数理方程_偏微分方程_分离变量法
+ 变系数常微分方程_广义幂级数解法_特殊函数

Partial Differential Equations

§3 Separation of variables on higher dimentional problems

Special functions,
appear as solutions of ODEs with variable coefficients, 2nd example:
Legendre Polynomial

weihuang[AT]mail.

Spring 2018, Hefei UST©-math-phys

wei huang at mail ustc copyright huang

曲线坐标系下分离变量

u = T(t) v(r, θ, φ)

- $T'' + a^2k^2T = 0$ 波动， $T' + a^2k^2T = 0$ 传导

球坐标 $\triangle_3 = \frac{1}{r^2}[\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}]$

- 球坐标下 $\frac{1}{r^2}[\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}]v + k^2v = 0$

进一步设变量分离形式的特解 $v = R(r)\Theta(\theta)\Phi(\varphi)$，逐层剥离

$$\frac{1}{r^2}\frac{(r^2R')'}{R} + \frac{1}{r^2}\cdot\ [\frac{1}{\sin\theta}\frac{(\sin\theta\Theta')'}{\Theta} + \frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi}] + k^2 = 0$$

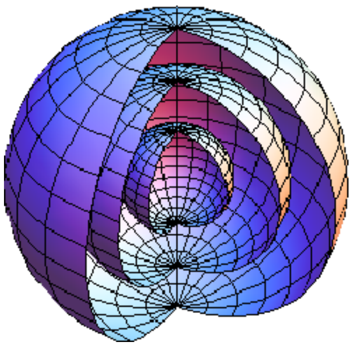
$= -\lambda$ $= -\mu$ $\equiv -m^2$

- $\frac{\Phi''}{\Phi} = -m^2$, $\Phi'' + m^2\Phi = 0$ 易解.
- $\frac{1}{r^2}(r^2R')' + (k^2 - \frac{\lambda}{r^2})R = 0$,
  称球 Bessel 方程，有技巧 $x = kr$, $z(x) \equiv \sqrt{x}R(\frac{x}{k})$, $\lambda \equiv l(l+1)$,
  化为 $l + \frac{1}{2}$ 阶 Bessel 方程 $x^2z'' + xz' + [x^2 - (l+\frac{1}{2})^2]z = 0$.
- $\frac{1}{\sin\theta}\frac{(\sin\theta\Theta')'}{\Theta} + \frac{(-m^2)}{\sin^2\theta} = -\lambda$, $\frac{1}{\sin\theta}(\sin\theta\Theta')' + (\lambda - \frac{m^2}{\sin^2\theta})\Theta = 0$,
  $\theta : 0 \to \pi$, $\sin\theta \in (0,1]$, $\cos\theta \in (1,-1)$，设 $x = \cos\theta$, $y(x) \equiv \Theta(\theta)$,
  得 $[(1-x^2)y']' + (\lambda - \frac{m^2}{1-x^2})y = 0$，称 “m 阶伴随 Legendre 方程”.
  我们只学习 m=0 情况，即 φ 方向轴对称，此时有 Legendre 方程:
  $[(1-x^2)y']' + \lambda y = 0$

huang

球问题 经线方向之固有值问题 的泛定方程:

general Legendre equation (m阶伴随勒让德方程)*



$$\Delta_3 u(r,\theta,\varphi)=0$$

径向 $(r^2R')'+(k^2\cdot r^2-\lambda)R=0$

经线 北极绕向南极 $\frac{1}{\sin\theta}(\sin\theta\Theta')'+(\lambda-\frac{m^2}{\sin^2\theta})\Theta=0 \xrightarrow{x=\cos\theta} [(1-x^2)y']'+(\lambda-\frac{m^2}{1-x^2})y=0$

纬线 周期角向 $\Phi''+m^2\Phi=0$

*<u>**轴对称**</u>球问题 经线方向固有值问题 的泛定方程:*

Legendre's differential equation (勒让德方程)

径向 $(\boldsymbol{r}^2\boldsymbol{R}')'+(\boldsymbol{k}^2\cdot\boldsymbol{r}^2-\boldsymbol{\lambda})\boldsymbol{R}=0$

北极绕向南极 经线方向 $\frac{1}{\sin\boldsymbol{\theta}}(\sin\boldsymbol{\theta}\Theta')'+\boldsymbol{\lambda}\Theta=0 \xrightarrow{\boldsymbol{x}=\cos\boldsymbol{\theta}} [(1-\boldsymbol{x}^2)\boldsymbol{y}']'+\boldsymbol{\lambda y}=0$

周期角向 纬线方向 $\Phi\equiv1,\quad \Phi''=0\quad(\boldsymbol{m}=0).$

$\boldsymbol{k}(\boldsymbol{x})=1-\boldsymbol{x}^2,\quad \boldsymbol{q}(\boldsymbol{x})=0,\quad \boldsymbol{\rho}(\boldsymbol{x})=1,$

$\boldsymbol{x}=\pm1$ 处 $\boldsymbol{k}(\pm1)=0,$ 是正则奇点.

Legendre方程幂级数形式通解 $\boldsymbol{y}=\boldsymbol{c}_1\boldsymbol{y}_1+\boldsymbol{c}_2\boldsymbol{y}_2$ 面临 $\boldsymbol{x}=\pm1$即北极南极处收敛问题？

wei huang at mail ustc copyright huang

$$y(x)=\sum_{n=0}^{+\infty}a_n x^n$$

$|x|<1$ 内解析，可逐项求导

$$y'=\sum_{n=1}^{+\infty}na_n x^{n-1}\qquad -2xy'=-\sum_{n=0}^{+\infty}2na_n x^n$$

$$y''=\sum_{n=2}^{+\infty}n(n-1)a_n x^{n-2}\qquad (1-x^2)y''=\sum_{k=0}^{+\infty}(k+2)(k+2-1)a_{k+2}x^k-\sum_{n=2}^{+\infty}n(n-1)a_n x^n$$

$$=\sum_{n=0}^{+\infty}[(n+2)(n+1)a_{n+2}-n(n-1)a_n]x^n$$

$$\sum_{n=0}^{+\infty}[(n+2)(n+1)a_{n+2}-n(n-1)a_n]x^n-\sum_{n=0}^{+\infty}2na_n x^n+\lambda\sum_{n=0}^{+\infty}a_n x^n=0$$

$$(n+2)(n+1)a_{n+2}-[n(n+1)-\lambda]a_n=0$$

$$a_{n+2}=\frac{n(n+1)-\lambda}{(n+1)(n+2)}a_n\equiv\frac{n(n+1)-l(l+1)}{(n+1)(n+2)}a_n=\frac{(n-l)(n+1+l)}{(n+2)(n+1)}a_n$$

偶串 $$y_1=a_0\left[1+\frac{(0-l)(0+1+l)}{2!}x^2+\frac{(2-l)(0-l)(0+1+l)(3+l)}{4!}x^4+\ldots\right]$$

$$=a_0\left[1+\frac{2^2(0-\frac{l}{2})(0+\frac{1+l}{2})}{2!}x^2+\frac{2^4(1-\frac{l}{2})(0-\frac{l}{2})(0+\frac{1+l}{2})(1+\frac{1+l}{2})}{4!}x^4+\ldots\right]=a_0\sum_{k=0}^{+\infty}\frac{2^{2k}}{(2k)!}\frac{\Gamma(k-\frac{l}{2})}{\Gamma(-\frac{l}{2})}\frac{\Gamma(k+\frac{1+l}{2})}{\Gamma(\frac{1+l}{2})}x^{2k}$$

奇串 $$y_2=a_1\left[1+\frac{(1-l)(1+1+l)}{2!}x^3+\frac{(3-l)(1-l)(1+1+l)(4+l)}{4!}x^4+\ldots\right]=a_1\sum_{k=0}^{+\infty}\frac{2^{2k}}{(2k+1)!}\frac{\Gamma(k+\frac{1-l}{2})}{\Gamma(\frac{1-l}{2})}\frac{\Gamma(k+\frac{2+l}{2})}{\Gamma(\frac{2+l}{2})}x^{2k+1}$$

$l=$ 整数 $\mathbf{n}=2m$：递推遇 $n-l=0$，$\Rightarrow a_{2m+2}=0$，$\Rightarrow a_{2m+4}=a_{2m+6}=\ldots=0$，偶串 y_1 被截断 $\sum_{k=0}^{n/2}$

$l=$ 整数 $\mathbf{n}=2m+1$：也遇 $n-l=0$，$\Rightarrow a_{2m+3}=0$，$\Rightarrow a_{2m+5}=a_{2m+7}=\ldots=0$，奇串 y_2 被截断 $\sum_{k=0}^{(n-1)/2}$

Legendre 方程幂级数形式通解 $y=c_1y_1+c_2y_2$ 面临 $x=\pm1$ 即北极南极处收敛问题？

当且仅当 $\lambda\equiv l(l+1)=n(n+1)$，$n$ 为整数时，存在有界解：y_1 或 y_2 被截断为多项式 P_n.

