1. Suppose you are given an arbitrary directed graph $G = (V, E)$ with arbitrary edge weights $\ell : E \to \mathbb{R}$. Each edge in G is colored either red, white, or blue to indicate how you are permitted to modify its weight:

   - You may increase, but not decrease, the length of any red edge.
   - You may decrease, but not increase, the length of any blue edge.
   - You may not change the length of any black edge.

   The ***cycle nullification*** problem asks whether it is possible to modify the edge weights—subject to these color constraints—so that *every cycle in G has length* 0. Assume that G is strongly connected.

   (a) Describe a linear program that is feasible if and only if it is possible to make every cycle in G have length 0.

   Solution: As suggested by the hint, our linear program has a variable $dist(v)$ for every vertex v of the graph, which represents the length of every walk from s to v *with the new edge lengths*.

$$\begin{array}{lrcll} \text{maximize} & & 0 & & \\ \text{subject to} & dist(v) - dist(u) & \geq & \ell(u\to v) & \text{for every red edge } u\to v \\ & dist(v) - dist(u) & = & \ell(u\to v) & \text{for every white edge } u\to v \\ & dist(v) - dist(u) & \leq & \ell(u\to v) & \text{for every blue edge } u\to v \\ & dist(s) & = & 0 & \end{array}$$

   Because we only care about feasibility, the objective function doesn't actually matter here; the objective function 0 is convenient for part (b).

   (The last constraint $dist(s) = 0$ is actually redundant.) ■

   Rubric: 5 points.

   (b) Construct the dual of the linear program from part (a). *[Hint: Choose a convenient objective function for your primal LP.]*

   Solution: We have a dual variable $f(u\to v)$ for each edge $u\to v$, corresponding to the primal constraints.

$$\begin{array}{lrcll} \text{minimize} & \displaystyle\sum_{u\to v} f(u\to v) \cdot \ell(u\to v) & & & \\ \text{subject to} & \displaystyle\sum_{u\to v} f(u\to v) - \sum_{v\to w} f(v\to w) & = & 0 & \text{for every vertex } v \neq s \\ & f(u\to v) & \leq & 0 & \text{for every red edge } u\to v \\ & f(u\to v) & \geq & 0 & \text{for every blue edge } u\to v \end{array}$$

   I called the dual variable f because the vertex constraints look like flow conservation; that's also why I chose the primal objective vector 0.

   (If we omit the redundant constraint $dist(s) = 0$ from the primal LP, the dual LP includes a redundant conservation constraint at s.) ■

   Rubric: 5 points.

(c) Give a self-contained description of the combinatorial problem encoded by the dual linear program from part (b), and prove *directly* that it is equivalent to the original cycle nullification problem. Do not use the words "linear", "program", or "dual".

Solution: Let H be the graph obtained from G by inserting the reversal $v{\to}u$ of every white or red edge $u{\to}v$, defining $\ell(v{\to}u) = -\ell(u{\to}v)$ for each reversed edge, and then deleting every original red edge. The dual LP is an uncapacitated minimum-cost flow problem.

I claim that ***all cycles in G can be nullified if and only if H does not contain a negative cycle.*** As usual, the proof has two parts.

($\Rightarrow$) Suppose all cycles in G can be nullified. Let $\ell' \colon E \to \mathbb{R}$ be any new length function such that all cycles in G have length 0.

Fix two vertices s and v, let α and β be two walks from s to v, and let γ be a walk from v to s (which must exist because G is strongly connected). The closed walks $\alpha \cdot \gamma$ and $\beta \cdot \gamma$ are composed of cycles and therefore have length zero. Thus α and β have the same length, namely the negation of the length of γ. We conclude that all walks from s to v have the same length.

Fix an arbitrary vertex s in G, and then for each vertex v, let $dist'(v)$ denote the common length of every walk from s to v in G with respect to the new edge lengths ℓ'. Think of each $dist'(v)$ as an estimated shortest-path distance in H.

To prove that H has no negative cycles (with the original edge lengths ℓ), it suffices to show that no edge in H is tense. Let $u{\to}v$ be an arbitrary edge in H; there are two cases to consider:

- If $u{\to}v$ is a (blue or white) edge in G, then

$$dist'(v) - dist'(u) \;=\; \ell'(u{\to}v) \;\le\; \ell(u{\to}v),$$

  which means $u{\to}v$ is not tense in H.
- If $v{\to}u$ is a (red or white) edge in G, then

$$dist'(u) - dist'(v) \;=\; \ell'(v{\to}u) \;\ge\; \ell(v{\to}u) \;=\; -\ell(u{\to}v)$$

  and thus $dist'(v) - dist'(u) \le \ell(u{\to}v)$, which means $u{\to}v$ is not tense in H.

($\Leftarrow$) Now suppose H does not contain a negative cycle. Then shortest-path distances in H are well-defined. Add a new vertex $\hat{s}$ with zero-length edges to every vertex in H, and then for each vertex v, let $dist(v)$ denote the shortest-path distance from $\hat{s}$ to v in H. (We need the extra vertex $\hat{s}$ because there might be no vertex that can reach every other vertex in H.) Finally, for every edge $u{\to}v$ in G, define $\ell'(u{\to}v) := dist(v) - dist(u)$.

Let $u{\to}v$ be an arbitrary edge in G. We need to verify that $\ell'(u{\to}v)$ is at least, at most, or equal to $\ell(u{\to}v)$, depending on the color of $u{\to}v$. There are three cases to consider.

- Suppose $u{\to}v$ is blue or white. Then

$$\ell'(u{\to}v) \;=\; dist(v) - dist(u) \;\le\; \ell(u{\to}v)$$

  because $u{\to}v$ is not tense in H.

– Suppose $u \to v$ is red or white. Then

$$\ell'(u \to v) \;=\; dist(v) - dist(u) \;\geq\; -\ell(v \to u) \;=\; \ell(u \to v)$$

because $v \to u$ is not tense in H.

– The previous two cases imply that if $u \to v$ is white, then $\ell'(u \to v) = \ell(u \to v)$.

We conclude that the new lengths are consistent with the edge colors.

Finally, any cycle $v_0 \to v_1 \to \ldots \to v_{k-1} \to v_0$ in G has length zero, because

$$\sum_{i=0}^{k-1} \ell'(v_i \to v_{i+1 \bmod k}) \;=\; \sum_{i=0}^{k-1} \big(dist(v_{i+1 \bmod k}) - dist(v_i)\big) \;=\; 0.$$

(Each term $dist(v_i)$ appears once positively and once negatively in the second sum.)

■

Rubric: 8 points = 2 for “no negative cycles in H + 3 for if proof + 3 for only if proof.

(d) Describe and analyze an algorithm to determine *in $O(EV)$ time* whether it is possible to make every cycle in G have length 0, using your dual formulation from part (c).

Solution: We can construct the graph H in $O(V + E)$ time, and then find negative cycles in H using a modification of the Bellman-Ford shortest-path algorithm, as described in the lecture notes, in $O(VE)$ time. ■

Rubric: 2 points.