5.(6) $\begin{cases} \Delta_2 u=0 & (a<r<b) \\ u(a,\theta)=1,\ u(b,\theta)=0 \end{cases}$

对于圆环域内 $\Delta_2 u=0$ 的边值问题：

$$u(r,\theta)=A_0+B_0\ln r+\sum_{k=1}^{+\infty}(A_k r^k+B_k r^{-k})(C_k\cos k\theta+D_k\sin k\theta)$$

将 $u(a,\theta)=1,\ u(b,\theta)=0$ 代入得：

$$A_0=\frac{-\ln b}{\ln a-\ln b}$$

$$B_0=\frac{1}{\ln a-\ln b}$$

$$\therefore u(r,\theta)=\frac{\ln r-\ln b}{\ln a-\ln b}$$

(7) $\begin{cases} \Delta_2 u=0 & (r<a,\ 0<\theta<\alpha) \\ u(r,0)=u(r,\alpha)=0 \\ u(a,\theta)=f(\theta) \end{cases}$

解：$\Delta_2 u=\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}$

令 $u=R(r)H(\theta)$

$\Rightarrow \begin{cases} r^2R''+rR'-\lambda R=0 \\ H''+\lambda H=0 \end{cases}$

$\Rightarrow \begin{cases} H''+\lambda H=0 \\ H(0)=H(\alpha)=0 \end{cases} \Rightarrow \begin{cases} H(\theta)=A_n\cos\sqrt{\lambda}\theta+B_n\sin\sqrt{\lambda}\theta \\ A_n=0,\ \sqrt{\lambda}=\dfrac{n^2\pi^2}{a^2},\ (n=1,2,3\cdots) \end{cases}$

$\therefore r^2R''+rR'-\lambda R=0 \rightarrow R_n(r)=C_n r^{\frac{n\pi}{a}}+D_n r^{-\frac{n\pi}{a}}\ (n=1,2,3\cdots)$

$\therefore u(r,\theta)=\sum_{n=1}^{+\infty}\left(C_n r^{\frac{n\lambda}{\alpha}}+D_n r^{-\frac{n\lambda}{\alpha}}\right)\sin\frac{n\lambda}{\alpha}\theta$

由解的有界性知：$D_n=0$

$\therefore u(r,\theta)=\sum_{n=1}^{+\infty}C_n r^{\frac{n\lambda}{\alpha}}\sin\frac{n\lambda}{\alpha}\theta$

代入 $u(a,\theta)=f(\theta)$ 得：

$\sum_{n=1}^{+\infty}C_n a^{\frac{n\lambda}{\alpha}}\sin\frac{n\lambda}{\alpha}\theta=f(\theta)$

$\therefore \quad C_n=a^{-\frac{n\lambda}{\alpha}}\ \frac{2}{\alpha}\int_0^{\alpha}f(\theta)\sin\frac{n\lambda\theta}{\alpha}d\theta$

$\therefore u(r,\theta)=\left[\frac{2}{\alpha}\int_0^{\alpha}f(\theta)\sin\frac{n\lambda\theta}{\alpha}d\theta\right]\left(\frac{r}{a}\right)^{\frac{n\lambda}{\alpha}}\sin\frac{n\lambda\theta}{\alpha}$

7. (1) $\begin{cases}u_t=a^2\Delta_3 u\\ u|_{r=R}=0,\ u(t,0)\text{有限}\\ u|_{t=0}=f(r)\end{cases}$

易知：采用球坐标系，$\because u|_{t=0}=f(r)$ $\therefore u$ 与 θ,φ 无关 $\therefore u=u(t,r)=R(r)T(t)$

$\because \Delta_3 u=\frac{1}{r^2}(r^2u_r)_r+\frac{1}{r^2\sin\theta}(u_\theta\sin\theta)_\theta+\frac{1}{r^2\sin^2\theta}u_{\varphi\varphi}$, $u=u(t,r)$

$\therefore \Delta_3 u=u_{rr}+\frac{2}{r}u_r$

$\therefore u_t=a^2(u_{rr}+\frac{2}{r}u_r)$

$\therefore$ 固有方程 $\begin{cases}T'+\lambda a^2T=0\\ rR''+2R'+\lambda Rr=0\end{cases}$

