CS124: Programming Assignment 3

Group Members: April 21, 2017
HUID: 80977816
HUID: 80983115

1 Problem Definition

In this programming assignment we implemented three heuristics for the Number Partition Problem which his NP-complete where the goal is the minimize the residue of the array. Residue is represented as u where

$$u = |\sum_{i=1}^{n} s_i \cdot a_i|$$

We implemented each of the three heuristics by calculating the residue of the generated array, and by splitting the set of numbers using the prepartitioning method.

2 Dynamic Programming Solution

The dynamic programming solution to the Number Partition problem runs in $O(ns)$ time and uses $O(ns)$ space, where s is the sum of the integers in A and n is the length of A. The algorithm will return 1 if the given array can be made into two subsets with equal sums (difference is 0), and 0 if it cannot. For a given array A, let $T(m, j)$ be 1 if there is a subset of the first j elements of A which sums to m. If the sum of the elements in A is s, we want to compute $T(\lceil s/2 \rceil, n)$. If s is even then this just equals $s/2$ but if s is odd (and thus there is no perfect partition), this will tell us if it's possible to split A into two subsets whose sums differ by 1, which is the best we can do in the odd case. We know that $T(0, i) = 1$ for $0 \leq i \leq n$ because the empty set is a subset of every set and has a sum of zero. The following recurrence will let us calculate $T(\lceil s/2 \rceil, n)$:

$$T(m, j) = \begin{cases} 1 & T(m, j-1) = 1 \text{ or } T(m - A[j], j-1) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Essentially if we need to figure out if there's a subset of the first j elements which have subset that sums to m, either the first $j-1$ elements could contain such a subset in which case we wouldn't include $A[j]$ in it or the first $j-1$ elements could a subset that sums to $m - A[j]$ and so we should include $A[j]$ in that subset. If neither of these cases are true, then it isn't possible for there to be a subset of the first j elements with a subset that sums to m. We can build up to $T(\lceil s/2 \rceil, n)$ by iterating over an array and storing all these results in a table. At the end of the process, to find the size of the largest possible subset we can make without exceeding $\lceil s/2 \rceil$, check the space that stored $T[\lceil s/2 \rceil][n]$ and if it's 1 we return 1, otherwise decrement the first argument which corresponds to the sum until we encounter a

1 and return that value.

In order to actually construct the partitions every time we calculate $T(m, j)$ we should include in our array whether or not we've included the jth element. So our table will actually store pairs of truth values that correspond to whether there is a subset of the first j elements which sum to s and whether that sum includes element j. We've shown above how to calculate the first truth value (that corresponds to the 0's and 1's). If $T(m, j)$ is 0, then set the second value to be false also. If $T(m, j-1)$ is true, then set the second value to false because we can't include $A[j]$ in the sum ($A[j]$ is non zero and positive). If $T(m - A[j], j-1)$ is true, then we include $A[j]$ and set the second value to true. Now to construct the partition we will basically traverse backwards through the array. So start with whatever the largest value of m is such that $T(m, n)$ is true. Then if the nth element was included in that subset (which we precomputed) go to $T(m - A[n], n-1)$. If the nth element was not included check what's in $T(m, n-1)$. Repeat this process until $T(0, 0)$ is reached, which must happen because we must have found some partition.

3 Algorithms and Results

If we assume that arithmetic operations can be done in constant time then the Karmarkar-Karp heuristic algorithm can be implemented in $O(n \log n)$ steps using a max-heap. First, transforming an array into a max heap takes $O(n \log n)$ time naively because each each insertion takes at worst $O(\log n)$ operations and this has to be done for all n elements in the list. Then to run Karmarkar-Karp, we can pop off the maximum element in constant time and then fix the heap in $\log n$ time. Then we can reinsert the difference into the heap and now the heap has 1 fewer element in it than before. We continue this process until the heap has 1 element left in it and that is the residue. Each step the number of elements in the heap decreases by 1 and we have to do $O(\log n)$ steps on each iteration and so this algorithm runs in $O(n \log n)$ time.

We ran the repeated random, hill climbing, and simulated annealing with and without pre-partitioning, as well as Karmarkar-Karp, 100 times on randomly generated arrays and calculated the average time to run each one as well as the average residual each algorithm found. The random, hill climbing, and simulated annealing each did 25000 iterations per trial.

