Stat 310A/Math 230A Theory of Probability

Homework 3

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Solutions should be complete and concisely written. Please, use a separate sheet (or set of sheets) per each problem. Staple sheets referring to the same problem, and write your name on each sheet.

You are welcome to discuss problems with your colleagues, but should write and submit your own solution.

In some cases, multiple homework options will be proposed (and indicated as 'Option 1', 'Option 2', etc.). You are welcome to work on all the problems proposed (solutions will be posted), but should submit only those corresponding to one 'Option'.

Exercises on inequalities and convergence

Solve Exercises [1.3.21], [1.3.36], [1.3.37] from Amir Dembo's lecture notes.

Exercises on L_p spaces

Fix a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Throughout this exercise, we will say that two random variables X, Y are equivalent if $\{\omega : X(\omega) \neq Y(\omega)\} \subseteq \Omega_0$ for some $\Omega_0 \in \mathcal{F}$ with $\mathbf{P}(\Omega_0) = 0$.

1. Show that the above indeed defines an equivalence relation.

Recall that for $p > 0$, $L_p(\Omega, \mathcal{F}, \mathbf{P})$ is the space of (equivalence classes of) random variables X such that $\mathbf{E}\{|X|^p\} < \infty$.

2. Show that, for $p \geq 1$, $||X||_p \equiv \mathbf{E}\{|X|^p\}^{1/p}$ is a norm on this space.

For $p = \infty$, $L_\infty(\Omega, \mathcal{F}, \mathbf{P})$ is the space of (equivalence classes of) random variables X such that there exists $M < \infty$ such that $\mathbf{P}(\{\omega : |X(\omega)| \leq M\}) = 1$.

3. Show that $||X||_\infty \equiv \inf\{M \ : \ \mathbf{P}(\{\omega : |X(\omega)| \leq M\}) = 1\}$ is a norm on $L_\infty(\Omega, \mathcal{F}, \mathbf{P})$.
4. For $X \in L_\infty(\Omega, \mathcal{F}, \mathbf{P})$, show that $X \in L_p(\Omega, \mathcal{F}, \mathbf{P})$ for any $p > 0$, and that $||X||_p \to ||X||_\infty$ as $p \to \infty$.
5. For X a random variable, let $S(X) \equiv \mathbf{P}(\{\omega : X(\omega) \neq 0\})$. Show that, if $X \in L_q(\Omega, \mathcal{F}, \mathbf{P})$ for some $q > 0$, then $\lim_{p\to 0} ||X||_p^p = S(X)$.
6. Show that the space of simple functions SF is dense in $L_p(\Omega, \mathcal{F}, \mathbf{P})$ for any $0 < p \leq \infty$

Optional: For the enthusiasts

This will not be graded, but is an interesting and well known fact: $L_p(\Omega, \mathcal{F}, \mathbf{P})$ is a *Banach space* for all $1 \leq p \leq \infty$. This means (beyond the fact of being a normed vector space) the following. If $\{X_n\}$ is a Cauchy sequence of (equivalence classes of) random variables in $L_p(\Omega, \mathcal{F}, \mathbf{P})$, then $\{X_n\}$ converges to a limit X_∞ in $L_p(\Omega, \mathcal{F}, \mathbf{P})$. *Cauchy* means that, for any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that, if $m, n \geq N(\varepsilon)$, then $||X_m - X_n||_p \leq \varepsilon$.

At this point of the course, you know all that is needed to prove this fact (or to read a proof ;-).

[Hint: Every Cauchy sequence converges if the following happens: for any sequence $\{Z_n\}$ such that $\sum_{k=1}^{\infty} ||Z_n||_p < \infty$ the sums $W_n = \sum_{k=1}^{n} Z_k$ converge (in the p-norm).]