$\Rightarrow \begin{cases}T'+\lambda a^2T=0\\ (R'r^2)'+\lambda r^2R=0\end{cases}$

$\Rightarrow \begin{cases} (R'r^2)' + \lambda r^2 R = 0 \\ R(R) = 0 \\ R(0)T(t) \text{有限} \end{cases}$ ①

由S-L定理：$k(r) = r,\ q(r) = 0,\ \rho(r) = r$，边界条件为$R(R)=0$为第一类

$\therefore \lambda > 0$

不妨设$\lambda = k^2,\ k > 0$，令$V(r) = rR(r)$

$\therefore R' = \left(\frac{V}{r}\right)' = \frac{V'r - V}{r^2}$

$\therefore$ 原固有值问题①化为：$\begin{cases} V'' + \lambda V = 0 \\ V(R) = 0 \\ V(0) = 0 \end{cases}$

$\Rightarrow V(r) = C_1 \cos kr + C_2 \sin kr$

代入$V(R) = V(0) = 0$得：$k = \frac{n\pi}{R}\ (n = 1, 2, \cdots)$

$\therefore V(r) = B_n \sin \frac{n\pi}{R} r$，$R_n(r) = \frac{B_n}{r} \sin \frac{n\pi}{R} r$

又$T' + \lambda a^2 T = 0 \Rightarrow T = C_n e^{-(\frac{n\pi}{R})^2 at}$

$\therefore u(r,t) = \sum_{n=1}^{+\infty} C_n \frac{1}{r} e^{-(\frac{n\pi}{R})^2 a^2 t} \sin \frac{n\pi}{R} r$

$\because u|_{t=0} = f(r) \quad \therefore \sum_{n=1}^{+\infty} C_n \cdot \sin \frac{n\pi}{R} r = r f(r)$

$\therefore C_n = \frac{2}{R} \int_0^R r f(r) \sin \frac{n\pi}{R} r \, dr$

$\therefore u(r,t) = \sum_{n=1}^{+\infty} \frac{2}{Rr} \left[\int_0^R r f(r) \sin \frac{n\pi}{R} r \, dr \right] e^{-(\frac{n\pi a}{R})^2 t} \sin \frac{n\pi}{R} r$

10.(1) $\begin{cases} u_t = a^2 u_{xx} \\ u(t,0)=u_0,\ u_x(t,l)=0 \\ u(0,x)=\varphi(x) \end{cases}$

(边界非齐次问题)

令 $V(x)=Ax+B$，有 $\begin{cases} V(0)=u_0=B \\ V_x(l)=A=0 \end{cases} \Rightarrow V(x)=u_0$

令 $u(t,x)=V(x)+W(t,x)$，则：

$$\begin{cases} w_t = a^2 w_{xx} \\ W(t,0)=0,\ W_x(t,l)=0 \\ w(0,x)=\varphi(x)-u_0 \end{cases} \quad (\mathrm{I})$$

对(I)用分离变量法求解，设 $W(t,x)=X(x)T(t)$，有

$$\begin{cases} X''+\lambda X=0 \\ X(0)=X_x(l)=0 \end{cases} \quad (\mathrm{II}) \quad 与 \quad T'+a^2\lambda T=0 \quad (\mathrm{III})$$

对(II)做固有值分析，得 $\lambda_n=\left(\frac{2n+1}{2l}\pi\right)^2$，$X_n(x)=B_n\sin\frac{2n+1}{2l}\pi x$

将 λ_n 代入(III)得 $T_n(t)=C_n e^{-a^2\left(\frac{2n+1}{2l}\pi\right)^2 t}$

$$\therefore W(t,x)=\sum_{n=0}^{+\infty} D_n e^{-a^2\left(\frac{2n+1}{2l}\pi\right)^2 t}\sin\frac{2n+1}{2l}\pi x$$

代入 $W(0,x)$ 得：$D_n=\frac{2}{l}\int_0^l(\varphi(x)-u_0)\sin\frac{2n+1}{2l}\pi x\,dx$

$\therefore u(t,x)=u_0+\sum_{n=0}^{+\infty} D_n e^{-a^2\left(\frac{2n+1}{2l}\pi\right)^2 t}\sin\frac{2n+1}{2l}\pi x$，其中 $D_n=\frac{2}{l}\int_0^l(\varphi(x)-u_0)\sin\frac{2n+1}{2l}\pi x\,dx$.

