Stat 310A/Math 230A Theory of Probability

Practice Midterm

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This is a practice midtetm (indeed it was the 2010 Midterm): rules described beow will be the same as for the real Midterm. Solutions wil be posted.

Solutions should be complete and concisely written. Please, use a separate booklet for each problem.

You have 3 hours but you are not required to solve all the problems!!!

Just solve those that you can solve within the time limit.
For any clarification on the text, one of the TA's will be outside the room.
You can consult the textbooks used as references and your notes, but no other books. You cannot use computers, and in particular you cannot use the web. You can cite theorems (propositions, corollaries, lemmas, etc.) from Amir Dembo's lecture notes by number, and exercises you have done as homework by number as well. Any other non-elementary statement must be proved!

Problem 1

Let λ_2 be the Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2})$. We know already that it is invariant under translation i.e. that $\lambda_2(B + x) = \lambda_2(B)$ for any Borel set B and $x \in \mathbb{R}^2$ (whereby $B + x = \{y \in \mathbb{R}^2 : y - x \in B\}$).

(a) Show that it is invariant under rotations as well, i.e. that for any $\alpha \in [0, 2\pi]$, and any Borel set $B \subseteq \mathbb{R}^2$, $\lambda_2(\mathsf{R}(\alpha)B) = \lambda_2(B)$ (whereby $\mathsf{R}(\alpha)$ denotes a rotation by an angle α and $\mathsf{R}(\alpha)B = \{x \in \mathbb{R}^2 : \mathsf{R}(-\alpha)x \in B\}$).

(b) For $s \in \mathbb{R}_+$, and $B \subseteq \mathbb{R}^2$ Borel, let $sB \equiv \{x \in \mathbb{R}^2 : s^{-1}x \in B\}$. Prove that $\lambda_2(sB) = s^2\lambda_2(B)$.

(c) For $r > 0$, $0 \le \alpha < \beta \le 2\pi$, let

$$C_{r,\alpha,\beta} \equiv \big\{x = (u\cos\theta, u\sin\theta) \,:\, u \in [0, r]\,, \theta \in [\alpha, \beta]\big\}\,. \tag{1}$$

Prove that $\lambda_2(C_{r,\alpha,\beta}) = (\beta - \alpha)r^2/2$.

(d) Let $\Omega \equiv [0, 2\pi] \times [0, \infty)$, $g : \Omega \to \mathbb{R}_+$ be given by $g(\theta, r) = r$, and define ρ to be the measure on $(\Omega, \mathcal{B}_\Omega)$ with density g with respect to the Lebesgue measure.

For any function $f \in L_1(\mathbb{R}^2, \mathcal{B}_{\mathbb{R}_2}, \lambda_2)$, let $\widehat{f} : \Omega \to \mathbb{R}$ be defined by $\widehat{f}(r, \theta) \equiv f(r\cos\theta, r\sin\theta)$. Prove that $f \in L_1(\Omega, \mathcal{B}_\Omega, \rho)$, and that

$$\int_\Omega \widehat{f}\; \mathrm{d}\rho = \int_{\mathbb{R}^2} f\; \mathrm{d}\lambda_2\,. \tag{2}$$

Problem 2

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $\{A_n\}_{n\in\mathbb{N}}$ a sequence of measurable sets and $f \in L_1(\Omega, \mathcal{F}, \mu)$. Assume that

$$\lim_{n\to\infty} \int |\mathbb{I}_{A_n} - f|\, \mathrm{d}\mu = 0\,. \tag{3}$$

Prove that there exists $A \in \mathcal{F}$ such that $f = \mathbb{I}_A$ almost everywhere.

Problem 3

Let $f_1, f_2 : [0,1] \to \mathbb{R}$ be two Borel functions with $f_1(x) \le f_2(x)$ for all $x \in [0,1]$, and define $A \subseteq \mathbb{R}^2$ by

$$A \equiv \big\{(x,y) \in [0,1] \times \mathbb{R} \,:\, f_1(x) \le y \le f_2(x) \,\big\} \tag{4}$$

(a) Prove that A is a Borel set.

