Hw 2

Sunday, October 13, 2024 10:31 PM

1. Find the time response y(t) of the following systems using Laplace transform table and partial fraction expansion. The input u(t) = 1(t) and the initial conditions are all zero. (2 pts each)

a. $Y(s) = \underbrace{\frac{1}{s^2+9}}_{G(s)} U(s)$

$1(t) \xrightarrow{\mathcal{L}} \frac{1}{s}$

$Y(s) = \frac{1}{(s^2+9)}\frac{1}{s}$ Case 3 complex poles

$= \frac{R_1}{s} + \frac{D_1 s + D_2}{s^2 + 3^2}$

$$Y(s) = \frac{R_1(s^2+3^2) + s(D_1 s + D_2)}{s(s^2+3^2)} = \frac{1}{s(s^2+9)}$$

$R_1 s^2 + 9R_1 + D_1 s^2 + D_2 s = 1$

$s^0: \quad 9R_1 = 1 \rightarrow R_1 = \frac{1}{9}$

$s: \quad D_2 = 0$

$s^2: \quad R_1 + D_1 = 0 \rightarrow D_1 = -\frac{1}{9}$

$$Y(s) = \frac{1}{9}\frac{1}{s} - \frac{1}{9}\frac{s}{s^2+3^2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \boxed{y(t) = \frac{1}{9} - \frac{1}{9}\cos(3t)}$$

b. $Y(s) = \frac{1}{s^2+4s+9}U(s)$

$u(t) = 1(t) \xrightarrow{\mathcal{L}} \frac{1}{s}$

$Y(s) = \frac{1}{s(s^2+4s+9)}$

$s = \frac{-4 \pm \sqrt{16-4\cdot 9}}{2} = -2 \pm \frac{\sqrt{20}}{2} j \rightarrow \sqrt{5}$

$\frac{1}{s(}$

$$Y(s) = \frac{1}{s\underbrace{(s+2+\sqrt{5}j)(s+2-\sqrt{5}j)}_{(s+a)^2+b^2}}$$

a bj

$$= \frac{1}{s\left((s+2)^2 + \sqrt{5}^2\right)}$$

s^2+4s+4

$$= \frac{R_1}{s} + \frac{D_1 s + D_2}{(s+2)^2 + \sqrt{5}^2} = \frac{R_1\left((s+2)^2+5\right) + s(D_1 s + D_2)}{s\left[(s+2)^2+\sqrt{5}^2\right]}$$

$\rightarrow R_1 s^2 + 4R_1 s + 9R_1 + D_1 s^2 + D_2 s = 1$

$s^0: \quad 9R_1 = 1 \rightarrow R_1 = \frac{1}{9}$

$s: \quad 4R_1 + D_2 = 0 \rightarrow D_2 = -\frac{4}{9}$

$s^2: \quad R_1 + D_1 = 0 \rightarrow D_1 = -\frac{1}{9}$

$$Y(s) = \frac{1}{9}\frac{1}{s} + \frac{-\frac{1}{9}s - \frac{4}{9}}{(s+2)^2+\sqrt{5}^2}$$

$$= \frac{1}{9}\frac{1}{s} - \frac{1}{9}\frac{s}{(s+2)^2+\sqrt{5}^2} - \frac{4}{9}\frac{1}{(s+2)^2+\sqrt{5}^2}$$

$$-\frac{1}{9}\frac{s+2-2}{(s+2)^2+\sqrt{5}^2} = -\frac{1}{9}\frac{s+2}{(s+2)^2+\sqrt{5}^2} + \frac{2}{9}\frac{1}{(s+2)^2+\sqrt{5}^2}\frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{1}{9}\frac{1}{s} - \frac{1}{9}\frac{s+2}{(s+2)^2+\sqrt{5}^2} + \frac{2}{9\sqrt{5}}\frac{\sqrt{5}}{(s+2)^2+\sqrt{5}^2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \boxed{y(t) = \frac{1}{9} - \frac{1}{9}e^{-2t}\cos(\sqrt{5}t) + \frac{2}{9\sqrt{5}}e^{-2t}\sin(\sqrt{5}t)}$$