10.(3) $\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - a^2 \dfrac{\partial u}{\partial t} + Ae^{-2x} = 0 \\ u(t,0)=u(t,l)=0 \\ u(0,x)=T_0 \end{cases}$

（非齐次方程，且该定方程的非齐次项只与x有关）

设 $u(t,x)=V(x)+W(t,x)$，其中

$$\begin{cases} V''+Ae^{-2x}=0 \\ V(0)=V(l)=0 \end{cases} \quad (\mathrm{I}) \qquad \begin{cases} W_{xx}=a^2 W_t \\ W(t,0)=W(t,l)=0 \\ W(0,x)=T_0-V(x) \end{cases} \quad (\mathrm{II})$$

对(I)：$V=-\frac{A}{4}e^{-2x}+Bx+C=\frac{A}{4}\left[e^{-2x}-\frac{x}{l}(e^{-2l}-1)-1\right]$

对(II)：利用分离变量法得：$W(t,x)=X(x)T(t)$

$$\begin{cases} X''+\lambda X=0 \\ X(0)=X(l)=0 \end{cases} \ (\mathrm{III}) \text{ 和 } T'(t)+\frac{\lambda}{a^2}T=0 \quad (\mathrm{IV})$$

$\Rightarrow \lambda_n=\left(\frac{n\pi}{l}\right)^2$，$X_n(x)=B_n \sin\frac{n\pi}{l}x$，$n=1,2,\cdots$

$T_n=C_n e^{-\left(\frac{n\pi}{la}\right)^2 t}$

$\therefore W(t,x)=\sum_{n=1}^{+\infty} D_n \exp\left\{-\left(\frac{n\pi}{la}\right)^2 t\right\}\cdot \sin\frac{n\pi}{l}x$

$\because W(0,x)=T_0-V(x)$

$\therefore \sum_{n=1}^{+\infty} D_n \sin\frac{n\pi}{l}x = T_0-\frac{A}{4}\left[e^{-2x}-\frac{x}{l}(e^{-2l}-1)-1\right]$

$\therefore D_n=\frac{2}{l}\int_0^l (T_0-V(x))\cdot\sin\frac{n\pi x}{l}dx=\frac{2}{l}\int_0^l T_0\sin\frac{n\pi x}{l}dx-\frac{2}{l}\int_0^l V(x)\sin\frac{n\pi x}{l}dx$

$=-\frac{2T_0}{n\pi}\left[(-1)^n-1\right]+\frac{A}{2l}\int_0^l\left(e^{-2x}-\frac{x}{l}(e^{-2l}-1)-1\right)\sin\frac{n\pi x}{l}dx=-\frac{2T_0}{n\pi}\left[(-1)^n-1\right]-\frac{2Al^2\left(1-(-1)^n e^{-2l}\right)}{n\pi(n^2\pi^2+4l^2)}$

令 $\frac{A}{2l}\int_0^l \left(e^{-2x}-\frac{x}{l}(e^{-2l}-1)-1\right)\sin\frac{n\pi x}{l}dx$

$$=\frac{A}{2l}\Big[\underbrace{\int_0^l e^{-2x}\sin\frac{n\pi x}{l}dx}_{I}-\underbrace{\int_0^l \frac{x}{l}(e^{-2l}-1)\sin\frac{n\pi x}{l}dx}_{II}-\underbrace{\int_0^l \sin\frac{n\pi x}{l}dx}_{III}\Big]$$

令 $\omega=\frac{n\pi}{l}$

$I=\int_0^l e^{-2x}\sin\omega x\,dx=$（利用分部积分法）$\cdots=\frac{1}{\omega}[1-e^{-2l}(-1)^n]+\frac{2}{\omega^2}\underbrace{\int_0^l(-2)e^{-2x}\sin\omega x\,dx}_{-2I}$

