Mitigation on the AIM Cryptanalysis (Preliminary Version)

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Abstract. Post-quantum signature schemes based on the MPC-in-the-Head (MPCitH) paradigm are recently attracting significant attention as their security solely depends on the one-wayness of the underlying primitive, providing diversity for the hardness assumption in post-quantum cryptography. Kim et al. proposed AIM as an MPCitH-friendly one-way function characterized by large algebraic S-boxes and parallel design, which lead to short signature size (CCS 2023).
Recently, Liu and Mahzoun proposed a fast exhaustive search attack on AIM (ePrint 2023), which degrades the security of AIM by upto 13 bits. While communicating with the authors, they pointed out another possible vulnerability on AIM. In this paper, we propose AIM2 which mitigates all the vulnerabilities, and analyze its security against algebraic attacks.

1 Introduction

MPC-in-the-Head (MPCitH), proposed by Ishai et al. [IKOS07], is a paradigm to construct a zero-knowledge proof (ZKP) system from a multiparty computation (MPC) protocol. Recently, the MPCitH paradigm is utilized as a building block of a post-quantum signature scheme since the security of MPCitH-based signature schemes solely depends on the security of the one-way function used in key generation.

Kim et al. [KHS+22] proposed an MPCitH-friendly one-way function AIM, and a signature scheme AIMer based on the BN++ proof [KZ22] of a preimage of a public key under AIM. AIM features a parallel structure and Mersenne S-boxes to fully enjoy repeated multipliers with high resistance to algebraic attacks. However, Liu and Mahzoun proposed a fast exhaustive search on AIM [LM23], which exploits the fact that AIM allows a low-degree system of equations in λ Boolean variables, where λ is the security parameter. Furthermore, Liu found a new low-degree system of equations in 2λ variables.[3] While it does not break AIM in a plausible assumption, it harms the original security claim in [KHS+22].

In this paper, we overview those two attacks and propose a new version of AIM, dubbed AIM2[4]. The main difference of AIM2 from AIM is three-fold:

1. Inverse Mersenne S-box: the S-box in the first round is placed in the opposite direction. In this way, we can make it harder to build a large number of equations compared to AIM.
2. Constant addition to the input of S-boxes: distinct constants are added to the inputs of first-round S-boxes. It differentiates the inputs of S-boxes with negligible cost.
3. Increasing exponents for S-boxes: we opt for larger exponents for some Mersenne S-boxes in order to make it harder to establish a low-degree system of equations in $\approx \lambda$ Boolean variables from a single evaluation of AIM.

We also analyze the security of AIM2 against various attacks. Finally, we will discuss how our patch affects efficiency of the resulting signature scheme.

[3] In private communication.
[4] This whitepaper is a preliminary version. The names and definitions of AIM2 are tentative and subject to change.

1.1 Notation

Throughout this paper, we denote (bit-)length of AIM and AIM2 as n. Unless stated otherwise, all logarithms are to the base 2. For two vectors a and b over a finite field, their concatenation is denoted by $a\|b$. For a positive integer m, we write $[m] = \{1, \ldots, m\}$. For an integer x and a boolean vector y, $\mathsf{hw}_n(x)$ and $\mathsf{hw}(y)$ denotes the Hamming weight of $x \bmod 2^n - 1$ in its binary representation and the Hamming weight of y, respectively. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_2^n$ and $x = (x_1, \ldots, x_n)$, monomial representation x^α means that $\prod_{i=1}^n x_i^{\alpha_i}$.

In this document, addition is usually operated on a binary field, which can be seen as bitwise exclusive-OR (XOR). When we want to emphasize this, we will write $\oplus$ to denote addition.

2 AIM and AIMer

AIM was proposed as an MPCitH-friendly symmetric primitive with high resistance to algebraic attacks [KHS+22]. AIMer is a signature scheme obtained by combining AIM with the BN++ proof system [KZ22].

Given the input/output size n and an $(\ell+1)$-tuple of exponents $(e_1, \ldots, e_\ell, e_*) \in \mathbb{Z}^{\ell+1}$,

$$\mathsf{AIM} : \{0,1\}^n \times \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$$

is defined by

$$\mathsf{AIM}(\mathsf{iv}, \mathsf{pt}) = \mathsf{Mer}[e_*] \circ \mathsf{Lin}[\mathsf{iv}] \circ \mathsf{Mer}[e_1, \ldots, e_\ell](\mathsf{pt}) \oplus \mathsf{pt}$$

where each function will be described below.[5] See Figure 2 for the pictorial description of AIM with $\ell = 3$.