值域有界要求 $\lambda=n(n+1)$，把 $c_1P_n+c_2y_2=c_1y_1+c_2P_n$ 改进写法后通解为 $y=CP_n+DQ_n$

然后丢弃 $x=\pm1$ $(\theta=0,\pi)$ 处发散的 Q_n，由自然边界条件(值域有界)取 $y=CP_n$

勒让德多项式的表示和性质

级数定义式： $\boldsymbol{P_n(x)=\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!}x^{n-2k}}$， $\boldsymbol{n}=0,1,2,\dots$

微分定义式： $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

$(x^2-1)^n=\sum_k C_n^k (x^2)^{n-k}(-1)^k \quad ,C_n^k=\frac{n!}{(n-k)!k!}$， 最高$x^{2n}$次；再求$n$次导,最高$x^n$次：

$$\frac{1}{2^n \cancel{n!}}\sum_k \frac{\cancel{n!}}{(n-k)!k!}\underbrace{(2n-2k)(2n-2k-1)\dots(n-2k+1)}_{n个}x^{n-2k}(-1)^k=\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k(2n-2k)!}{2^n(n-k)!k!(n-2k)!}x^{n-2k}$$

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$
6	$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7	$\frac{1}{16}(429x^7-693x^5+315x^3-35x)$
8	$\frac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35)$
9	$\frac{1}{128}(12155x^9-25740x^7+18018x^5-4620x^3+315x)$
10	$\frac{1}{256}(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63)$

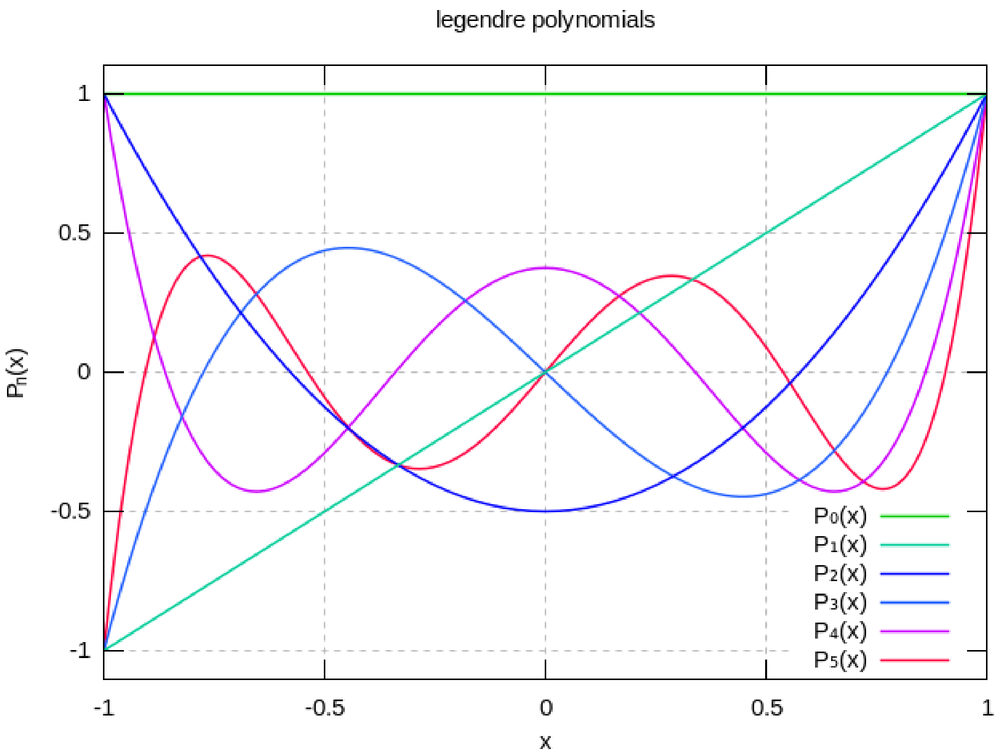
$$P_0(\cos\theta)=1$$

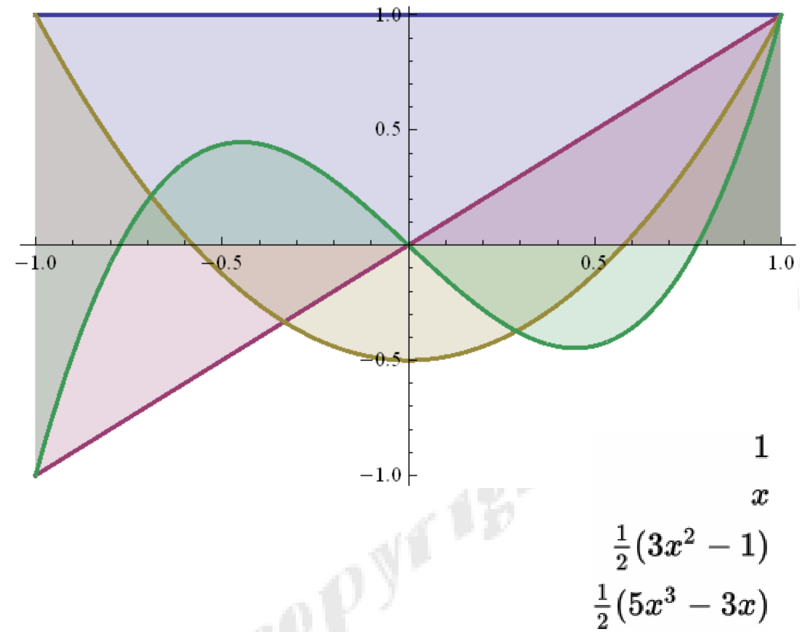
$$P_1(\cos\theta)=\cos\theta$$

$$P_2(\cos\theta)=\frac{1}{2}(3\cos^2\theta-1)$$

$$P_3(\cos\theta)=\frac{1}{2}(5\cos^3\theta-3\cos\theta)$$

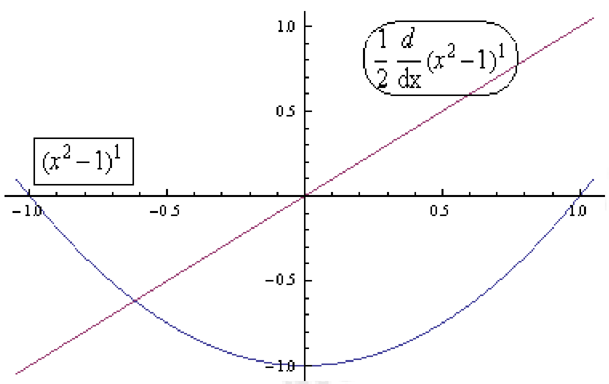
ustc copyright huang

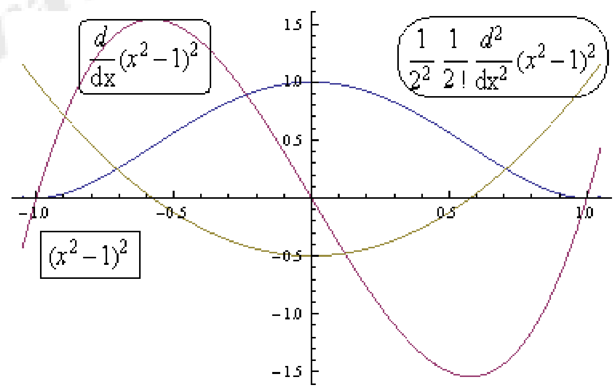




零点: *Rolle定理 f(a)=f(b), 则存在c∈[a,b], 使f'(c)=0.

