Basics of combinatorics

binomial catalan combination dearrangement inclusion-exclusion pascal permutation

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Original Article:

- [Basics of Combinatorics (by x-ray)]
- [Catalan Numbers]

Combinatorial primitives

The rule of sum

$$A \cap B = \phi \implies |A| + |B| = |A \cup B|$$

The rule of product

$$|A|.|B| = |AxB|$$

Overview

Permutation without repetition

When we choose k objects from n-element set in such a way that the order matters and each object can be chosen only once:

$$^nP_k = \frac{n!}{(n-k)!}$$

Permutation (variation) with repetition

The number of possible choices of k objects from a set of n objects when order is important and one object can be chosen more than once:

$$n^k$$

Permutation with repetition

The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2,…, and n_k indistinguishable objects of type k

$(n_1 + n_2 + \ldots + n_k = n)$, is:

$$^{(n_1, n_2, \ldots, n_k)}P_k = \frac{n!}{n_1! n_2! \ldots n_k!}$$

Combinations without repetition

In combinations we choose a set of elements (rather than an arrangement, as in permutations) so the order doesn't matter. The number of different k-element subsets (when each element can be chosen only once) of n-element set is:

$$^{n}C_k = \frac{n!}{(n-k)!k!}$$

Combination with repetition

Let's say we choose k elements from an n-element set, the order doesn't matter and each element can be chosen more than once. In that case, the number of different combinations is:

$$^{n+k-1}C_k$$

This is also the number of solutions for the following integer equation:

$$x_1 + x_2 + \ldots + x_n = k, (x_i >= 0 i \in [1, n]$$

The number of such solutions are calculated by finding the power of x_k in the polynomial expansion of the LHS. Which is the product of sum of raising the power of each x to its each possible value for every x_i.

Inclusion-Exclusion Principle

Let's take a look at a very interesting and useful formula called the inclusion-exclusion principle (also known as the sieve principle):

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} (-1)^{i-1} \sum_{\substack{I \subseteq \overline{1,n}, \\ |I|=i}} \left|\bigcap_{j \in I} A_j\right|$$

This formula is a generalization of:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Dearrangement Theorem

One of its apt applications is to the problem of dearrangement. The idea is to count the number of possibilities such that no element is present in its original location. For example, for the set $X = \{1, 2, 3\}$, $Y = \{2, 3, 1\}$ is a possible dearrangement but $Z = \{1, 3, 2\}$ is not.

The idea is to use the inclusion exclusion principle:

First add all possible arrangements $\implies n!$
Then subtract all arrangements having one element in its original location $\implies {}^{n}C_1 * (n-1)!$
Then add all arrangements having two elements in their original location $\implies {}^{n}C_2 * (n-2)!$
...
...
Finally add/subtract all arrangements having all elements in their original location $\implies {}^{n}C_n * 0!$

$$N_d = n! - {}^{n}C_1(n-1)! + {}^{n}C_2(n-2)! + \ldots + (-1)^i \times {}^{n}C_2(n-2)! + \ldots$$

$$N_d = n![1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots]$$

Catalan Numbers

These numbers show up in multiple counting problems:

- Count the number of expressions containing n pairs of parentheses which are correctly matched.
- Count the number of possible Binary Search Trees with n keys.
- Count the number of full binary trees (A rooted binary tree is full if every vertex has either two children or no children) with n+1 leaves.
- **Counting Dyck words**: It’s a string consisting only of n X’s and n Y’s, and matching this criteria: each prefix of this string has more X’s than Y’s. For example, “XXYY” and “XYXY” are Dyck words, but “XYYX” and “YYXX” are not.
- **Diagonal avoiding paths:** Counting number of staircase (only move right or up) paths from (0,0) to (n,n) in a square grid that don't cross the diagonal line from (0,0) to (n,n).
- Counting the number of *"mountain ranges"* that can be constructed using n upstrokes and n downstrokes such that the *"mountain range"* always stays above the original line.
- **Polygon Triangulation:** Counting the number of ways in which a regular polygon of n + 2 sides can be triangulated.
- **Non-Crossing Handshakes:** If 2n people are seated around a circular table, in how many ways can all of them be simultaneously shaking hands with another person at the table in such a way that none of the arms cross each other.

Figure 1: Corresponding Path and Range

The closed form solution to this problem are the Catalan numbers (C_n):

$$C_n = \frac{{}^{2n}C_n}{n+1}$$

Base case $C_0 = 1$.
Other recurrences (try using the balanced parantheses problem as an example to come up with this recurrence):

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

To see how this works assume let's calculate the number of orderings for n pairs of parantheses. Assume that the first i of these n pairs are balanced that means the number of such ordering are $C_{i-1} C_{n-i}$. Since one pair would be removed as it was enclosing the first (i - 1) pairs.