§3 Higher dimentional PDEs ($\xrightarrow{\textit{Separation of variables}}$ODEs)
and Special functions 特殊函数
(appear as solutions of ODEs with variable coefficients)

1st example, cylindrical coordinates: Bessel function
柱坐标: 贝塞尔函数

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曲线坐标系

* 补充内容

- $ds^2 = (1 \cdot dx)^2 + (1 \cdot dy)^2 + (1 \cdot dy)^2$
  $= (1 \cdot dr)^2 + (r \cdot d\theta)^2 + (r \sin\theta \cdot d\varphi)^2$ 球坐标
  $= (1 \cdot dr)^2 + (r \cdot d\theta)^2 + (1 \cdot dz)^2$ 柱坐标
  $\equiv (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2$ Lame 系数
- $\nabla = \frac{\vec{e_1}}{h_1}\frac{\partial}{\partial q_1} + \frac{\vec{e_2}}{h_2}\frac{\partial}{\partial q_2} + \frac{\vec{e_3}}{h_3}\frac{\partial}{\partial q_3}$
- $\nabla \cdot \vec{F} = \frac{1}{h_1 h_2 h_3}[\frac{\partial}{\partial q_1}(F_1 h_2 h_3) + \frac{\partial}{\partial q_2}(h_1 F_2 h_3) + \frac{\partial}{\partial q_3}(h_1 h_2 F_3)]$
- $\nabla \times \vec{F} = \frac{\vec{e_1}}{h_2 h_3}[\frac{\partial}{\partial q_2}(h_3 F_3) - \frac{\partial}{\partial q_3}(h_2 F_2)]$
  $+ \frac{\vec{e_2}}{h_3 h_1}[\frac{\partial}{\partial q_3}(h_1 F_1) - \frac{\partial}{\partial q_1}(h_3 F_3)] + \frac{\vec{e_3}}{h_1 h_2}[\frac{\partial}{\partial q_1}(h_2 F_2) - \frac{\partial}{\partial q_2}(h_1 F_1)]$
- $\triangle = \nabla \cdot \nabla =$
  $\frac{1}{h_1 h_2 h_3}[\frac{\partial}{\partial q_1}(h_1^{-1} h_2 h_3 \frac{\partial}{\partial q_1}) + \frac{\partial}{\partial q_2}(h_1 h_2^{-1} h_3 \frac{\partial}{\partial q_2}) + \frac{\partial}{\partial q_3}(h_1 h_2 h_3^{-1} \frac{\partial}{\partial q_3})]$
- 球坐标 $\triangle_3 = \frac{1}{r^2}[\frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}]$
- 柱坐标 $\triangle_3 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$

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曲线坐标系下分离变量

- $$\begin{cases} u_{tt} = a^2 \triangle u \quad PDE \quad u_t = a^2 \triangle u \\ BC \\ IC \end{cases}$$
- 设变量分离形式的特解 $u(t, q_1, q_2, q_3) = T(t) v(q_1, q_2, q_3)$,
  则泛定方程可分离变量为
  $T_{tt} v = a^2 T \triangle v$ 波动振动, $T_t v = a^2 T \triangle v$ 传导扩散
  $\frac{T_{tt}}{a^2 T} = \frac{\triangle v}{v} = -k^2$ 常数 $\frac{T_t}{a^2 T} = \frac{\triangle v}{v} = -k^2$
  $T'' + a^2 k^2 T = 0$ $T' + a^2 k^2 T = 0$
- $\triangle v + k^2 v = 0$ Helmholtz Equation
- 球坐标下 $\frac{1}{r^2}[\frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r}) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}(\sin\theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2}] v + k^2 v = 0$
- 柱坐标下 $[\frac{1}{r} \frac{\partial}{\partial r}(r \frac{\partial}{\partial r}) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}] v + k^2 v = 0$