(b) Denoting by λ_d the Lebesgue measure on $\mathbb{R}^d$, prove that

$$\lambda_2(A) = \int_{[0,1]} [f_2(x) - f_1(x)]\, \mathrm{d}\lambda_1(x)\,. \tag{5}$$

(c) For a Borel function $f : [0,1] \to \mathbb{R}$, and $y \in \mathbb{R}$, let

$$A_y \equiv \big\{x \in [0,1] \,:\, y = f(x) \,\big\}\,. \tag{6}$$

Prove that $\lambda_1(A_y) = 0$ for almost every y.

Problem 4

Let $\Omega = \{\mathsf{red}, \mathsf{blue}\}^{\mathbb{Z}^2}$ be the set of all possible ways to color the vertices of $\mathbb{Z}^2$ (the infinite 2-dimensional lattice) with two colors (red and blue). Denote $\mathcal{F}$ the σ-algebra generated sets of the type An element of this space is an assignment of colors $\omega : x \mapsto \omega_x \in \{\mathsf{red}, \mathsf{blue}\}$ for all $x \in \mathbb{Z}^2$.

Let A_x be the set of conficurations such that vertex x is red: $A_x = \{\omega : \omega_x = \mathsf{red}\}$, and consider the σ-algebra $\mathcal{F} \equiv \sigma(\{A_x : x \in \mathbb{Z}^2\})$.

Given a coloring ω, a *red cluster* R is a connected subset of red vertices. By 'connected' we mean that for any two vertices $x, y \in R$, there exists a nearest-neighbors path of red vertices connecting them (i.e. a sequence $x_1, x_2, \dots, x_n \in \mathbb{Z}^2$ such that $x_1 = x$, $x_n = y$, $\|x_{i+1} - x_i\| = 1$ and $\omega_{x_i} = \mathsf{red}$ for all i.

(a) Let $C \subseteq \Omega$ be the subset of configurations defined by

$$C = \big\{\omega \,:\, \omega \text{ contains an infinite red cluster } \big\}\,. \tag{7}$$

Prove that $C \in \mathcal{F}$.

(b) Let $p \in [0,1]$ be given and define $\mathbb{P}$ to be the probability measure on $(\Omega, \mathcal{F})$ such that the collection of events $\{A_x \,:\, x\mathbb{Z}^2\}$ are mutually independent, with $\mathbb{P}(A_x) = p$ for all $x \in \mathbb{Z}^2$.

Prove that either $\mathbb{P}(C) = 1$ or $\mathbb{P}(C) = 0$.

Problem 5

Consider the measurable space $(\Omega, \mathcal{F})$, with: $\Omega = \{0,1\}^{\mathbb{N}}$ the set of (infinite) binary sequences $\omega = (\omega_1, \omega_2, \omega_3, \dots)$; $\mathcal{F}$ the σ-algebra generated by cylindrical sets (equivalently the σ-algebra generated by sets of the type $A_{i,x} = \{\omega : \omega_i = x\}$ for $i \in \mathbb{N}$ and $x \in \{0,1\}$).

Let $\mathbb{P}$ be the probability measure on $(\Omega, \mathcal{F})$ such that for all n

$$\mathbb{P}\big(\{\omega \,:\, (\omega_1, \dots, \omega_n) = (x_1, \dots, x_n)\}\big) = \frac{1}{2} \prod_{i=1}^{n-1} p_i(x_i, x_{i+1})\,, \tag{8}$$

where

$$p_i(x_i, x_{i+1}) = \begin{cases} 1 - \frac{1}{i^2} & \text{if } x_i = x_{i+1}, \\ \frac{1}{i^2} & \text{otherwise.} \end{cases} \tag{9}$$

(a) Prove that a probability measure satisfying (8) does indeed exist.

(b) Let $X_i(\omega) = \omega_i$ and consider the tail σ-algebra $\mathcal{T} = \cap_{n=1}^{\infty} \sigma(\{X_m : m \geq n\})$. Is $\mathcal{T}$ trivial? Prove your answer.