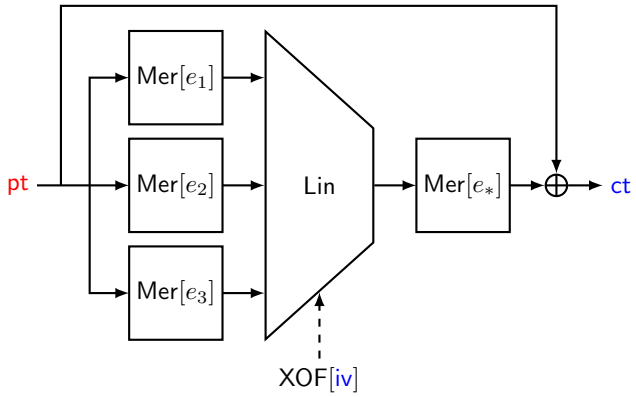

Fig. 1: The AIM-V one-way function with $\ell = 3$. The input pt (in red) is the secret key of the signature scheme, and $(\mathsf{iv}, \mathsf{ct})$ (in blue) is the corresponding public key.

NON-LINEAR COMPONENTS. In AIM, S-boxes are exponentiation by Mersenne numbers over a large field. More precisely, for $x \in \mathbb{F}_{2^n}$,

$$\mathsf{Mer}[e](x) = x^{2^e - 1}$$

for some e. Note that this map is a permutation if $\gcd(e, n) = 1$. As an extension, $\mathsf{Mer}[e_1, \ldots, e_\ell] : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}^\ell$ is defined by

$$\mathsf{Mer}[e_1, \ldots, e_\ell](x) = \mathsf{Mer}[e_1](x) \| \ldots \| \mathsf{Mer}[e_\ell](x).$$

[5] In the AIMer scheme, the initial vector iv is public and we claim the one-wayness of AIM for a fixed iv.

LINEAR COMPONENTS. AIM includes two types of linear components: an affine layer and feed-forward. The affine layer consists of multiplication by an $n \times \ell n$ random binary matrix A_{iv} and addition by a random constant $b_{\mathsf{iv}} \in \mathbb{F}_2^n$. The matrix

$$A_{\mathsf{iv}} = \left[A_{\mathsf{iv},1} \middle| \ldots \middle| A_{\mathsf{iv},\ell}\right] \in (\mathbb{F}_2^{n \times n})^\ell$$

is composed of ℓ random invertible matrices $A_{\mathsf{iv},i}$. The matrix A_{iv} and the vector b_{iv} are generated by an extendable-output function (XOF) with the initial vector iv. Each matrix $A_{\mathsf{iv},i}$ can be equivalently represented by a linearized polynomial $L_{\mathsf{iv},i}$ on $\mathbb{F}_{2^n}$. For $x = (x_1, \ldots, x_\ell) \in (\mathbb{F}_{2^n})^\ell$,

$$\mathsf{Lin}[\mathsf{iv}](x) = \sum_{1 \le i \le \ell} L_{\mathsf{iv},i}(x_i) \oplus b_{\mathsf{iv}}.$$

By abuse of notation, we will write Ax to denote $\sum_{1 \le i \le \ell} L_{\mathsf{iv},i}(x_i)$. Feed-forward operation, which is addition by the input itself, makes the entire function non-invertible.

RECOMMENDED PARAMETERS. Recommended sets of parameters for $\lambda \in \{128, 192, 256\}$ are given in Table 1. The irreducible polynomials for extension fields $\mathbb{F}_{2^{128}}$, $\mathbb{F}_{2^{192}}$, and $\mathbb{F}_{2^{256}}$ are the same as those used in Rain [DKR+22].

Scheme	λ	n	ℓ	e_1	e_2	e_3	e_*
AIM-I	128	128	2	3	27	-	5
AIM-III	192	192	2	5	29	-	7
AIM-V	256	256	3	3	53	7	5

Table 1: Recommended sets of parameters of AIM.