柱坐标系下分离变量

- $$u_{tt} = a^2\triangle_3 u \quad \text{波动,} \qquad u_t = a^2\triangle_3 u \quad \text{传导}$$
  $$\frac{T''}{a^2 T} = \frac{\triangle_3 v}{v} = -k^2 \qquad \frac{T'}{a^2 T} = \frac{\triangle_3 v}{v}$$
  $$T'' + a^2k^2T = 0 \qquad T' + a^2k^2T = 0$$
  $$\triangle_3 v + k^2 v = 0 \qquad \triangle_3 v + k^2 v = 0$$
- 柱坐标系下 $[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}]v + k^2 v = 0$,
  进一步设存在变量分离形式的特解 $v = R(r)\Theta(\theta)Z(z)$,
  逐层剥离
  $$\frac{1}{r}(rR')'/R + \frac{1}{r^2}\cdot \quad \Theta''/\Theta \quad + Z''/Z \quad + k^2 = 0$$
  $$= -\sigma \qquad = -\mu$$
  $$\equiv -\nu^2$$
- $\Theta''/\Theta = -\nu^2$, $Z''/Z = -\mu$ 易解.
- $(rR')' - \frac{\nu^2}{r}R + (-\mu + k^2)rR = 0$ 如何解? 是变系数常微分方程

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径向出现的变系数 ODE

注意$\Theta''/\Theta = -\nu^2$, $Z''/Z = -\mu$

- $(rR')' - \frac{\nu^2}{r}R + (-\mu + k^2)rR = 0$,
  像 S-L 型 $k = r, q = \frac{\nu^2}{r}, \rho = r$
  定义 $\lambda = k^2 - \mu$, 当 $\lambda \geq 0$ 时是 S-L 型能组成固有值问题.
- * 不考虑 $k^2 - \mu < 0$ 情况, 是需另作研究的虚宗量贝塞尔方程.
- 当且仅当 $\lambda = k^2 - \mu \geq 0$ 时, 是贝塞尔方程

$$\xrightarrow{Trick\ x=\sqrt{\lambda}r} x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad Bessel\ eq.$$

- 特例 $k^2 - \mu = 0$, 可按 Bessel 知识求解, 也可按最简变系数 ODE 即 Euler 方程技巧解:

$$r^2R'' + rR' - \nu^2R = 0 \xrightarrow{r=e^t} R_0 = \left\{ \begin{array}{c} 1 \\ \ln r \end{array} \right\}, R_\nu \overset{\nu \geq 1}{=} \left\{ \begin{array}{c} r^\nu \\ r^{-\nu} \end{array} \right\}$$

- $y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2})y = 0$, Bessel 方程有正则奇点 $x = 0$,
  在 $0 < |x| < +\infty$ 去心圆域解析.
  Regular singularity at x=0: 1/x has a pole of first order, when $\nu \neq 0$, $1 - \frac{\nu^2}{x^2}$ has a pole of second order.
  Möbius trans w=1/x, $\eta'' + \frac{1}{w}\eta' + (\frac{1}{w^4} - \frac{\nu^2}{w^2})\eta = 0$, found irregular singularity at w=0, $x \to \infty$.

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变系数 ODE Bessel 方程的广义幂级数解 →Bessel 函数

§3.1 $x^2y'' + xy' + (x^2 - \nu^2)y = 0$

- **广义幂级数**: $y = x^\rho \sum_{n=0}^{\infty} a_n x^n = a_0 x^\rho + a_1 x^{\rho+1} + \ldots,\ a_0 \neq 0.$

	x^ρ	$x^{\rho+1}$	$x^{\rho+2}$	$x^{\rho+n}$
x^2y''	$\rho(\rho-1)a_0$	$(\rho+1)\rho a_1$	$(\rho+2)(\rho+1)a_2$	$(\rho+n)(\rho+n-1)a_n$
xy'	ρa_0	$(\rho+1)a_1$	$(\rho+2)a_2$	$(\rho+n)a_n$
x^2y			a_0	a_{n-2}
$-\nu^2 y$	$-\nu^2 a_0$	$-\nu^2 a_1$	$-\nu^2 a_2$	$-\nu^2 a_n$
	$\rho^2 - \nu^2{}_{=0}$			$a_n = \frac{-a_{n-2}}{(\rho+n)^2-\nu^2}$
	指标方程	特例 $\rho = \frac{-1}{2}$		递推式
	$\rho = \pm\nu$	$(1+2\rho)a_{1=0}$		
		$a_1 = 0$		

- (1) $\rho_1 = +\nu \geq 0,\ a_n = \frac{-a_{n-2}}{n(n+2\nu)}$

  这时 $\rho \neq \frac{-1}{2}$, **奇串** $a_1 = 0 \longrightarrow a_{2k+1} = 0$

  偶串 $a_0 \longrightarrow a_{2k} = \frac{-1}{2k(2k+2\nu)} a_{2k-2} = \frac{-1}{2^2k(k+\nu)} a_{2(k-1)} = \ldots$

