \\ 0 \end{bmatrix} = \dot{\theta}_1 \cdot \begin{bmatrix} -l_1 \cdot s_1 \\ l_1 \cdot c_1 \\ 0 \end{bmatrix}$$

Example



- $v_{i+1} = v_i + \omega_i \times P_{i+1}$
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$$ {}^0v_{P_2} = 0 + \begin{bmatrix} 0 & -\dot{\theta}_1 & 0 \\ \dot{\theta}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} l_1 \cdot c_1 \\ l_1 \cdot s_1 \\ 0 \end{bmatrix} = \dot{\theta}_1 \cdot \begin{bmatrix} -l_1 \cdot s_1 \\ l_1 \cdot c_1 \\ 0 \end{bmatrix} $$

- $v_{P_3} = v_{P_2} + \omega_2 \times P_3$

$$ {}^0v_{P_3} = \dot{\theta}_1 \cdot \begin{bmatrix} -l_1 \cdot s_1 \\ l_1 \cdot c_1 \\ 0 \end{bmatrix} + (\dot{\theta}_1 + \dot{\theta}_2) \cdot \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot {}^0P_3 $$

Example

- $v_{P_3} = v_{P_2} + \omega_2 \times P_3$

$$
{}^0v_{P_3} = \dot{\theta}_1 \cdot \begin{bmatrix} -l_1 \cdot s_1 \\ l_1 \cdot c_1 \\ 0 \end{bmatrix} + (\dot{\theta}_1 + \dot{\theta}_2) \cdot \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} l_2 \cdot c_{12} \\ l_2 \cdot s_{12} \\ 0 \end{bmatrix}
$$

$$
= \underbrace{\begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{J_v} \cdot \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
$$

Example

- $v_{P_3} = v_{P_2} + \omega_2 \times P_3$

$$^0v_{P_3} = \dot{\theta}_1 \cdot \begin{bmatrix} -l_1 \cdot s_1 \\ l_1 \cdot c_1 \\ 0 \end{bmatrix} + (\dot{\theta}_1 + \dot{\theta}_2) \cdot \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} l_2 \cdot c_{12} \\ l_2 \cdot s_{12} \\ 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{J_v} \cdot \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

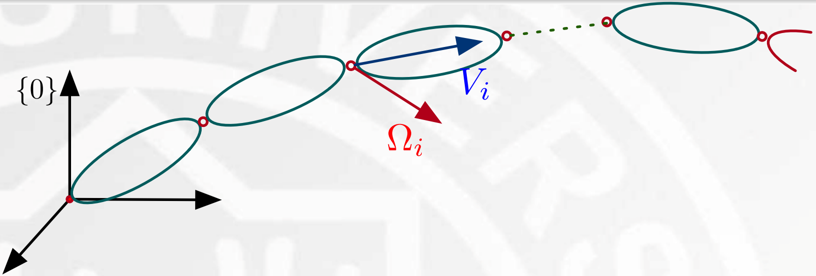
- $^0\omega_3 = (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) \cdot {}^0Z_0$

$$^0\omega_3 = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{J_\omega} \cdot \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

Jacobian

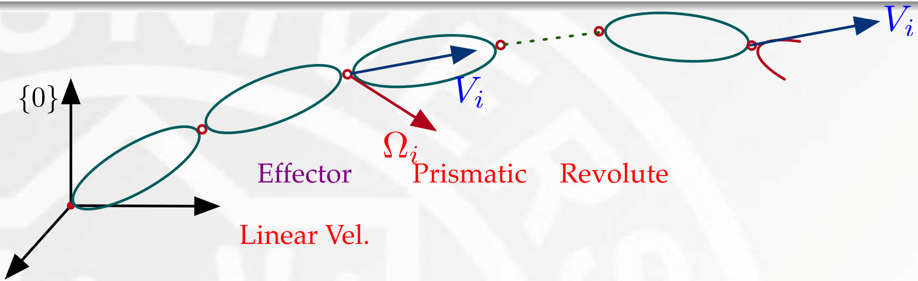
- Differential Motion
- Linear & Angular Motion
- Velocity Propagation
- Explicit Form
- Static Forces

The Jacobian (EXPLICIT FORM)



Revolute Joint	$\Omega_i = Z_i \dot{q}_i$
Prismatic Joint	$V_i = Z_i \dot{q}_i$

The Jacobian (EXPLICIT FORM)



Effector

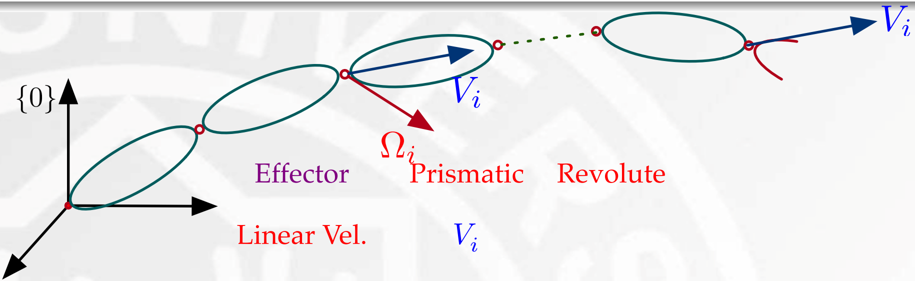
Prismatic

Revolute

Linear Vel.

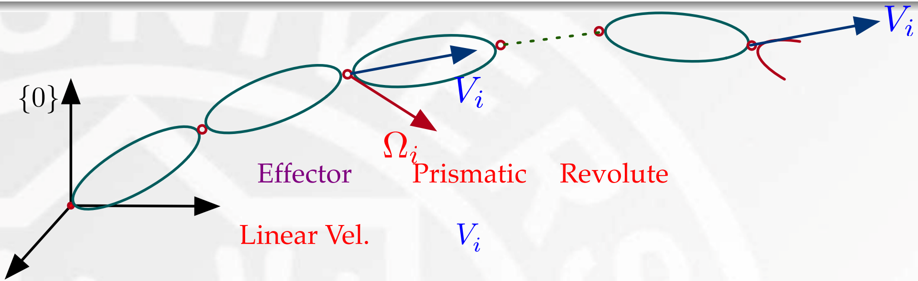
Angular Vel.

The Jacobian (EXPLICIT FORM)



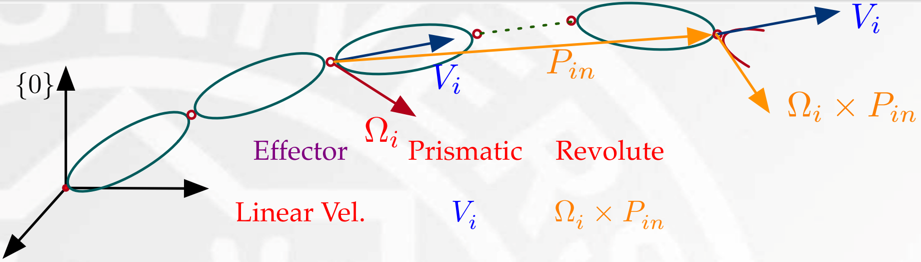
Effector	Prismatic	Revolute
Linear Vel.	V_i	
Angular Vel.		

The Jacobian (EXPLICIT FORM)



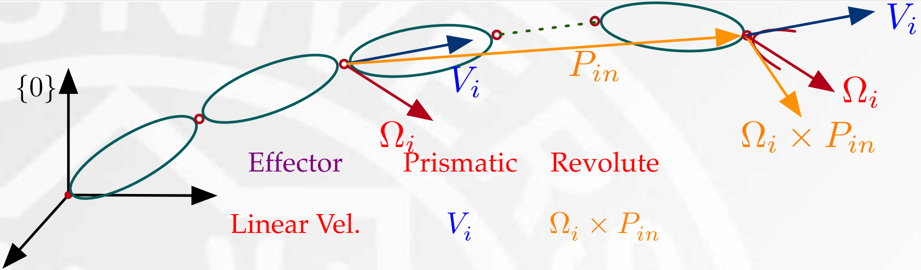
Effector	Prismatic	Revolute
Linear Vel.	V_i	
Angular Vel.	0	

The Jacobian (EXPLICIT FORM)



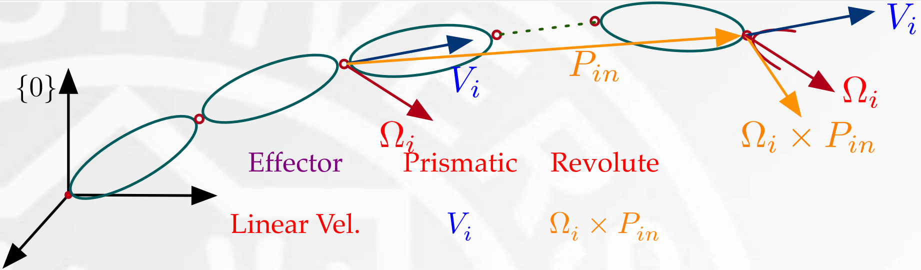
Effector	Prismatic	Revolute
Linear Vel.	V_i	$\Omega_i \times P_{in}$
Angular Vel.	0	

The Jacobian (EXPLICIT FORM)



Effector	Prismatic	Revolute
Linear Vel.	V_i	$\Omega_i \times P_{in}$
Angular Vel.	0	Ω_i

The Jacobian (EXPLICIT FORM)



Effector	Prismatic	Revolute
Linear Vel.	V_i	$\Omega_i \times P_{in}$
Angular Vel.	0	Ω_i

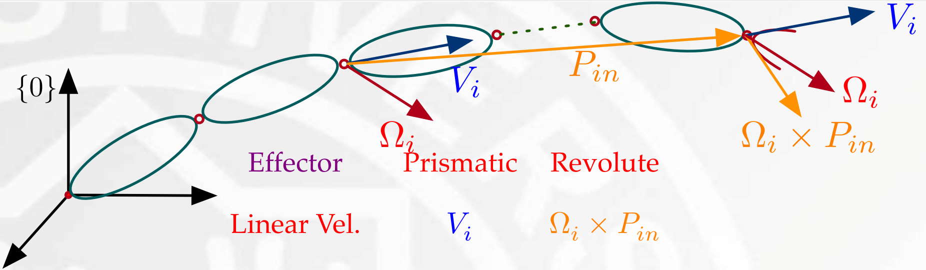
Effector Linear Velocity

$$v = \sum_{i=1}^{n} [\epsilon_i V_i + \bar{\epsilon}_i(\Omega_i \times P_{in})]$$

Effector Augular Velocity

$$\omega = \sum_{i=1}^{n} \bar{\epsilon}_i \Omega_i$$

The Jacobian (EXPLICIT FORM)



Effector	Prismatic	Revolute
Linear Vel.	V_i	$\Omega_i \times P_{in}$
Angular Vel.	0	Ω_i