Algorithm	Average Residual	Average time (s)
Karmarkar-Karp	193170.18	0.000047
Repeated Random	294757958.45	0.038612
Hill Climbing	272129392.28	0.024985
Simulated Annealing	243988833.34	0.039531
Repeated Random Pre-partition	188.18	1.671885
Hill Climbing Pre-partition	671.81	1.503473
Simulated Annealing Pre-partition	186.00	2.239715

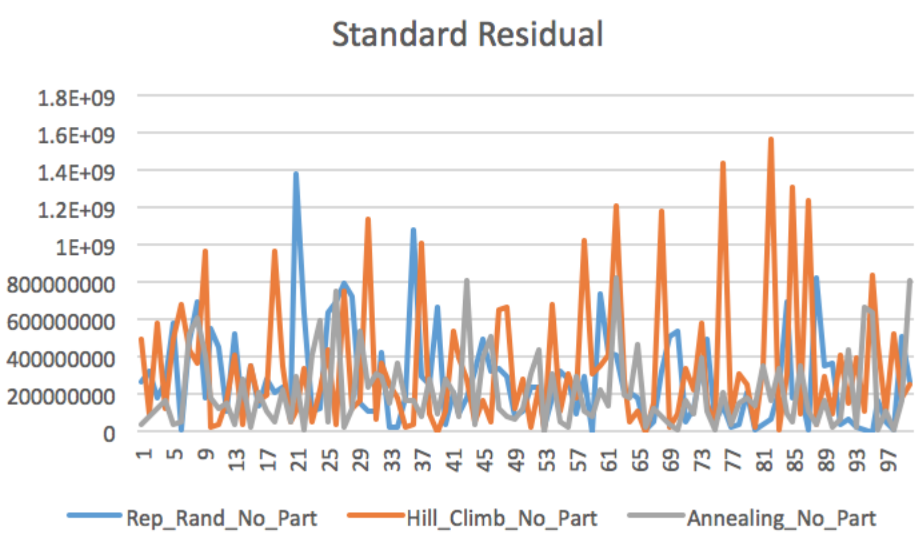
This next table shows the residues from 100 different trials of the same set of algorithms. For a given trial, the same array was used for each of the algorithms and so was the same starting solution - Rep_Rand No_Part, Hill_Climb No_Part, and Annealing No_Part were all based the same array of signs and then the other the prepartitioned methods were given the same prepartitioning scheme to start.

Note: To generate 64-bit integers, we used the Mersenne Twister random number generator which we found online and seems to be an accepted standard for generating large random numbers.

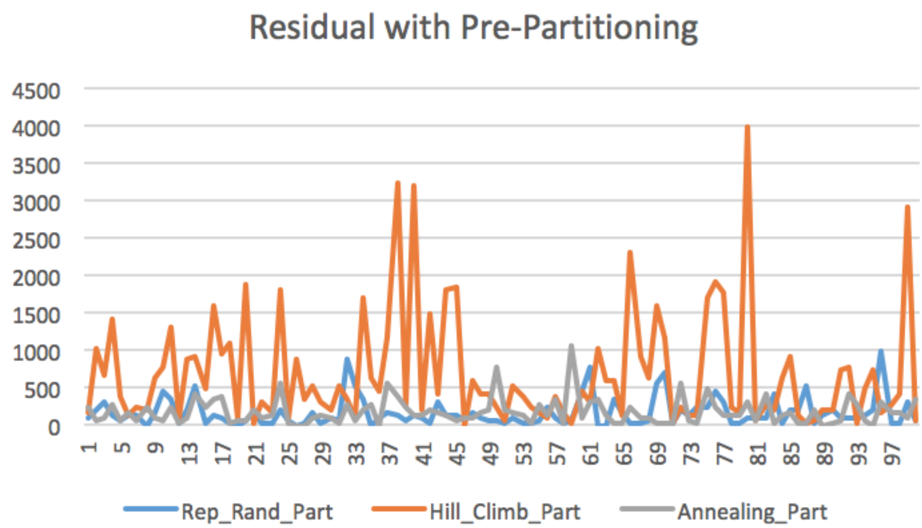
Trial	KK	Rep_Rand No_Part	Hill_Climb No_Part	Annealing No_Part	Rep_Rand Pre_Part	Hill_Climb Pre_Part	Annealing Pre_Part
1	430442	274633482	496820058	38374468	88	186	248
2	87086	325140718	97914116	81362666	212	1024	48
3	570210	176431450	585743004	130788918	324	678	114
4	724262	274670874	130658862	171004958	118	1416	288
5	167954	586659654	498472618	38247932	54	390	58
6	269454	8997842	682304554	58052848	124	128	162
7	338240	477682406	442740212	524715786	138	234	48
8	57065	696671019	370602217	606656921	5	201	227
9	223568	175839854	967863282	406766868	178	630	90
10	1574876	550641814	24559176	178673732	440	780	50
11	600813	449630603	39664245	119669809	337	1323	251
12	1637909	105602461	165278383	156198303	43	85	25
13	209931	526278519	403831399	33310499	215	893	117
14	109317	116946859	45723411	285232379	545	933	415
15	59374	352742974	347692946	21521324	30	484	228
16	19803	146182125	169470351	213893151	125	1597	345
17	182833	278655283	195573607	105355221	117	945	375
18	154148	215123734	964647780	59634362	30	1112	40
19	371565	244639803	353776429	226079083	1	89	53
20	24951	145580139	59976559	58595577	59	1873	69
21	49885	1376849515	112747229	293471665	183	21	211
22	669675	638165965	341781411	4500953	45	313	87
23	370730	114302658	52801076	411821876	14	154	146
24	12114	128613550	225805434	594337510	220	1818	548
25	182939	638081315	434796479	56540289	77	103	17
26	219767	699403313	42504665	755647539	5	881	1
27	514369	790897105	753413863	22879789	31	339	9
28	111579	719080525	128444585	128859175	179	529	93
29	9266	156844374	168069346	534267730	44	318	144
30	111484	111003668	1136442422	235672114	104	224	82
31	43950	114354224	69774170	315526652	80	528	28
32	78530	425429166	374506276	301577164	882	358	270
33	350969	18714931	248812833	147277489	527	99	81