  $= (\frac{1}{2})^{2k} \frac{(-1)^k}{k(k+\nu)(k-1)(k-1+\nu)\ldots 1\cdot(1+\nu)} a_0 = \frac{(-1)^k}{2^{2k}k!} \frac{\Gamma(\nu+1)a_0}{\Gamma(k+\nu+1)}$

- $y_1 = x^{\nu}\sum_{k=0}^{+\infty} a_{2k}x^{2k} = \sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}[2^{\nu}\Gamma(\nu+1)a_0]$

  找特解，可设与 k 无关的常数项 $[2^{\nu}\Gamma(\nu+1)a_0]$ 为 1. 得

  第一类 ν 阶 **Bessel** 函数 $J_{\nu}(x) \equiv \sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}$

- $\lim_{k\to+\infty}\left|\frac{a_{2k}}{a_{2k-2}}\right| = \lim_{k\to+\infty}\left|\frac{1}{4k(k+\nu)}\right| = 0$, J_{ν} 收敛半径 $+\infty$.

- 另一端性质 $\lim_{x\to 0} J_{\nu} = \begin{cases} 1 & \nu=0, \\ 0 & \nu>0. \end{cases} \Rightarrow$ 总之 J_{ν} 处处有界.

- (2) $\rho_2 = -\nu < 0$, $a_n = \frac{-a_{n-2}}{n(n-2\nu)}$

- $n-2\nu$ 避开等 0 问题的情况，2ν 不为整数，$y_2 = \sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(k-\nu+1)}\left(\frac{x}{2}\right)^{2k-\nu} \equiv J_{-\nu}$

- $n-2\nu$ 出现等 0 问题，$2\nu = 2m+1$, 奇串 $a_1 = 0 \to a_{2m-1} = 0$, 把无法递推的 a_{2m+1} 补充约定为 0, 则可以继续推串为 0. $y_2 = J_{-\nu}$ 可用.

- $n-2\nu$ 出现等 0 问题，$2\nu = 2m$, x^{-m}, $x^{-m+2}, \cdots x^{m-2}$ 的系数出现问题:

- $J_{-m}=\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(k-m+1)}\left(\frac{x}{2}\right)^{2k-m}=$

  $\frac{\dots}{\Gamma(-m+1)}x^{-m}+\dots+\frac{\dots}{\Gamma(0)}x^{m-2}+\sum_{k=m}^{+\infty}\frac{(-1)^k}{k!\Gamma(k-m+1)}\left(\frac{x}{2}\right)^{2k-m}$

- $\Gamma(z+1)=z\Gamma(z)$, $\Gamma(1)=0!=1\Rightarrow\ \Gamma(z)=\frac{1}{z}\Gamma(z+1)\overset{z\to 0}{\sim}\frac{1}{z}\xrightarrow{z\to 0}\infty$,
  $\Gamma(z)|_{z\to -n}=\frac{1}{z(z+1)\dots(z+n-1)}\Gamma(z+n)|_{z\to -n}\sim\frac{1}{(-n)(-n+1)\dots(-1)}\frac{1}{z+n}$
  $\sim\frac{(-1)^n}{n!}\frac{1}{z+n}\xrightarrow{z\to -n}\infty$, 零和负整数是 Γ 函数的单极点.

- $J_{-m}=0+\dots+0+\sum_{l=k-m=0}^{+\infty}\frac{(-1)^{l+m}}{(l+m)!\Gamma(l+1)}\left(\frac{x}{2}\right)^{2l+2m-m}$

  $=(-1)^m\sum_{l=0}^{+\infty}\frac{(-1)^l}{l!\Gamma(k+m+1)}\left(\frac{x}{2}\right)^{2l+m}=(-1)^mJ_m$, 非独立解.

- Liouville 公式法从 J_m 求另一独立解

- Fuchs 定理 $y_2(x)=\alpha y_1(x)ln(x-0)+x^{-m}\sum_{n=0}^{+\infty}b_nx^n$, 待定广义幂级数展开系数法

  见梁昆淼《数学物理方法》第四版 P204-P210 和附录十三.