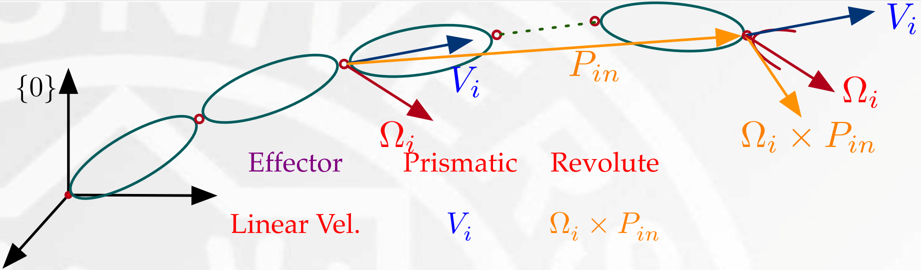
Effector Linear Velocity

$$v = \sum_{i=1}^{n} [\epsilon_i V_i + \bar{\epsilon}_i (\Omega_i \times P_{in})] \qquad \Leftarrow \qquad V_i = Z_i \dot{q}_i$$

Effector Augular Velocity

$$\omega = \sum_{i=1}^{n} \bar{\epsilon}_i \Omega_i \qquad \Leftarrow \qquad \Omega_i = Z_i \dot{q}_i$$

The Jacobian (EXPLICIT FORM)



Effector	Prismatic	Revolute
Linear Vel.	V_i	$\Omega_i \times P_{in}$
Angular Vel.	0	Ω_i

Effector Linear Velocity

$$v = \sum_{i=1}^{n} [\epsilon_i V_i + \bar{\epsilon}_i (\Omega_i \times P_{in})] = \sum_{i=1}^{n} [\epsilon_i Z_i + \bar{\epsilon}_i (Z_i \times P_{in})] \dot{q}_i \quad \Leftarrow \quad V_i = Z_i \dot{q}_i$$

Effector Augular Velocity

$$\omega = \sum_{i=1}^{n} \bar{\epsilon}_i \Omega_i = \sum_{i=1}^{n} (\bar{\epsilon}_i Z_i) \dot{q}_i \quad \Leftarrow \quad \Omega_i = Z_i \dot{q}_i$$

$$v = [\epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n})]\dot{q}_1 + \cdots + [\epsilon_{n-1} Z_{n-1} + \bar{\epsilon}_{n-1}(Z_{n-1} \times P_{(n-1)n})]\dot{q}_{n-1} + \epsilon_n Z_n \dot{q}_n$$

$$v = [\epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n})]\dot{q}_1 + \cdots + [\epsilon_{n-1} Z_{n-1} + \bar{\epsilon}_{n-1}(Z_{n-1} \times P_{(n-1)n})]\dot{q}_{n-1} + \epsilon_n Z_n \dot{q}_n$$

$$v = \begin{bmatrix} \epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n}) & \cdots & \epsilon_{n-1} Z_{n-1} + \bar{\epsilon}_{n-1}(Z_{n-1} \times P_{(n-1)n}) & \epsilon_n Z_n \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_{n-1} \\ \dot{q}_n \end{bmatrix}$$

$$v = [\epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n})]\dot{q}_1 + \cdots + [\epsilon_{n-1} Z_{n-1} + \bar{\epsilon}_{n-1}(Z_{n-1} \times P_{(n-1)n})]\dot{q}_{n-1} + \epsilon_n Z_n \dot{q}_n$$

$$v = \begin{bmatrix} \epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n}) & \cdots & \epsilon_{n-1} Z_{n-1} + \bar{\epsilon}_{n-1}(Z_{n-1} \times P_{(n-1)n}) & \epsilon_n Z_n \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_{n-1} \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{v = J_v \dot{q}}$$

$$v = [\epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n})]\dot{q}_1 + \cdots + [\epsilon_{n-1} Z_{n-1} + \bar{\epsilon}_{n-1}(Z_{n-1} \times P_{(n-1)n})]\dot{q}_{n-1} + \epsilon_n Z_n \dot{q}_n$$

$$v = \begin{bmatrix} \epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n}) & \cdots & \epsilon_{n-1} Z_{n-1} + \bar{\epsilon}_{n-1}(Z_{n-1} \times P_{(n-1)n}) & \epsilon_n Z_n \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_{n-1} \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{v = J_v \dot{q}}$$

$$\omega = \bar{\epsilon}_1 Z_1 \dot{q}_1 + \bar{\epsilon}_2 Z_2 \dot{q}_2 + \cdots + \bar{\epsilon}_n Z_n \dot{q}_n$$

$$v = [\epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n})]\dot{q}_1 + \cdots + [\epsilon_{n-1} Z_{n-1} + \bar{\epsilon}_{n-1}(Z_{n-1} \times P_{(n-1)n})]\dot{q}_{n-1} + \epsilon_n Z_n \dot{q}_n$$

$$v = \begin{bmatrix} \epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n}) & \cdots & \epsilon_{n-1} Z_{n-1} + \bar{\epsilon}_{n-1}(Z_{n-1} \times P_{(n-1)n}) & \epsilon_n Z_n \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_{n-1} \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{v = J_v \dot{q}}$$

$$\omega = \bar{\epsilon}_1 Z_1 \dot{q}_1 + \bar{\epsilon}_2 Z_2 \dot{q}_2 + \cdots + \bar{\epsilon}_n Z_n \dot{q}_n$$

$$\omega = \begin{bmatrix} \bar{\epsilon}_1 Z_1 & \bar{\epsilon}_2 Z_2 & \cdots & \bar{\epsilon}_n Z_n \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$v = [\epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n})]\dot{q}_1 + \cdots + [\epsilon_{n-1} Z_{n-1} + \bar{\epsilon}_{n-1}(Z_{n-1} \times P_{(n-1)n})]\dot{q}_{n-1} + \epsilon_n Z_n \dot{q}_n$$

$$v = \begin{bmatrix} \epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n}) & \cdots & \epsilon_{n-1} Z_{n-1} + \bar{\epsilon}_{n-1}(Z_{n-1} \times P_{(n-1)n}) & \epsilon_n Z_n \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_{n-1} \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{v = J_v \dot{q}}$$

$$\omega = \bar{\epsilon}_1 Z_1 \dot{q}_1 + \bar{\epsilon}_2 Z_2 \dot{q}_2 + \cdots + \bar{\epsilon}_n Z_n \dot{q}_n$$

$$\omega = \begin{bmatrix} \bar{\epsilon}_1 Z_1 & \bar{\epsilon}_2 Z_2 & \cdots & \bar{\epsilon}_n Z_n \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{\omega = J_\omega \dot{q}}$$

Jacobian in a Frame

$$J = \left[\begin{array}{c} J_v \\ \hdashline J_\omega \end{array}\right]$$

Jacobian in a Frame

$$J = \begin{bmatrix} J_v \\ \hdashline J_\omega \end{bmatrix}$$

Matrix J_v (direct differentiation)

$$v = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \dot{x}_P = \frac{\partial x_P}{\partial q_1} \cdot \dot{q}_1 + \frac{\partial x_P}{\partial q_2} \cdot \dot{q}_2 + \cdots + \frac{\partial x_P}{\partial q_n} \cdot \dot{q}_n$$

$$J_v = \begin{bmatrix} \frac{\partial x_P}{\partial q_1} & \frac{\partial x_P}{\partial q_2} & \cdots & \frac{\partial x_P}{\partial q_N} \end{bmatrix}$$

Jacobian in a Frame

$$J = \begin{bmatrix} J_v \\ \hdashline J_\omega \end{bmatrix}$$

Matrix J_v (direct differentiation)

$$v = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \dot{x}_P = \frac{\partial x_P}{\partial q_1} \cdot \dot{q}_1 + \frac{\partial x_P}{\partial q_2} \cdot \dot{q}_2 + \cdots + \frac{\partial x_P}{\partial q_n} \cdot \dot{q}_n$$

$$J_v = \begin{bmatrix} \frac{\partial x_P}{\partial q_1} & \frac{\partial x_P}{\partial q_2} & \cdots & \frac{\partial x_P}{\partial q_N} \end{bmatrix}$$

Vector Respesentation:

$$J = \begin{bmatrix} \frac{\partial x_P}{\partial q_1} & \frac{\partial x_P}{\partial q_2} & \cdots & \frac{\partial x_P}{\partial q_N} \\ \bar{\epsilon}_1 Z_1 & \bar{\epsilon}_2 Z_2 & \cdots & \bar{\epsilon}_n Z_n \end{bmatrix}$$

Jacobian in a Frame $\{0\}$

$$
{}^0J = \begin{bmatrix} \frac{\partial {}^0 x_P}{\partial q_1} & \frac{\partial {}^0 x_P}{\partial q_2} & \cdots & \frac{\partial {}^0 x_P}{\partial q_N} \\ \bar{\epsilon}_1 {}^0 Z_1 & \bar{\epsilon}_2 {}^0 Z_2 & \cdots & \bar{\epsilon}_n {}^0 Z_n \end{bmatrix}
$$

Jacobian in a Frame $\{0\}$

$$^0J = \begin{bmatrix} \frac{\partial {}^0x_P}{\partial q_1} & \frac{\partial {}^0x_P}{\partial q_2} & \cdots & \frac{\partial {}^0x_P}{\partial q_N} \\ \bar{\epsilon}_1 {}^0Z_1 & \bar{\epsilon}_2 {}^0Z_2 & \cdots & \bar{\epsilon}_n {}^0Z_n \end{bmatrix}$$

$$^0Z_i = {}^0_iR \cdot {}^iZ_i$$

Jacobian in a Frame $\{0\}$

$$
{}^{0}J = \begin{bmatrix} \dfrac{\partial {}^{0}x_P}{\partial q_1} & \dfrac{\partial {}^{0}x_P}{\partial q_2} & \cdots & \dfrac{\partial {}^{0}x_P}{\partial q_N} \\ \bar{\epsilon}_1 {}^{0}Z_1 & \bar{\epsilon}_2 {}^{0}Z_2 & \cdots & \bar{\epsilon}_n {}^{0}Z_n \end{bmatrix}
$$

$$
{}^{0}Z_i = {}^{0}_{i}R \cdot {}^{i}Z_i \qquad {}^{i}Z_i = Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

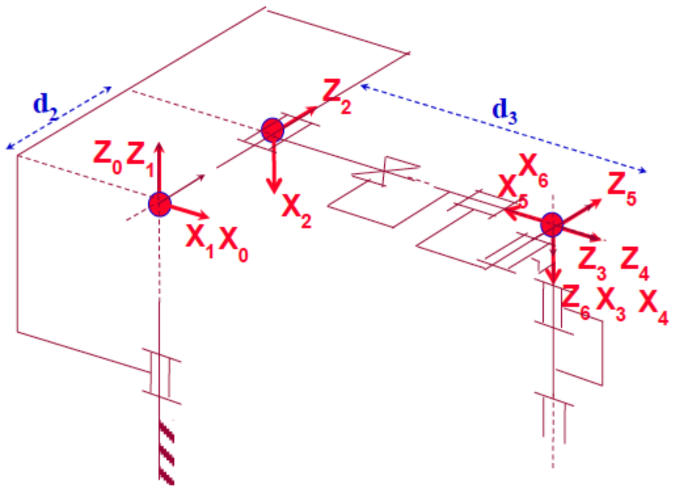
Jacobian in a Frame {0}

$$
{}^0J = \begin{bmatrix} \dfrac{\partial {}^0x_P}{\partial q_1} & \dfrac{\partial {}^0x_P}{\partial q_2} & \cdots & \dfrac{\partial {}^0x_P}{\partial q_N} \\ \bar{\epsilon}_1 {}^0Z_1 & \bar{\epsilon}_2 {}^0Z_2 & \cdots & \bar{\epsilon}_n {}^0Z_n \end{bmatrix}
$$