c. $Y(s) = \frac{1}{s^2+6s+9}U(s) \rightarrow \frac{1}{s} = \mathcal{L}\{1(t)\}$

$= \frac{1}{(s^2+6s+9)s}$

Case 2 repeated poles

$= \frac{1}{s(s+3)^2} = \frac{R_1}{s} + \frac{C_1}{(s+3)^2} + \frac{C_2}{s+3}$

$R_1 = \frac{1}{(s+3)^2}\Big|_{s=0} = \frac{1}{9}$

$C_1 = \frac{1}{s}\Big|_{s=-3} = -\frac{1}{3}$

$C_2 = \frac{d}{ds}\left[\frac{1}{s}\right]\Big|_{s=-3} = -\frac{1}{s^2}\Big|_{s=-3} = -\frac{1}{9}$

$$Y(s) = \frac{1}{9}\frac{1}{s} - \frac{1}{3}\frac{1}{(s+3)^2} - \frac{1}{9}\frac{1}{(s+3)}$$

$\downarrow \mathcal{L}^{-1}$

$$\boxed{y(t) = \frac{1}{9} - \frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t}}$$

d. $Y(s) = \frac{1}{s^2+10s+9}U(s) \rightarrow u(t) = 1(t)$

$U(s) = \frac{1}{s}$

$Y(s) = \frac{1}{s(s^2+10s+9)} = \frac{1}{s(s+9)(s+1)}$

$= \frac{R_1}{s} + \frac{R_2}{s+9} + \frac{R_3}{s+1}$

$R_1 = \frac{1}{(s+9)(s+1)}\Big|_{s=0} = \frac{1}{9}$

$R_2 = \frac{1}{s(s+1)}\Big|_{s=-9} = \frac{1}{72}$

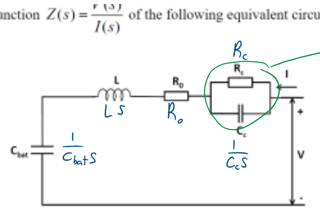
$R_3 = \frac{1}{s(s+9)}\Big|_{s=-1} = -\frac{1}{8}$

$$Y(s) = \frac{1}{9}\frac{1}{s} + \frac{1}{72}\frac{1}{(s+9)} - \frac{1}{8}\frac{1}{(s+1)}$$

$\downarrow \mathcal{L}^{-1}$

$$\boxed{y(t) = \frac{1}{9} + \frac{1}{72}e^{-9t} - \frac{1}{8}e^{-t}}$$

2. Find the transfer function $Z(s) = \frac{V(s)}{I(s)}$ of the following equivalent circuit model for the lithium ion battery: (3 pts)



$R_c // \frac{1}{C_c s} = \frac{R_c \frac{1}{C_c s}}{R_c + \frac{1}{C_c s}}$

$$Z(s)_{tot} = \frac{R_c}{C_c s\left(R_c + \frac{1}{C_c s}\right)} + R_0 + Ls + \frac{1}{C_{bat}s}$$

$$\boxed{Z(s) = \frac{R_c}{R_c C_c s + 1} + R_0 + Ls + \frac{1}{C_{bat}s}}$$

3. Linearize the following nonlinear system around operating point x=0: (3 pts)

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + x = \sin(x)$$ Non linear

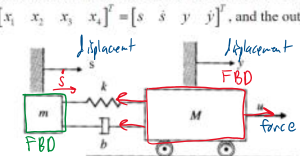
$x_0 = 0$

$x = x_0 + \delta x$

$f(x) = \sin x \simeq f(x) + \frac{df}{dx}\Big|_{x=x_0} \cdot \delta x$

$= \sin x_0 + \cos x_0 \cdot \delta x$

4. Find a state space model for the following mechanical system. The input is the applied force u, the states are $[x_1 \ x_2 \ x_3 \ x_4]^T = [s \ \dot{s} \ y \ \dot{y}]^T$, and the output is y. (3 pts)