- 约定第二类 Bessel 函数 (又叫 Neumann 函数, Weber 函数):
  $N_\nu(x)\equiv\frac{\cos\nu\pi J_\nu-J_{-\nu}}{\sin\nu\pi}\xrightarrow{\nu\to m}N_m=\frac{2J_m}{\pi}(\ln\frac{x}{2}+\gamma)+$ 负幂 $+$ 正幂.

Bessel 方程的通解

- $\nu \geq 0$, 在去心圆域 $0 < |x| < +\infty$, 通解皆可表示为
$$y(x) = CJ_\nu + DN_\nu$$
$x = 0$ 处, J_ν 有界; $N_m \sim \ln\frac{x}{2} \xrightarrow{x\to 0} -\infty$, m=0,1,...;
$$N_\nu \sim \frac{-\Gamma(\nu)}{x^\nu} \xrightarrow[\nu\notin\mathbb{Z}]{x\to 0} \pm\infty.$$
- 类比初等函数 $\cos x$, $\sin x$ 组合 $\cos x \pm i\sin x = e^{\pm ix}$,
J_ν, N_ν 可组合出第三类 Bessel 函数:
第一类 Hankel 函数 $H_\nu^{(1)}(x) = J_\nu + iN_\nu$,
第二类 Hankel 函数 $H_\nu^{(2)}(x) = J_\nu - iN_\nu$.
$$y(x) = C_1 H_\nu^{(1)}(x) + C_2 H_\nu^{(2)}(x)$$
- 能写成初等函数的特例
$$\left(\begin{array}{c|c} J_{\frac{1}{2}} & J_{-\frac{1}{2}} \\ \hline N_{\frac{1}{2}} & N_{-\frac{1}{2}} \end{array}\right) = \left(\begin{array}{c|c} \sqrt{\frac{2}{\pi x}}\sin x & \sqrt{\frac{2}{\pi x}}\cos x \\ \hline -\sqrt{\frac{2}{\pi x}}\cos x & \sqrt{\frac{2}{\pi x}}\sin x \end{array}\right)$$

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贝塞尔函数的表示和性质

1. 级数定义式 $J_\nu(x) \equiv \sum\limits_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}$
2. 生成函数 $e^{\frac{x}{2}(t-\frac{1}{t})} = \sum\limits_{n=-\infty}^{+\infty} J_n(x)t^n \qquad t \in \mathbb{C}\backslash\{0\}$
3. 积分表示 取围道 $t = e^{i\theta}$ 积生成函数，得 $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)d\theta$

   * Differ from Anger function $\mathbf{J}_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu\theta - z\sin\theta)d\theta$, only when $n \in \mathbb{Z}$, $\mathbf{J}_n = J_n$.
4. 微分及递推关系 $\left(\frac{J_\nu}{x^\nu}\right)' = [\Sigma_{k=0}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{1}{2}\right)^{2k+\nu}x^{2k}]' = \Sigma_{k=1}\frac{(-1)^k 2k}{k!\Gamma(k+\nu+1)}\left(\frac{1}{2}\right)^{2k+\nu}$ $x^{2k-1} \overset{k=l+1}{=} \frac{-1}{x^\nu}\Sigma_{l=0}\frac{(-1)^l}{l!\Gamma(l+1+\nu+1)}\left(\frac{1}{2}\right)^{2l+1+\nu}x^{2l+1+\nu} = -\frac{J_{\nu+1}}{x^\nu}$

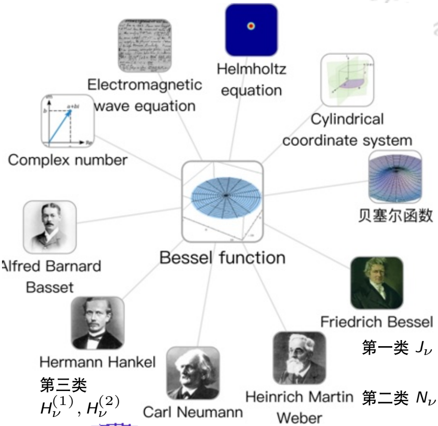
   $\Rightarrow \left(\frac{1}{x}\frac{d}{dx}\right)^l\left[\frac{J_\nu}{x^\nu}\right] = (-1)^l\frac{J_{\nu+l}}{x^{\nu+l}}$, $\left(\frac{1}{x}\frac{d}{dx}\right)^l[x^\nu J_\nu] = x^{\nu-l}J_{\nu-l}$, 掌握常用特例 $J_0{}' = -J_1$.
5. $x = \sqrt{\lambda}r = 0$ 处 J_ν 有界，N_ν 无界; ∞ 处都有界，大 $x \gg |\nu^2 - \frac{1}{4}|$ 处有衰减震荡性