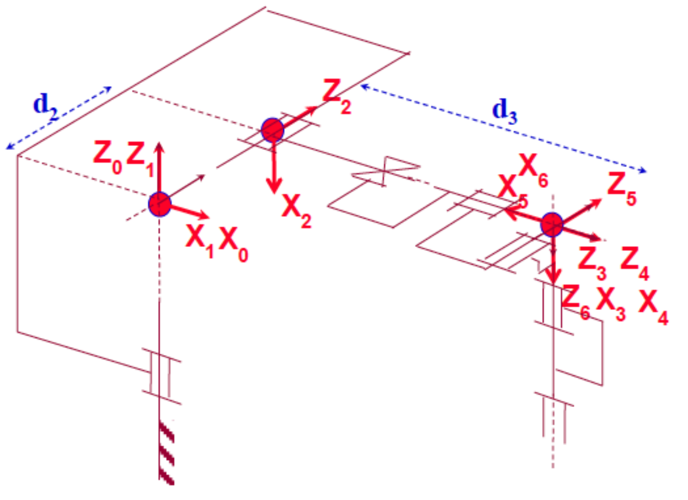
$$
{}^0Z_i = {}^0_iR \cdot {}^iZ_i \qquad {}^iZ_i = Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
\boxed{{}^0J = \begin{bmatrix} \dfrac{\partial}{\partial q_1}({}^0x_P) & \dfrac{\partial}{\partial q_2}({}^0x_P) & \cdots & \dfrac{\partial}{\partial q_N}({}^0x_P) \\ \bar{\epsilon}_1({}^0_1R \cdot Z) & \bar{\epsilon}_2({}^0_2R \cdot Z) & \cdots & \bar{\epsilon}_n({}^0_nR \cdot Z) \end{bmatrix}}
$$

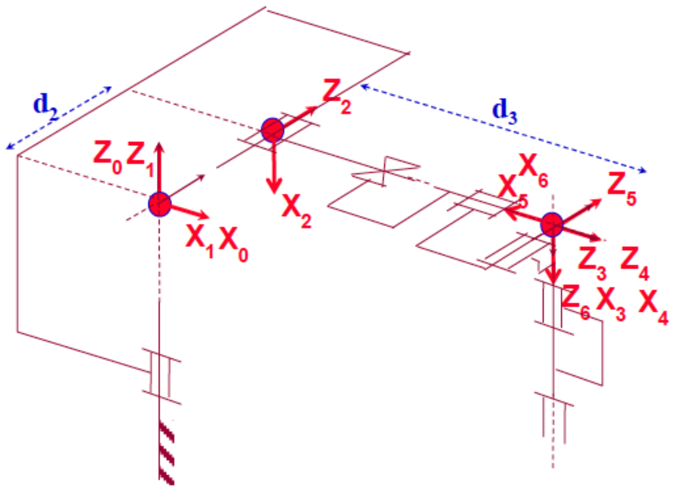
Stanford Scheinman Arm



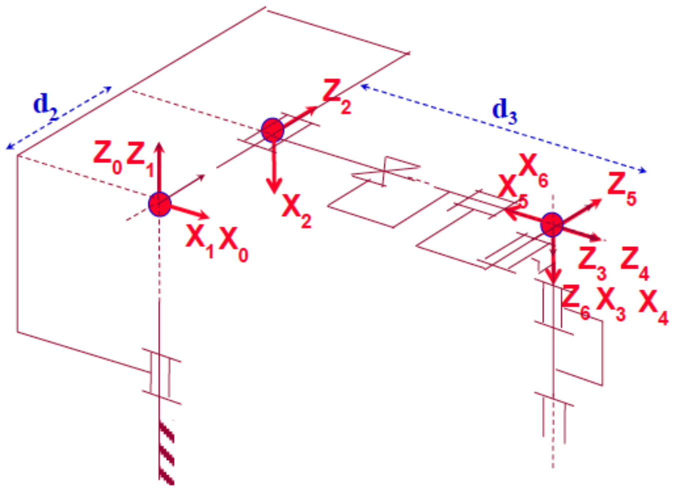
$$J = \left[\begin{array}{c:c:c:c:c:c} \quad & \quad & \quad & \quad & \quad & \quad \\ \hdashline \quad & \quad & \quad & \quad & \quad & \quad \end{array} \right]$$



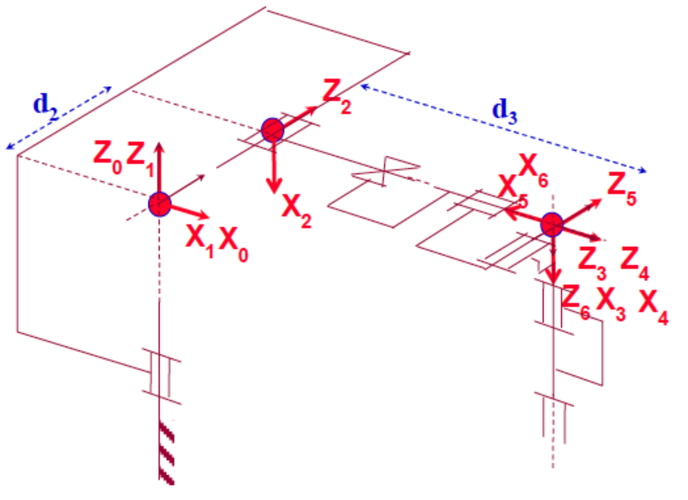
$$J = \left[\begin{array}{c:c:c:c:c:c} Z_1 \times P_{13} & & & & & \\ \hdashline & & & & & \end{array}\right]$$



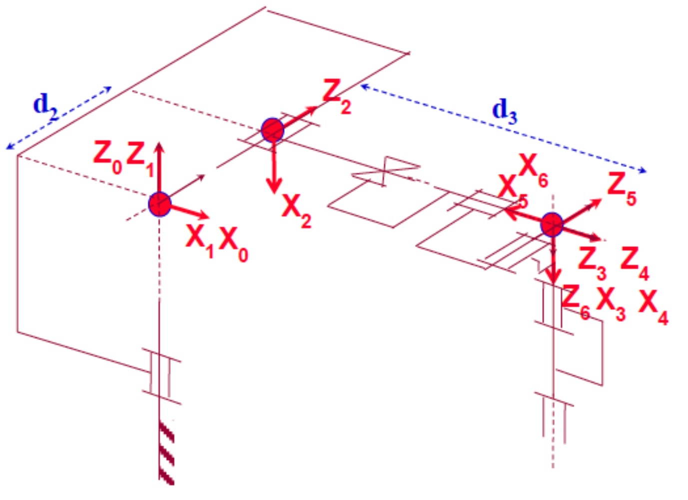
$$J = \left[\begin{array}{c:c:c:c:c:c} Z_1 \times P_{13} & & & & & \\ \hdashline Z_1 & & & & & \end{array} \right]$$



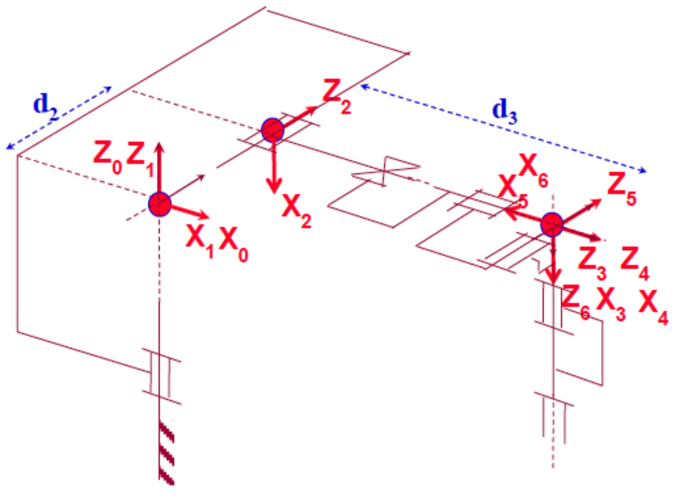
$$J = \left[\begin{array}{c:c:c:c:c:c} Z_1 \times P_{13} & Z_2 \times P_{23} & & & & \\ \hdashline Z_1 & & & & & \end{array}\right]$$



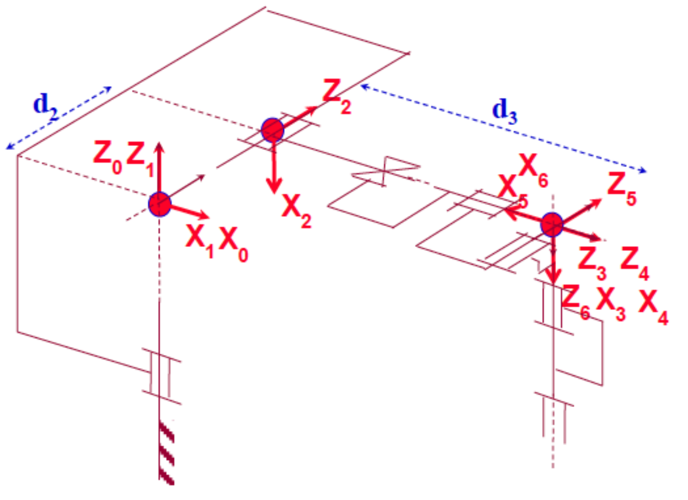
$$J = \left[\begin{array}{c:c:c:c:c:c} Z_1 \times P_{13} & Z_2 \times P_{23} & & & & \\ \hdashline Z_1 & Z_2 & & & & \end{array}\right]$$



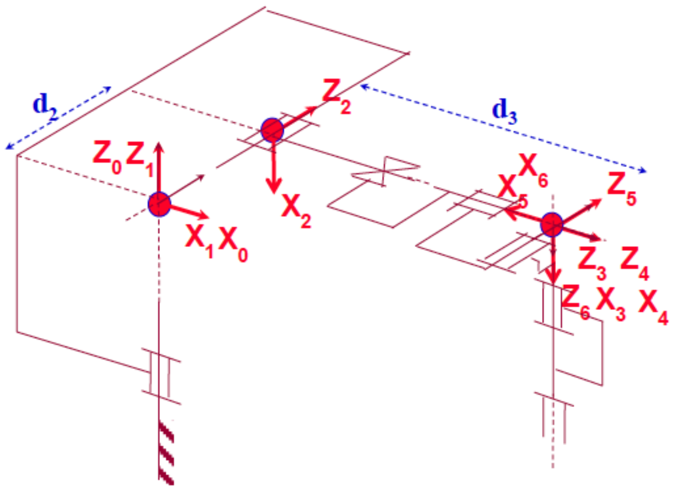
$$J = \left[\begin{array}{c:c:c:c:c:c} Z_1 \times P_{13} & Z_2 \times P_{23} & Z_3 & & & \\ \hdashline Z_1 & Z_2 & & & & \end{array}\right]$$