3 Algebraic Attack Models

In this section, we briefly introduce some algebraic attack models and their complexities. Throughout this section, we will focus on constructing an overdetermined system of m equations in n Boolean variables where the degree of each equation is denoted as d_i for $i = 1, \ldots, m$.

3.1 XL Algorithm with Independent Equations Model

The XL algorithm [CKPS00] is a generalization of the relinearization attack [KS99]. The XL algorithm extends the system by multiplying all the monomials of degree $D - d_i$ to the equation of degree d_i, resulting in $\sum_{i=1}^{m}(\sum_{j=0}^{D-d_i}\binom{n}{j})$ equations of degrees at most D. As the extended system is of degrees at most D, at most $\sum_{i=1}^{D}\binom{n}{i}$ monomials appear in the extended system. When the number of linearly independent equations becomes greater than the number of monomials as D grows, one can solve the extended system of equations by linearization.

The complexity of the XL attack depends on the number of linearly independent equations obtained from the XL algorithm, while we can loosely upper bound the number of linearly independent equations by $\sum_{i=1}^{m}\sum_{j=0}^{D-d_j}\binom{n}{j}$.

Assumption 1 *All the equations obtained while running the XL algorithm are linearly independent.*

Under Assumption 1, which is in favor of the attacker, we can search for the (smallest) degree D such that

$$\sum_{i=1}^{m} \sum_{j=0}^{D-d_i} \binom{n}{j} \geq T_D \tag{1}$$

where T_D denotes the exact number of monomials appearing in the extended system of equations, which is upper bounded by $\sum_{i=1}^{D} \binom{n}{i}$. Once D is fixed, the extended system of equations can be solved by trivial linearization whose time complexity is given as $O\left(T_D^{\omega}\right)$, where the constant ω is the matrix multiplication exponent.

In literature, Assumption 1 is not widely-used to estimate the security of a cryptosystem since it is regarded as too strong. The equations obtained while running the XL algorithm are linearly dependent with non-negligible probability, and the degree D is much higher than one computed from Assumption 1. Ars et al. [AFI+04] showed that the XL algorithm is in fact a redundant variant of the F_4 algorithm [Fau99]. AIM was claimed to be secure even if Assumption 1 is true [KHS+22].

3.2 Gröbner Basis Attack Model

The Gröbner basis attack is to solve a system of equations by computing its Gröbner basis. The attack consists of the following steps.

1. Compute a Gröbner basis in the *grevlex* (graded reverse lexicographic) order.
2. Change the order of terms to obtain a Gröbner basis in the *lex* (lexicographic) order.
3. Find a univariate polynomial in this basis and solve it.
4. Substitute this solution into the Gröbner basis and repeat Step 3.

When a system of equations has only finitely many solutions in its algebraic closure, its Gröbner basis in the *lex* order always contains a univariate polynomial. When a single variable of the polynomial is replaced by a concrete solution, the Gröbner basis still remains a Gröbner basis of the "reduced" system, allowing one to obtain a univariate polynomial again for the next variable. We refer to [SS21] for more details on Gröbner basis computation.

The security of a cryptosystem against the Gröbner basis attack is usually estimated by the complexity of the first step, which is the Gröbner basis computation in the *grevlex* order using F_4/F_5 algorithm or its variants [Fau99, Fau02]. The complexity of Gröbner basis computation can be estimated using the *degree of regularity* of the system of equations [BFS04]. Consider a system of m homogeneous equations $\{f_i(x_1, \ldots, x_n) = 0\}_{i=1}^{m}$ in n Boolean variables. Let d_i denote the degree of f_i for $i = 1, 2, \ldots, m$. If the system of equations is overdetermined, i.e., $m > n$, then the degree of regularity can be estimated by the smallest degree of the terms with non-positive coefficients appearing in the Hilbert series

$$\frac{(1+z)^n}{\prod_{i=1}^{m}(1+z^{d_i})}$$

under Assumption 2.