$\therefore I=\frac{n\pi l(1-e^{-2l}(-1)^n)}{n^2\pi^2+4l^2}$

$II=\int_0^l\frac{x}{l}(e^{-2l}-1)\sin\frac{n\pi x}{l}dx=\frac{e^{-2l}-1}{l}\int_0^l-\frac{1}{\omega}x\,d\cos\omega x=\frac{e^{-2l}-1}{l}\left[-\frac{1}{\omega}x\cos x\Big|_0^l+\int_0^l\cos\omega x\cdot\frac{1}{\omega}dx\right]$

$=-\frac{l(e^{-2l}-1)}{n\pi}(-1)^n$

$III=\int_0^l\sin\omega x\,dx=\frac{l}{n\pi}(1-(-1)^n)$

$\therefore I-II-III=-\frac{4l^3(1-e^{-2l}(-1)^n)}{n\pi(n^2\pi^2+4l^2)}$

$\therefore$ 原积分 $=\frac{A}{2l}(I-II-III)=-\frac{2Al^2(1-(-1)^ne^{-2l})}{n\pi(n^2\pi^2+4l^2)}$

$\therefore D_n=-\frac{2T_0}{n\pi}[(-1)^n-1]-\frac{2Al^2(1-(-1)^ne^{-2l})}{n\pi(n^2\pi^2+4l^2)}$

$\therefore u(t,x)=\frac{A}{4}[e^{-2x}-\frac{x}{l}(e^{-2l}+1)-1)+\sum_{n=1}^{+\infty}D_n\exp\{-(\frac{n\pi}{la})^2t\}\sin\frac{n\pi x}{l}$，$D_n$ 如上.

10.(6) $\begin{cases} \Delta_2 u = a+b(x^2-y^2) & (a,b\text{为常数}, r<R) \\ u(R,\theta)=c & (c\text{为常数}) \end{cases}$

令 $u = V(x,y)+w(r,\theta)$，则：

$$\Delta_2 V = a+b(x^2-y^2) \quad \& \quad \begin{cases} \Delta_2 w = 0 \\ w(R,\theta) = c - V(R,\theta) \end{cases}$$

令 $V(x,y) = C_1 x^2 + C_2(x^4-y^4)$

$\Rightarrow \Delta_2 V = 2C_1 + 12C_2(x^2-y^2) = a+b(x^2-y^2)$

$\therefore V = \frac{a}{2}x^2 + \frac{b}{12}(x^4-y^4) = \frac{a}{2}(r\cos\theta)^2 + \frac{b}{12}r^4(\cos^4\theta-\sin^4\theta) = \frac{a}{4}r^2(\cos2\theta+1)+\frac{b}{12}r^4\cos2\theta$

$\therefore \begin{cases} \Delta_2 w = 0 \\ w(R,\theta) = c - \frac{b}{12}R^4\cos2\theta - \frac{a}{4}R^2(\cos2\theta+1) \end{cases}$

$\because w = A_0 + B_0\ln r + \sum_{n=1}^{+\infty}(A_k r^k + B_k r^{-k})(C_k\cos k\theta + D_k\sin k\theta)$

$\because$ 定义在圆内

$\therefore B_0 = 0,\ B_k = 0$

$\therefore A_0 + \sum_{n=1}^{+\infty} A_k R^k (C_k\cos k\theta + D_k \sin k\theta) = c - \frac{b}{12}R^4\cos2\theta - \frac{a}{4}R^2\cos2\theta - \frac{a}{4}R^2$

$\therefore A_0 = c - \frac{a}{4}R^2,\ A_2R^2C_2 = (-\frac{b}{12}R^2+\frac{a}{4})R^2$，其余 $A_i=0$

$\therefore u(r,\theta) = V+w = \frac{a}{4}r^2(\cos2\theta+1)+\frac{b}{12}r^4\cos2\theta + c - \frac{a}{4}R^2 + (-\frac{b}{12}R^2+\frac{a}{4})r^2\cos2\theta$

$$= c + \frac{a}{4}(r^2-R^2) + \frac{b}{12}(r^2-R^2)r^2\cos2\theta.$$

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1. 柱坐标系中：$\frac{1}{r}u_r+u_{rr}+\frac{1}{r^2}u_{\theta\theta}+u_{zz}=0$（先分离 z，再 θ，最后 r）