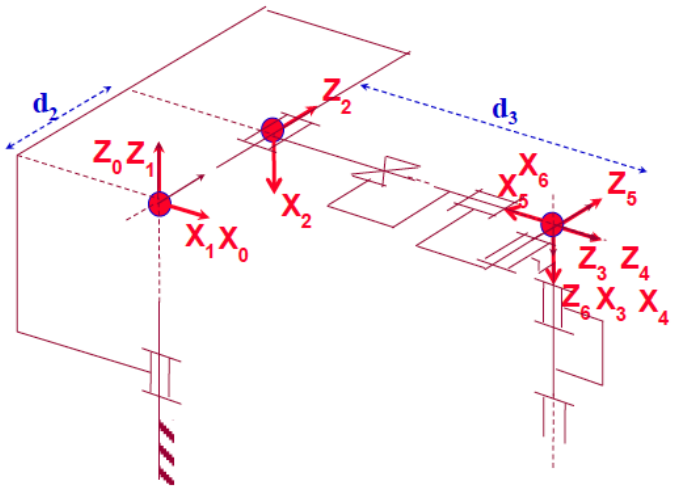
$$J = \left[\begin{array}{c:c:c:c:c:c} Z_1 \times P_{13} & Z_2 \times P_{23} & Z_3 & & & \\ \hdashline Z_1 & Z_2 & 0 & & & \end{array}\right]$$



$$J = \left[\begin{array}{c:c:c:c:c:c} Z_1 \times P_{13} & Z_2 \times P_{23} & Z_3 & 0 & & \\ \hdashline Z_1 & Z_2 & 0 & & & \end{array}\right]$$



$$J = \left[\begin{array}{c:c:c:c:c:c} Z_1 \times P_{13} & Z_2 \times P_{23} & Z_3 & 0 & & \\ \hdashline Z_1 & Z_2 & 0 & Z_4 & & \end{array}\right]$$



$$J = \left[\begin{array}{c:c:c:c:c:c} Z_1 \times P_{13} & Z_2 \times P_{23} & Z_3 & 0 & 0 & 0 \\ \hdashline Z_1 & Z_2 & 0 & Z_4 & Z_5 & Z_6 \end{array}\right]$$

Stanford Scheinman Arm

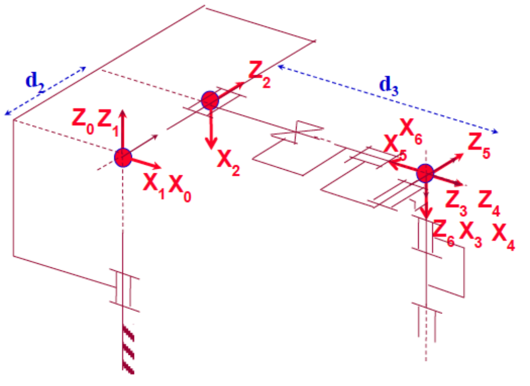


Table: DH 参数

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	d_2	θ_2
3	90	0	d_3	0
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6

Stanford Scheinman Arm

$$
{}^{i-1}_{i}T = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ \sin\theta_i\cos\alpha_{i-1} & \cos\theta_i\cos\alpha_{i-1} & -\sin\alpha_{i-1} & -d_i\sin\alpha_{i-1} \\ \sin\theta_i\sin\alpha_{i-1} & \cos\theta_i\sin\alpha_{i-1} & \cos\alpha_{i-1} & d_i\cos\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Forward Kinematics: ${}^{0}_{N}T = {}^{0}_{1}T \cdot {}^{1}_{2}T \cdots {}^{N-1}_{N}T$

$$
{}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^{0}_{2}T = \begin{bmatrix} c_1c_2 & -c_1s_2 & -s_1 & -s_1d_2 \\ s_1c_2 & -s_1s_2 & c_1 & c_1d_2 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^{0}_{3}T = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & c_1s_2d_3 - s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & s_1s_2d_3 + c_1d_2 \\ -s_2 & 0 & c_2 & c_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^{0}_{4}T = \begin{bmatrix} X & X & c_1s_2 & c_1s_2d_3 - s_1d_2 \\ X & X & s_1s_2 & s_1s_2d_3 + c_1d_2 \\ X & X & c_2 & c_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Stanford Scheinman Arm

$$
{}^0_5T = \begin{bmatrix} X & X & -c_1c_2s_4 - s_1c_4 & X \\ X & X & -s_1c_2s_4 + c_1c_4 & X \\ X & X & s_2s_4 & X \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; {}^0_6T = \begin{bmatrix} X & X & c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2s_5 & X \\ X & X & s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c_5 & X \\ X & X & -s_2c_4s_5 + c_2c_5 & X \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
X = \begin{bmatrix} X_P \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} d_3c_1s_2 - d_2s_1 \\ d_3s_1s_2 + d_2c_1 \\ d_3c_2 \\ c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) \\ s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) \\ -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 \\ c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) \\ s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6) + c_1(-s_4c_5s_6 + c_4c_6) \\ s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 \\ c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 \\ s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 \\ -s_2c_4s_5 + c_2c_5 \end{bmatrix}
$$

Stanford Scheinman Arm Jacobian

$$
{}^0J = \begin{bmatrix} \frac{\partial}{\partial q_1}({}^0x_P) & \frac{\partial}{\partial q_2}({}^0x_P) & \frac{\partial}{\partial q_3}({}^0x_P) & 0 & 0 & 0 \\ {}^0Z_1 & {}^0Z_2 & 0 & {}^0Z_4 & {}^0Z_5 & {}^0Z_6 \end{bmatrix}
$$

$$
\begin{bmatrix}
-(d_3s_1s_2+d_2c_1) & c_1c_2d_3 & c_1s_2 & 0 & 0 & 0 \\
d_3c_1s_2-d_2s_1 & s_1c_2d_3 & s_1s_2 & 0 & 0 & 0 \\
0 & -s_2d_3 & c_2 & 0 & 0 & 0 \\
0 & -s_1 & 0 & c_1s_2 & -c_1c_2s_4-s_1c_4 & c_1c_2c_4s_5-s_1s_4s_5+c_1s_2c_5 \\
0 & c_1 & 0 & s_1s_2 & -s_1c_2s_4+c_1c_4 & s_1c_2c_4s_5+c_1s_4s_5+s_1s_2c_5 \\
1 & 0 & 0 & c_2 & s_2s_4 & -s_2c_4s_5+c_2c_5
\end{bmatrix}
$$

Kinematic Singularity and Singular Configurations

Singular Direction

The Effector Locality loses the ability to move in a direction or to rotate about a direction.

Kinematic Singularity and Singular Configurations

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$${}^BJ = \begin{bmatrix} {}^B_AR & 0 \\ 0 & {}^B_AR \end{bmatrix} {}^AJ$$

$$\boxed{\det({}^BJ) \equiv \det({}^AJ)}$$

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$$\det({}^iJ) = \det({}^jJ)$$

$$\Rightarrow \text{Singular Configurations}$$

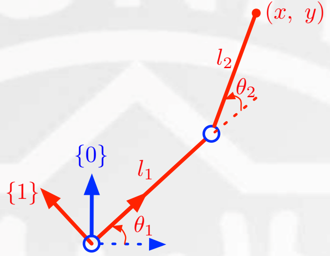
$$\det[J(q)] = S_1(q) \cdot S_2(q) \cdots S_s(q) = 0$$

$${}^BJ = \begin{bmatrix} {}^B_AR & 0 \\ 0 & {}^B_AR \end{bmatrix} {}^AJ$$

$$\boxed{\det({}^BJ) \equiv \det({}^AJ)}$$

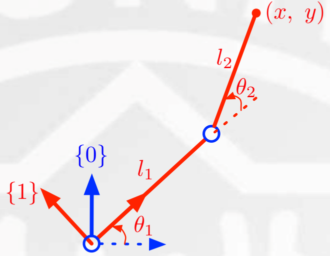
$$\Rightarrow \boxed{\begin{array}{c} S_1(q) = 0 \\ S_2(q) = 0 \\ \vdots \\ S_s(q) = 0 \end{array}}$$

Example (Kinematic Singularities)



$$x = l_1 c_1 + l_2 c_{12}$$
$$y = l_1 s_1 + l_2 s_{12}$$

Example (Kinematic Singularities)



$$x = l_1 c_1 + l_2 c_{12}$$
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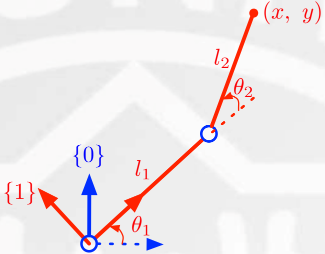
$$J = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

Example (Kinematic Singularities)



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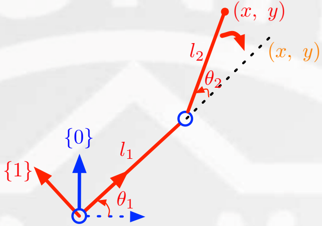
$$J = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

$$\det(J) = l_1 l_2 s_2$$

Sigularity at $q_2 = k\pi$

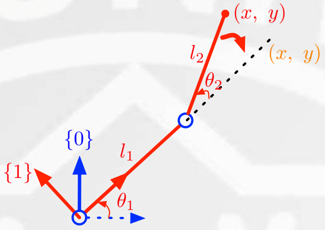
Example (Kinematic Singularities)



$$^1J = {}^1_0R\,{}^0J$$

$$^0J = \begin{bmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{bmatrix} \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

Example (Kinematic Singularities)



$$^1J = {}^1_0R\,^0J$$

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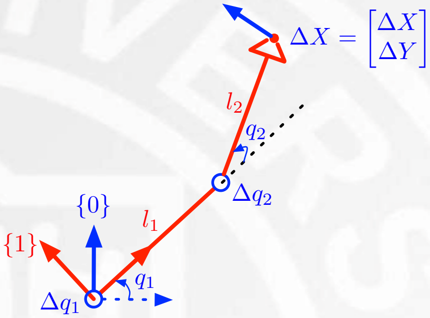
At singularity:

$$^1J = \begin{bmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{bmatrix}$$

$$\begin{cases} ^1\delta x = 0 \\ ^1\delta y = (l_1 + l_2)\delta\theta_1 + l_2\delta\theta_2 \end{cases}$$