$\Sigma F = Ma$

$-k(\Delta \text{displacement}) - b(\Delta \text{velocity}) + u = Ma$ ①

$x_0 = 0$

$x = x_0 + \delta x$

$$f(x) = \sin x \simeq f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} \cdot \delta x$$

$$= \sin x_0 + \cos x_0 \cdot \delta x$$

$$= \sin(0) + \cos(0) \cdot \delta x$$

$$f(x) \simeq \delta x$$

$$\rightarrow \frac{d^2x}{dt^2} + 5\frac{dx}{dt} + x = \sin(x)$$

$x_0 = 0$

$x = \cancel{x_0} + \delta x$

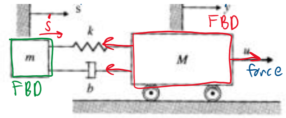
$\frac{dx}{dt} = \frac{d\delta x}{dt}$

$\frac{d^2x}{dt^2} = \frac{d^2\delta x}{dt^2}$

$\downarrow \simeq$

$$\rightarrow \frac{d^2\delta x}{dt^2} + 5\frac{d\delta x}{dt} + \cancel{\delta x} = \cancel{\delta x}$$

$$\boxed{\frac{d^2\delta x}{dt^2} + 5\frac{d\delta x}{dt} = 0}$$



$\Sigma F = Ma$

$-k(\Delta\text{displacement}) - b(\Delta\text{velocity}) + u = Ma$ ①

$\rightarrow -k(y-s) - b(\dot{y} - \dot{s}) + u = M\ddot{y}$

$$\frac{-k(x_3 - x_1) - b(x_4 - x_2) + u}{\cancel{M}} = \cancel{M}\dot{x}_4$$

$x_1 = s$

$x_2 = \dot{s} = \dot{x}_1$

$x_3 = y$

$x_4 = \dot{x}_3$

$\Sigma F = ma$

$k(\Delta\text{displacement}) + b(\Delta\text{velocity}) + ma = 0$ ②

$k(y-s) + b(\dot{y} - \dot{s}) + m\ddot{s} = 0$

$$\frac{-k(x_3 - x_1) - b(x_4 - x_2)}{m} = \cancel{m}\dot{x}_2$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{k}{m} & \frac{b}{m} & -\frac{k}{m} & -\frac{b}{m} \\ 0 & 0 & 0 & 1 \\ \frac{k}{M} & \frac{b}{M} & -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M \end{bmatrix} u$$

5. Find the transfer functions for the following state space models: (2 pts each)

a.

$$\dot{x} = \begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

b.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & 1 \end{bmatrix} x$$

a) $\dot{x} = \begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$

$\mathcal{L} \downarrow$

$$sX(s) = \begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix} X(s) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} U(s)$$

$$\left(sI - \begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix}\right) X(s) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} U(s)$$

$\downarrow$

$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$

$$X(s) = \Bigg(\underbrace{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix}}\Bigg)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} U(s)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} s+1 & 3 \\ 0 & s+2 \end{bmatrix}^{-1} = \frac{1}{(s+1)(s+2) - 0} \begin{bmatrix} s+2 & -3 \\ 0 & s+1 \end{bmatrix}$$

$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$

$\mathcal{L} \downarrow$

$Y(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} X(s)$ ②

① $X(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+2 & -3 \\ 0+ & s+1 \end{bmatrix} U(s)$

① and ②

$$Y(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 \\ s+1 \end{bmatrix} \frac{1}{(s+1)(s+2)} U(s)$$

$$Y(s) = \boxed{\frac{s-1}{(s+1)(s+2)}} U(s)$$

transfer function

b) $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$\mathcal{L} \downarrow$

$$sX(s) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U$$

$$\left(sI - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right) X(s) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(s)$$

$\downarrow$

$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$

$$X(s) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(s)$$

$$= \frac{1}{s(s+3) - (-2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(s)$$

$$X(s) = \frac{1}{\underbrace{s^2 + 3s + 2}_{(s+2)(s+1)}} \begin{bmatrix} 1 \\ s \end{bmatrix} U(s) \quad ①$$