Assumption 2 ([Frö85]) *Almost all polynomial sequences are semi-regular.*

For nonhomogeneous equations, the degree of regularity is computed from the following Hilbert series obtained by homogenization [BFSS13]:

$$\frac{(1+z)^n}{(1-z)\prod_{i=1}^{m}(1+z^{d_i})}. \tag{2}$$

Given the degree of regularity d_{reg}, the complexity of computing a Gröbner basis of the system of equations is known to be

$$O\left(\binom{n}{d_{\mathrm{reg}}}^{\omega}\right).$$

In [KHS+22], the degree of regularity has been wrongly computed using the Hilbert series

$$\frac{1}{(1-z)^n}\prod_{i=1}^{m}(1-z^{d_i}).$$

and the complexity formula

$$O\left(\binom{n+d_{\mathrm{reg}}}{d_{\mathrm{reg}}}^{\omega}\right)$$

for the field of characteristic larger than n. As far as we check, this discrepancy leads to no significant difference in the attack complexity.

3.3 Hybrid Wiedemann XL Algorithm Model

The state-of-the-art model of solving a system of polynomial equations is to use the hybrid Wiedemann XL algorithm [BFP09, YCBC07]. This model is based on the following three techniques:

1. XL algorithm with termination at the degree of regularity (also known as the operating degree),
2. hybrid approach with the guess-and-determine attack [BFP09],
3. sparse linear system solving algorithm which is called the Wiedemann algorithm [Wie86].

Nowadays, the XL algorithm has been proved to terminate at degree d_{reg} defined by the Hilbert series (2) [YC04, YCBC07] under Assumption 2. So, the complexity of the hybrid Wiedemann XL algorithm on a system of Boolean equations is upper bounded by

$$\min_k 3\cdot 2^k\cdot\binom{n-k}{d_{\mathrm{reg}}(n,k)}^2\cdot\binom{n-k}{\max_i d_i} \tag{3}$$

where the degree of regularity $d_{\mathrm{reg}}(n,k)$ is the smallest degree of the terms with non-positive coefficients of the Hilbert series

$$\frac{(1+z)^{n-k}}{(1-z)\prod_{i=1}^{m}(1+z^{d_i})}. \tag{4}$$

3.4 Complexity Model in this Paper

In the previous sections, we introduced three complexity models for algebraic attacks (XL and Gröbner basis computation). Although the hybrid Wiedemann XL algorithm is the most widely-deployed model, we use the Gröbner basis attack model with $\omega = 2$ and hybrid approach [BFP09] since the complexity of this model lower bounds that of the hybrid Wiedemann XL model. Specifically, we use the complexity formula

$$\min_k 2^k\cdot\binom{n-k}{d_{\mathrm{reg}}(n,k)}^2 \tag{5}$$

where $d_{\mathrm{reg}}(n,k)$ is the smallest degree of the terms with non-positive coefficients of (4).

4 Cryptanalysis on AIM

4.1 Fast Exhaustive Search

Exhaustive search is the most basic attack for any keyed function $f_k(\cdot)$. For some given pairs (x_i, y_i) such that $f_k(x_i) = y_i$, an attacker checks whether $f_{\bar{k}}(x_i) = y_i$ or not for all i over all possible keys $\bar{k}$ in the key

space. Fast exhaustive search improves concrete efficiency of exhaustive search when the keyed function can be represented by a set of low-degree polynomials.

For a degree-d system in n variables, Bouillaguet et al. proposed a fast exhaustive search with time complexity $4d\log(n)2^n$ in Boolean operations and memory complexity $O(n^{2d})$ [BCC+10]. Bouillaguet also proposed a memory-efficient version of the fast exhaustive search with the same time complexity and memory complexity $n^2 \cdot \sum_{i=0}^{d} \binom{n}{i}$ in bits [Bou22]. We refer to the original papers for more details.

Liu and Mahzoun proposed a low-degree representation of AIM, and applied the fast exhaustive search algorithm to it [LM23]. The low-degree representation is described as follows.

Let z be the output of Lin. Then, pt can be represented in terms of z as follows.

$$\mathsf{pt} = z^{2^{e_*}-1} + \mathsf{ct}$$

Denoting the output of Mer$[e_i]$ by t_i, one has

$$t_i = \left(z^{2^{e_*}-1} + \mathsf{ct}\right)^{2^{e_i}-1}.$$

Let d_i be the degree of t_i with respect to z, and let $d_{\max} = \max_{i\neq 2} d_i$. The exponent e_2 is the largest from $\{e_1, \ldots, e_\ell\}$ (for the sets of recommended parameters), and t_2 can also be expressed as

$$t_2 = A_{\mathsf{iv},2}^{-1}\left(b_{\mathsf{iv}} + z + A_{\mathsf{iv},1}(t_1) + A_{\mathsf{iv},3}(t_3)\right)$$

where $A_{\mathsf{iv},3}(t_3)$ does not appear for AIM-I or AIM-III. Now we obtain an equation of degree at most $d_{\max}+e_*$ from $\mathsf{pt}\cdot t_2 = \mathsf{pt}^{2^{e_*}}$ as follows.