Small Displacements Δq, ΔX

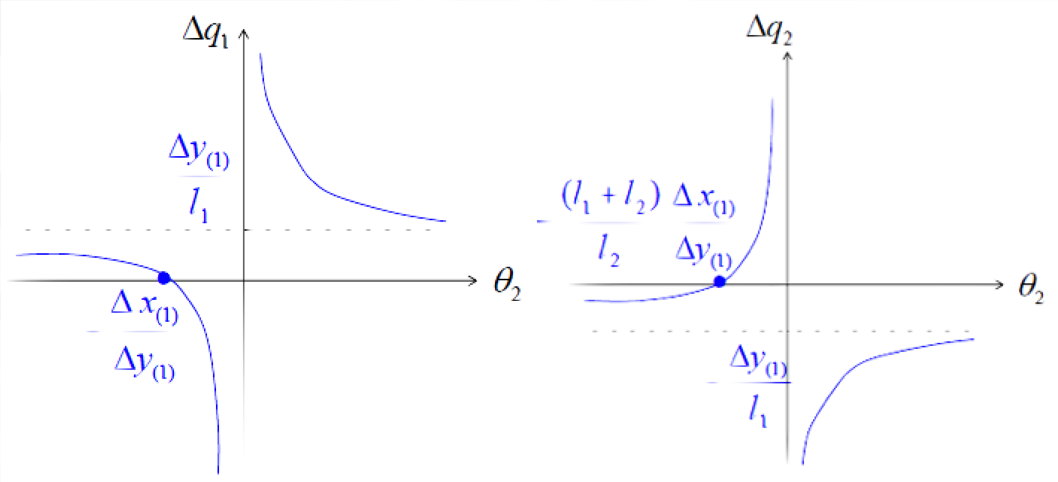


$$\Delta q = J^{-1} \Delta X$$

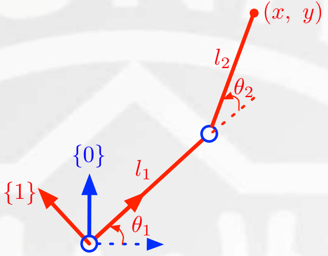
Small θ_2 (i.e. $\cos\theta_2 \approx 1$, $\sin\theta_2 \approx \theta_2$)

$$J_{(1)}^{-1} \cong \begin{bmatrix} \dfrac{1}{l_1\theta_2} & \dfrac{1}{l_1} \\ -\dfrac{l_1+l_2}{l_1 l_2 \theta_2} & -\dfrac{1}{l_1} \end{bmatrix} \Rightarrow \begin{cases} \Delta q_1 = \dfrac{\Delta x_{(1)}}{l_1} \cdot \dfrac{1}{\theta_2} + \dfrac{\Delta y_{(1)}}{l_1} \\ \Delta q_2 = \dfrac{(l_1+l_2)\Delta x_{(1)}}{l_1 l_2} \cdot \dfrac{1}{\theta_2} + \dfrac{\Delta y_{(1)}}{l_1} \end{cases}$$

Small Displacements Δq, ΔX

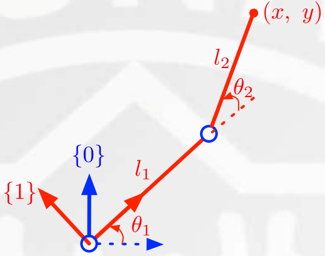


Kinematic Singularities (reduced matrix)



$$J = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Kinematic Singularities (reduced matrix)



$$J = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

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$$\det(J) = l_1 l_2 s_2$$

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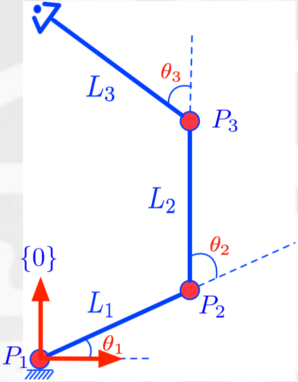
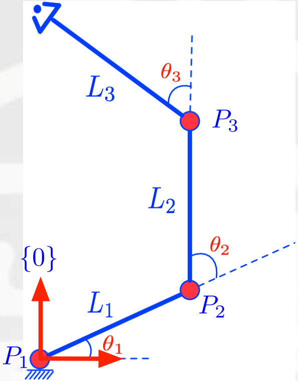


$$
{}^0J_E = \begin{bmatrix} -l_1s_1 - l_2s_{12} - l_3s_{123} & -l_2s_{12} - l_3s_{123} & -l_3s_{123} \\ l_1c_1 + l_2c_{12} + l_3c_{123} & l_2c_{12} + l_3c_{123} & l_3c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}
$$

Kinematic Singularities (reduced matrix)



$$^0J_E = \begin{bmatrix} -l_1s_1 - l_2s_{12} - l_3s_{123} & -l_2s_{12} - l_3s_{123} & -l_3s_{123} \\ l_1c_1 + l_2c_{12} + l_3c_{123} & l_2c_{12} + l_3c_{123} & l_3c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$^0J_E = \begin{bmatrix} -l_1s_1 - l_2s_{12} - l_3s_{123} & -l_2s_{12} - l_3s_{123} & -l_3s_{123} \\ l_1c_1 + l_2c_{12} + l_3c_{123} & l_2c_{12} + l_3c_{123} & l_3c_{123} \\ 1 & 1 & 1 \end{bmatrix}$$

Stanford Scheinman Arm Jacobian

$$\begin{bmatrix} -(d_3s_1s_2+d_2c_1) & c_1c_2d_3 & c_1s_2 & 0 & 0 & 0 \\ d_3c_1s_2-d_2s_1 & s_1c_2d_3 & s_1s_2 & 0 & 0 & 0 \\ 0 & -s_2d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1s_2 & -c_1c_2s_4-s_1c_4 & c_1c_2c_4s_5-s_1s_4s_5+c_1s_2c_5 \\ 0 & c_1 & 0 & s_1s_2 & -s_1c_2s_4+c_1c_4 & s_1c_2c_4s_5+c_1s_4s_5+s_1s_2c_5 \\ 1 & 0 & 0 & c_2 & s_2s_4 & -s_2c_4s_5+c_2c_5 \end{bmatrix}$$

$$\theta_5 = k\pi$$

Stanford Scheinman Arm Jacobian

$$\begin{bmatrix} -(d_3 s_1 s_2 + d_2 c_1) & c_1 c_2 d_3 & c_1 s_2 & 0 & 0 & 0 \\ d_3 c_1 s_2 - d_2 s_1 & s_1 c_2 d_3 & s_1 s_2 & 0 & 0 & 0 \\ 0 & -s_2 d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ 0 & c_1 & 0 & s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\ 1 & 0 & 0 & c_2 & s_2 s_4 & -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}$$

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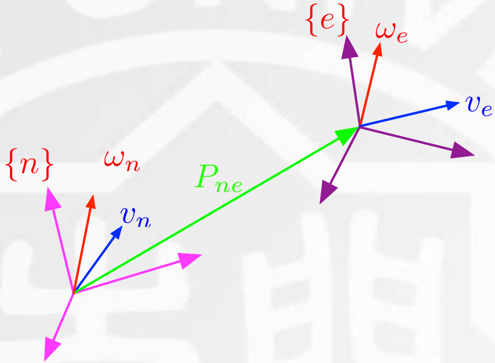
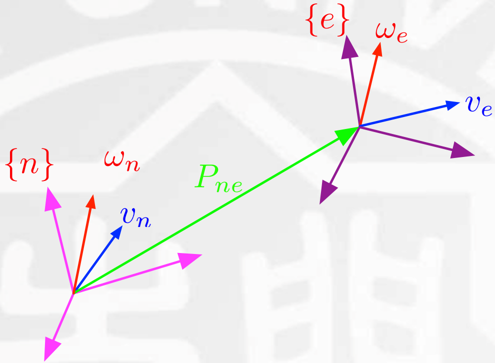
$$\begin{bmatrix} -(d_3 s_1 s_2 + d_2 c_1) & c_1 c_2 d_3 & c_1 s_2 & 0 & 0 & 0 \\ d_3 c_1 s_2 - d_2 s_1 & s_1 c_2 d_3 & s_1 s_2 & 0 & 0 & 0 \\ 0 & -s_2 d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & c_1 s_2 \\ 0 & c_1 & 0 & s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & s_1 s_2 \\ 1 & 0 & 0 & c_2 & s_2 s_4 & c_2 \end{bmatrix}$$

Jacobian at the End-Effector



Jacobian at the End-Effector



$$v_e = v_n + \omega_n \times P_{ne}$$

$$\begin{cases} v_e = v_n - P_{ne} \times \omega_n \\ \omega_e = \omega_n \end{cases}$$

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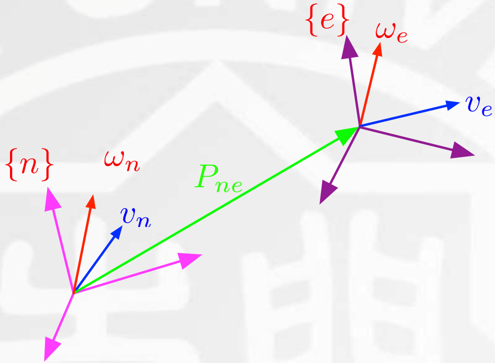

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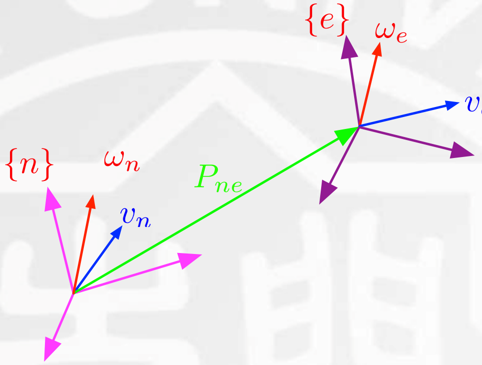
$$\begin{bmatrix} v_e \\ \omega_e \end{bmatrix} = \left[\begin{array}{c:c} I & -\hat{P}_{ne} \\ \hdashline 0 & I \end{array}\right] \begin{bmatrix} v_n \\ \omega_n \end{bmatrix}$$

Jacobian at the End-Effector



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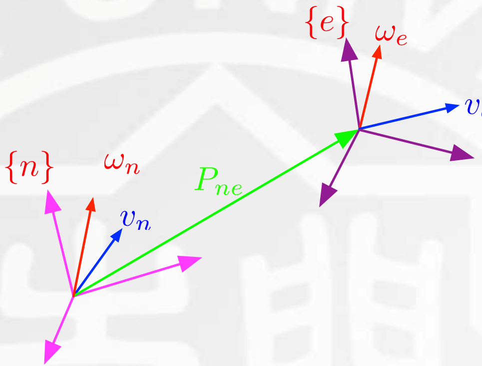
$$J_e \dot{q} = \left[\begin{array}{c:c} I & -\hat{P}_{ne} \\ \hdashline 0 & I \end{array}\right] J_n \dot{q} \qquad \Rightarrow \qquad J_e = \left[\begin{array}{c:c} I & -\hat{P}_{ne} \\ \hdashline 0 & I \end{array}\right] J_n$$

Cross Product Operator (in diff. frames)



