Optimal Piecewise Linear Approximation of Convex Functions

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***Abstract*— The optimal minimax solution to the N segment piecewise linear approximation of arbitrary convex differentiable functions over a finite range is described. The optimal solution is uniquely described by the derivatives at N distinct points. The optimality of the solution is proven and a recursive algorithm is proposed. The efficacy of the algorithm and optimality of the solution are demonstrated in example solutions for common functions such as 1/x and sin(πx).**

***Index Terms*— approximation linear minimax optimal piecewise**

I. Introduction

Approximation of functions by piecewise linear (PWL) functions are widely used in many hardware and software design applications. PWL functions allow the representation of arbitrary functions to any accuracy by simply increasing the number of segments until the desired accuracy is met. The purpose of this paper is to present a methodology for obtaining an optimal PWL function approximation to a convex (or concave) function over a finite range, where the PWL function is constrained to a fixed number of segments, and the optimization criteria is to minimize the maximum absolute error over the range, i.e. the minimax solution.

II. Background

A function $f(x)$ is approximated by a linear function $g(x)$ defined over the range $\alpha_i \le x \le \alpha_{i+1}$ by the segment connection endpoints (α_i, β_i) and $(\alpha_{i+1}, \beta_{i+1})$. The minimax solution is defined as the function g(x) such that it minimizes the error ε given by

$$\varepsilon = \max_{\alpha_i \le x \le \alpha_{i+1}} \left| g(x) - f(x) \right| . \quad (1)$$

Figure 1 shows the construct of the minimax solution where f(x) is a convex, continuously differentiable function. The segment connecting the endpoints $(\alpha_i, f(\alpha_i))$ and $(\alpha_{i+1}, f(\alpha_{i+1}))$ has a maximum error of 2ε. The segment connecting the endpoints $(\alpha_i, f(\alpha_i)+\varepsilon)$ and $(\alpha_{i+1}, f(\alpha_{i+1})+\varepsilon)$ has a maximum error of ε and is the minimax solution.

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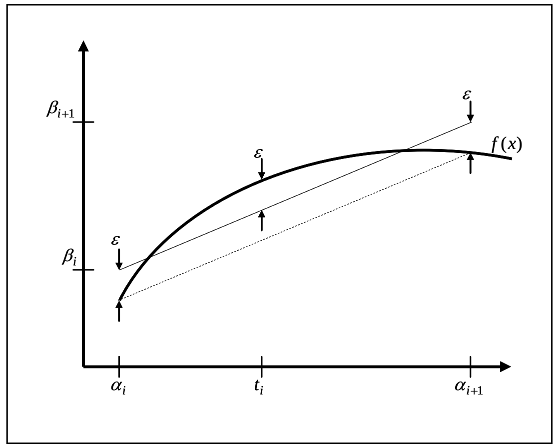

Figure 1 − 1 Segment Minimax Solution

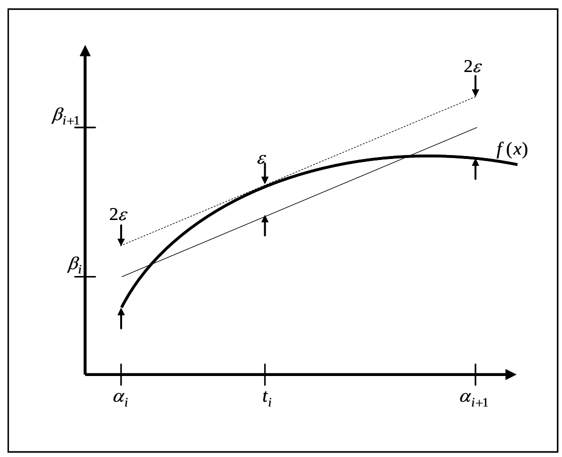

Figure 2 − Minimax Solution, Alternate Construct

Figure 2 shows an alternate construct of the minimax solution, which we will then use to construct the N segment piecewise linear solution. We define t_i as the segment *pivot*, where $f'(t_i) = \dfrac{\beta_{i+1} - \beta_i}{\alpha_{i+1} - \alpha_i}$. The minimax solution can then be defined as $g(x) = f'(t_i)(x - t_i) + f(t_i) - \varepsilon$, where the maximum error is ε. The tangent at t_i, given by $\tilde{g}(x) = f'(t_i)(x - t_i) + f(t_i)$, has the property that the error function $e(x) = \tilde{g}(x) - f(x)$ is strictly positive, convex, with maximum error ε at the endpoints α_i and α_{i+1}.