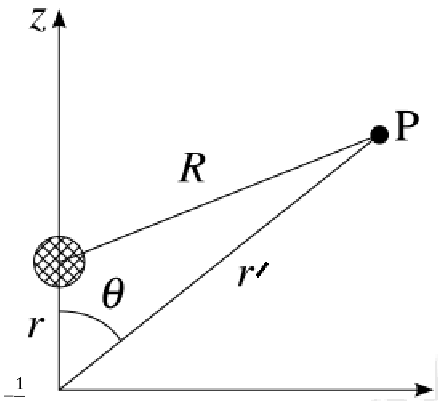
例如函数两根间f(a)=f(b)=0,其导数至少存在一根f'(c)= 0.





$P_n(x)$在[-1,1]有且仅有n个单零点.

$$\frac{1}{R}=\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}}\xrightarrow{\cos\theta=x,\ t=\frac{r'}{r}}\begin{cases}\frac{1}{r'}\frac{1}{\sqrt{1+(\frac{1}{t})^2-2\frac{1}{t}x}}, & |t|>1\\ \frac{1}{r}\frac{1}{\sqrt{1+t^2-2tx}}, & |t|<1\end{cases}$$



无量纲量 $(1-2xt+t^2)^{-\frac{1}{2}}$:

$$f(t)=(1-2xt+t^2)^{-\frac{1}{2}}=\sum_{n=0}^{+\infty}C_n(x)t^n,\quad \text{其中}C_n(x)=\left.\frac{f^{(n)}(t)}{n!}\right|_{t=0}=\frac{1}{2\pi i}\oint_C\frac{(1-2x\xi+\xi^2)^{-\frac{1}{2}}}{(\xi-0)^{n+1}}d\xi$$

$\boxed{\text{复分析知识}: g^{(n)}(z)=\frac{n!}{2\pi i}\oint_C\frac{g(\xi)}{(\xi-z)^{n+1}}d\xi}$

变换 $\sqrt{1-2xt+t^2}\equiv 1-tz,$

反解出 $t=2\frac{(z-x)}{z^2-1},\quad \frac{1}{1-tz}=\frac{1}{\frac{z^2-1-2(z-x)z}{z^2-1}},\quad \frac{dt}{dz}=2[\frac{1}{(z^2-1)}+\frac{(z-x)(-2z)}{(z^2-1)^2}],$

$$C_n(x)=\frac{1}{2\pi i}\oint_C\frac{(1-2xt+t^2)^{-\frac{1}{2}}}{t^{n+1}}dt=\frac{1}{2\pi i}\oint_C\frac{\frac{1}{1-tz}}{t^{n+1}}\frac{dt}{dz}dz=\frac{1}{2\pi i}\oint_C\frac{\frac{z^2-1}{z^2-1-2(z-x)z}}{[2\frac{(z-x)}{z^2-1}]^{n+1}}2[\frac{(z^2-1)-2(z-x)z}{(z^2-1)^2}]dz$$

$$=\frac{1}{2^n}\frac{1}{2\pi i}\oint_C\frac{(z^2-1)^n}{[z-x]^{n+1}}dz=\frac{1}{2^n}\frac{g^{(n)}(x)}{n!}=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$= P_n(x)$ **Schläfli**复积分表示

实积分表示

$$P_n(x)=\frac{1}{2^n}\frac{1}{2\pi i}\oint_C\frac{(z^2-1)^n}{[z-x]^{n+1}}dz\xrightarrow{z=x+i\sqrt{1-x^2}e^{i\theta}}$$

$$=\frac{1}{2^n}\frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{[x^2-1+2ix\sqrt{1-x^2}-(1-x^2)e^{2i\theta}]^n}{(i\sqrt{1-x^2}e^{i\theta})^{n+1}}(i\sqrt{1-x^2}ie^{i\theta}d\theta)$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}[x+i\sqrt{1-x^2}\frac{e^{-i\theta}+e^{i\theta}}{2}]^n d\theta=\frac{1}{\pi}\int_0^{\pi}[x+i\sqrt{1-x^2}\cos\theta]^n d\theta$$

$$P_n(\pm1)=\frac{1}{\pi}\int_0^{\pi}[x+i\sqrt{1-x^2}\cos\theta]^n d\theta=\frac{1}{\pi}\int_0^{\pi}[\pm1]^n d\theta=\begin{cases}1, & x=1\\ (-1)^n, & x=-1\end{cases}$$

“生成函数”:

$$(1-2xt+t^2)^{-\frac{1}{2}}=\begin{cases}\sum_{n=0}^{+\infty}P_n(x)t^n, & \left|\frac{r'}{r}\right|=|t|<1\\ \frac{1}{t}\sum_{n=0}^{+\infty}P_n(x)\left(\frac{1}{t}\right)^n, & \left|\frac{r'}{r}\right|=|t|>1\end{cases}$$

Wei huang at mail.ustc copyright

勒让德多项式递推关系

$$(1-2xt+t^2)^{-\frac{1}{2}}=\sum_{n=0}^{+\infty}\mathbf{P}_{\mathrm{n}}(x)t^n \xrightarrow{\text{对 t 求导}} \frac{-1}{2}(-2x+2t)(1-2xt+t^2)^{-\frac{3}{2}}=\sum \mathbf{nP}_{\mathrm{n}}(x)t^{n-1}$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-\frac{1}{2}}=(x-t)\sum \mathbf{P}_{\mathrm{n}}(x)t^n=(1-2xt+t^2)\sum \mathbf{nP}_{\mathrm{n}}(x)t^{n-1}$$

$$\Rightarrow x\sum \mathbf{P}_{\mathrm{n}}(x)t^n \underline{-\sum \mathbf{P}_{\mathbf{n-1}}(x)t^n}=\sum(\mathbf{n}+\mathbf{1})\mathbf{P}_{\mathbf{n+1}}(x)t^n-2x\sum \mathbf{nP}_{\mathrm{n}}(x)t^n+\underline{\sum(\mathbf{n}-\mathbf{1})\mathbf{P}_{\mathbf{n-1}}(x)t^n}$$

$$\xrightarrow{\text{拣出}\mathbf{t}^{n}\text{系数}}(2n+1)x\mathbf{P}_{\mathrm{n}}(x)=(n+1)\mathbf{P}_{\mathbf{n+1}}(x)+\mathbf{nP}_{\mathbf{n-1}}(x) \qquad (3)$$

$$(1-2xt+t^2)^{-\frac{1}{2}}=\sum_{n=0}^{+\infty}\mathbf{P}_{\mathrm{n}}(x)t^n \xrightarrow{\text{对x求导}} \frac{-2t}{-2}(1-2xt+t^2)^{-\frac{3}{2}}=\sum \mathbf{P}'_{\mathrm{n}}(x)t^n$$

$$\Rightarrow t(1-2xt+t^2)^{-\frac{1}{2}}=t\sum \mathbf{P}_{\mathrm{n}}(x)t^n=(1-2xt+t^2)\sum \mathbf{P}'_{\mathrm{n}}(x)t^n$$

$$\xrightarrow{\text{拣出}\mathbf{t}^{n+1}\text{系数}}\mathbf{P}_{\mathrm{n}}(x)=\mathbf{P}'_{\mathbf{n+1}}(x)-2x\mathbf{P}'_{\mathrm{n}}(x)+\mathbf{P}'_{\mathbf{n-1}}(x) \qquad \bigstar$$

