14. Probabilistic Reasoning

14.1 We have a bag of three biased coins a, b, and c with probabilities of coming up heads of 20%, 60%, and 80%, respectively. One coin is drawn randomly from the bag (with equal likelihood of drawing each of the three coins), and then the coin is flipped three times to generate the outcomes X_1, X_2, and X_3.

1. Draw the Bayesian network corresponding to this setup and define the necessary CPTs.
2. Calculate which coin was most likely to have been drawn from the bag if the observed flips come out heads twice and tails once.

14.2 We have a bag of three biased coins a, b, and c with probabilities of coming up heads of 30%, 60%, and 75%, respectively. One coin is drawn randomly from the bag (with equal likelihood of drawing each of the three coins), and then the coin is flipped three times to generate the outcomes X_1, X_2, and X_3.

1. Draw the Bayesian network corresponding to this setup and define the necessary CPTs.
2. Calculate which coin was most likely to have been drawn from the bag if the observed flips come out heads twice and tails once.

14.3 [cpt-equivalence-exercise] Equation (parameter-joint-repn-equation) on page parameter-joint-repn-equation defines the joint distribution represented by a Bayesian network in terms of the parameters $\theta(X_i \mid Parents(X_i))$. This exercise asks you to derive the equivalence between the parameters and the conditional probabilities $\mathbf{P}(X_i \mid Parents(X_i))$ from this definition.

1. Consider a simple network $X \rightarrow Y \rightarrow Z$ with three Boolean variables. Use Equations (conditional-probability-equation) and (marginalization-equation) (pages conditional-probability-equation and marginalization-equation) to express the conditional probability $P(z \mid y)$ as the ratio of two sums, each over entries in the joint distribution $\mathbf{P}(X, Y, Z)$.
2. Now use Equation (parameter-joint-repn-equation) to write this expression in terms of the network parameters $\theta(X)$, $\theta(Y \mid X)$, and $\theta(Z \mid Y)$.
3. Next, expand out the summations in your expression from part (b), writing out explicitly the terms for the true and false values of each summed variable. Assuming that all network parameters satisfy the constraint $\sum_{x_i} \theta(x_i \mid parents(X_i)) = 1$, show that the resulting expression reduces to $\theta(z \mid y)$.
4. Generalize this derivation to show that $\theta(X_i \mid Parents(X_i)) = \mathbf{P}(X_i \mid Parents(X_i))$ for any Bayesian network.

14.4 The **arc reversal** operation of in a Bayesian network allows us to change the direction of an arc $X \rightarrow Y$ while preserving the joint probability distribution that the network represents @Shachter:1986. Arc reversal may require introducing new arcs: all the parents of X also become parents of Y, and all parents of Y also become parents of X.

1. Assume that X and Y start with m and n parents, respectively, and that all variables have k values. By calculating the change in size for the CPTs of X and Y, show that the total number of parameters in the network cannot decrease during arc reversal. (*Hint*: the parents of X and Y need not be disjoint.)
2. Under what circumstances can the total number remain constant?
3. Let the parents of X be $\mathbf{U} \cup \mathbf{V}$ and the parents of Y be $\mathbf{V} \cup \mathbf{W}$, where $\mathbf{U}$ and $\mathbf{W}$ are disjoint. The formulas for the new CPTs after arc reversal are as follows:

$$\begin{aligned} \mathbf{P}(Y \mid \mathbf{U}, \mathbf{V}, \mathbf{W}) &= \sum_x \mathbf{P}(Y \mid \mathbf{V}, \mathbf{W}, x)\mathbf{P}(x \mid \mathbf{U}, \mathbf{V}) \\ \mathbf{P}(X \mid \mathbf{U}, \mathbf{V}, \mathbf{W}, Y) &= \mathbf{P}(Y \mid X, \mathbf{V}, \mathbf{W})\mathbf{P}(X \mid \mathbf{U}, \mathbf{V}) / \mathbf{P}(Y \mid \mathbf{U}, \mathbf{V}, \mathbf{W}) \ . \end{aligned}$$

   Prove that the new network expresses the same joint distribution over all variables as the original network.

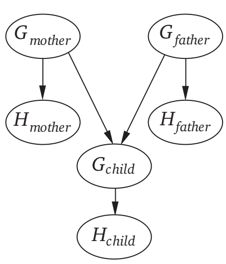
14.5 Consider the Bayesian network in Figure burglary-figure.