（利用 $x=r\cos\theta$，$y=r\sin\theta$，$z=z$ 得到）

令 $u(r,\theta,z)=RHZ \Rightarrow -\frac{Z''}{Z}=\frac{\frac{R'H}{r}+R''H+\frac{RH''}{r^2}}{RH}=-\lambda$.

$\therefore Z''-\lambda Z=0$

又令 $-\frac{H''}{H}=\frac{rR'+r^2R''+\lambda r^2R}{R}=k$

记 $k=m^2$，则：

$$\begin{cases} Z''-\lambda Z=0 \\ H''+m^2H=0 \\ r^2R''+rR'+(\lambda r^2-m^2)R=0 \end{cases}$$

13. ω_n 是 $J_1(x)=0$ 的正实根 $\Rightarrow$ 第一类边界条件 $\Rightarrow N_{01}^2=\frac{1}{2}J_2^2(\omega_n)$

$$f(x)=\sum_{n=1}^{+\infty}f_nJ_1(\omega_nx),\quad f_n=\frac{1}{N_{01}^2}\int_0^1x^2J_1(\omega_nx)dx \xlongequal{t=\omega_nx}\frac{2}{J_2^2(\omega_n)}\int_0^{\omega_n}\frac{t^2}{\omega_n^3}J_1(t)dt$$

$$=\frac{2}{J_2^2(\omega_n)}\cdot\frac{1}{\omega_n^3}\,t^2J_2(x)\Big|_0^{\omega_n}=\frac{2}{\omega_nJ_2(\omega_n)}\quad\left[(x^{\nu}J_{\nu}(x))'=x^{\nu}J_{\nu-1}(x)\right]$$

$$\therefore f(x)=\sum_{n=1}^{+\infty}\frac{2}{\omega_nJ_2(\omega_n)}J_1(\omega_nx).$$

16. $$\begin{cases} u_t = a^2\Delta_3 u & (x^2+y^2<R^2,\ -\infty<z<+\infty) \\ u|_{r=R} = u_0 \\ u|_{t=0} = 0 \end{cases}$$

(边界条件非齐次)

易知：u不依赖于z, θ（对称性）∴设$u=u(r,t)$

令$u=u_0+w(r,t)$，则：$\begin{cases} w_t = a^2\Delta_3 w \\ w|_{r=R}=0 \\ w|_{t=0}=-u_0 \end{cases}$

利用分离变量法：$w(r,t)=R(r)T(t)$，则

$$\begin{cases} T'+\lambda a^2 T=0 \\ r^2R''+rR'+\lambda r^2 R=0 \end{cases}$$

∵$R(R)=0$（边界条件）∴$\begin{cases} r^2R''+rR'+\lambda r^2R=0 \\ R(R)=0 \end{cases}$ (I)

方程(I)的有界解为：$R_n(r)=J_0(\omega_n r)$

∵$R(R)=0$ ∴$J_0(\omega R)=0$，记此方程所有非负根为$\omega_1, \omega_2, \cdots$

将$\lambda=\omega_n^2$代入$T'+\lambda a^2T=0$得：$T_n=Ce^{-(\omega_n a)^2 t}$

$$\therefore w=\sum_{n=1}^{+\infty} C_n e^{-(\omega_n a)^2 t} J_0(\omega_n r)$$

由初始条件$w(r,0)=-u_0$得：

$$\sum_{n=1}^{+\infty} C_n J_0(\omega_n r)=-u_0$$

第一类边界条件：$N_{01}^2=\frac{R^2}{2}J_1^2(\omega_n R)$

$$\therefore C_n=\frac{1}{N_{01}^2}\int_0^R -u_0 x J_0(\omega_n x)\,dx$$

$$=\frac{-2u_0}{R^2 J_1^2(\omega_n R)}\int_0^R xJ_0(\omega_n x)\,dx$$

$$=\frac{-2u_0}{\omega_n R J_1(\omega_n R)}$$

$$\therefore u=u_0+\sum_{n=1}^{+\infty}\frac{-2u_0}{\omega_n R J_1(\omega_n R)}\exp\{-(\omega_n a)^2 t\}J_0(\omega_n r),$$