34	330820	29063844	180471182	368280808	354	1712	210
35	360012	186285830	32093716	162392472	2	624	292
36	190772	1084784302	34810458	165637692	100	466	8
37	60282	300775766	1012776420	89068900	164	1160	550
38	103020	249895596	97821648	284645508	126	3242	370
39	311401	664235139	1708739	91307501	57	285	241
40	233845	34473475	106716577	286012925	137	3219	123
41	93870	216882298	531905596	207359410	98	218	140
42	25876	100539910	397017404	87013904	14	1478	206
43	545935	178567063	285025175	808384429	299	409	157
44	103395	322709717	50455669	39386027	131	1817	141
45	2020078	490293984	174258692	399577840	120	1852	68
46	449250	321949218	52987672	505166384	48	6	96
47	215044	339506580	650422928	126026336	158	590	84
48	47716	300287758	672107762	79481254	82	418	158
49	15094	80847880	121483986	69217590	48	434	222
50	466550	111275470	282575096	132496874	70	256	776
51	9081	236184269	28000177	309788455	45	65	213
52	274189	238445121	245923609	439224989	91	527	161
53	22716	27640722	14365542	1600930	32	376	118
54	312220	186044132	679830944	308902926	12	256	32
55	919458	329868272	114502878	60072406	54	182	270
56	71539	279349777	314499909	23554501	241	115	131
57	449103	96675739	183703465	293029617	93	387	357
58	35806	296144964	1018385892	109778648	42	178	26
59	547760	942276	315195158	88508092	52	30	1052
60	111972	733950184	349054630	218502490	450	464	94
61	14907	422914681	405537053	143841665	783	323	311
62	188243	405563359	1205888259	828514327	1	1021	339
63	165464	213902722	285038558	193658056	8	602	150
64	69797	217731447	48551123	187818611	359	585	11
65	10188	180907978	111878344	465895584	160	130	16
66	41331	12708883	1809455	32560223	19	2325	259
67	161090	48616168	152517976	121723360	26	920	92
68	35398	329222594	1178181630	82510906	58	636	84
69	1200713	513963675	28052751	34578285	563	1581	11
70	296513	537711671	98106813	12135783	719	1169	27
71	43441	57898611	345634177	161934433	25	13	17
72	98512	128432436	221208852	97031492	214	228	566
73	45307	351311733	584018251	403218099	127	119	49
74	432549	503190453	155004079	104027171	231	119	23
75	46087	60458247	28517619	8708919	231	1713	505
76	50137	134248419	1436741785	214905619	461	1935	231
77	133483	31198279	87093135	37958583	311	1775	149
78	66670	32697288	304113442	148016740	12	226	126