1. If no evidence is observed, are $Burglary$ and $Earthquake$ independent? Prove this from the numerical semantics and from the topological semantics.
2. If we observe $Alarm = true$, are $Burglary$ and $Earthquake$ independent? Justify your answer by calculating whether the probabilities involved satisfy the definition of conditional independence.

14.6 Suppose that in a Bayesian network containing an unobserved variable Y, all the variables in the Markov blanket $MB(Y)$ have been observed.

1. Prove that removing the node Y from the network will not affect the posterior distribution for any other unobserved variable in the network.
2. Discuss whether we can remove Y if we are planning to use (i) rejection sampling and (ii) likelihood weighting.

Figure [handedness-figure] Three possible structures for a Bayesian network describing genetic inheritance of handedness.

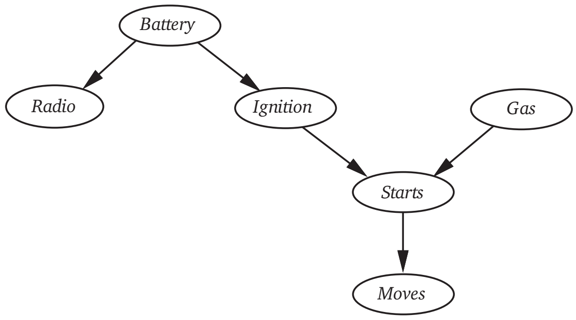


14.7 [handedness-exercise] Let H_x be a random variable denoting the handedness of an individual x, with possible values l or r. A common hypothesis is that left- or right-handedness is inherited by a simple mechanism; that is, perhaps there is a gene G_x, also with values l or r, and perhaps actual handedness turns out mostly the same (with some probability s) as the gene an individual possesses. Furthermore, perhaps the gene itself is equally likely to be inherited from either of an individual's parents, with a small nonzero probability m of a random mutation flipping the handedness.

1. Which of the three networks in Figure handedness-figure claim that $\mathbf{P}(G_{father}, G_{mother}, G_{child}) = \mathbf{P}(G_{father})\mathbf{P}(G_{mother})\mathbf{P}(G_{child})$?
2. Which of the three networks make independence claims that are consistent with the hypothesis about the inheritance of handedness?
3. Which of the three networks is the best description of the hypothesis?
4. Write down the CPT for the G_{child} node in network (a), in terms of s and m.
5. Suppose that $P(G_{father} = l) = P(G_{mother} = l) = q$. In network (a), derive an expression for $P(G_{child} = l)$ in terms of m and q only, by conditioning on its parent nodes.
6. Under conditions of genetic equilibrium, we expect the distribution of genes to be the same across generations. Use this to calculate the value of q, and, given what you know about handedness in humans, explain why the hypothesis described at the beginning of this question must be wrong.

14.8 [markov-blanket-exercise] The **Markov blanket** of a variable is defined on page markov-blanket-page. Prove that a variable is independent of all other variables in the network, given its Markov blanket and derive Equation (markov-blanket-equation) (page markov-blanket-equation).

Figure [car-starts-figure] A Bayesian network describing some features of a car's electrical system and engine. Each variable is Boolean, and the *true* value indicates that the corresponding aspect of the vehicle is in working order.



14.9 Consider the network for car diagnosis shown in Figure car-starts-figure.

1. Extend the network with the Boolean variables $IcyWeather$ and $StarterMotor$.
2. Give reasonable conditional probability tables for all the nodes.
3. How many independent values are contained in the joint probability distribution for eight Boolean nodes, assuming that no conditional independence relations are known to hold among them?
4. How many independent probability values do your network tables contain?
5. The conditional distribution for $Starts$ could be described as a **noisy-AND** distribution. Define this family in general and relate it to the noisy-OR distribution.

14.10 Consider a simple Bayesian network with root variables $Cold$, Flu, and $Malaria$ and child variable $Fever$, with a noisy-OR conditional distribution for $Fever$ as described in Section canonical-distribution-section. By adding appropriate auxiliary variables for inhibition events and fever-inducing events, construct an equivalent Bayesian network whose CPTs (except for root variables) are deterministic. Define the CPTs and prove equivalence.

14.11 [LG-exercise] Consider the family of linear Gaussian networks, as defined on page LG-network-page.