($\omega_1, \omega_2\cdots$为$J_0(\omega R)$的非负根)

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18 (1) $\begin{cases} u_{rr}+\frac{1}{r}u_r+u_{zz}=0 & (0<r<a,\ 0<z<l) \\ u(a,z)=0 \\ u(r,0)=0,\ u(r,l)=T_0\ (\text{常数}) \end{cases}$

令 $u=R(r)Z(z)$，变量分离后：$\begin{cases} r^2R''+rR'+\lambda r^2R=0 \\ Z''-\lambda Z=0 \end{cases}$

记 $\lambda=\omega^2$，则有界解 $R(r)=J_0(\omega r)$

由边界条件 $R(a)=0$，即 $J_0(\omega a)=0$，记此方程所有正实根为 ω_n，则固有值 $\lambda_n=\omega_n^2$，固有函数 $J_0(\omega_n r)$

将 $\lambda_n=\omega_n^2$ 代入 $Z''-\lambda Z=0$ 得：$Z_n(z)=C_n\operatorname{ch}\omega_n z+D_n\operatorname{sh}\omega_n z$ $\left[\operatorname{sh}x=\frac{e^x-e^{-x}}{2},\ \operatorname{ch}x=\frac{e^x+e^{-x}}{2}\right]$

$\therefore u(r,z)=\sum_{n=1}^{+\infty}(C_n\operatorname{ch}\omega_n z+D_n\operatorname{sh}\omega_n z)J_0(\omega_n r)$

由 $u(r,0)=0$ 得：$\sum_{n=1}^{+\infty}C_nJ_0(\omega_n r)=0 \Rightarrow C_n=0$

由 $u(r,l)=T_0$ 得：$\sum_{n=1}^{+\infty}(D_n\operatorname{sh}\omega_n l)J_0(\omega_n r)=T_0 \Rightarrow D_n=\frac{T_0}{N_{01}^2\operatorname{sh}\omega_n l}\int_0^a rJ_0(\omega_n r)\,dr$

$$=\frac{2T_0}{a\omega_n\operatorname{sh}\omega_n l\,J_1(a\omega_n)}$$

$\therefore u(r,z)=2T_0\sum_{n=1}^{+\infty}\frac{1}{a\omega_nJ_1(a\omega_n)\operatorname{sh}\omega_n l}\operatorname{sh}\omega_n z\,J_0(\omega_n r)$，其中 ω_n 为 $J_0(\omega a)=0$ 的正实根.

18.(2) $\begin{cases} u_{tt}+2hu_t=a^2(u_{rr}+\frac{1}{r}u_r) & (h<<1) \\ u(0,t)=\text{有限},\ u_r(l,t)=0 \\ u(r,0)=\varphi(r),\ u_t(r,0)=0. \end{cases}$

令 $u=R(r)T(t)$. 代入方程得:

$$T''R+2hT'R=a^2(R''T+\frac{1}{r}R'T)$$

$$\therefore \frac{T''+2hT'}{a^2T}=\frac{R''+\frac{1}{r}R'}{R}=-\lambda.$$

$\therefore \begin{cases} T''+2hT'+\lambda a^2T=0 \\ r^2R''+rR'+\lambda r^2R=0 \end{cases} \Rightarrow R(r)=J_0(\omega r)$

由边界条件: $R(0)T$有限, $R'(l)T=0$

$\therefore R'(l)=0$, 即 $J_1(\omega_n l)=0$, 记此方程所有非负根为 $\omega_1,\omega_2,\cdots$

$\therefore$ 固有值为 $\lambda_0=0$, $\lambda_n=\omega_n^2$

代入 $T''+2hT'+\lambda a^2T=0$ 得: $T_0=B_1e^{-2ht}+B_2$

$T_n=e^{-ht}(C_n\cos g_nt+D_n\sin g_nt)$, $g_n=\sqrt{a^2\omega_n^2-h^2}$, $n=1,2,\cdots$

