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第四章 1.

20. $P_n(x)=\sum_{k=0}^{[\frac{n}{2}]}\frac{(2n-2k)!}{2^n\cdot k!(n-k)!(n-2k)!}x^{n-2k}$

$P_n'(x)=\sum_{k=0}^{[\frac{n}{2}]}\frac{(2n-2k)!}{2^n\cdot k!(n-k)!(n-2k-1)!}x^{n-2k-1}$

$\therefore P_n(0)=\sum_{k=0}^{M}\frac{(2n-2k)!}{2^n\cdot k!(n-k)!(n-2k)!}=emmm\cdots=\begin{cases}0, & n=2m+1\\ \frac{(-1)^m(2m-1)!!}{(2m)!!}, & n=2m\\ 1, & n=0\end{cases}\quad(m=1,2,3\cdots)$

$P_n'(0)=\sum_{k=0}^{m}\frac{(2n-2k)!}{2^n k!(n-k)!(n-2k-1)!}=emmm\cdots=\begin{cases}0, & n=2k\\ \frac{(-1)^k(2k+1)!!}{(2k)!!}, & n=2k+1\end{cases}\quad(k=0,1,2,\cdots)$

22 (1) $\int_{-1}^{1}x^m P_n(x)dx$

令 $f(m,n)=\int_{-1}^{1}x^m P_n(x)dx$

① $m\geq n$ 时，$f(m,n)=\frac{m}{m+n+1}f(m-1,n-1)$ [书上P280-281结论]

$\therefore f(m,n)=\frac{m}{m+n+1}f(m-1,n-1)=\frac{m}{m+n+1}\cdot\frac{m-1}{m+n-1}f(m-2,n-2)=\cdots=\frac{m!}{(m-n)!}\frac{(m-n+1)!!}{(m+n+1)!!}f(m-n,0)$

$\because f(m-n,0)=\int_{-1}^{1}x^{m-n}P_0(x)dx=\int_{-1}^{1}x^{m-n}dx=\frac{1-(-1)^{m-n+1}}{m-n+1}$

$\therefore f(m,n)=\frac{m![1+(-1)^{m-n}]}{(m-n)!!(m+n+1)!!}$

② $m<n$ 时，$f(m,n)=m!\frac{(n-m+1)!!}{(m+n+1)!!}f(0,n-m)$

$\because f(0,n-m)=\int_{-1}^{1}P_{n-m}(x)dx=0$ [ $\int_{-1}^{1}P_n(x)dx=0$，$n\neq0$时 ]

$\therefore f(m,n)=0$

24(1) $f(x)=x^3$

由22(1)知，$n>3$时，$\int_{-1}^{1}x^3P_n(x)dx=0$

$\therefore$ 可令 $x^3=C_0P_0(x)+C_1P_1(x)+C_2P_2(x)+C_3P_3(x)$

由 $P_n(x)$ 奇偶性知. $x^3 = C_1 P_1(x) + C_3 P_3(x)$

$\therefore \; C_n = \frac{2n+1}{2}\int_{-1}^{1} x^3 P_n(x)\,dx = \begin{cases} \frac{3}{5}, & n=1 \\ \frac{2}{5}, & n=3 \end{cases}$

$\therefore \; x^3 = \frac{3}{5}P_1(x) + \frac{2}{5}P_3(x)$

25. $\begin{cases} \Delta_3 u = 0 \\ u|_{r=a} = \cos^2\theta \end{cases}$

在球内解题, $r \to 0$ 时 $u$ 有界 $\therefore B_n = 0$

$\therefore \; u(r,\theta) = \sum_{n=0}^{+\infty} A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta)$

$u|_{r=a} = \sum_{n=0}^{+\infty} A_n P_n(\cos\theta) = \cos^2\theta$

令 $x = \cos\theta$. 则 $\sum_{n=0}^{+\infty} A_n P_n(x) = x^2 \Rightarrow A_0 = \frac{1}{3}$, $A_2 = \frac{2}{3}$

$\therefore \; u(r,\theta) = \frac{1}{3} + \frac{2}{3}\left(\frac{r}{a}\right)^2 P_2(\cos\theta)$

26. $\begin{cases} \Delta_3 u = 0 \\ u|_{r=1} = 3\cos 2\theta + 1 \end{cases}$

$u(r,\theta) = \sum_{n=0}^{+\infty} A_n r^n P_n(\cos\theta)$

$u|_{r=1} = \sum_{n=0}^{+\infty} A_n P_n(\cos\theta) = 6\cos^2\theta - 2$

令 $x = \cos\theta$, 则 $\sum_{n=0}^{+\infty} P_n(x) A_n = 6x^2 - 2$

$\Rightarrow A_0 = 0$

$A_n = \begin{cases} 0, & n=1 \\ 4, & n=2 \\ 0, & n>2 \end{cases}$ 书上 P282.