$$\left(z^{2^{e_*}-1} + \mathsf{ct}\right)\cdot A_{\mathsf{iv},2}^{-1}\left(b_{\mathsf{iv}} + z + A_{\mathsf{iv},1}(t_1) + A_{\mathsf{iv},3}(t_3)\right) = \left(z^{2^{e_*}-1} + \mathsf{ct}\right)^{2^{e_2}}$$

The degree $d_{\max}+e_*$ is known to be 10/14/15 for AIM-I, III, V, respectively. As the time complexity of the fast exhaustive search is $4d(\log n)2^n$, the (bitwise) gate-count complexity becomes $2^{136.2}/2^{200.7}/2^{265.0}$ for AIM-I,III,V, respectively, while straightforward exhaustive search requires $2^{146.4}/2^{211.9}/2^{277.0}$, respectively.[6]

4.2 Possible Algebraic Vulnerability on AIM

While communicating with the authors of [LM23], Liu pointed out that introducing a new variable results in an easier system of equations than expected. In this section, we briefly introduce how to make such a system.

We introduce a new variable $w = \mathsf{pt}^{-1}$, and let t_i be the output of Mer$[e_i]$ for $i \in \{1, \ldots, \ell\}$. Then, we have

$$t_i = \mathsf{pt}^{2^{e_i}} w$$

for all $i = 1, \ldots, \ell$. Then we can establish three types of equations

$$\mathsf{pt}\cdot w = 1, \tag{6}$$

$$\mathsf{Lin}\left(\mathsf{pt}^{2^{e_1}}w, \ldots, \mathsf{pt}^{2^{e_\ell}}w\right)\cdot(\mathsf{pt}+\mathsf{ct}) = \mathsf{Lin}\left(\mathsf{pt}^{2^{e_1}}w, \ldots, \mathsf{pt}^{2^{e_\ell}}w\right)^{2^{e_*}}, \tag{7}$$

$$\mathsf{Lin}\left(\mathsf{pt}^{2^{e_1}}w, \ldots, \mathsf{pt}^{2^{e_\ell}}w\right)\cdot(1+w\cdot\mathsf{ct}) = \mathsf{Lin}\left(\mathsf{pt}^{2^{e_1}}w, \ldots, \mathsf{pt}^{2^{e_\ell}}w\right)^{2^{e_*}}\cdot w. \tag{8}$$

Since the inverse S-box of n-bit input produces $5n$ linearly independent quadratic equations, we obtain $5n$ quadratic equations from (6). For (7) and (8), multiplying pt and w results in n more cubic equations,

[6] The complexity of straightforward exhaustive search has been slightly revised in the submission to the NIST PQC project [KCC+23].

respectively. Moreover, we have

$$\begin{aligned}
&\mathsf{Lin}\left(\mathsf{pt}^{2^{e_1}}w,\ldots,\mathsf{pt}^{2^{e_\ell}}w\right)^2\cdot(\mathsf{pt}+\mathsf{ct})+\mathsf{Lin}\left(\mathsf{pt}^{2^{e_1}}w,\ldots,\mathsf{pt}^{2^{e_\ell}}w\right)^2\cdot(1+w\cdot\mathsf{ct})\cdot\mathsf{ct}\\
&=\mathsf{Lin}\left(\mathsf{pt}^{2^{e_1}}w,\ldots,\mathsf{pt}^{2^{e_\ell}}w\right)^{2^{e_*+1}}+\mathsf{Lin}\left(\mathsf{pt}^{2^{e_1}}w,\ldots,\mathsf{pt}^{2^{e_\ell}}w\right)^{2^{e_*+1}}\cdot w\cdot\mathsf{ct}\\
&=\mathsf{Lin}\left(\mathsf{pt}^{2^{e_1}}w,\ldots,\mathsf{pt}^{2^{e_\ell}}w\right)^{2^{e_*+1}}\cdot w
\end{aligned}$$

which produces n more cubic equations. Overall, we have a system of $5n$ quadratic equations and $5n$ cubic equations in $2n$ Boolean variables regardless of ℓ.