$y = \begin{bmatrix} -1 & 1 \end{bmatrix} x$

$\mathcal{L} \downarrow$

$Y(s) = \begin{bmatrix} -1 & 1 \end{bmatrix} X(s)$ ②

① and ②

$$Y(s) = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} \frac{1}{(s+2)(s+1)} U(s)$$

$$Y(s) = \boxed{\frac{-1+s}{(s+2)(s+1)}} U(s)$$

transfer function

6. Find a state space model of the following transfer function: (3 pts)

$$Y(s) = \frac{2s^3+7}{s^4+3s^2+4s}U(s)$$

$$u \rightarrow \boxed{\frac{1}{s^4+3s^2+4s}} \xrightarrow{X_1(s)} \boxed{2s^3+7} \xrightarrow{Y(s)}$$

$m=3$

$a_0=0 \quad a_1=4 \quad a_2=3 \quad a_3=0 \qquad b_0=7 \quad b_1=0 \quad b_2=0 \quad b_3=2$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4 & -3 & 0 \end{bmatrix} \qquad C = [b_0\, b_1\, b_2\, b_3] = [7 \ 0 \ 0 \ 2]$$

$\dot{x} = Ax + Bu$

$y = Cx + Du$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad D = 0$$

$$\boxed{\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \qquad y = [7 \ 0 \ 0 \ 2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}$$

State space Model

7. Consider the car with a cruise control system in Homework 1. The input is the gas pedal position u, the states are $[x_1 \quad x_2]^T = [v \quad f]^T$, and the output is v.

input

a. Write the state space model of the system; (3 pts)

b. Derive the transfer function of the system based on the state space model. (3 pts)

$$\frac{df}{dt} = -\frac{1}{\tau}f + bu$$

a) $x_1 = v = y$ ①

$x_2 = f$

Dynamics:

$\Sigma F = ma$

$f - Rv = m\dot{v}$

$\dot{v} = \frac{f - rv}{m}$

$\dot{f} = -\frac{1}{\tau}f + bu$ input

$\dot{x}_2 = -\frac{1}{\tau}x_2 + bu$ ③

$\dot{x}_1 = \dot{v} = \frac{f - rv}{m}$

$\dot{x}_1 = \frac{x_2}{m} - \frac{r}{m}x_1$ ②

$$\boxed{\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{r}{m} & \frac{1}{m} \\ 0 & -\frac{1}{\tau} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u \qquad y = [1 \ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cancel{[0]\,u}^{\,0}}$$

state space model

b) $\dot{x} = \begin{bmatrix} -\frac{r}{m} & \frac{1}{m} \\ 0 & -\frac{1}{\tau} \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t)$

$\mathcal{L} \downarrow$

$sX(s) = \begin{bmatrix} -\frac{r}{m} & \frac{1}{m} \\ 0 & -\frac{1}{\tau} \end{bmatrix} X(s) + \begin{bmatrix} 0 \\ b \end{bmatrix} U(s)$

$X(s)\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -\frac{r}{m} & \frac{1}{m} \\ 0 & -\frac{1}{\tau} \end{bmatrix}\right) = \begin{bmatrix} 0 \\ b \end{bmatrix} U(s)$

$X(s) = \begin{bmatrix} s+\frac{r}{m} & -\frac{1}{m} \\ 0 & s+\frac{1}{\tau} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ b \end{bmatrix} U(s)$

$= \frac{1}{(s+\frac{r}{m})(s+\frac{1}{\tau}) - 0} \begin{bmatrix} s+\frac{1}{\tau} & \frac{1}{m} \\ 0 & s+\frac{r}{m} \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} U(s)$

$X(s) = \frac{1}{(s+\frac{r}{m})(s+\frac{1}{\tau})} \begin{bmatrix} b/m \\ b(s+\frac{r}{m}) \end{bmatrix} U(s)$ ④

$y = x_1 \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$y(t) = [1 \quad 0]\, x(t)$

$\mathcal{L} \downarrow$

$Y(s) = [1 \quad 0]\, X(s)$ ⑤

④ into ⑤

$Y(s) = [1 \ 0] \begin{bmatrix} b/m \\ b(s+\frac{r}{m}) \end{bmatrix} \frac{U(s)}{(s+\frac{r}{m})(s+\frac{1}{\tau})}$

$Y(s) = \boxed{\frac{b/m}{(s+\frac{r}{m})(s+\frac{1}{\tau})}}\, U(s)$

transfer function