$\therefore \; u(r,\theta) = 4r^2 P_2(\cos\theta) = 2r^2(3\cos^2\theta - 1)$

27. $\begin{cases} \Delta_3 u=0 \\ u|_{r=1}=\cos^2\theta \end{cases}$

在球外解题. $u(r,\theta)=\sum_{n=0}^{+\infty} B_n r^{-(n+1)} P_n(\cos\theta)$

$u|_{r=1}=\cos^2\theta=\sum_{n=0}^{+\infty} B_n P_n(\cos\theta)$

令 $x=\cos\theta$, 则 $\sum_{n=0}^{+\infty} B_n P_n(x)=x^2 \Rightarrow B_0=\frac{1}{3}$, $B_2=\frac{2}{3}$

$\therefore u(r,\theta)=\frac{1}{3}r^{-1}+\frac{2}{3}r^{-3}P_2(\cos\theta)$

$=\frac{1}{3}r^{-1}+r^{-3}\cos^2\theta-\frac{1}{3}r^{-3}$

29. $\begin{cases} \Delta_3 u=0 \\ u|_{r=R}=u|_{r=\frac{R}{2}}=A\sin^2\frac{\theta}{2}=\frac{A}{2}-\frac{A}{2}\cos\theta \\ u|_{\theta=\frac{\pi}{2}}=\frac{A}{2} \quad (0\le\theta\le\frac{\pi}{2}) \end{cases}$

设 $u=\frac{A}{2}+V$, 则 $V$ 满足: $\begin{cases} \Delta_3 V=0 \\ V|_{r=R}=V|_{r=\frac{R}{2}}=-\frac{A}{2}\cos\theta \\ V|_{\theta=\frac{\pi}{2}}=0 \quad (0\le\theta\le\frac{\pi}{2}) \end{cases}$

$\Rightarrow V(r,\theta)=\sum_{n=0}^{+\infty}\left[A_n r^n+B_n r^{-(n+1)}\right]P_n(\cos\theta)$

由 $V|_{\theta=\frac{\pi}{2}}=0 \Rightarrow P_n(\cos\theta)\big|_{\theta=\frac{\pi}{2}}=0 \Rightarrow P_n(0)=0$

$\therefore n=\cancel{2k+1}$ 为奇数, $\lambda_n=(2n+1)(2n+2)$ 为该问题固有值.

$\therefore V(r,\theta)=\sum_{n=0}^{+\infty}\left[A_n r^{2n+1}+B_n r^{-(2n+2)}\right]P_{2n+1}(\cos\theta)$

$\begin{cases} V|_{r=R}=\sum_{n=0}^{+\infty}\left[A_n R^{2n+1}+B_n R^{-(2n+2)}\right]P_{2n+1}(\cos\theta)=-\frac{A}{2}\cos\theta \\ V|_{r=\frac{R}{2}}=\sum_{n=0}^{+\infty}\left[A_n (\frac{R}{2})^{2n+1}+B_n (\frac{R}{2})^{-(2n+2)}\right]P_{2n+1}(\cos\theta)=-\frac{A}{2}\cos\theta \end{cases}$

$\begin{cases} A_n R^{2n+1}+B_n R^{-(2n+2)}=\frac{1}{\|P_{2n+1}(\cos\theta)\|^2}\int_0^{\frac{\pi}{2}}-\frac{A}{2}\cos\theta P_{2n+1}(\cos\theta)d(\cos\theta) \\ A_n(\frac{R}{2})^{2n+1}+B_n(\frac{R}{2})^{-(2n+2)}=\frac{1}{\|P_{2n+1}(\cos\theta)\|^2}\int_0^{\frac{\pi}{2}}-\frac{A}{2}\cos\theta P_{2n+1}(\cos\theta)d(\cos\theta) \end{cases}$

$\because \|P_{2n+1}(\cos\theta)\| = \int_0^1 P_{2n+1}^2(x)\,dx = \frac{1}{2}\int_{-1}^1 P_{2n+1}^2(x)\,dx = \frac{1}{4n+3}$ $\left[\|P_{2n+1}(\cos\theta)\| = \int_0^{\frac{\pi}{2}} P_{2n+1}^2(\cos\theta)\sin\theta\,d\theta\right]$

$(x=\cos\theta)$

$\int_0^{\frac{\pi}{2}} -\frac{A}{2}\cos\theta\, P_{2n+1}(\cos\theta)\, d(\cos\theta)$

$\xlongequal{x=\cos\theta} \int_0^1 -\frac{A}{2} x P_{2n+1}(x)\,dx = -\frac{A}{2}\cdot\left[\frac{2n+2}{4n+3}\int_0^1 P_{2n+2}(x)\,dx + \frac{2n+1}{4n+3}\int_0^1 P_{2n}(x)\,dx\right]$ (P279 (3))

$= \begin{cases} -\frac{A}{2}\cdot\frac{1}{3}, & n=0 \\ 0, & n\geq 1 \end{cases}$

$\therefore \begin{cases} A_0 R + B_0 R^{-2} = -\frac{A}{2} \\ A_0 \frac{R}{2} + B_0 \left(\frac{R}{2}\right)^{-2} = -\frac{A}{2} \end{cases} \Rightarrow \begin{cases} A_0 = -\frac{3A}{7R} \\ B_0 = -\frac{A}{14}R^2 \end{cases}$

$\therefore u(r,\theta) = \frac{A}{2} - \left(\frac{3r}{7R} + \frac{R^2}{14r^2}\right)A\cos\theta$