Furthermore, since the error function is convex, the minimax solution over an alternate range $\alpha_i \le x \le \alpha^*_{i+1}$ where $\alpha^*_{i+1} < \alpha_{i+1}$ results in $\varepsilon^* < \varepsilon$ and $t^*_i < t_i$. Similarly, if $\alpha^*_{i+1} > \alpha_{i+1}$, then $\varepsilon^* > \varepsilon$ and $t^*_i > t_i$.

III. PROPOSED SOLUTION

A convex function $f(x)$ is approximated by an N segment continuous piecewise linear function $g(x)$ defined over the range $\alpha_0 \le x \le \alpha_N$ by the set of points or *knots* $(\alpha_i, \beta_i)_{i=0}^N$ connected by N segments given by

$$g(x) = \sum_{i=0}^{N-1} g_i(x) \text{, where} \tag{2}$$

$$g_i(x) = \begin{cases} m_i x + b_i & \text{for} \quad \alpha_i \le x \le \alpha_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$m_i = \frac{\beta_{i+1} - \beta_i}{\alpha_{i+1} - \alpha_i} \tag{4}$$

$$b_i = \beta_i - m_i \alpha_i, \quad \text{for i=0 to N-1} \tag{5}$$

Furthermore, we define the set of pivots $(t_0, t_1, \ldots t_{N-1})$ where

$$f'(t_i) = m_i, \ \alpha_i \le t_i \le t_{i+1} \tag{6}$$

We propose that g(x) is the minimax optimal N segment piecewise linear approximation, i.e.

$$\varepsilon = \max_{\alpha_0 \le x \le \alpha_N} |g(x) - f(x)| \text{ is minimized} \tag{7}$$

if for i= 0 to N, and j = 0 to N-1:

$$|g(\alpha_i) - f(\alpha_i)| = |g(t_j) - f(t_j)| = \varepsilon. \tag{8}$$

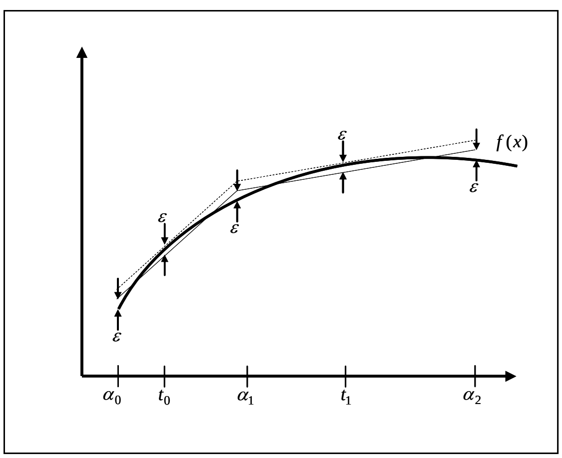

Figure 3 − 2 Segment Minimax Solution

Fig. 3 shows a representation of a 2 segment solution. Note that the set of pivots $(t_0, t_1, \ldots t_{N-1})$ fully defines g(x) and that g(x) may be alternately written as

$$g(x) = \max_{i=0}^{N-1} f'(t_i)(x - t_i) + f(t_i) - \varepsilon. \tag{9}$$

IV. PROOF OF OPTIMALITY

Define $g_i(x)$, α_i, and t_i such that

$$|g_i(\alpha_i) - f(\alpha_i)| = \varepsilon \text{, for i=0 to N.} \tag{10}$$

Assume $g^*(x)$, α^*, and t^* exist to represent a better solution (i.e. one with a smaller ε) such that

$$|g_i^*(\alpha_i^*) - f(\alpha_i^*)| < \varepsilon \text{, for i=0 to N} \tag{11}$$

and $\alpha_0^* = \alpha_0$ and $\alpha_N^* = \alpha_N$.