1. In a two-variable network, let X_1 be the parent of X_2, let X_1 have a Gaussian prior, and let $\mathbf{P}(X_2 \mid X_1)$ be a linear Gaussian distribution. Show that the joint distribution $P(X_1, X_2)$ is a multivariate Gaussian, and calculate its covariance matrix.
2. Prove by induction that the joint distribution for a general linear Gaussian network on $X_1, \ldots, X_n$ is also a multivariate Gaussian.

14.12 [multivalued-probit-exercise] The probit distribution defined on page probit-page describes the probability distribution for a Boolean child, given a single continuous parent.

1. How might the definition be extended to cover multiple continuous parents?
2. How might it be extended to handle a *multivalued* child variable? Consider both cases where the child's values are ordered (as in selecting a gear while driving, depending on speed, slope, desired acceleration, etc.) and cases where they are unordered (as in selecting bus, train, or car to get to work). (*Hint*: Consider ways to divide the possible values into two sets, to mimic a Boolean variable.)

14.13 In your local nuclear power station, there is an alarm that senses when a temperature gauge exceeds a given threshold. The gauge measures the temperature of the core. Consider the Boolean variables A (alarm sounds), F_A (alarm is faulty), and F_G (gauge is faulty) and the multivalued nodes G (gauge reading) and T (actual core temperature).

1. Draw a Bayesian network for this domain, given that the gauge is more likely to fail when the core temperature gets too high.
2. Is your network a polytree? Why or why not?
3. Suppose there are just two possible actual and measured temperatures, normal and high; the probability that the gauge gives the correct temperature is x when it is working, but y when it is faulty. Give the conditional probability table associated with G.
4. Suppose the alarm works correctly unless it is faulty, in which case it never sounds. Give the conditional probability table associated with A.

5. Suppose the alarm and gauge are working and the alarm sounds. Calculate an expression for the probability that the temperature of the core is too high, in terms of the various conditional probabilities in the network.

14.14 [telescope-exercise] Two astronomers in different parts of the world make measurements M_1 and M_2 of the number of stars N in some small region of the sky, using their telescopes. Normally, there is a small possibility e of error by up to one star in each direction. Each telescope can also (with a much smaller probability f) be badly out of focus (events F_1 and F_2), in which case the scientist will undercount by three or more stars (or if N is less than 3, fail to detect any stars at all). Consider the three networks shown in Figure telescope-nets-figure.

1. Which of these Bayesian networks are correct (but not necessarily efficient) representations of the preceding information?
2. Which is the best network? Explain.
3. Write out a conditional distribution for $\mathbf{P}(M_1 \mid N)$, for the case where $N \in \{1, 2, 3\}$ and $M_1 \in \{0, 1, 2, 3, 4\}$. Each entry in the conditional distribution should be expressed as a function of the parameters e and/or f.
4. Suppose $M_1 = 1$ and $M_2 = 3$. What are the *possible* numbers of stars if you assume no prior constraint on the values of N?
5. What is the *most likely* number of stars, given these observations? Explain how to compute this, or if it is not possible to compute, explain what additional information is needed and how it would affect the result.

14.15 Consider the network shown in Figure telescope-nets-figure(ii), and assume that the two telescopes work identically. $N \in \{1, 2, 3\}$ and $M_1, M_2 \in \{0, 1, 2, 3, 4\}$, with the symbolic CPTs as described in Exercise telescope-exercise. Using the enumeration algorithm (Figure enumeration-algorithm on page enumeration-algorithm), calculate the probability distribution $\mathbf{P}(N \mid M_1 = 2, M_2 = 2)$.

Figure [telescope-nets-figure] Three possible networks for the telescope problem.

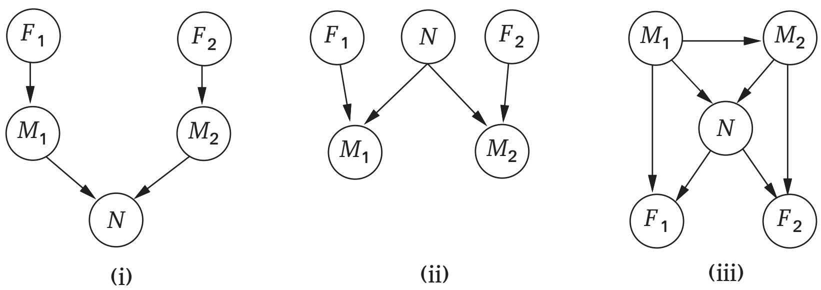
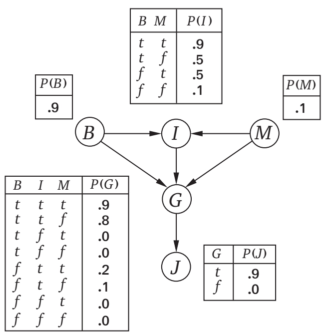


Figure [politics-figure] A simple Bayes net with Boolean variables B = {BrokeElectionLaw}, I = {Indicted}, M = {PoliticallyMotivatedProsecutor}, G= {FoundGuilty}, J = {Jailed}.