   $$J_m(\sqrt{\lambda}r) \sim \sqrt{\frac{2}{\pi\sqrt{\lambda}r}}\cos\left(\sqrt{\lambda}r - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

   $$N_m(\sqrt{\lambda}r) \sim \sqrt{\frac{2}{\pi\sqrt{\lambda}r}}\sin\left(\sqrt{\lambda}r - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

   $$e^{-i\frac{E}{\hbar}t}H_m^{(1),(2)}(\sqrt{\lambda}r) \sim \sqrt{\frac{2}{\pi\sqrt{\lambda}r}}e^{\pm i(\sqrt{\lambda}r \mp \frac{E}{\hbar}t - \frac{m\pi}{2} - \frac{\pi}{4})}$$
6. 图像
7. 零点

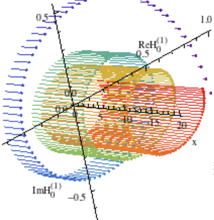
plot Bessel functions

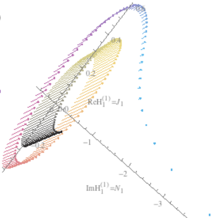


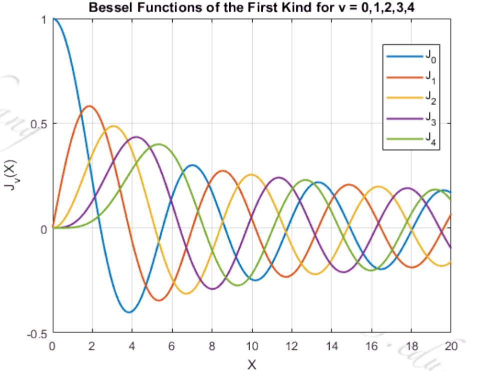
第一类 J_ν

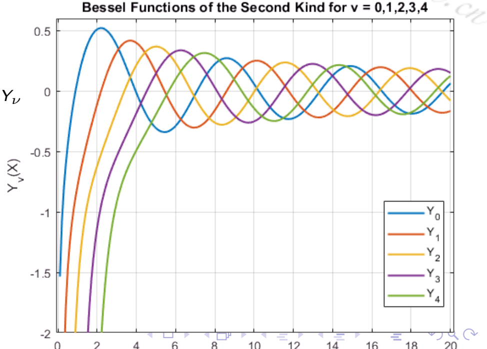
第二类 $N_\nu = Y_\nu$

第三类 $H_\nu^{(1)}, H_\nu^{(2)}$









例 2 轴对称稳恒柱问题 $\begin{cases} \triangle_3 u = 0 & r < a,\ 0 < z < h \\ \frac{\partial u}{\partial r}|_{r=a} = 0 & \text{绝热是齐次}\mathcal{II}\text{类}BC \\ u|_{z=0} = f_1(r), & u|_{z=h} = f_2(r) \end{cases}$

解:(一) 设$u = R(r) \cdot 1 \cdot Z(z)$
泛定 $0 = \triangle_3 u = [\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{\partial^2}{\partial z^2}]u \to$ $\begin{cases} \Theta \equiv 1 \xrightarrow{\Theta''+0\cdot\Theta=0} \nu = 0 & Q1 \\ \frac{\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial R}{\partial r})}{R} = -\lambda & Q2 \\ Z'' - \lambda Z = 0 & Q3 \end{cases}$

齐次定解条件可配上, 组固有值问题Q2: $\begin{cases} (rR')' + \lambda rR = 0 \\ \mathrm{R}'(\mathrm{a}) = 0\ \mathrm{II}\text{齐}\mathrm{BC},\ |\mathrm{R}(0)| < \infty \end{cases}$

固有值问题用不上的条件, 终极定解条件 QC: $u|_{z=0} = f_1(r), u|_{z=h} = f_2(r)$.