$$0=(n+1)\mathbf{P}_{\mathbf{n+1}}(x)-(2n+1)x\mathbf{P}_{\mathrm{n}}(x)+\mathbf{nP}_{\mathbf{n-1}}(x) \qquad \text{(3)再对x求导得:}$$

$$0=(n+1)\mathbf{P}'_{\mathbf{n+1}}(x)-(2n+1)x\mathbf{P}'_{\mathrm{n}}(x)-(2n+1)\mathbf{P}_{\mathrm{n}}(x)+\mathbf{nP}'_{\mathbf{n-1}}(x) \qquad \blacklozenge$$

$$\text{消}x\mathbf{P}'_{\mathrm{n}}(x)\xrightarrow{\blacklozenge\times 2-\bigstar\times(2n+1)}(2n+1)\mathbf{P}_{\mathrm{n}}(x)=\mathbf{P}'_{\mathbf{n+1}}(x)-\mathbf{P}'_{\mathbf{n-1}}(x) \qquad (6)$$

$$\text{消}\mathbf{P}'_{\mathbf{n+1}}(x)\xrightarrow{\blacklozenge-\bigstar\times(n+1)}n\mathbf{P}_{\mathrm{n}}(x)=x\mathbf{P}'_{\mathrm{n}}(x)-\mathbf{P}'_{\mathbf{n-1}}(x) \qquad (4)$$

$$\text{消}\mathbf{P}'_{\mathbf{n-1}}(x)\xrightarrow{\blacklozenge-\bigstar\times n}(n+1)\mathbf{P}_{\mathrm{n}}(x)=\mathbf{P}'_{\mathbf{n+1}}(x)-x\mathbf{P}'_{\mathrm{n}}(x) \qquad (5')$$

weihuang@mail.ustc copyright huang

Separation of variables
Legendre polynomials

λ_n	I X(b)=0	II X'(b)=0	III	自然
I X(a)=0				
II X'(a)=0				
III				
自然				≥0

$$k=0:\quad r^2R''+2rR'-l(l+1)R=0$$

$$Euler: r^2y''+bry'+cy=0$$

$$\xleftrightarrow{r=e^t}\ddot{Y}+(b-1)\dot{Y}+cY=0$$

$$\xrightarrow{e^{\kappa t}}\kappa^2+(b-1)\kappa+c=0\rightarrow\begin{Bmatrix}e^{\kappa_1 t}\\ e^{\kappa_2 t}\end{Bmatrix}\sim\begin{Bmatrix}r^{\kappa_1}\\ r^{\kappa_2}\end{Bmatrix},or\begin{Bmatrix}e^{\kappa_1 t}\\ te^{\kappa_1 t}\end{Bmatrix}$$

$$\kappa^2+\kappa-l(l+1)=0\xrightarrow{\kappa_1=l,\kappa_2=-(l+1)}R(r)=Cr^l+Dr^{-(l+1)}$$

$$\Delta_3 u(r,\theta,\varphi)=0$$

$$(r^2R')'+(0\cdot r^2-\lambda)R=0$$

$$\frac{1}{\sin\theta}(\sin\theta\Theta')'+(\lambda-\frac{m^2}{\sin^2\theta})\Theta=0$$

$$\Phi''+m^2\Phi=0$$

$$[(1-x^2)y']'+(\lambda-\frac{m^2}{1-x^2})y=0$$

轴对称$\Phi\equiv 1, m=0$:

$$\begin{cases}[(1-x^2)y']'+\lambda y=0\\ \quad |y(\pm 1)|<+\infty\end{cases}$$

$$\lambda_n=n(n+1)$$

$$n=0,1,2,\ldots$$

$$\Theta_n=P_n(\cos\theta)$$

wei huang at mail ustc copyright huang

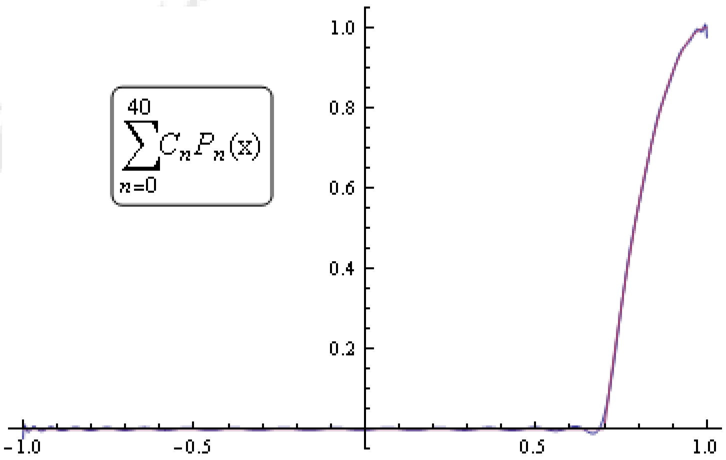
With Inner product,
Sturm-Liouville theory: orthogonal 正交?
Sturm-Liouville theory: completeness 完备?

$$m \neq n:$$

$$\int_{-1}^{1} P_m P_n dx = 0$$

$$f(x) = \sum_n C_n y_n$$

$$C_n = \frac{1}{\int_a^b y_n^2 \rho dx} \int_a^b f(x) y_n \rho dx$$



$$(\sin\theta\Theta')' + (\lambda\sin\theta - \frac{m^2}{\sin\theta})\Theta = 0$$

$$\int_0^{\pi} P_n^2(\cos\theta)\sin\theta d\theta = \int_{-1}^{1} P_n^2 dx = \frac{2}{2n+1}$$

wei huang at mai copyright huang

$$I=\int_{-1}^{1}\frac{\mathrm{d}x}{\sqrt{1-2xt+t^2}\sqrt{1-2xs+s^2}}.$$

作代换

$$u^2=\frac{1+t^2}{2t}-x,\qquad v^2=\frac{1+s^2}{2s}-x,$$

将积分变量 x 换为 u 和 v(当然 u 和 v 不是互相独立的)．

$$u\mathrm{d}u=v\mathrm{d}v,\qquad \mathrm{d}x=-2u\mathrm{d}u=-2v\mathrm{d}v=-u\mathrm{d}u-v\mathrm{d}v$$

所以

$$\frac{\mathrm{d}u}{v}=\frac{\mathrm{d}v}{u}=\frac{\mathrm{d}(u+v)}{u+v}.$$

于是就得到

$$\frac{\mathrm{d}x}{\sqrt{1-2xt+t^2}\sqrt{1-2xs+s^2}}=-\frac{u\mathrm{d}u+v\mathrm{d}v}{2\sqrt{tsuv}}=-\frac{1}{2\sqrt{ts}}\left(\frac{\mathrm{d}u}{v}+\frac{\mathrm{d}v}{u}\right)=-\frac{1}{\sqrt{ts}}\frac{\mathrm{d}(u+v)}{u+v}.$$

这样就能算出积分

$$I=-\frac{1}{\sqrt{ts}}\ln|u+v|\Big|_{x=-1}^{x=1}.$$

注意根据 u 和 v 的定义，当 $|t|<1$， $|s|<1$ 时，应该有

$$u\big|_{x=1}=\frac{1-t}{\sqrt{2t}},\qquad u\big|_{x=-1}=\frac{1+t}{\sqrt{2t}},$$

① 引自 H. Sagan, Boundary & eigenvalue problems in mathemat
John Wiley & Sons, Inc., New York, 1961.

*生成函数技巧证**正交性**,*
同时能算出 ||模平方||.