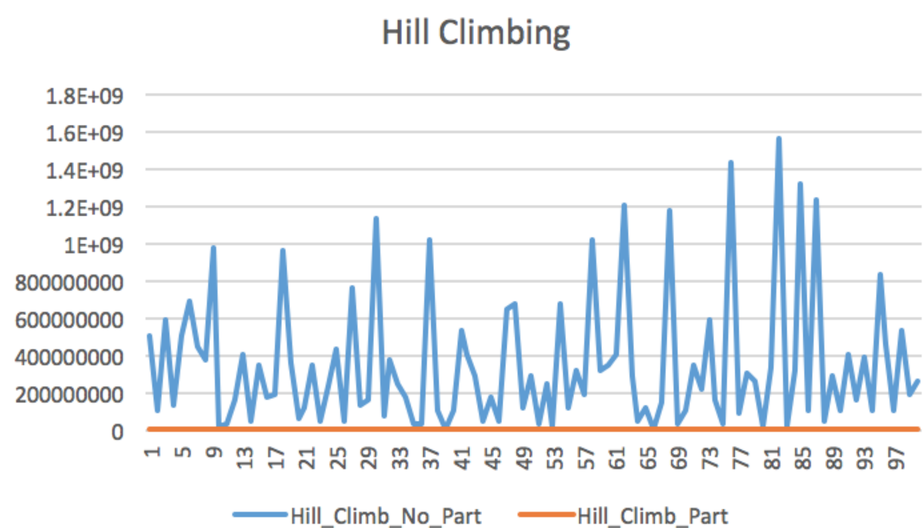
79	287632	217605564	254542474	180429272	40	170	150
80	89513	11091565	25512999	142040463	95	3971	305
81	177	41320897	335895879	359092339	103	73	53
82	297127	67527317	1562014079	174245787	95	283	413
83	23440	218819004	16621518	334908018	404	218	40
84	75116	703019356	312647132	89873994	46	644	132
85	60032	178851880	1315946664	55292324	214	926	166
86	513385	219919051	101749773	351138125	203	127	25
87	271952	16768590	1234511086	110543904	516	30	46
88	43821	822438065	42953921	43837465	39	55	197
89	259952	350702054	290509446	161174548	140	224	8
90	399393	374065175	97986767	21602685	195	201	39
91	380323	35825705	405200753	66036589	95	745	65
92	354018	64103050	155566506	440676302	90	780	410
93	182006	28730594	394316852	21825366	116	46	290
94	144083	14848437	105358307	663029397	137	509	71
95	18075	3056405	835270923	642917019	207	727	7
96	14102	162730976	464435470	12032102	990	168	300
97	4232	50720652	101416792	111545420	24	276	186
98	16994	17467590	529520882	5872488	14	434	172
99	140549	510162347	188794897	171721341	301	2919	95
100	240949	268457317	259644117	806665137	71	65	347
Average	257825.63	280590214.4	347896244.6	213164716	171.37	704.63	177.47