(二) (2-1) 解唯一的固有值问题 Q2, 轴对称情形 $\nu = 0$, 这是 0 阶 Bessel 配齐次 $\mathcal{II}$ 类边条. 柱内通解为 $R(r) = AJ_0(\omega r) + BN_0(\omega r)$. 适应自然边条, 柱内问题舍去无界 N_0. 适应柱面边条 $0 = \mathrm{R}'(\mathrm{a}) = \omega \mathrm{J}_0'(\omega \mathrm{a}) \sim -\mathrm{J}_1(\omega \mathrm{a})$, ω 需处能使 $\mathrm{J}_0' = -\mathrm{J}_1$ 等于零的离散点: $\omega_{0\mathrm{II}0} = 0, \omega_{0\mathrm{II}1} = \frac{x_{1\mathrm{II}1}}{a}, \omega_{0\mathrm{II}2} = \frac{x_{1\mathrm{II}2}}{a}, \ldots$
设 ω_n 标记 $\mathrm{J}_0'(\omega \mathrm{a}) = 0$ 或 $\mathrm{J}_1(\omega \mathrm{a}) = 0$ 的第 n 个非负根, 固有值 $\lambda_n = \omega_n^2$, 固有函数$R_n(r) = J_0(\omega_n r)$, n=0,1,2,...; 有特例 $\lambda_0 = 0, R_0(r) = J_0(0) \equiv 1$.

(2-2) 解组不成固有值问题的问题 Q3 $Z'' - \omega_n^2 Z = 0$. 对不同的 n, 通解为 $Z_n(z) = c_n e^{\omega_n z} + d_n e^{-\omega_n z}$; 有特例, n=0 时 $e^0\ e^{-0} = 1$ 重合为仅一个独立解, 另单列 $Z'' = 0$ 解得此特例通解 $Z_0 = c_0 + d_0 z$.

也可写为双曲函数 $Z_n(z) = C_n ch\omega_n z + D_n sh\omega_n z$, n>1; $Z_0 = C_0 + D_0 z$.

(2-3)PDE 和部分 BC 约束下，存在特解族 $\left\{\begin{array}{c} ch\omega_n z \\ sh\omega_n z \end{array}\right\} \oplus \left\{\begin{array}{c} 1 \\ z \end{array}\right\} J_0(\omega_n r)$.

(三) 满足 PDE 和柱面 BC 的任一个可能解，可由正交基 “叠加” 出来:

$$u = (C_0 + D_0 z)\cdot 1 + \sum_{n=1}^{\infty}(C_n ch\omega_n z + D_n sh\omega_n z) J_0(\omega_n r)$$

(太具体条件 QC，对局限于变量分离形式的可数个特解/驻波/特定分布里某个，不能用；但对其协奏出的任意可能解，能用.) 用剩余条件 QC“定系数”:

$$u|_{z=0} = C_0\cdot 1 + \sum_{n=1}^{\infty}(C_n ch0 + D_n sh0) J_0(\omega_n r) = C_0\cdot 1 + \sum_{n=1}^{\infty} C_n J_0(\omega_n r) = f_1 = 0$$

$$u|_{z=h} = (C_0 + D_0 h)\cdot 1 + \sum_{n=1}^{\infty}(C_n ch\omega_n h + D_n sh\omega_n h) J_0(\omega_n r) = f_2 = 1 - \frac{r^2}{a^2}$$

$$C_0 = \frac{\int_0^a f_1(r) 1\, rdr}{\int_0^a 1 rdr} \stackrel{\text{def}}{=} f_{10}, \qquad C_n = \frac{\int_0^a f_1(r) J_0(\omega_n r)\, rdr}{\int_0^a J_0^2(\omega_n r) rdr} \stackrel{\text{def}}{=} f_{1n},$$

$$C_0 + D_0 h = \frac{\int_0^a f_2(r) 1\, rdr}{\int_0^a 1 rdr} \stackrel{\text{def}}{=} f_{20}, \qquad C_n ch\omega_n h + D_n sh\omega_n h = \frac{\int_0^a f_2(r) J_0(\omega_n r)\, rdr}{\int_0^a J_0^2(\omega_n r) rdr} \stackrel{\text{def}}{=} f_{2n},$$

$$\frac{f_{20} - f_{10}}{h} = D_0, \qquad \frac{f_{2n} - f_{1n} ch\omega_n h}{sh\omega_n h} = D_n.$$

分母是正交函数系某基函数的模的平方，本题$\mathcal{II}$ b.c. $N_{0\mathcal{II}n}^2 = \frac{1}{2}(a^2 - \frac{0}{\omega_n^2}) J_0^2(\omega_n a)$