(*)验算 $\longrightarrow$ $\ln(1+x)=\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{x^n}{n}$

$$\frac{1}{\sqrt{ts}}[\ln(1+\sqrt{ts})-\ln(1-\sqrt{ts})]$$
$$=\frac{1}{\sqrt{ts}}[\sqrt{ts}-\frac{(\sqrt{ts})^2}{2}+\frac{(\sqrt{ts})^3}{3}-\cdots]$$
$$-\frac{1}{\sqrt{ts}}[-\sqrt{ts}-\frac{(-\sqrt{ts})^2}{2}+\frac{(-\sqrt{ts})^3}{3}-\cdots]$$
$$=2+2\frac{ts}{3}+2\frac{(ts)^2}{5}\cdots=\sum_l\frac{2}{2l+1}t^ls^l$$

$$v\big|_{x=1}=\frac{1-s}{\sqrt{2s}},\qquad u\big|_{x=-1}=\frac{1+s}{\sqrt{2s}}.$$

代入即得

$$(*)\qquad I=\frac{1}{\sqrt{ts}}\ln\frac{1+\sqrt{ts}}{1-\sqrt{ts}}=\sum_{l=0}^{\infty}\frac{2}{2l+1}(ts)^l.$$

另一方面，由 Legendre 多项式的生成函数，又应该有

$$I=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}t^ls^k\int_{-1}^{1}\mathrm{P}_l(x)\mathrm{P}_k(x)\mathrm{d}x.$$

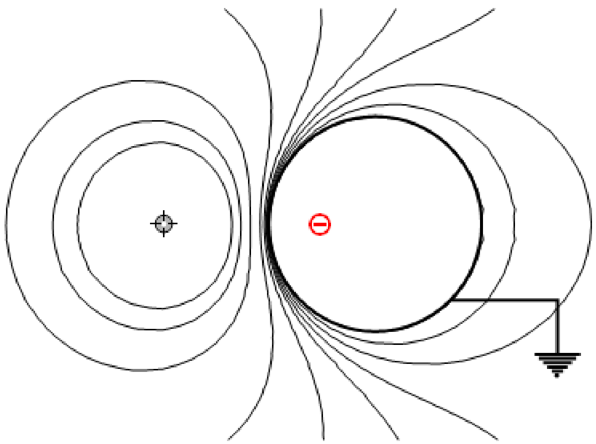
比较系数，就求得

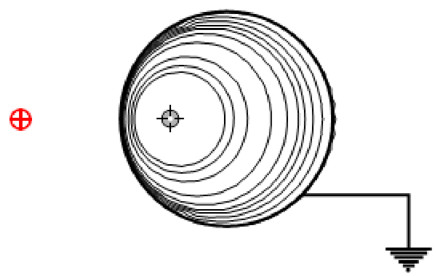
$$\int_{-1}^{1}\mathrm{P}_l(x)\mathrm{P}_k(x)\mathrm{d}x=\frac{2}{2l+1}\delta_{kl}.$$

场位方程 球坐标 边值问题

↓

§5 镜像法, 对称点





由自然边界条件, *球内问题*: $D_n=0$ (n≥0).

球外问题: $C_n=0(n\geq 1)$; C_0待定, D_n待定.

§3 采用球坐标, 求轴对称情形下的三维球外边值问题

$$\begin{cases} \Delta_3 u(r,\theta)=0 \quad (\mathrm{r}>\mathrm{R},\ 0\le\theta\le\pi), \\ u|_{\mathrm{r=R}}=\sin^2\theta, \quad u|_{\mathrm{r}=+\infty}=0. \end{cases}$$

①分离 与φ无关: 轴对称 $u=R(r)\Theta(\theta)$

$$\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right\}u=0$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\lambda$$

$$\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-\lambda$$

$$x=\cos\theta\to[(1-x^2)y']'+\lambda y=0$$

②(2-1)解固有值问题(Legendre方程)

$$\lambda_n=n(n+1),\ n=0,1,2\cdots$$

$$\Theta_n=P_n(\cos\theta)$$

(2-2)Euler方程 $R_n=C_nr^n+D_nr^{-(n+1)}$

③叠加: 轴对称情形下解为

$$\boxed{\mathbf{u}(\mathbf{r},\theta)=\sum_{\mathbf{n=0}}^{+\infty}[\mathbf{C_n r^n}+\mathbf{D_n r^{-(n+1)}}]\mathbf{P_n}(\cos\theta)}$$

球外: 自然边条 无穷远有界 (漏C_0扣分)

$$u(r,\theta)=C_0+\sum_{n=0}^{+\infty}[D_nr^{-(n+1)}]P_n(\cos\theta)$$

定系数 $u|_{r=+\infty}=C_0=0$

$$u|_{r=R}=(C_0+\frac{D_0}{R})P_0+\frac{D_1}{R^2}P_1+\frac{D_2}{R^3}P_2(\cos\theta)+\ldots$$

$$=\sin^2\theta=1-\cos^2\theta$$

组配: 偶⇒$D_1=0$; 正交⇒与$P_n, n>2$无关

$$D_2=-\frac{2}{3}R^3,\ D_0=\frac{2}{3}R$$

$$u=\frac{2}{3}\frac{R}{r}P_0-\frac{2}{3}\frac{R^3}{r^3}P_2=\frac{2}{3}\frac{R}{r}-\frac{1}{3}\frac{R^3}{r^3}(3\cos^2\theta-1)$$

Chapter3, homework22(1) 从被积函数的奇偶性可以判断，

情况(1)

$$\int_{-1}^{1} x^k \mathrm{P}_l(x)\mathrm{d}x = 0, \qquad \text{当 } k \pm l = \text{奇数}. \tag{16.24}$$

情况(2)

当 $k \pm l$ 为偶数时，可将 $\mathrm{P}_l(x)$ 用它的微分表示代入，于是有

$$\begin{aligned}\int_{-1}^{1} x^k \mathrm{P}_l(x)\mathrm{d}x &= \frac{1}{2^l\, l!}\int_{-1}^{1} x^k \frac{\mathrm{d}^l}{\mathrm{d}x^l}\left(x^2-1\right)^l \mathrm{d}x \\ &= \frac{1}{2^l\, l!}\left[x^k \frac{\mathrm{d}^{l-1}}{\mathrm{d}x^{l-1}}\left(x^2-1\right)^l \Big|_{-1}^{1} \right. \\ &\qquad \left. - \int_{-1}^{1} \frac{\mathrm{d}x^k}{\mathrm{d}x}\frac{\mathrm{d}^{l-1}}{\mathrm{d}x^{l-1}}\left(x^2-1\right)^l \mathrm{d}x\right].\end{aligned}$$

由于 $\dfrac{\mathrm{d}^{l-1}}{\mathrm{d}x^{l-1}}\left(x^2-1\right)^l$ 中一定含有因子 $\left(x^2-1\right)$，所以在代入上下限 $x=\pm 1$ 后，分部积分出来的项一定为 0，于是就有

$$\int_{-1}^{1} x^k \mathrm{P}_l(x)\mathrm{d}x = \frac{1}{2^l\, l!}\int_{-1}^{1} (-)^1 \frac{\mathrm{d}x^k}{\mathrm{d}x}\frac{\mathrm{d}^{l-1}}{\mathrm{d}x^{l-1}}\left(x^2-1\right)^l \mathrm{d}x.$$

这样，分部积分一次，其效果就表现在三方面：(1) 改变一次正负号；(2) 对函数 $\left(x^2-1\right)^l$ 的微商减少一次；(3) 对函数 x^k 的微商增加一次．这样，分部积分 l 次后，微商运算就全部转移到函数 x^k 上．结果就变为

$$\int_{-1}^{1} x^k \mathrm{P}_l(x)\mathrm{d}x = \frac{1}{2^l\, l!}\int_{-1}^{1} (-)^l \frac{\mathrm{d}^l x^k}{\mathrm{d}x^l}\left(x^2-1\right)^l \mathrm{d}x.$$

这时有两种可能，一是 $k<l$，函数 x^k 微商 l 次一定为 0，

情况(2-1)

$$\int_{-1}^{1} x^k \mathrm{P}_l(x)\mathrm{d}x = 0, \qquad \text{当} k<l.$$

另一种可能是 $k>l$，不妨令 $k=l+2n$，于是

情况(2-2)

$$\begin{aligned}\int_{-1}^{1} x^{l+2n}\mathrm{P}_l(x)\mathrm{d}x &= \frac{1}{2^l\, l!}\int_{-1}^{1}(-)^l \frac{\mathrm{d}^l x^{l+2n}}{\mathrm{d}x^l}\left(x^2-1\right)^l \mathrm{d}x \\ &= \frac{1}{2^l\, l!}\frac{(l+2n)!}{(2n)!}\int_{-1}^{1} x^{2n}\left(1-x^2\right)^l \mathrm{d}x.\end{aligned}$$