14.16 Consider the Bayes net shown in Figure politics-figure.

1. Which of the following are asserted by the network *structure*?
   A. $\mathbf{P}(B, I, M) = \mathbf{P}(B)\mathbf{P}(I)\mathbf{P}(M)$.
   B. $\mathbf{P}(J \mid G) = \mathbf{P}(J \mid G, I)$.
   C. $\mathbf{P}(M \mid G, B, I) = \mathbf{P}(M \mid G, B, I, J)$.
2. Calculate the value of $P(b, i, \neg m, g, j)$.
3. Calculate the probability that someone goes to jail given that they broke the law, have been indicted, and face a politically motivated prosecutor.
4. A **context-specific independence** (see page CSI-page) allows a variable to be independent of some of its parents given certain values of others. In addition to the usual conditional independences given by the graph structure, what context-specific independences exist in the Bayes net in Figure politics-figure?
5. Suppose we want to add the variable $P = PresidentialPardon$ to the network; draw the new network and briefly explain any links you add.

14.17 Consider the Bayes net shown in Figure politics-figure.

1. Which, if any, of the following are asserted by the network *structure* (ignoring the CPTs for now)?
   A. $\mathbf{P}(B, I, M) = \mathbf{P}(B)\mathbf{P}(I)\mathbf{P}(M)$.
   B. $\mathbf{P}(J \mid G) = \mathbf{P}(J \mid G, I)$.
   C. $\mathbf{P}(M \mid G, B, I) = \mathbf{P}(M \mid G, B, I, J)$.
2. Calculate the value of $P(b, i, m, \neg g, j)$.
3. Calculate the probability that someone goes to jail given that they broke the law, have been indicted, and face a politically motivated prosecutor.

4. A **context-specific independence** (see page CSI-page) allows a variable to be independent of some of its parents given certain values of others. In addition to the usual conditional independences given by the graph structure, what context-specific independences exist in the Bayes net in Figure politics-figure?
5. Suppose we want to add the variable $P = \mathit{PresidentialPardon}$ to the network; draw the new network and briefly explain any links you add.

14.18 [VE-exercise] Consider the variable elimination algorithm in Figure elimination-ask-algorithm (page elimination-ask-algorithm).

1. Section exact-inference-section applies variable elimination to the query
$$\mathbf{P}(\mathit{Burglary} \mid \mathit{JohnCalls} = \mathit{true}, \mathit{MaryCalls} = \mathit{true})\ .$$
Perform the calculations indicated and check that the answer is correct.
2. Count the number of arithmetic operations performed, and compare it with the number performed by the enumeration algorithm.
3. Suppose a network has the form of a *chain*: a sequence of Boolean variables $X_1, \ldots, X_n$ where $\mathit{Parents}(X_i) = \{X_{i-1}\}$ for $i = 2, \ldots, n$. What is the complexity of computing $\mathbf{P}(X_1 \mid X_n = \mathit{true})$ using enumeration? Using variable elimination?
4. Prove that the complexity of running variable elimination on a polytree network is linear in the size of the tree for any variable ordering consistent with the network structure.

14.19 [bn-complexity-exercise] Investigate the complexity of exact inference in general Bayesian networks:

1. Prove that any 3-SAT problem can be reduced to exact inference in a Bayesian network constructed to represent the particular problem and hence that exact inference is NP-hard. (*Hint*: Consider a network with one variable for each proposition symbol, one for each clause, and one for the conjunction of clauses.)
2. The problem of counting the number of satisfying assignments for a 3-SAT problem is #P-complete. Show that exact inference is at least as hard as this.

14.20 [primitive-sampling-exercise] Consider the problem of generating a random sample from a specified distribution on a single variable. Assume you have a random number generator that returns a random number uniformly distributed between 0 and 1.

