Stat. Phys. II: 2-d random walk problems

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Two models for a 2D random walk

We define a N-step random walk as the set of N positions $\{x_k, y_k\}_{k=0,..,N-1}$, starting at (x_0, y_0) such that each step from position (x_k, y_k) to (x_{k+1}, y_{k+1}) is of fixed length d and random direction θ_k :

$$x_{k+1} = x_k + d \cos\theta_k$$
$$y_{k+1} = y_k + d \sin\theta_k$$

or equivalenty

$$x_N = x_0 + d \sum_{k=1}^{N} \cos\theta_k$$
$$y_N = y_0 + d \sum_{k=1}^{N} \sin\theta_k$$

Two models are considered, each characterized by the probability distribution of θ_k :

1. θ_k is a continuous variable uniformly distributed in $[0, 2pi[$.
2. θ_k may only take four values $\theta_k = 0$, $\theta_k = \frac{\pi}{2}$, $\theta_k = \pi$ or $\theta_k = -\frac{\pi}{2}$, each with equal probability.

Preliminary tasks (common to all students)

— For each model, plot the trajectories of M = 10 random walks (on the same plot) for N = 100 steps.

— For each model, verify that $\sqrt{< x_N^2 + y_N^2 >} = \sqrt{N}d$ by plotting $\sqrt{< x_N^2 + y_N^2 >}$ as a function of N.

Guidelines for the individual project

Each student must hand out a .pdf report, providing an answer to one project taken from the list that follows. Each report must contain some graphical illustrations of random walks relevant to the specific project being investigated. Whenever possible, propose a theoretical law that fit the numerical results.

"type 1" projects

1. We consider random walks all starting at $(x_0, y_0) = 0$, with θ_k following model 1. Determine the average number of steps needed for a random walk to hit the boundary of a circular area of radius R and center $(0, 0)$, as a function of $\frac{R}{d}$. Plot a few examples of random walks, for a few relevant values of $\frac{R}{d}$.

2. We consider random walks all starting at $(x_0, y_0) = 0$, with θ_k following model 2. Determine the average number of steps needed for a random walk to hit the boundary of a circular area of radius R and center $(0,0)$, as a function of $\frac{R}{d}$. Plot a few examples of random walks, for a few relevant values of $\frac{R}{d}$.
3. We consider random walks all starting at $(x_0, y_0) = 0$, with θ_k following model 1. Determine the average number of steps needed for a random walk to hit the boundary of a square area of side a and center $(0,0)$, as a function of $\frac{a}{d}$. Plot a few examples of random walks, for a few relevant values of $\frac{a}{d}$.
4. We consider random walks all starting at $(x_0, y_0) = 0$, with θ_k following model 2. Determine the average number of steps needed for a random walk to hit the boundary of a square area of side a and center $(0,0)$, as a function of $\frac{a}{d}$. Plot a few examples of random walks, for a few relevant values of $\frac{a}{d}$.

"type 2" projects

5. We consider random walks all starting at $(x_0, y_0) = 0$, with θ_k following model 1, inside a disk of radius $R = 25 \times d$ and center $(0,0)$. The walk is stopped when the walker crosses the boundary of the disk. Determine the probability density function for the exit angle, defined as the angle between the direction of the outgoing step and the surface normal at the exit point. Do you expect this probability distribution to depend on $\frac{R}{D}$?
6. We consider random walks all starting at $(x_0, y_0) = 0$, with θ_k following model 2, inside a disk of radius $R = 25 \times d$ and center $(0,0)$. The walk is stopped when the walker crosses the boundary of the disk. Determine the probability density function for the exit angle, defined as the angle between the direction of the outgoing step and the surface normal at the exit point. Do you expect this probability distribution to depend on $\frac{R}{D}$?
7. We consider random walks all starting at $(x_0, y_0) = 0$, with θ_k following model 1, inside a square of side $a = 50 \times d$ and center $(0,0)$. The walk is stopped when the walker crosses the boundary of the square. Determine the probability density function for the exit angle, defined as the angle between the direction of the outgoing step and the surface normal at the exit point. Do you expect this probability distribution to depend on $\frac{R}{D}$?

"type 3" projects

8. We consider random walks all starting at $(x_0, y_0) = 0$, with θ_k following model 1, stuck inside a square box of side $a = 50 \times d$. The behaviour of the walker in the box is the following : if a step would lead to hit a boundary of the square, the step is cancelled and a new step is generated. There is however a hole of size $5 \times d$ in the square boundary, located at $(x_h = \frac{a}{2}) \times d, y_h = 0)$, which the walker is allowed to go through and leave the box. Determine the probability density function for the exit angle, defined as the angle between the direction of the outgoing step and the x axis.
9. We consider random walks starting at random positions (x_0, y_0), with θ_k following model 1, stuck inside a square box of side $a = 50 \times d$. The initial position (x_0, y_0) is uniformely distributed inside the box. The behaviour of the walker in the box is the following : if a step would lead to hit a boundary of the square, the step is cancelled and a new step is generated. There is however a hole of size $5 \times d$ in the square boundary, located at $(x_h = \frac{a}{2}) \times d, y_h = 0)$, which the walker is allowed to go through and leave the box. Determine the probability density function for the exit angle, defined as the angle between the direction of the outgoing step and the x axis.

"type 4" projects

10. We consider random walks all starting at $(x_0, y_0) = 0$, with θ_k following model 2, stuck inside a square box of side $a = (2k + 1) \times d$. The behaviour of the walker in the box is the following : if a

step would lead to hit a boundary of the square, the step is cancelled and a new step is generated. There is however a hole of size d in the square boundary, located at $(x_h = (k + \frac{1}{2}) \times d, y_h = 0)$, which the walker is allowed to go through and leave the box. Determine the average number of steps to leave the box, as a function of k.

11. We consider random walks starting at random positions (x_0, y_0), with θ_k following model 2, stuck inside a square box of side $a = (2k+1) \times d$. The initial position (x_0, y_0) is uniformely distributed inside the box. The behaviour of the walker in the box is the following : if a step would lead to hit a boundary of the square, the step is cancelled and a new step is generated. There is however a hole of size d in the square boundary, located at $(x_h = (k + \frac{1}{2}) \times d, y_h = 0)$, which the walker is allowed to go through and leave the box. Determine the average number of steps to leave the box, as a function of k.