作变换 $x^2=t$，并利用 B 函数就可以算出积分

$$\begin{aligned}\int_{-1}^{1} x^{l+2n}\mathrm{P}_l(x)\mathrm{d}x &= \frac{1}{2^l\, l!}\frac{(l+2n)!}{(2n)!}\int_{0}^{1} t^{n-1/2}(1-t)^l \mathrm{d}t \\ &= \frac{1}{2^l\, l!}\frac{(l+2n)!}{(2n)!}\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma(l+1)}{\Gamma\left(n+l+\frac{3}{2}\right)} \\ &= \frac{(l+2n)!}{2^{l+2n}\, n!}\frac{\sqrt{\pi}}{\Gamma\left(n+l+\frac{3}{2}\right)} \\ &= 2^{l+1}\frac{(l+2n)!\,(l+n)!}{n!\,(2l+2n+1)!}.\end{aligned}$$

特别是 $k=l$，即 $n=0$ 时，

$$\int_{-1}^{1} x^l \mathrm{P}_l(x)\mathrm{d}x = \frac{l!}{2^l}\frac{\sqrt{\pi}}{\Gamma\left(l+\frac{3}{2}\right)} = 2^{l+1}\frac{l!\,l!}{(2l+1)!}.$$

$$u|_{r=R} = \left(C_0+\frac{D_0}{R}\right)P_0 + \frac{D_1}{R^2}P_1 + \frac{D_2}{R^3}P_2(\cos\theta) + \cdots$$
$$= \sin^2\theta = 1-\cos^2\theta \equiv \varphi = 1-x^2$$

也可投影定系数: $\varphi_n = \dfrac{\int_{-1}^{1}\varphi\cdot \mathbf{P_n}dx}{\int_{-1}^{1}\mathbf{P_n}\cdot\mathbf{P_n}dx} = \dfrac{2n+1}{2}\int_{-1}^{1}\varphi\cdot\mathbf{P_n}dx$

$$C_0+\frac{D_0}{R} = \frac{0+1}{2}\left[\int_{-1}^{1} 1\cdot \mathbf{1}dx - \int_{-1}^{1} x^2\cdot \mathbf{1}dx\right] = \frac{1}{2}\left[2-\frac{2}{3}\right] = \frac{2}{3}$$

$$\frac{D_1}{R^2} = \frac{2+1}{2}\left[\int_{-1}^{1} 1\cdot \mathbf{x}dx - \int_{-1}^{1} x^2\cdot \mathbf{x}dx\right] = 0$$ 奇函数积得0

$$\frac{D_2}{R^3} = \frac{4+1}{2}\int_{-1}^{1}\left(1-x^2\right)\cdot\frac{\mathbf{3x^2-1}}{\mathbf{2}}dx = \frac{5}{2}\Big(\int_{-1}^{1}\frac{2}{3}\cdot\frac{\mathbf{3x^2-1}}{\mathbf{2}}dx$$
$$+\int_{-1}^{1}\frac{-2}{3}\cdot\frac{3x^2-1}{2}\cdot\frac{\mathbf{3x^2-1}}{\mathbf{2}}dx\Big) = 0+\frac{5}{2}\cdot\frac{-2}{3}\cdot\frac{2}{2\cdot 2+1} = \frac{-2}{3}$$

$$[(1-x^2)y']'+(\lambda-\frac{m^2}{1-x^2})y=0$$

*(非轴对称的一般情形) m≠0, 面临伴随勒让德方程

*Associated Legendre polynomials (functions)

$$[(1-x^2)v']'+\lambda v=0$$

$$(1-x^2)v''-2x\cdot 1v'+\lambda v=0$$

$$(1-x^2)v'''-2x\cdot(1+1)v''+[\lambda-2\cdot 1]v'=0$$

$$(1-x^2)v^{(2+2)}-2x\cdot(1+2)v^{(1+2)}+[\lambda-2\cdot 3]v^{(2)}=0$$

$$(1-x^2)v^{(2+3)}-2x\cdot(1+3)v^{(1+3)}+[\lambda-4\cdot 3]v^{(3)}=0$$

$$(1-x^2)v^{(2+m)}-2x\cdot(1+m)v^{(1+m)}+[\lambda-m(m+1)]v^{(m)}=0$$

$$y=(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}v(x)$$

$$=(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}[CP_n(x)+DQ_n(x)]$$

$$\equiv CP_n^m+DQ_n^m$$

$$u\equiv v^{(m)},\quad [(1-x^2)^{\frac{m}{2}}u]'=(1-x^2)^{\frac{m}{2}}u'+(1-x^2)^{\frac{m}{2}-1}\frac{m(-2x)}{2}u$$

$$Try\quad \{(1-x^2)[(1-x^2)^{\frac{m}{2}}u]'\}'+(\lambda-\frac{m^2}{1-x^2})[(1-x^2)^{\frac{m}{2}}u]$$

$$=\{(1-x^2)^{\frac{m}{2}+1}u'+(1-x^2)^{\frac{m}{2}}\frac{m(-2x)}{2}u\}'+(\lambda-\frac{m^2}{1-x^2})(1-x^2)^{\frac{m}{2}}u$$

$$=(1-x^2)^{\frac{m}{2}+1}u''+(1-x^2)^{\frac{m}{2}}[\frac{(m+2)(-2x)}{2}+\frac{m(-2x)}{2}]u'$$

$$+[(1-x^2)^{\frac{m}{2}-1}(\frac{m(-2x)}{2})^2+(1-x^2)^{\frac{m}{2}}\frac{m(-2)}{2}]u+(\lambda-\frac{m^2}{1-x^2})(1-x^2)^{\frac{m}{2}}u$$

$$=(1-x^2)^{\frac{m}{2}}\{(1-x^2)u''-2x(m+1)u'+(\lambda-\frac{m^2-m^2x^2+m(1-x^2)}{1-x^2})u$$

$$=(1-x^2)^{\frac{m}{2}}\{(1-x^2)v^{(m+2)}-2x(m+1)v^{(m+1)}+[\lambda-m(m+1)]v^{(m)}=0$$