Under Assumption 1 and the condition $\omega = 2$, the time complexity of the XL algorithm is $2^{124.8}/2^{157.5}/2^{188.9}$, respectively, which harms the original security claim in [KHS+22]. However, this assumption is usually regarded *too strong* as it is not plausible to expect that all the expanded equations are linearly independent. This assumption estimates the complexity much lower than the real computation of the XL algorithm [AFI+04].

If we estimate the complexity in the hybrid Gröbner basis attack model with Assumption 2 which is regarded as a more realistic assumption, the time complexity of the XL algorithm is $2^{158.3}/2^{226.5}/2^{290.2}$. Those values imply all the instances are secure against the XL algorithm.

The main reason of this vulnerability is insufficient difference between S-boxes in the first round. Since the exponents are simple and similar to each other, it is possible to set a new variable from a common factor. In the next section, we introduce our patch to AIM which differentiates the S-boxes much more than the original AIM.

5 Mitigation on the Cryptanalysis

5.1 AIM2[7]: Overall Patch

Given input/output size n and an $(\ell+1)$-tuple of exponents $(e_1,\ldots,e_\ell,e_*)\in\mathbb{Z}^{\ell+1}$, $\mathsf{AIM2}:\{0,1\}^n\times\mathbb{F}_{2^n}\to\mathbb{F}_{2^n}$ is defined by

$$\mathsf{AIM2}(\mathsf{iv},\mathsf{pt})=\mathsf{Mer}[e_*]\circ\mathsf{Lin}[\mathsf{iv}]\circ\mathsf{Mer}[e_1,\ldots,e_\ell]^{-1}\circ\mathsf{AddConst}(\mathsf{pt})\oplus\mathsf{pt}$$

where each function will be described below. See Figure 2 for the pictorial description of AIM2 with $\ell=3$.

NON-LINEAR COMPONENTS. AIM2 uses two types of S-boxes: Mersenne S-box $\mathsf{Mer}[e]$, and its inverse $\mathsf{Mer}[e]^{-1}$. These two S-boxes are defined by exponentiation over a large field as follows. For $x\in\mathbb{F}_{2^n}$,

$$\begin{aligned}
\mathsf{Mer}[e](x)&=x^{2^e-1},\\
\mathsf{Mer}[e]^{-1}(x)&=x^{\mathrm{inv}}\quad\text{where }\mathrm{inv}=(2^e-1)^{-1}\pmod{2^n-1}
\end{aligned}$$

for some e. To follow the spirit of AIM, the exponents e in AIM2 are selected for $\mathsf{Mer}[e]^{-1}$ to have $3n$ quadratic equations. We remark that the exponents e are chosen such that $\gcd(e,n)=1$, and hence the inverse exponent inv is well-defined. As an extension, $\mathsf{Mer}[e_1,\ldots,e_\ell]^{-1}:\mathbb{F}_{2^n}^\ell\to\mathbb{F}_{2^n}^\ell$ is defined by

$$\mathsf{Mer}[e_1,\ldots,e_\ell]^{-1}(x_1,\ldots,x_\ell)=\mathsf{Mer}[e_1]^{-1}(x_1)\|\ldots\|\mathsf{Mer}[e_\ell]^{-1}(x_\ell).$$

LINEAR COMPONENTS. AIM2 includes three types of linear components: constant addition, an affine layer, and feed-forward. For fixed constants $c_1,\ldots,c_\ell$, $\mathsf{AddConst}:\mathbb{F}_{2^n}\to\mathbb{F}_{2^n}^\ell$ is defined by

$$\mathsf{AddConst}(x)=(x+c_1)\|\ldots\|(x+c_\ell)$$

where the constants are defined in Table 2.

[7] This whitepaper is a preliminary version. The names and definitions of **AIM2** are tentative and subject to change.