$\therefore u(r,t)=B_1e^{-2ht}+B_2+\sum_{n=1}^{+\infty}e^{-ht}(C_n\cos g_nt+D_n\sin g_nt)J_0(\omega_nr)$

由初始条件 $u(r,0)=\varphi(r)$, $u_t(r,0)=0$ 得:

$$\begin{cases} B_1+B_2+\sum_{n=1}^{+\infty}C_nJ_0(\omega_nr)=\varphi(r) \Rightarrow C_n=\frac{2}{l^2J_0^2(\omega_nl)}\int_0^l r\varphi(r)J_0(\omega_nr)dr,\ B_1+B_2=\frac{2}{l^2}\int_0^l r\varphi(r)dr \\ -2hB_1+\sum_{n=1}^{+\infty}(-hC_n+g_nD_n)J_0(\omega_nr)=0 \Rightarrow B_1=0,\ D_n=\frac{h}{g_n}C_n \end{cases}$$

$\therefore u=\frac{2}{l^2}\int_0^l r\varphi(r)dr+\sum_{n=1}^{+\infty}e^{-ht}C_n(\cos g_nt+\frac{h}{g_n}\sin g_nt)$, $g_n=\sqrt{a^2\omega_n^2-h^2}$, C_n如上所示

19. 问题归结为定解问题：
$$\begin{cases} u_{rr}+\frac{1}{r}u_r+u_{zz}=0 \quad (u_t=a^2\Delta_3 u,\ \text{稳定温度场},\ u_t=0) \\ u(0,z)\text{有限},\quad u_r(R,z)+ku(R,z)=0 \\ u(r,0)=0,\quad u(r,h)=f(r) \end{cases}$$

用分离变量法，令 $u(r,z)=R(r)Z(z)$，则：$\begin{cases} R''+\frac{1}{r}R'+\lambda R=0 \\ R(0)Z(z)\text{有限},\ R'+kR|_{r=R}=0 \end{cases}$ (I) & $Z''-\lambda Z=0$ (II)

解(I)有：$R_n(r)=J_0(\omega_n r)$，$\lambda_n=\omega_n^2$，ω_n为方程 $-\omega J_1(\omega R)+kJ_0(\omega R)=0$ 的根.

将 $\lambda_n=\omega_n^2$ 代入Z方程得：$Z_n(z)=C_n ch\,\omega_n z+D_n sh\,\omega_n z$

$$\therefore u(r,z)=\sum_{n=1}^{+\infty}(C_n ch\,\omega_n z+D_n sh\,\omega_n z)J_0(\omega_n r)$$

又有 $u(r,0)=0$，$u(r,h)=f(r)$

$$\therefore \sum_{n=1}^{+\infty}C_n J_0(\omega_n r)=0 \Rightarrow C_n=0$$

$$\sum_{n=1}^{+\infty}(D_n sh\,\omega_n h)J_0(\omega_n r)=f(r) \Rightarrow D_n=\frac{1}{sh\,\omega_n h}\frac{1}{N_{03}^2}\int_0^R rf(r)J_0(\omega_n r)dr$$

$\because$ 第三类边界条件 $\therefore N_{03}^2=\frac{1}{2}\left[R^2+\left(\frac{kR}{\omega_n}\right)^2\right]J_0^2(\omega_n R)$

$$\therefore D_n=\frac{2}{sh\,\omega_n h}\frac{1}{\left[1+\left(\frac{k}{\omega_n}\right)^2\right]R^2J_0^2(\omega_n R)}\int_0^R rf(r)J_0(\omega_n r)dr$$

$$\therefore u(r,z)=\sum_{n=1}^{+\infty}\frac{2}{R^2}\frac{sh\,\omega_n z}{sh\,\omega_n h}\frac{J_0(\omega_n r)}{\left(1+\frac{k^2}{\omega_n^2}\right)J_0^2(\omega_n R)}\int_0^R rf(r)J_0(\omega_n r)dr.$$