$$P_n^m(x)\equiv(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x)$$

$$=\frac{1}{2^n n!}(1-x^2)^{\frac{m}{2}}[(x^2-1)^n]^{(n+m)},m\le n$$

$$P_n^0=P_n$$

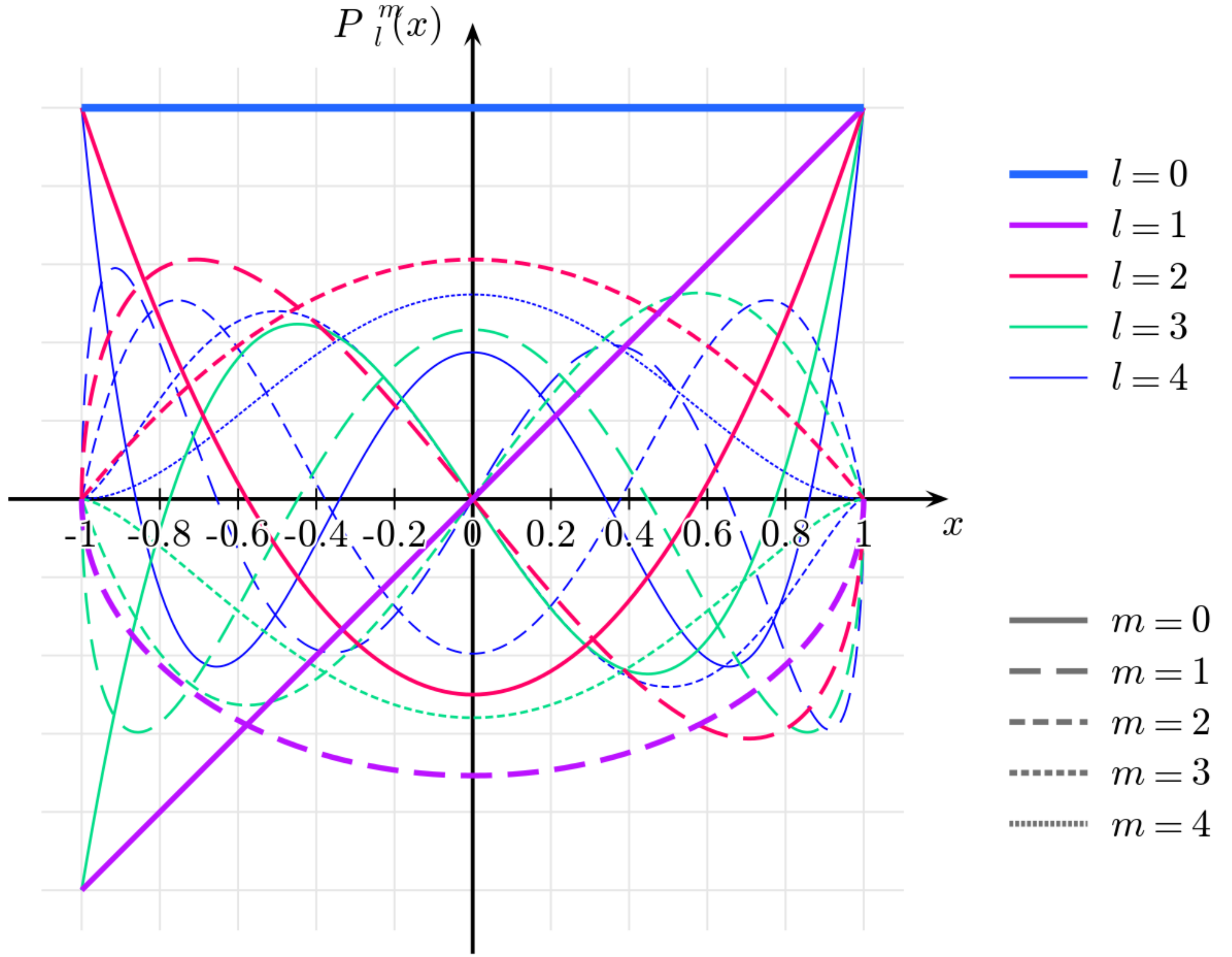
$$P_1^1=\frac{1}{2}(1-x^2)^{\frac{1}{2}}2\Rightarrow\sin\theta$$

$$P_2^1=3x(1-x^2)^{\frac{1}{2}}=\frac{3}{2}\sin 2\theta$$

$$P_2^2=3(1-x^2)=\frac{3}{2}(1-\cos 2\theta)$$

wei huang at mail ustc copyright huang

associated legendre functions (normalized)



*(一般情形) 球谐函数 spherical harmonics

$$u(r,\theta,\varphi)=\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}[A_n r^n+B_n r^{-(n+1)}]P_n^m(\cos\theta)(C_{nm}\cos m\varphi+D_{nm}\sin m\varphi)$$

m阶伴随Legendre函数

$$P_n^m(x)=(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x)$$

$$[(1-x^2)y']'+\left(\lambda-\frac{m^2}{1-x^2}\right)y=0$$

$$\begin{aligned}u&=R(r)\Theta(\theta)\Phi(\phi)\\&\equiv R(r)Y(\theta,\phi)\end{aligned}$$

$$Y_n^m=P_n^m(\cos\theta)\ e^{im\varphi}$$

球函数 n=0,1,2…,

m=0, ±1, ±2, …, ±n

λ_n对应的解2n+1重简并

$$Y_n^m=P_n^m(\cos\theta)\begin{Bmatrix}\cos m\varphi\\ \sin m\varphi\end{Bmatrix}$$

n=0,1,2…, m=0,1…n

本征值λ_n对应2n+1重简并态

$k=0$ $\begin{Bmatrix}r^l\\ r^{-(l+1)}\end{Bmatrix}$ $P_l^m(\cos\theta)$ $\begin{Bmatrix}\cos m\varphi\\ \sin m\varphi\end{Bmatrix}$

$m\overset{k=0}{=}0$ $\begin{Bmatrix}r^l\\ r^{-(l+1)}\end{Bmatrix}$ $P_l(\cos\theta)$ 1

$k\neq0$ $T_{k_p}(t)$ $j_l(k_p r)$ $P_l^m(\cos\theta)$ $\begin{Bmatrix}\cos m\varphi\\ \sin m\varphi\end{Bmatrix}$

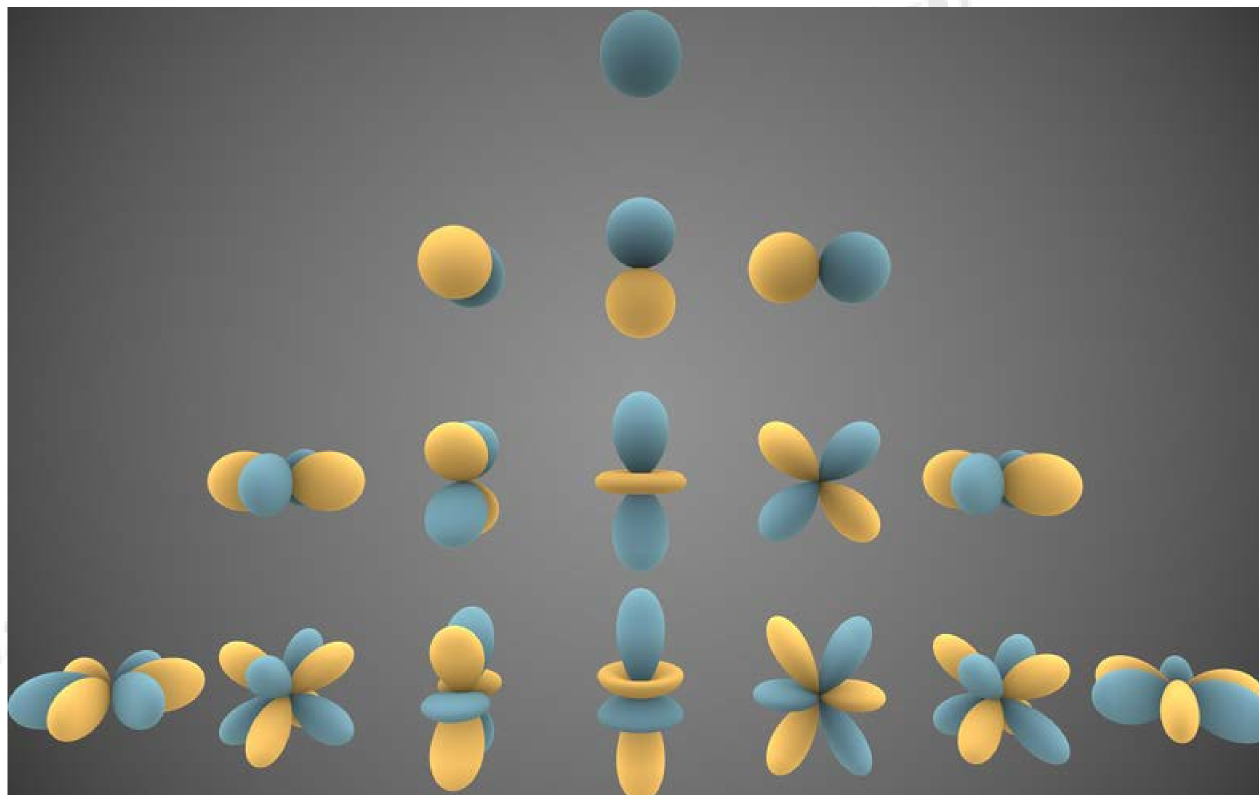
$m\overset{k\neq0}{=}0$ $T_{k_p}(t)$ $j_l(k_p r)$ $P_l(\cos\theta)$ 1