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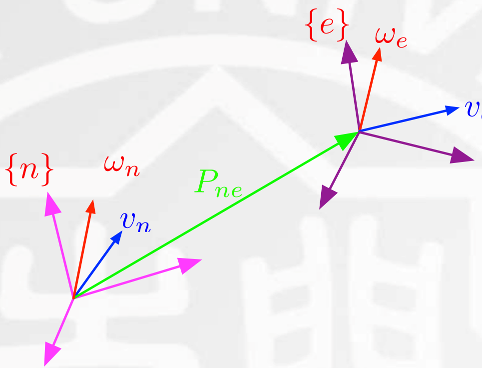
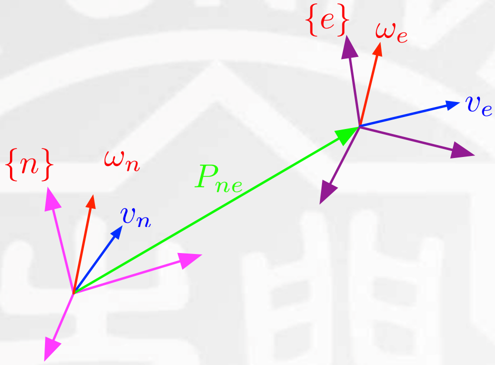


$$J_e = \left[\begin{array}{c:c} I & -\hat{P}_{ne} \\ \hdashline 0 & I \end{array}\right] J_n$$

$${}^0\hat{P}_{ne} = {}^0_nR \cdot {}^n\hat{P}_{ne} \quad ?$$

Cross Product Operator (in diff. frames)

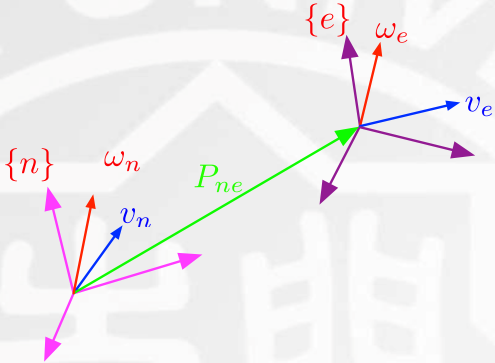


$$J_e = \left[\begin{array}{c:c} I & -\hat{P}_{ne} \\ \hdashline 0 & I \end{array}\right] J_n$$

$${}^0\hat{P}_{ne} = {}^0_nR \cdot {}^n\hat{P}_{ne} \quad ?$$

$${}^0\hat{P}_{ne} \times {}^0\omega = {}^0_nR \cdot ({}^nP_{ne} \times {}^n\omega)$$

Cross Product Operator (in diff. frames)



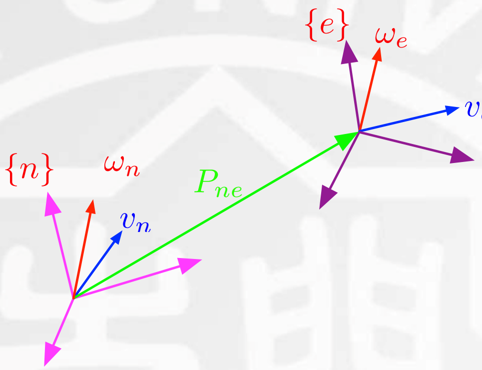
$$J_e = \left[\begin{array}{c:c} I & -\hat{P}_{ne} \\ \hdashline 0 & I \end{array}\right] J_n$$

$${}^0\hat{P}_{ne} = {}^0_nR \cdot {}^n\hat{P}_{ne} \quad ?$$

$${}^0\hat{P}_{ne} \times {}^0\omega = {}^0_nR \cdot ({}^nP_{ne} \times {}^n\omega)$$

$${}^0\hat{P}_{ne} \cdot {}^0\omega = {}^0_nR \cdot ({}^n\hat{P}_{ne} \cdot {}^n\omega) = {}^0_nR \cdot ({}^n\hat{P}_{ne} \cdot {}^0_nR^T \cdot {}^0\omega$$

Cross Product Operator (in diff. frames)



$$J_e = \left[\begin{array}{c:c} I & -\hat{P}_{ne} \\ \hdashline 0 & I \end{array}\right] J_n$$

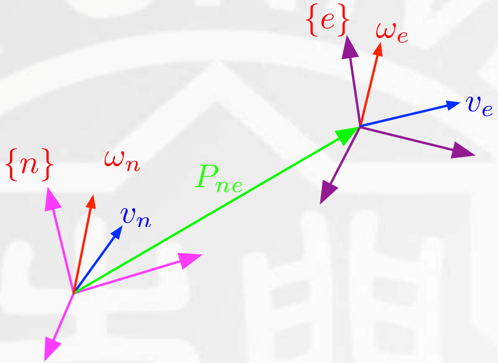
$${}^0\hat{P}_{ne} = {}^0_nR \cdot {}^n\hat{P}_{ne} \quad ?$$

$${}^0\hat{P}_{ne} \times {}^0\omega = {}^0_nR \cdot ({}^nP_{ne} \times {}^n\omega)$$

$${}^0\hat{P}_{ne} \cdot {}^0\omega = {}^0_nR \cdot ({}^n\hat{P}_{ne} \cdot {}^n\omega) = {}^0_nR \cdot ({}^n\hat{P}_{ne} \cdot {}^0_nR^T \cdot {}^0\omega$$

$$\boxed{{}^0\hat{P}_{ne} = {}^0_nR \cdot {}^n\hat{P}_{ne} \cdot {}^0_nR^T}$$

Cross Product Operator (in diff. frames)



$$J_e = \left[\begin{array}{c:c} I & -\hat{P}_{ne} \\ \hdashline 0 & I \end{array}\right] J_n$$

$${}^0\hat{P}_{ne} = {}^0_nR \cdot {}^n\hat{P}_{ne} \quad ?$$

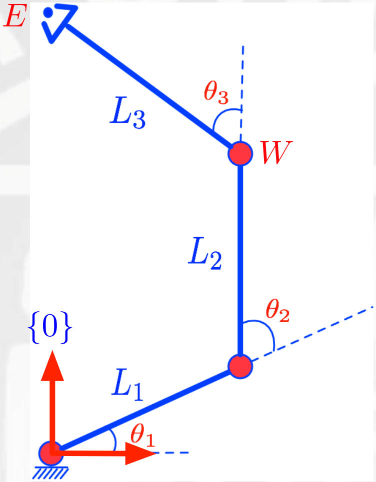
$${}^0P_{ne} \times {}^0\omega = {}^0_nR \cdot ({}^nP_{ne} \times {}^n\omega)$$

$${}^0\hat{P}_{ne} \cdot {}^0\omega = {}^0_nR \cdot ({}^n\hat{P}_{ne} \cdot {}^n\omega) = {}^0_nR \cdot ({}^n\hat{P}_{ne} \cdot {}^0_nR^T \cdot {}^0\omega$$

$$\boxed{{}^0\hat{P}_{ne} = {}^0_nR \cdot {}^n\hat{P}_{ne} \cdot {}^0_nR^T}$$

$${}^iJ = \begin{bmatrix} {}^i_jR & 0 \\ 0 & {}^i_jR \end{bmatrix} {}^jJ \quad \Rightarrow \quad {}^0J_e = \begin{bmatrix} {}^i_jR & -{}^0_nR \cdot {}^n\hat{P}_{ne} \cdot {}^0_nR^T \\ 0 & {}^i_jR \end{bmatrix} {}^nJ_n$$

Example: RRR Arm



Wrist Point:

$$\begin{cases} x = l_1 c_1 + l_2 c_{12} \\ y = l_1 s_1 + l_2 s_{12} \end{cases}$$

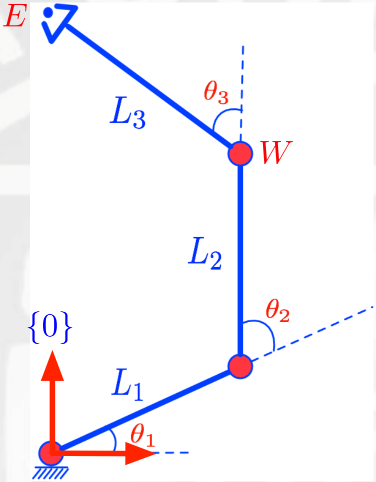
End-Effector Point:

$$\begin{cases} x = l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ y = l_1 s_1 + l_2 s_{12} + l_3 s_{123} \end{cases}$$

Jacobian (W):

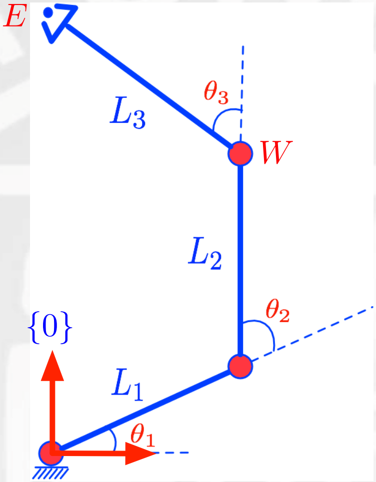
$$J_W = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Example: RRR Arm



$$
{}^0J_e = \begin{bmatrix} I & -{}^0\hat{P}_{ne} \\ 0 & I \end{bmatrix} {}^0J_W
$$

Example: RRR Arm

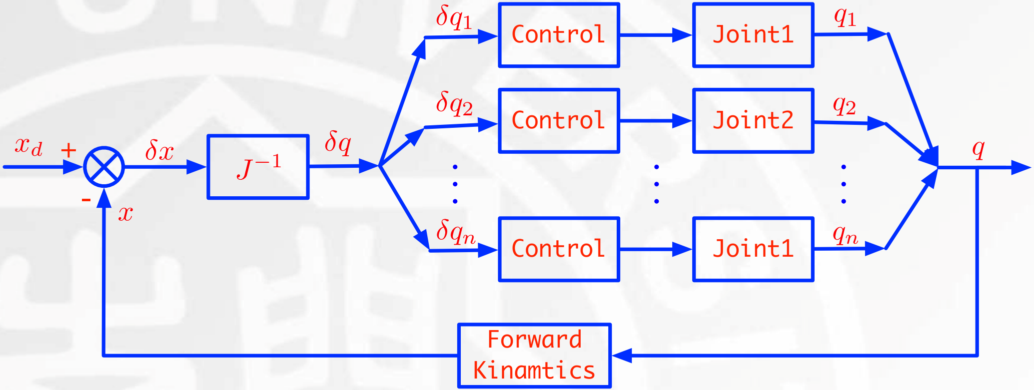


$$^0J_e = \begin{bmatrix} I & -^0\hat{P}_{ne} \\ 0 & I \end{bmatrix} {^0J_W}$$

$$^0P_{ne} = \begin{bmatrix} l_3c_{123} \\ l_3s_{123} \\ 0 \end{bmatrix} \Rightarrow {^0\hat{P}_{ne}} = \begin{bmatrix} 0 & 0 & l_3s_{123} \\ 0 & 0 & -l_3c_{123} \\ -l_3s_{123} & l_3c_{123} & 0 \end{bmatrix}$$

$$J_E = \begin{bmatrix} -l_1s_1 - l_2s_{12} - l_3s_{123} & -l_2s_{12} - l_3s_{123} & -l_3s_{123} \\ l_1c_1 + l_2c_{12} + l_3c_{123} & l_2c_{12} + l_3c_{123} & l_3c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Resolved Motion Rate Control



$$\delta x = J(\theta)\delta\theta$$

Outside singularities:

$$\delta\theta = J(\theta)^{-1}\delta x$$

Arm at Configuration θ:

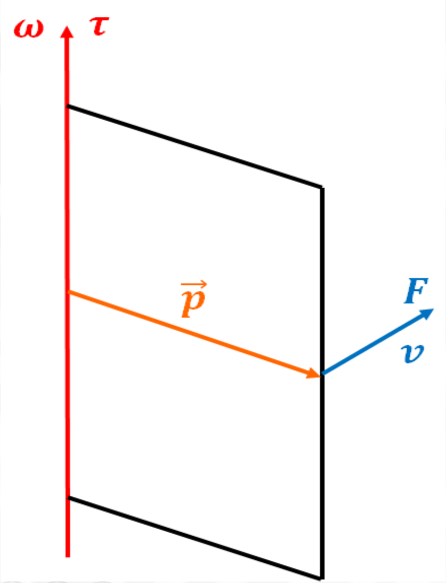
$$x = f(\theta) \Rightarrow \delta x = x_d - x \Rightarrow \delta\theta = J(\theta)^{-1}\delta x$$

$$\boxed{\theta^{+} = \theta + \delta\theta}$$

Jacobian

- Differential Motion
- Linear & Angular Motion
- Velocity Propagation
- Explicit Form
- Static Forces

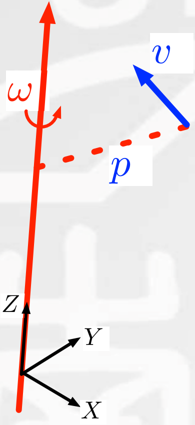
Angular / Torque -Velocities / Forces



$$v = \omega \times p$$

$$\tau = p \times F$$

Angular/Torque -Velocities/Forces

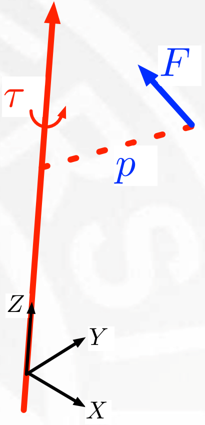


$$v = \omega \times p$$

$$v = -\hat{p}\omega$$

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} -p_y \\ p_x \end{bmatrix} \dot{\theta}$$

$$\boxed{v = J\dot{\theta}}$$



$$\tau = p \times F$$

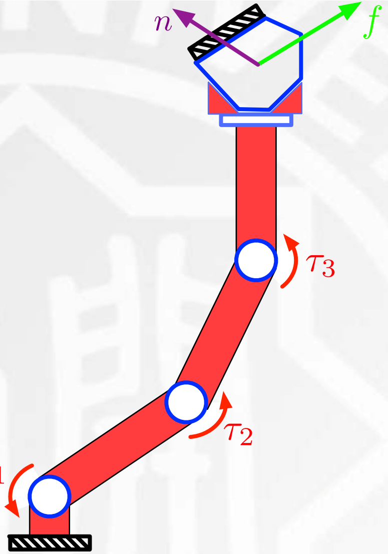
$$\tau = \hat{p}F$$

$$\tau = -\hat{p}^T F$$

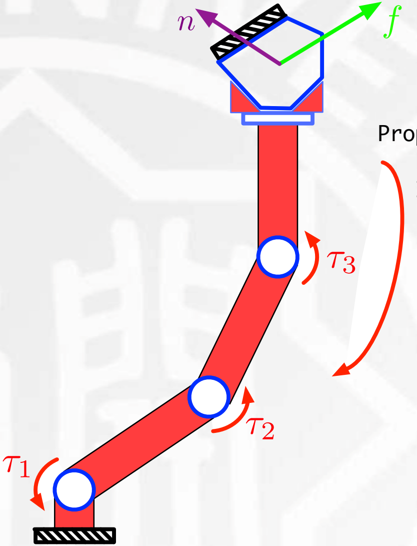
$$\tau = \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

$$\boxed{\tau = J^T F}$$

Static Force Propagation



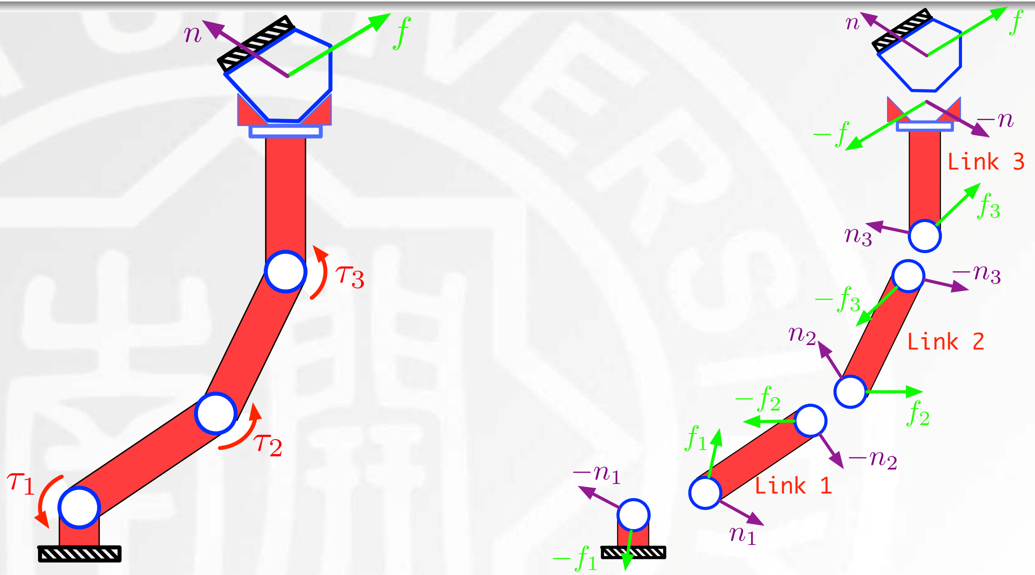
Static Force Propagation



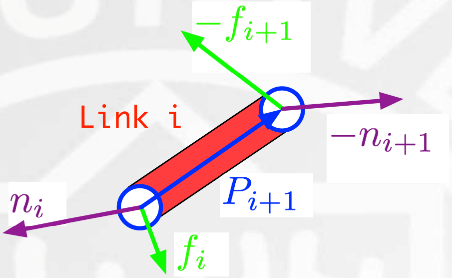
Propagation:
Elimination of
Internal forces

Energy Analysis
Virtual Work
Static Equilibrium

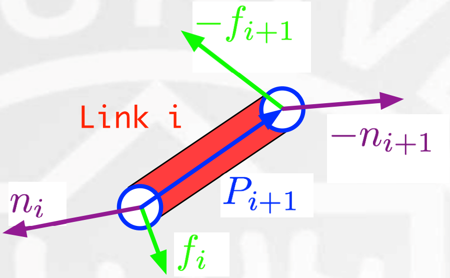
Static Force Propagation



Static Force Propagation



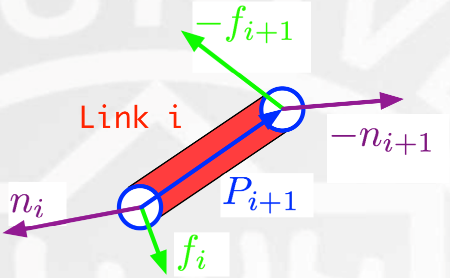
Static Force Propagation



- Static Equilibrium:

$$\sum \text{forces} = 0$$
$$\sum \text{moments at a point} = 0$$

Static Force Propagation



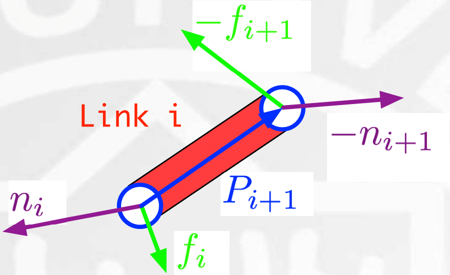
- Static Equilibrium:

$$\sum \text{forces} = 0$$
$$\sum \text{moments at a point} = 0$$

- About origin $\{i\}$

$$f_i + (-f_{i+1}) = 0$$
$$n_i + (-n_{i+1}) + P_{i+1} \times (-f_{i+1}) = 0$$

Static Force Propagation



- Static Equilibrium:

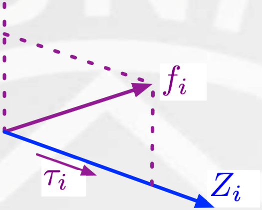
$$\sum \text{forces} = 0$$
$$\sum \text{moments at a point} = 0$$

- About origin $\{i\}$

$$f_i + (-f_{i+1}) = 0$$
$$n_i + (-n_{i+1}) + P_{i+1} \times (-f_{i+1}) = 0$$

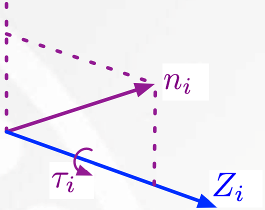
$$\boxed{\begin{aligned} f_i &= f_{i+1} \\ n_i &= n_{i+1} + P_{i+1} \times f_{i+1} \end{aligned}}$$

Static Force Propagation



Prismatic Joint

$$\tau_i = f_i^T Z_i$$



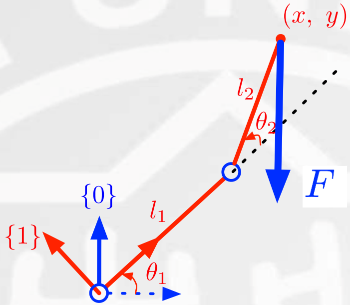
Revolute Joint

$$\tau_i = n_i^T Z_i$$

Algorithm:

$$
\begin{aligned}
{}^n f_n &= {}^n f \\
{}^n n_n &= {}^n n + {}^n P_{n+1} \times {}^n f \\
{}^i f_i &= {}^{i}_{i+1}R \cdot {}^{i+1} f_{i+1} \\
{}^i n_i &= {}^{i}_{i+1}R \cdot {}^{i+1} n_{i+1} + {}^i P_{i+1} \times {}^i f_i
\end{aligned}
$$

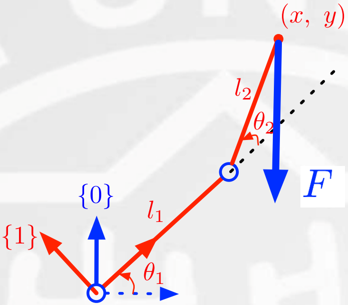
Example (Static Forces)



$$
\begin{aligned}
{}^n f_n &= {}^n f \\
{}^n n_n &= {}^n n + {}^n P_{n+1} \times {}^n f \\
{}^i f_i &= {}^i_{i+1}R \cdot {}^{i+1} f_{i+1} \\
{}^i n_i &= {}^i_{i+1}R \cdot {}^{i+1} n_{i+1} + {}^i P_{i+1} \times {}^i f_i
\end{aligned}
$$

- When $l_1 = l_2 = 1$, $\theta_1 = 0$, $\theta_2 = 60^0$, and $F = \begin{bmatrix} 0 & -1N \end{bmatrix}^T$