If

$$|g_0^*(\alpha_0) - f(\alpha_0)| < \varepsilon \tag{12}$$

and

$$|g_1^*(\alpha_1^*) - f(\alpha_1)| < \varepsilon, \tag{13}$$

then $t_0^* < t_0$, $\alpha_1^* < \alpha_1$.

Similarly, $\alpha_i^* < \alpha_i$, $t_i^* < t_i$ for i=1 to N-1.

Since $\alpha_N^* = \alpha_N$, if $t_{N-1}^* < t_{N-1}$, $\alpha_{N-1}^* < t_{N-1}$, then

$$|g_{N-1}^*(\alpha_N) - f(\alpha_N)| > \varepsilon. \tag{14}$$

Equation (14) shows that $|g_i^*(\alpha_i^*) - f(\alpha_i^*)| < \varepsilon$ cannot be true for N. Therefore, a better solution with a smaller ε does not exist and $g_i(x)$ as defined in (10) is the optimal minimax PWL solution.

V. RECURSIVE ALGORITHM

Recursive solutions to finding the minimax solution of piecewise linear approximations have been previously proposed in [1] and [2]. However, these solutions are based on splitting algorithms, where a search algorithm is used to find the segments that meet a certain error criteria. An upper bound and lower bound is used, and the algorithm searches for improved approximations by reducing the lower and upper bounds until it converges to an acceptable accuracy.

We propose a recursive algorithm for finding the optimal N segment PWL approximation directly by finding the solution to (8), the locations of the pivots $t_{i,j}$.

The algorithm, shown in Figure 4, starts with an estimate of the pivots $t_{i,j}$. The initial estimate $t_{i,0}$ can simply be uniformly spaced on $\alpha_0 < x < \alpha_N$. The tangent segments $g_{i,j}(x)$ and the knot locations $\alpha_{i,j}(x)$ at the intersection of the tangent segments are then computed based on the pivots. The errors at the knots, $\varepsilon_{i,j} = g_{i,j}(\alpha_{i,j}) - f(\alpha_{i,j})$, are computed and then used to provide a better estimate of the pivot locations $t_{i,j+1}$.

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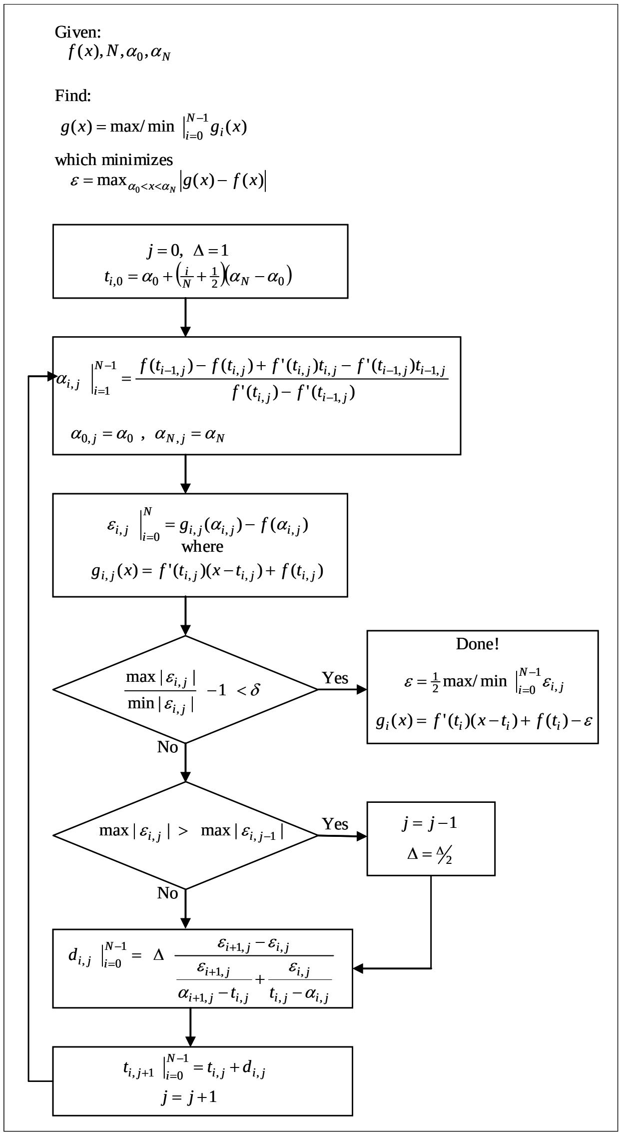

Figure 4 − PWL Search Algorithm

The computation of $t_{i,j+1}$ is depicted graphically in Figure 5. To ensure convergence, step control Δ is used to ensure that the error decreases with each iteration. The algorithm converges to t_i when the errors are essentially equal within a desired accuracy δ.

