2. (2) 令 $y=rR$. 则 $y'=R+rR'$, $y''=2R'+rR''$

$\therefore (r^2R')'+\lambda r^2R=0 \Rightarrow y''+\lambda y=0 \ (0<r<a)$

① 若 $\lambda=0$. $y=Ar+B$. $R=A+\frac{B}{r}$

$|R(0)|<+\infty \quad \therefore B=0$

$R(a)=A=0 \quad \therefore A=B=0$, 舍去

② 若 $\lambda<0$. $y=Ae^{\sqrt{-\lambda}r}+Be^{-\sqrt{-\lambda}r}$, $R=\frac{y}{r}$

$|R(0)|<+\infty \quad \therefore A+B=0$

$R(a)==0 \quad \therefore A=B=0$. 舍去

③ $\lambda>0$. $y=A\cos\sqrt{\lambda}r+B\sin\sqrt{\lambda}r$. $R=\frac{y}{r}$

$|R(0)|<+\infty \quad \therefore A=0$

$R(a)=\frac{1}{a}B\sin\sqrt{\lambda}a=0 \quad \because B\neq0 \quad \therefore \sqrt{\lambda}a=n\pi$

$\therefore \lambda_n=\left(\frac{n\pi}{a}\right)^2$. $n=1,2,3,\cdots$

$\therefore R_n=\frac{B_n}{r}\sin\frac{n\pi r}{a}$, $n=1,2,3,\cdots$

$$\begin{cases} u_t = a^2 u_{xx} \ (0<x<l,\ t>0) \\ u(t,0)=u(t,l)=0 \\ u(0,x)=x(l-x) \end{cases}$$

① 令 $u(t,x)=T(t)X(x)$

$\therefore \quad XT' = a^2 X'' T$

$\therefore \dfrac{T'}{a^2 T} = \dfrac{X''}{X} = -\lambda.$

$$\therefore \begin{cases} X''+\lambda X=0 \\ X(0)=X(l)=0. \end{cases}$$

固有值问题的解为:

$\lambda_n = \left(\dfrac{n\pi}{l}\right)^2$

$X_n = B_n \sin \dfrac{n\pi x}{l}$

$T_n = C_n e^{-\lambda a^2 t}$

$u_n = C_n e^{-\lambda a^2 t} \sin \dfrac{n\pi x}{l},\ n=1,2,\cdots$

③ $u(t,x) = \sum\limits_{n=1}^{+\infty} u_n$

$= \sum\limits_{n=1}^{+\infty} C_n e^{-\lambda a^2 t} \sin \dfrac{n\pi x}{l}$

④ $u(0,x) = \sum\limits_{n=1}^{+\infty} C_n \sin \dfrac{n\pi x}{l} = -x^2 + lx$

$\therefore C_n = \dfrac{2}{l}\int_0^l (-x^2+lx)\sin\dfrac{n\pi x}{l}\,dx$

$= \dfrac{2}{l}\int_0^l \dfrac{-l}{n\pi}(-x^2+lx)\,d\left(\cos\dfrac{n\pi x}{l}\right)$

$= -\dfrac{2}{n\pi}\left[(-x^2+lx)\cos\dfrac{n\pi x}{l}\Big|_0^l - \int_0^l \cos\dfrac{n\pi x}{l}\cdot(-2x+l)\,d\right.$

$= \dfrac{2}{n\pi}\int_0^l \dfrac{l}{n\pi}(-2x+l)\,d\left(\sin\dfrac{n\pi x}{l}\right)$

$= \dfrac{2l}{(n\pi)^2}\left[(-2x+l)\sin\dfrac{n\pi x}{l}\Big|_0^l - \int_0^l \sin\dfrac{n\pi x}{l}\cdot(-2)\,dx\right]$

$= \dfrac{4l}{(n\pi)^2}\int_0^l \sin\dfrac{n\pi x}{l}\,dx = \dfrac{4l}{(n\pi)^2}\cdot\dfrac{-l}{n\pi}\cos\dfrac{n\pi x}{l}\Big|_0^l = \dfrac{-4l^2}{(n\pi)^3}[(-1)$

$$= \begin{cases} 0\ , & n\text{为偶数} \\ \dfrac{8l^2}{(n\pi)^3}\ , & n\text{为奇数} \end{cases}$$

$\therefore u(t,x) = \sum\limits_{n=0}^{+\infty} \dfrac{8l^2}{\pi^3}\dfrac{1}{(2n+1)^3} e^{-\lambda a^2 t} \sin\dfrac{(2n+1)\pi x}{l},\ \lambda = \left[\dfrac{(2n+1)\pi}{l}\right]^2$

(3) $\begin{cases} u_{tt}=a^2u_{xx}-2hu_t & (0<x<l,\ t>0,\ 0<h<\frac{\pi a}{l},\ h\text{为常数}) \\ u(t,0)=u(t,l)=0 \\ u(0,x)=\varphi(x) \\ u_t(0,x)=\psi(x) \end{cases}$

令 $u(t,x)=T(t)X(x)$

$$T''X=a^2TX''-2hT'X$$

$$\Rightarrow \frac{X''}{X}=\frac{T''+2hT'}{a^2T}=-\lambda$$

固有值问题: $\begin{cases} X''+\lambda X=0 \\ X(0)=0 \\ X(l)=0 \end{cases}$

$\lambda_n=\left(\frac{n\pi}{l}\right)^2$, $X_n(x)=B_n\sin\frac{n\pi x}{l}$, $n=1,2,3,\cdots$

$$\therefore\ T''+2hT'+\lambda a^2T=0$$

$$\Rightarrow\ T_n(t)=C_ne^{(-h+\sqrt{h^2-\lambda a^2})t}+D_ne^{(-h-\sqrt{h^2-\lambda a^2})t}$$

$$=e^{-ht}\left(C_n\sin kt+D_n\cos kt\right),\ h^2-\lambda a^2=-k^2$$

$$\therefore\ u(t,x)=\sum_{n=1}^{+\infty}X_nT_n=\sum_{n=1}^{+\infty}e^{-ht}\left(C_n\sin kt+D_n\cos kt\right)\sin\frac{n\pi x}{l}$$

由初始条件得: $\begin{cases} \sum_{n=1}^{+\infty}D_n\sin\frac{n\pi x}{l}=\varphi(x) \\ \sum_{n=1}^{+\infty}(-hD_n+kC_n)\sin\frac{n\pi x}{l}=\psi(x) \end{cases}$

$$\therefore\ D_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}\,dx$$

$$C_n = \frac{h}{k}D_n + \frac{2}{kl}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx$$