$\boxed{\text{模平方}}\ \|\sin m\theta\|^2 \equiv \int_0^{2\pi} \sin^2 m\theta\, d\theta = \int_0^{2\pi} \frac{1-\cos 2m\theta}{2} d\theta = \int_0^{2\pi} \frac{d\theta}{2} = \pi,\quad m>0$

$$\|\cos m\theta\|^2 \equiv \int_0^{2\pi} \cos^2 m\theta\, d\theta = \int_0^{2\pi} \frac{1+\cos 2m\theta}{2} d\theta = \begin{cases} \int_0^{2\pi} 1\, d\theta = 2\pi, & m=0 \\ \int_0^{2\pi} \frac{1}{2} d\theta = \pi, & m>0 \end{cases}$$

对种类繁多的 Bessel 固有函数系，第 m 阶、第 $\mathcal{I}$ 类边条下第 n 个基的模平方:

$$N^2_{m\mathcal{I}n} \equiv \|J_m(\omega_{mn}r)\|^2 \equiv \int_0^{r_0} J_m^2(\omega_{mn}r)\, r dr$$

Trick: 凑 $rJ^2 = rR^2$ 被积项? $(rR_n')' + (\omega_{mn}^2 r - \frac{\nu^2}{r})R_n = 0$ 同乘以 rR_n' 可实现.

$$\int_0^{r_0} (rR_n')'(rR_n')dr + \int_0^{r_0} (\omega_{mn}^2 r^2 - \nu^2) R_n R_n' dr = 0$$

$$\int d\frac{(rR_n')^2}{2} + \int d[(\omega_{mn}^2 r^2 - \nu^2)\frac{R_n^2}{2}] - \int \frac{R_n^2}{2} d(\omega_{mn}^2 r^2 - \nu^2) = 0$$

$$\frac{(rR_n')^2}{2}\Big|_0^{r_0} + (\omega_{mn}^2 r^2 - \nu^2)\frac{R_n^2}{2}\Big|_0^{r_0} = \omega_{mn}^2 \int_0^{r_0} R_n^2 r dr$$

$$\boxed{\frac{r_0^2}{2} J_m'^2(\omega_{mn}r_0) + \frac{1}{2}(r_0^2 - \frac{\nu^2}{\omega_{mn}^2}) J_m^2(\omega_{mn}r_0) = \int_0^{r_0} J_m^2(\omega_{mn}r) r dr}$$

$$\xrightarrow[J_m(\omega_{mn}r_0)=0]{\mathcal{I}\text{ b.c.}} N^2_{m\mathcal{I}n} = \frac{r_0^2}{2} J_m'^2(\omega_{mn}r_0) \xrightarrow[J_m' - \frac{m}{r_0}J_m(\omega_{mn}r_0) = J_{m+1}]{(3.4.2b)} \frac{r_0^2}{2} J_{m+1}^2(\omega_{mn}r_0)$$

$$\xrightarrow[J_m'(\omega_{mIIn}r_0)=0]{\mathcal{II}\text{ b.c.}} N^2_{m\mathcal{II}n} = \frac{1}{2}(r_0^2 - \frac{\nu^2}{\omega_{mIIn}^2}) J_m^2(\omega_{mIIn}r_0)$$

$$\xrightarrow[\alpha J_m(\omega_{m3n}r_0) + \beta\omega_{m3n} J_m'(\omega_{m3n}r_0) = 0]{\mathcal{III}\text{ b.c.}} N^2_{m\mathcal{III}n} = \frac{1}{2}(\frac{\alpha^2}{\beta^2\omega_{m3n}^2} r_0^2 + r_0^2 - \frac{\nu^2}{\omega_{m3n}^2}) J_m^2(\omega_{m3n}r_0)$$

例 3 恒温零环境中无限长圆柱无源冷却问题

$$\begin{cases} u_t = a^2\triangle u & 0 \le r < r_0,\ t > 0 & PDE \\ u|_{r=r_0} = 0 & \mathcal{I}\text{齐}\,b.c.(homogeneous\ Dirichlet) & BC \\ u|_{t=0} = \varphi(x,y) = \varphi(r\cos\theta, r\sin\theta) \equiv \Phi(r,\theta) & & IC \end{cases}$$