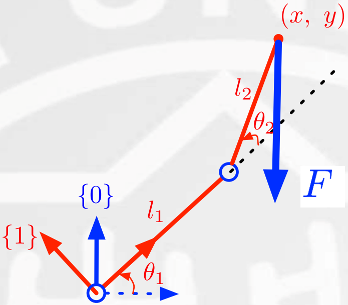
Example (Static Forces)



$$
\begin{aligned}
{}^n f_n &= {}^n f \\
{}^n n_n &= {}^n n + {}^n P_{n+1} \times {}^n f \\
{}^i f_i &= {}^i_{i+1} R \cdot {}^{i+1} f_{i+1} \\
{}^i n_i &= {}^i_{i+1} R \cdot {}^{i+1} n_{i+1} + {}^i P_{i+1} \times {}^i f_i
\end{aligned}
$$

- When $l_1 = l_2 = 1$, $\theta_1 = 0$, $\theta_2 = 60^0$, and $F = \begin{bmatrix} 0 & -1N \end{bmatrix}^T$
- 对于连杆 2：

Example (Static Forces)

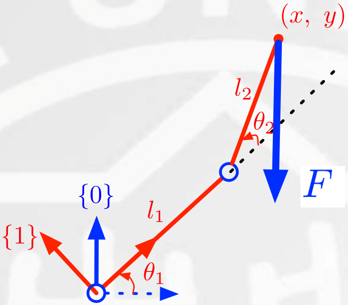


$$
\begin{aligned}
{}^{n}f_n &= {}^{n}f \\
{}^{n}n_n &= {}^{n}n + {}^{n}P_{n+1} \times {}^{n}f \\
{}^{i}f_i &= {}^{i}_{i+1}R \cdot {}^{i+1}f_{i+1} \\
{}^{i}n_i &= {}^{i}_{i+1}R \cdot {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i
\end{aligned}
$$

- When $l_1 = l_2 = 1$, $\theta_1 = 0$, $\theta_2 = 60^0$, and $F = \begin{bmatrix} 0 & -1N \end{bmatrix}^T$
- 对于连杆 2：

$$f_2 = F, \quad n_2 = n + \vec{l_2} \times F = \vec{l_2} \times F = \begin{bmatrix} 0 & 0 & -l_2\cos(\theta_1 + \theta_2)N \end{bmatrix}^T$$

Example (Static Forces)



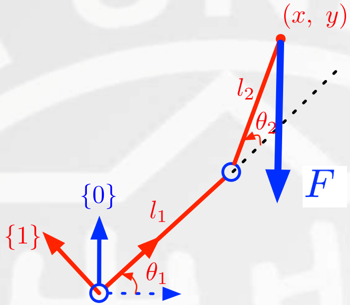
$$
\begin{aligned}
{}^n f_n &= {}^n f \\
{}^n n_n &= {}^n n + {}^n P_{n+1} \times {}^n f \\
{}^i f_i &= {}^i_{i+1}R \cdot {}^{i+1} f_{i+1} \\
{}^i n_i &= {}^i_{i+1}R \cdot {}^{i+1} n_{i+1} + {}^i P_{i+1} \times {}^i f_i
\end{aligned}
$$

- When $l_1 = l_2 = 1$, $\theta_1 = 0$, $\theta_2 = 60^0$, and $F = \begin{bmatrix} 0 & -1N \end{bmatrix}^T$
- 对于连杆 2：

$$f_2 = F, \quad n_2 = n + \vec{l_2} \times F = \vec{l_2} \times F = \begin{bmatrix} 0 & 0 & -l_2 \cos(\theta_1 + \theta_2) N \end{bmatrix}^T$$

- 对于连杆 1：

Example (Static Forces)



$$
\begin{aligned}
{}^{n}f_n &= {}^{n}f \\
{}^{n}n_n &= {}^{n}n + {}^{n}P_{n+1} \times {}^{n}f \\
{}^{i}f_i &= {}^{i}_{i+1}R \cdot {}^{i+1}f_{i+1} \\
{}^{i}n_i &= {}^{i}_{i+1}R \cdot {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i
\end{aligned}
$$

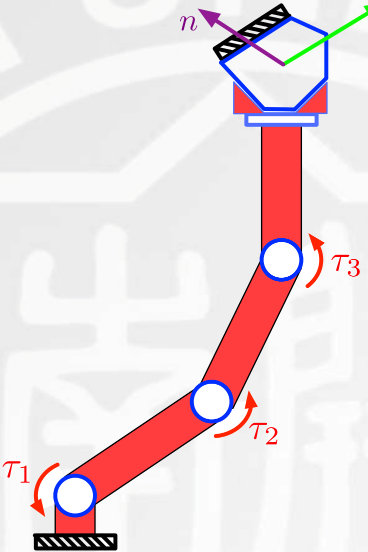
- When $l_1 = l_2 = 1, \theta_1 = 0, \theta_2 = 60^0$, and $F = [0 \quad -1N]^T$
- 对于连杆 2：

$$f_2 = F, \quad n_2 = n + \vec{l_2} \times F = \vec{l_2} \times F = [0 \quad 0 \quad -l_2 \cos(\theta_1 + \theta_2)N]^T$$

- 对于连杆 1：

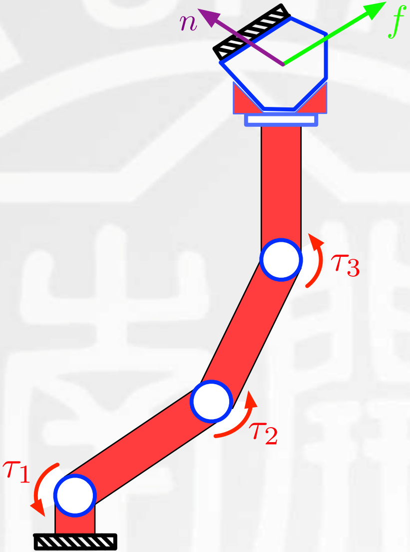
$$f_1 = f_2 = F, \quad n_1 = n_2 + \vec{l_1} \times f_2 = [0 \quad 0 \quad -[l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)]N]^T$$

Virtual Work Principal



$$\delta W = \sum_i \underbrace{f_i}_{\text{applied forces}} \underbrace{\delta x_i}_{\text{virtual displacements}}$$

Virtual Work Principal



$$\delta W = \sum_i f_i \, \delta x_i$$

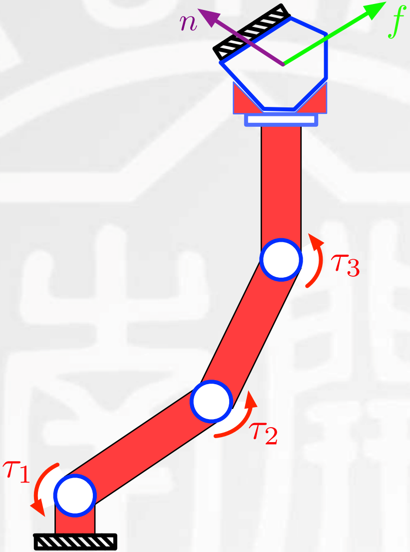
applied forces (f_i) virtual displacements (δx_i)

Static Equilibrium:

If the virtual work done by applied forces is zero in displacements consistent with constraints.

$$\tau^T \delta Q + (-F)^T \delta x = 0$$

Virtual Work Principal



$$\delta W = \sum_i \underbrace{f_i}_{\text{applied forces}} \underbrace{\delta x_i}_{\text{virtual displacements}}$$

Static Equilibrium:

If the virtual work done by applied forces is zero in displacements consistent with constraints.

$$\tau^T \delta Q + (-F)^T \delta x = 0$$

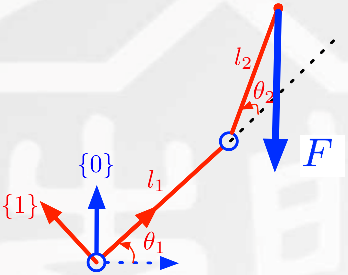
Using $\delta x = J \delta q$

$$\boxed{\tau = J^T F}$$

Example (Static Forces)

$$\dot{x} = J\dot{\theta}$$

$$\tau = J^T F$$



$$J = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

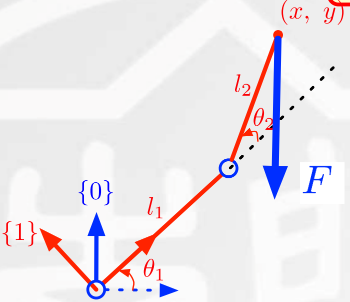
$$J^T = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & l_1 c_1 + l_2 c_{12} \\ -l_2 s_{12} & l_2 c_{12} \end{bmatrix}$$

- When $l_1 = l_2 = 1$, $\theta_1 = 0$, $\theta_2 = 60^0$, and $F = [0 \quad -1N]^T$

Example (Static Forces)

$$\dot{x} = J\dot{\theta}$$

$$\tau = J^T F$$



$$J = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

$$J^T = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & l_1 c_1 + l_2 c_{12} \\ -l_2 s_{12} & l_2 c_{12} \end{bmatrix}$$

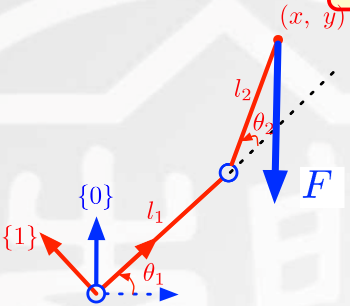
- When $l_1 = l_2 = 1$, $\theta_1 = 0$, $\theta_2 = 60^0$, and $F = [0 \quad -1N]^T$

$$\tau = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & l_1 c_1 + l_2 c_{12} \\ -l_2 s_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} 0 \\ -1N \end{bmatrix} = -\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

Example (Static Forces)

$$\dot{x} = J\dot{\theta}$$

$$\tau = J^T F$$



$$J = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

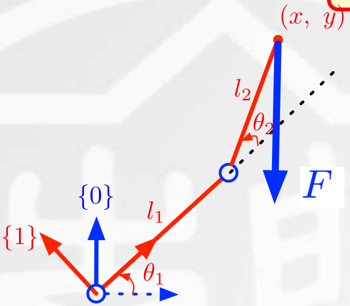
$$J^T = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & l_1 c_1 + l_2 c_{12} \\ -l_2 s_{12} & l_2 c_{12} \end{bmatrix}$$

- When $l_1 = l_2 = 1$, $\theta_1 = 90^0$, $\theta_2 = 0^0$, and $F = [0 \quad -1000N]^T$

Example (Static Forces)

$$\dot{x} = J\dot{\theta}$$

$$\tau = J^T F$$



$$J = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

$$J^T = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & l_1 c_1 + l_2 c_{12} \\ -l_2 s_{12} & l_2 c_{12} \end{bmatrix}$$

- When $l_1 = l_2 = 1$, $\theta_1 = 90^0$, $\theta_2 = 0^0$, and $F = [0 \quad -1000N]^T$

$$\tau = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & l_1 c_1 + l_2 c_{12} \\ -l_2 s_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} 0 \\ -1KN \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

第五章 机器人速度与雅可比

Chapter 5 Instantaneous Kinematics

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2021 年 4 月 9 日

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机器人与信息自动化研究所 Institute of Robotics and Automatic Information System