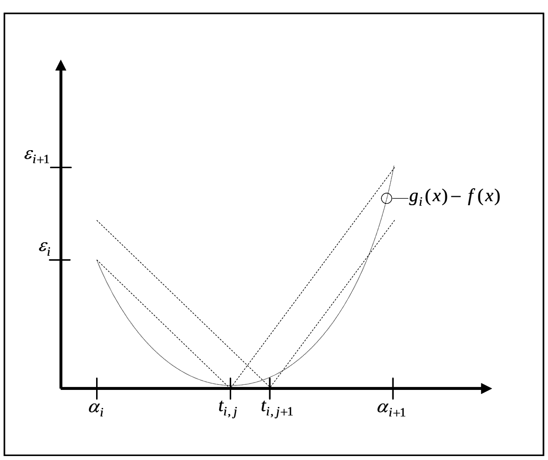

Figure 5 − Updating Pivot Estimates

VI. EXAMPLES

The efficacy of the recursive algorithm is shown in the following examples.

Table 1 shows the 8 segment solution for the function $f(x) = \frac{1}{x}$ from $1 \le x \le 2$. The pivots (t_i), knots (α_i, β_i), and segments $(m_i x + b_i)$ are shown. As discussed previously, the pivots t_i uniquely describe the solution and the knots and segments are easily derived once the pivot locations are known.

TABLE 1
8 SEGMENT PWL APPROXIMATION
$\frac{1}{x},\ 1 \le x \le 2\ ,\ \varepsilon = 0.00067$

i	t_i	α_i	β_i	m_i	b_i
0	1.0380	1.0000	0.9987	-0.9281	1.9274
1	1.1200	1.0775	0.9268	-0.7972	1.7863
2	1.2121	1.1643	0.8576	-0.6806	1.6506
3	1.3161	1.2620	0.7911	-0.5773	1.5202
4	1.4341	1.3726	0.7272	-0.4862	1.3953
5	1.5687	1.4984	0.6661	-0.4064	1.2756
6	1.7231	1.6423	0.6076	-0.3368	1.1614
7	1.9015	1.8079	0.5518	-0.2766	1.0524
8		2.0000	0.4987		

Figure 6 shows the piecewise linear approximation g(x) and the error e(x) = g(x)-1/x over the range $1 \le x \le 2$. The equal maximum errors in each segment clearly demonstrate that g(x) is the minimax solution and that the error of the approximation is 0.00067.

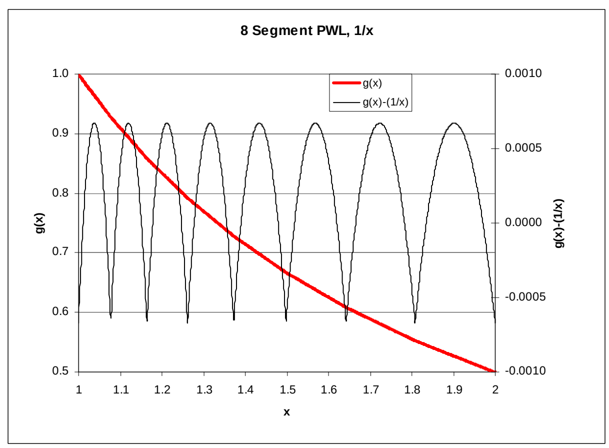

Figure 6 − 8 Segment PWL Error
$\frac{1}{x},\ 1 \le x \le 2\ ,\ \varepsilon = 0.00067$

Figure 7 shows the construction of the PWL using the 8 segments displayed over the entire range, showing the relationship between the pivots and tangent segments in creating the piecewise linear approximation.