解: (一) 设变量分离的特解 $u = T(t)R(r)\Theta(\theta)\cdot 1$

泛定 $\frac{u_t}{a^2} = \triangle_3 u = [\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r}\frac{\partial^2}{\partial\theta^2}]u \to \begin{cases} T' + a^2k^2T = 0 & Q4 \\ Z \equiv 1 \to Z'' + 0\cdot Z = 0 & Q1 \\ \frac{\partial^2\Theta}{\partial\theta^2} = -\nu^2\Theta & Q2 \\ \frac{\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial R}{\partial r})}{R} + (\frac{-\nu^2}{r} + k^2) = 0 & Q3 \end{cases}$

配上边条, Q2: $\begin{cases} \frac{\partial^2\Theta}{\partial\theta^2} + \nu^2\Theta = 0 \\ \Theta(\theta + 2\pi) = \Theta(\theta) \end{cases}$, Q3: $\begin{cases} \frac{\partial}{\partial r}(r\frac{\partial R}{\partial r}) + (k^2 r - \frac{\nu^2}{r})R = 0 \\ |R(0)| < +\infty \\ R(r_0) = 0 \quad \mathcal{I}\text{齐}\,b.c. \end{cases}$

固有值问题用不上的 (非齐次) 条件, 终极定解条件 QC: $u|_{t=0} = \Phi(r,\theta)$.
(二) (2-1) 解第一个固有值问题 Q2, 固有值为 $\nu_m^2 = m^2,\ m = 0, 1, 2\ldots,$
固有函数为 $\Theta_m(\theta) = c_m\cos m\theta + d_m\sin m\theta$.
(2-2) 对每个不同的 m, 解第二个固有值问题 Q3, 这是 m 阶 Bessl 方程配上齐次的第一类边界条件.

ODE Bessel 的通解为 $R(r) = AJ_m(\omega r) + BN_m(\omega r)$
用自然边界条件：舍掉无界的 N_m
用柱面边界条件：$R(r_0) = J_m(\omega r_0) = 0$，对每个不同的 m 总能找到使
$J_m(\omega r_0) = J_m(x)$ 等于零的那些零点：$\omega_{m1} = \frac{x_{ml1}}{r_0}, \omega_{m2} = \frac{x_{ml2}}{r_0}, \ldots$
"设 ω_{mn} 标记 $J_m(\omega r_0) = 0$ 的第 n 个正根，n=1,2,...,"
固有值为 $\lambda_{mn} = \omega_{mn}^2$，固有函数为 $R_{mn}(r) = J_m(\omega_{mn} r)$，$n = 1, 2, \ldots$
(2-3) 解组不成固有值问题的问题 Q4，注意残余方向分离的单列问题没有定解条件!!! 只列得出泛定方程. $k^2 = \lambda + \mu = \omega_{mn}^2 + 0$，ODE:
$\frac{T'}{T} = (\ln T)' = -a^2\omega_{mn}^2$. 对不同的 m,n，通解为 $T_{mn}(t) = ce^{-a^2\omega_{mn}^2 t}$.
(2-4) 存在一族特解，一特解是 $e^{-a^2\omega_{mn}^2 t} J_m(\omega_{mn} r) \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} 1$, $\begin{array}{l} m = 0, 1 \ldots \\ n = 1, 2 \ldots \end{array}$
(三) 满足 PDE 和 BC 的所有可能的解，可由完备正交函数系 "叠加" 出来 (本问题由二元基 $\{J_m(\omega_{mn} r)\Theta_m\}$ 组配)：

$$u = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (C_{nm} \cos m\theta + D_{nm} \sin m\theta) J_m(\omega_{mn} r) e^{-a^2\omega_{mn}^2 t}$$

用终极条件 QC"定系数"：$u|_{t=0} = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (C_{nm} cos m\theta + D_{nm} sin m\theta) J_m(\omega_{mn} r) = \Phi$

$$C_{mn} = \frac{\int_0^{r_0} \int_0^{2\pi} \Phi(r,\theta) J_m(\omega_{mn} r) cos m\theta \, d\theta rdr}{\int_0^{r_0} J_m^2(\omega_{mn} r) rdr \int_0^{2\pi} \cos^2 m\theta d\theta}, \qquad D_{mn} = \frac{\int_0^{r_0} \int_0^{2\pi} \Phi(r,\theta) J_m(\omega_{mn} r) sin m\theta \, d\theta rdr}{\int_0^{r_0} J_m^2(\omega_{mn} r) rdr \int_0^{2\pi} \sin^2 m\theta d\theta}$$