$l\overset{k\neq0,m=0}{=}0$ $T_{k_p}(t)$ $j_0(k_p r)=\dfrac{\sin k_p r}{k_p r}$ 1 1

wei huang at mail.ustc copyright huang

n	m	P_n^m	Φ_m	Y_n^m
0	0	1	1	
1	0	$\cos\theta$	1	
1	1	$\sin\theta$	$\cos\varphi$	
1	1	$\sin\theta$	$\sin\varphi$	
2	0	$\frac{3\cos^2\theta-1}{2}$	1	
2	1	$\frac{3\sin 2\theta}{2}$	$\cos\varphi$	
2	1	$\frac{3\sin 2\theta}{2}$	$\sin\varphi$	
2	2	$\frac{3(1-\cos 2\theta)}{2}$	$\cos 2\varphi$	
2	2	$\frac{3(1-\cos 2\theta)}{2}$	$\sin 2\varphi$	

n	m=-3	m=-2	m=-1	m=0	m=1	m=2	m=3
0				球对称			
1			θ π/2最大 φ π/2最大 φ 3π/2最负	θ=0最大 θ=π最负 轴对称	θ π/2最大 φ 0最大 φ π最负		
2		θ π/2最大 0, π为零 φ π/4,5π/4大 3π/4,7π/4负	θ π/4最大 3π/4最负 φ π/2最大 φ 3π/2最负	θ0, π最大 π/2最负	θ π/4最大 3π/4最负 φ 0最大 φ π最负	θ π/2最大 0, π为零 φ 0, π最大 π/2,3π/2负	
3				θ 0最正,渐变负,π/2后又变正, 最负于π			



*球谐函数：量子力学、原子分子、物化、计算图形学、光照中广泛应用

$$\boldsymbol{E} \leftrightarrow \mathrm{i}\hbar\frac{\partial}{\partial t}, \quad \boldsymbol{E} = \frac{\vec{p}^2}{2m} + V(t,\vec{r}), \quad \vec{p} \leftrightarrow -\mathrm{i}\hbar\nabla$$

- 薛定谔方程 (Schrödinger)

$$i\hbar\frac{\partial}{\partial t}\psi(t,\vec{\mathbf{r}}) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(t,\vec{\mathbf{r}})\right]\psi(t,\vec{\mathbf{r}})$$

外势场不显含t时，能量$\boldsymbol{E} = \frac{\vec{p}^2}{2m} + V(\vec{r})$守恒，
可分离变量$\psi(t,\vec{r}) = T(t)\Psi(\vec{r}) = e^{-\mathrm{i}Et/\hbar}\Psi(\vec{r})$
转化为定态问题

$$\left[\frac{-\hbar^2}{2m}\nabla^2 + V(\vec{\mathbf{r}})\right]\Psi(\vec{\mathbf{r}}) = E\Psi(\vec{\mathbf{r}})$$

- 类氢原子的单电子波函数

$$-\frac{\hbar^2}{2\mu r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin^2\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\phi^2}\right]\right\}\Psi - \frac{Ze^2}{4\pi\epsilon_0 r}\Psi = E\Psi$$

$$\Theta(\theta)\Phi(\phi) = Y(\theta,\phi)$$

$$-\frac{1}{\sin^2\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\phi^2}\right]Y_l^m(\theta,\phi) = l(l+1)Y_l^m(\theta,\phi)$$

* Z=1, Hydrogen Atom $\psi(t,r,\theta,\varphi)=e^{-iEt/\hbar}\Psi(r,\theta,\varphi)$, $\Psi=R(r)Y(\theta,\varphi)$

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}\right]R=-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2(rR)}{dr^2}+[...]R=ER$$

$$\xrightarrow[Trick\ \chi(r)=rR(r)]{} -\frac{\hbar^2}{2m}\frac{d^2\chi}{dr^2}+\left[\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}\right]\chi=E\chi$$

Tidy up notation: $\rho\equiv\frac{\sqrt{-2mE}}{\hbar}r$, $\rho_0\equiv\frac{me^2}{2\pi\epsilon_0\hbar\sqrt{-2mE}}$, $\frac{d^2\chi}{d\rho^2}=\left[1-\frac{\rho_0}{\rho}+\frac{l(l+1)}{\rho^2}\right]\chi$

(bound state of a proton and an electron, $0\xleftarrow{0\leftarrow r}\chi\xrightarrow{r\to\infty}0$)

$\rho\to\infty$: $\frac{d^2\chi}{d\rho^2}=1\cdot\chi$. General solution: $\chi=Ae^{-\rho}+Be^{\rho}\xrightarrow{e^\rho\ blows\ up}Ae^{-\rho}$.

$\rho\to 0$: Euler $\frac{d^2\chi}{d\rho^2}=\frac{l(l+1)}{\rho^2}\chi$. Gen sol: $\chi=C\rho^{-l}+D\rho^{l+1}\xrightarrow{\rho^{-l}\ blows\ up}D\rho^{l+1}$.

Peer off the asymptotic behavior: $\chi=\rho^{l+1}e^{-\rho}v(\rho)$

$$\rho\frac{d^2v}{d\rho^2}+2(l+1-\rho)\frac{dv}{d\rho}+[\rho_0-2(l+1)]v=0$$

Generalized/associated Laguerre eq.: $xy''+(\alpha+1-x)y'+ny=0$

ConFluent HyperGeometric eq.(Kummer): $zw''+(\gamma-z)w'-\alpha w=0$

Power series in ρ: $v=\sum_{j=0}^{\infty}a_j\rho^j$, $\frac{dv}{d\rho}=\Sigma_{j=0,1}^{\infty}ja_j\rho^{j-1}=\Sigma_{j=0}^{\infty}(j+1)a_{j+1}\rho^j$,

$\frac{d^2v}{d\rho^2}=\Sigma j(j+1)a_{j+1}\rho^{j-1}$, yields a recursion formula: $a_{j+1}=\frac{2(j+l+1)-\rho_0}{(j+1)(j+2l+2)}a_j$

v blows up at large ρ, the seris must terminate: $2(j_{max}+l+1)-\rho_0=0$,

$a_0\to a_1\to a_2\to\cdots\to a_{j_{max}}\to a_{j_{max}+1}=0\to\cdots 0$

$j_{max}\in\mathbb{N}$, Legendre $\Rightarrow l=0,1,2...$, DEFINE principal $n\equiv j_{max}+l+1$

$$2n=\rho_0=\frac{me^2}{2\pi\epsilon_0\hbar\sqrt{-2mE}}\Longrightarrow E_n=-\frac{me^4}{2\hbar^2(4\pi\epsilon_0)^2}\frac{1}{n^2}=\frac{E_1}{n^2}=\frac{-13.6eV}{n^2},\ n=1,2,...$$

With Bohr radius $a\equiv\frac{\hbar^2(4\pi\epsilon_0)}{me^2}=0.529\times10^{-10}m$,

$$\Psi_{nlm}=R_{nl}Y_l^m=\sqrt{\left(\frac{2}{na}\right)^3\frac{(n-l-1)!}{2n[(n+l)!]^3}}e^{-\frac{r}{na}}\left(\frac{2r}{na}\right)^lL_{n-l-1}^{2l+1}\left(\frac{2r}{na}\right)Y_l^m(\theta,\varphi)$$

$$=\frac{2\sqrt{\frac{(n+l)!}{(n-l-1)!}}}{a^{3/2}n^2(2l+1)!}e^{-\frac{r}{na}}\left(\frac{2r}{na}\right)^lF\left(-n+l+1,2l+2,\frac{2r}{na}\right)Y_l^m(\theta,\varphi),$$

where gen Laguerre $L_n^\alpha(x)=\frac{\Gamma(\alpha+1+n)}{n!\Gamma(\alpha+1)}F(-n,\alpha+1,x)$ confluent hypergeo.