1. Let X be a discrete variable with $P(X = x_i) = p_i$ for $i \in \{1, \ldots, k\}$. The **cumulative distribution** of X gives the probability that $X \in \{x_1, \ldots, x_j\}$ for each possible j. (See also Appendix [math-appendix].) Explain how to calculate the cumulative distribution in $O(k)$ time and how to generate a single sample of X from it. Can the latter be done in less than $O(k)$ time?
2. Now suppose we want to generate N samples of X, where $N \gg k$. Explain how to do this with an expected run time per sample that is *constant* (i.e., independent of k).
3. Now consider a continuous-valued variable with a parameterized distribution (e.g., Gaussian). How can samples be generated from such a distribution?
4. Suppose you want to query a continuous-valued variable and you are using a sampling algorithm such as LIKELIHOODWEIGHTING to do the inference. How would you have to modify the query-answering process?

14.21 Consider the query $\mathbf{P}(\mathit{Rain} \mid \mathit{Sprinkler} = \mathit{true}, \mathit{WetGrass} = \mathit{true})$ in Figure rain-clustering-figure(a) (page rain-clustering-figure) and how Gibbs sampling can answer it.

1. How many states does the Markov chain have?
2. Calculate the **transition matrix** $\mathbf{Q}$ containing $q(\mathbf{y} \rightarrow \mathbf{y}')$ for all $\mathbf{y}, \mathbf{y}'$.
3. What does $\mathbf{Q}^2$, the square of the transition matrix, represent?
4. What about $\mathbf{Q}^n$ as $n \rightarrow \infty$?
5. Explain how to do probabilistic inference in Bayesian networks, assuming that $\mathbf{Q}^n$ is available. Is this a practical way to do inference?

14.22 [gibbs-proof-exercise] This exercise explores the stationary distribution for Gibbs sampling methods.

1. The convex composition $[\alpha, q_1; 1 - \alpha, q_2]$ of q_1 and q_2 is a transition probability distribution that first chooses one of q_1 and q_2 with probabilities α and $1 - \alpha$, respectively, and then applies whichever is chosen. Prove that if q_1 and q_2 are in detailed balance with π, then their convex composition is also in detailed balance with π. (*Note*: this result justifies a variant of GIBBS-ASK in which variables are chosen at random rather than sampled in a fixed sequence.)
2. Prove that if each of q_1 and q_2 has π as its stationary distribution, then the sequential composition $q = q_1 \circ q_2$ also has π as its stationary distribution.

14.23 [MH-exercise] The **Metropolis--Hastings** algorithm is a member of the MCMC family; as such, it is designed to generate samples $\mathbf{x}$ (eventually) according to target probabilities $\pi(\mathbf{x})$. (Typically we are interested in sampling from $\pi(\mathbf{x}) = P(\mathbf{x} \mid \mathbf{e})$.) Like simulated annealing, Metropolis–Hastings operates in two stages. First, it samples a new state $\mathbf{x'}$ from a **proposal distribution** $q(\mathbf{x'} \mid \mathbf{x})$, given the current state $\mathbf{x}$. Then, it probabilistically accepts or rejects $\mathbf{x'}$ according to the **acceptance probability**

$$\alpha(\mathbf{x'} \mid \mathbf{x}) = \min\left(1, \frac{\pi(\mathbf{x'})q(\mathbf{x} \mid \mathbf{x'})}{\pi(\mathbf{x})q(\mathbf{x'} \mid \mathbf{x})}\right).$$

If the proposal is rejected, the state remains at $\mathbf{x}$.

1. Consider an ordinary Gibbs sampling step for a specific variable X_i. Show that this step, considered as a proposal, is guaranteed to be accepted by Metropolis–Hastings. (Hence, Gibbs sampling is a special case of Metropolis–Hastings.)
2. Show that the two-step process above, viewed as a transition probability distribution, is in detailed balance with π.

14.24 [soccer-rpm-exercise]Three soccer teams A, B, and C, play each other once. Each match is between two teams, and can be won, drawn, or lost. Each team has a fixed, unknown degree of quality—an integer ranging from 0 to 3—and the outcome of a match depends probabilistically on the difference in quality between the two teams.

1. Construct a relational probability model to describe this domain, and suggest numerical values for all the necessary probability distributions.
2. Construct the equivalent Bayesian network for the three matches.
3. Suppose that in the first two matches A beats B and draws with C. Using an exact inference algorithm of your choice, compute the posterior distribution for the outcome of the third match.
4. Suppose there are n teams in the league and we have the results for all but the last match. How does the complexity of predicting the last game vary with n?
5. Investigate the application of MCMC to this problem. How quickly does it converge in practice and how well does it scale?