Solutions to Homework One

CSE 101

1. (a) By the usual formula for a geometric series, $1 + 2 + \cdots + 2^n = 2^{n+1} - 1 \leq 2 \cdot 2^n$. Thus it is $O(2^n)$.
   (b) $1 + \frac{1}{2} + \cdots + \frac{1}{2^n} \leq 2$, and thus the sum is $O(1)$.
   (c) By the geometric series formula,
   $$S(n) = \frac{c^{n+1} - 1}{c - 1}.$$
   When $c < 1$, the formula above can more conveniently be written as
   $$S(n) = \frac{1 - c^{n+1}}{1 - c},$$
   from which we see that $S(n)$ lies between 1 and $1/(1-c)$ and is hence $\Theta(1)$. For $c = 1$, $S(n) = n+1$, which is $\Theta(n)$. For $c > 1$, by the formula, $S(n)$ lies between c^n and $\frac{c}{c-1} \cdot c^n$ and is thus $\Theta(c^n)$.

2. Let $S(n)$ denote the statement $1 + 3 + \cdots + (2n-1) = n^2$. We'll show by induction that $S(n)$ is true for all positive integers n.

   Base case: $n = 1$. This is immediate.

   Inductive step. Suppose $S(1), \ldots, S(k)$ are true. We'll show that $S(k+1)$ is also true. This is because
   $$1 + 3 + \cdots + (2k+1) \;=\; (1 + 3 + \cdots + (2k-1)) + (2k+1) \;=\; k^2 + (2k+1)$$
   where the last equation uses the inductive hypothesis to simplify $1 + 3 + \cdots + (2k-1)$. We finish by simplifying $k^2 + (2k+1) = (k+1)^2$.

3. Define $S(n) = 1^c + \cdots + n^c$.

   (a) Each term in the series is at most n^c. Therefore, $S(n) \leq n^c + \cdots + n^c = n \cdot n^c = n^{c+1}$.
   (b) Each term in the 2nd half of the series is at least $(n/2)^c$, so $S(n) \geq (n/2) \cdot (n/2)^c = n^{c+1}/2^{c+1}$.

4. (a) Using m bits, we can write down all the numbers $< 2^m$. Therefore, writing down a number n requires $O(\log n)$ bits.
   (b) n needs to be halved $O(\log n)$ times before it goes below 1; therefore "hello" is printed this often.
   (c) Subroutine A is invoked on $n, n/2, n/4, n/8, \ldots$. The time for division is dominated by the time taken by A. Thus the total running time is
   $$O\left(n^3 + \left(\frac{n}{2}\right)^3 + \left(\frac{n}{4}\right)^3 + \left(\frac{n}{8}\right)^3 + \cdots\right) = O(n^3) \times \left(1 + \frac{1}{2^3} + \frac{1}{4^3} + \frac{1}{8^3} \cdots\right) = O(n^3).$$

5. (a) *True.* $\log_2 n = (\log_2 3)(\log_3 n)$, which is $O(\log_3 n)$.
   (b) *False.* $2^{\log_2 n} = n$ whereas $2^{\log_3 n} = 2^{(\log_3 2)(\log_2 n)} = (2^{\log_2 n})^{\log_3 2} = n^{\log_3 2}$, which is much smaller than n.
   (c) *True.* $(\log_2 n)^2 = ((\log_2 3)(\log_3 n))^2 = (\log_2 3)^2(\log_3 n)^2$.

6. (a) $100n^3 + 3^n = O(3^n)$.
   (b) $200n \log(200n) = O(n \log n)$.
   (c) $100n^2 2^n + 3^n = O(3^n)$.
   (d) $100n \log n + 20n^3 + \sqrt{n} = O(n^3)$.
   (e) $1^3 + 2^3 + \cdots + n^3 = O(n^4)$.

7. *d-ary tree.* Here $d \geq 2$.

   (a) The maximum number of nodes at level j is d^j.
   (b) The maximum number of nodes in a tree of depth k is at most $1 + d + \cdots + d^k \leq d^{k+1}$.
   (c) If the tree has n nodes, its depth k must satisfy $n \leq d^{k+1}$, and thus k is $\Omega(\log_d n)$.