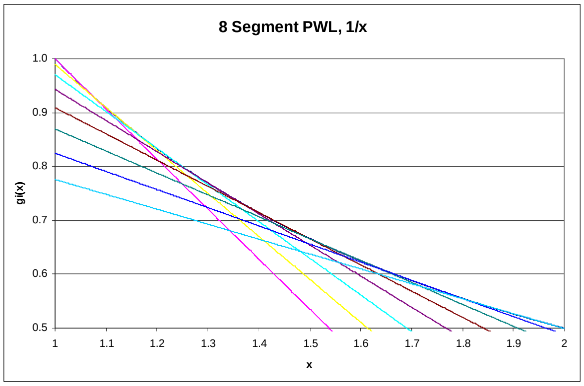

Figure 7 – 8 Segment PWL

$\frac{1}{x},\ 1 \le x \le 2\ ,\ g_i(x)\big|_{i=0}^{7}$

A second example shows the algorithms' easy scalability to more complex solutions. Table 2 shows the pivots for the 64 segment approximation of $\sin(\pi x)$ from $0 \le x \le 0.5$. The pivot locations are an efficient method of describing the solution and the knots and segments are easily computed from the pivots.

TABLE 2

64 Segment PWL Approximation, pivots t_i

$\sin(\pi x),\ 0 \le x \le 0.5\ ,\ \varepsilon = 0.000022$

0.016198	0.191817	0.305997	0.406857
0.037175	0.199673	0.312562	0.412945
0.053101	0.207396	0.319083	0.419018
0.066903	0.214999	0.325561	0.425075
0.079413	0.222489	0.331999	0.431119
0.091019	0.229877	0.338399	0.437150
0.101943	0.237168	0.344764	0.443169
0.112327	0.244370	0.351095	0.449179
0.122270	0.251488	0.357395	0.455179
0.131841	0.258529	0.363665	0.461171
0.141095	0.265496	0.369906	0.467156
0.150074	0.272395	0.376122	0.473136
0.158811	0.279229	0.382312	0.479111
0.167333	0.286003	0.388480	0.485082
0.175663	0.292720	0.394625	0.491050
0.183819	0.299384	0.400750	0.497017

The PWL approximation and error of the approximation are shown in Figure 8. The maximum error for 64 segments is 0.000022. Figure 9 shows the construction of the PWL approximation from the 64 segments.

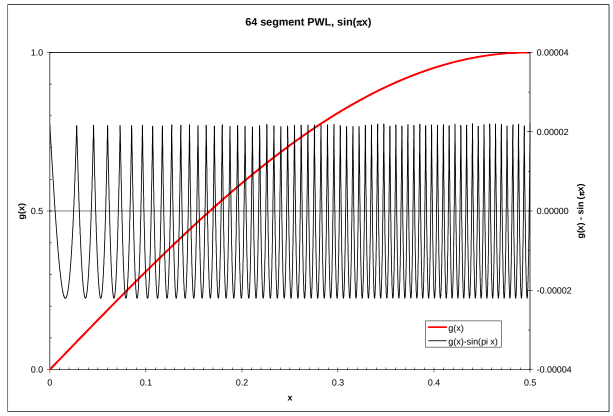

Figure 8 – 64 Segment PWL Error

$\sin(\pi x),\ 0 \le x \le 0.5\ ,\ \varepsilon = 0.000022$

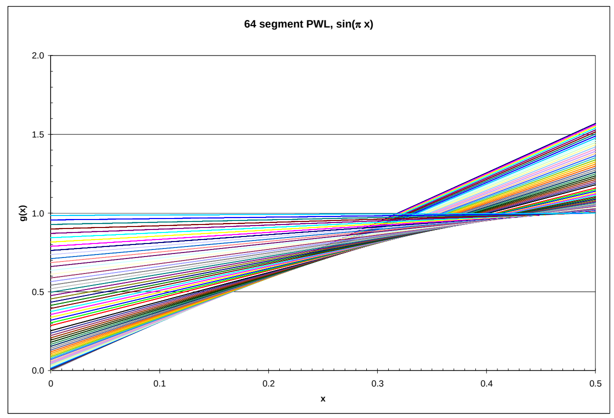

Figure 9 – 64 Segment PWL

$\sin(\pi x),\ 0 \le x \le 0.5\ ,\ g_i(x)\big|_{i=0}^{63}$

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