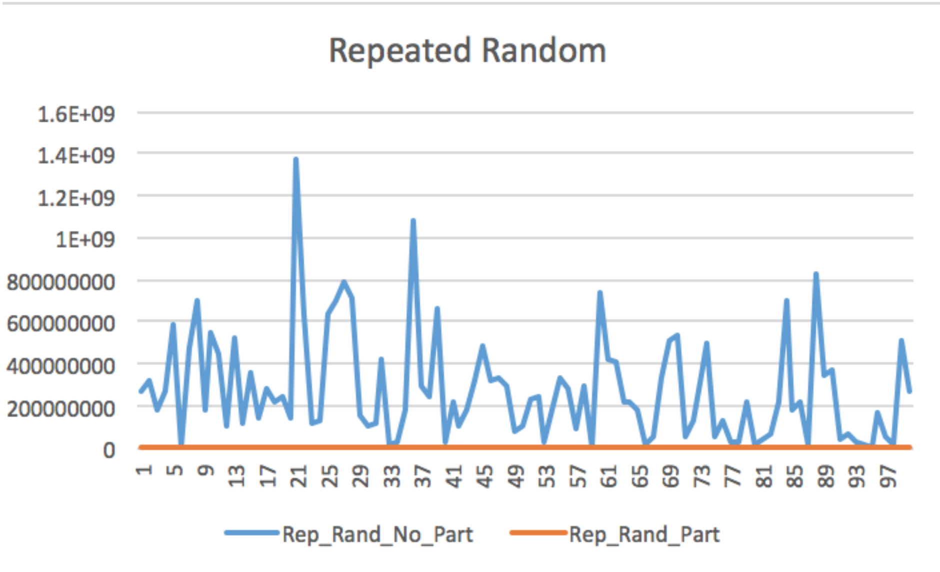


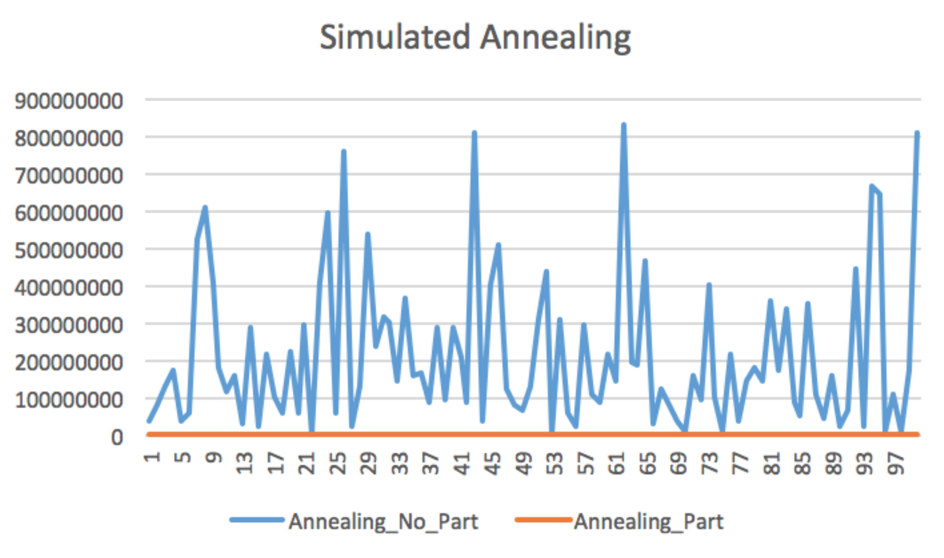
Y-Axis representing residue for 100 trials shows hill climbing (orange line) is overall worse than the other heuristics. Hill climbing has the highest spikes because it gets stuck in local optima



Y-Axis representing residue for 100 trials are significantly lower for all heuristics. Around the 80th trial, the large spike in hill climbing is due to the solution state getting stuck in a local minima. By and large hill climbing is more likely to perform more poorly than the other methods because it's performance depends much more on its initial condition and can thus get 'stuck' in these local optima.







4 Analysis

In a local search algorithm only one state is tracked at a time and each proceeding state it moves to is based on one of many heuristics. While these solutions are not exact, they tend to be close to optimal.

The randomized methods (all but KK) had extremely large variation in how well they performed as indicated by the graphs above. The KK algorithm is deterministic in that it always produces the same output (by differencing) regardless of its input type. Of all the methods we tried, the pre-partitioned random method and the pre-partitioned simulated annealing performed the best by far. Because the state space for the number partitioning problem in non-convex, often the solution we find is a local, not a global optima, and so, at least for hill climbing methods, there are some initial conditions that we can start at which will find us a good, but non-optimal solution. Pre-partitioning has a lower residual because it calls KK heuristic on each iteration while the standard algorithm does not. For this reason the pre-partition heuristics were slower overall but more accurate. Using Karmarkar-Karp is approximately 1000 times better than the non-prepartitioned methods and so because the preparitioned methods rely on it, they are given a huge leg up in terms of finding better parititons.

If the solution from the Karmarkar Karp algorithm was used as a starting point rather than a random starting point, we would see significant improvements and much better solutions using the simulated annealing and hill climbing heuristics because using Karmarkar-Karp would give us a much better starting place in much less time. In essence, hill climbing or the simulated annealing would start much closer to a local maximum than just by guessing one at random. Our data show that the residual determined by KK is far less than that determined by either hill climbing or simulated annealing with no pre-partitioning. In repeated random, random solutions are repeatedly generated to the problem and so any starting point would have no effect on future values unless the KK algorithm returned a residual that was worse than the randomly chosen states, which we've shown in out experiments almost never happens. Given how low the probability a randomly generated partition (for a given array) will have a lower residual than the one KK will give is, the repeated random method would likely do no better than what KK would start with.

Likewise running KK before the pre-partitioned algorithms would also improve their result, if not with regards to how small a residual they found, then likely how many iterations it would take to achieve a comparable residual. Instead of beginning with a random pre-partition, using the Karmarkar-Karp algorithm will give an assignment of each value. The lower bounds on each solution state is tighter and closer to the optimal because the starting value is more accurate than a randomly generated solution. This would have the largest impact on the fewer number of trials that are run because there are fewer iterations to improve the solution state.