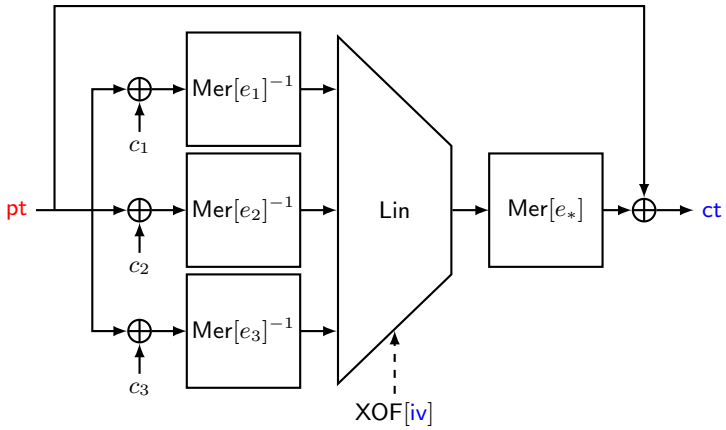

Fig. 2: The AIM2-V one-way function with $\ell = 3$. The input pt (in red) is the secret key of the signature scheme, and $(\mathsf{iv}, \mathsf{ct})$ (in blue) is the corresponding public key.

AIM2-I	c_1	0x243f6a8885a308d3
	c_2	0x13198a2e03707344
AIM2-III	c_1	0xa4093822299f31d0082efa98
	c_2	0xec4e6c89452821e638d01377
AIM2-V	c_1	0xbe5466cf34e90c6cc0ac29b7c97c50dd
	c_2	0x3f84d5b5b54709179216d5d98979fb1b
	c_3	0xd1310ba698dfb5ac2ffd72dbd01adfb7

Table 2: Constants $c_1, \ldots, c_\ell$ in AddConst are written in hexadecimal. These constants are taken from the numbers below the decimal point of the π ratio.

The affine layer in AIM2 is exactly the same as AIM. It consists of multiplication by an $n \times \ell n$ random binary matrix A_{iv} and addition by a random constant $b_{\mathsf{iv}} \in \mathbb{F}_2^n$. The matrix

$$A_{\mathsf{iv}} = \left[A_{\mathsf{iv},1} \middle| \ldots \middle| A_{\mathsf{iv},\ell}\right] \in (\mathbb{F}_2^{n \times n})^\ell$$

is composed of ℓ random invertible matrices $A_{\mathsf{iv},i}$. The matrix A_{iv} and the vector b_{iv} are generated by an extendable-output function (XOF) with the initial vector iv. Each matrix $A_{\mathsf{iv},i}$ can be equivalently represented by a linearized polynomial $L_{\mathsf{iv},i}$ on $\mathbb{F}_{2^n}$. For $x = (x_1, \ldots, x_\ell) \in (\mathbb{F}_{2^n})^\ell$,

$$\mathsf{Lin}[\mathsf{iv}](x) = \sum_{1 \le i \le \ell} L_{\mathsf{iv},i}(x_i) \oplus b_{\mathsf{iv}}.$$

By abuse of notation, we will write Ax to denote $\sum_{1 \le i \le \ell} L_{\mathsf{iv},i}(x_i)$. Feed-forward operation, which is addition by the input itself, makes the entire function non-invertible.

RECOMMENDED PARAMETERS. Recommended sets of parameters for $\lambda \in \{128, 192, 256\}$ are given in Table 1. The irreducible polynomials for extension fields $\mathbb{F}_{2^{128}}$, $\mathbb{F}_{2^{192}}$, and $\mathbb{F}_{2^{256}}$ are the same as those used in Rain [DKR+22].

Scheme	λ	n	ℓ	e_1	e_2	e_3	e_*
AIM2-I	128	128	2	49	99	-	3
AIM2-III	192	192	2	17	37	-	5
AIM2-V	256	256	3	191	219	7	3

Table 3: Recommended sets of parameters of AIM2.

5.2 Algebraic Attacks on AIM2

VARIOUS SYSTEMS OF AIM2. There are multiple ways of building a system of equations from an evaluation of AIM2. We can categorize them according to the number of (Boolean) variables and find the optimal choice of variables to obtain a system of the lowest degree. Since $\ell \in \{2, 3\}$ is recommended, we consider 4 types of systems of equations as follows.

1. Systems in n variables.
2. Systems in $2n$ variables.
3. Systems in $3n$ variables.
4. Systems in $4n$ variables (only for $\ell = 3$).

With $(\ell + 1)n$ variables, we can establish a system S_{quad} of *quadratic* equations. The variables are denoted as follows.

- x: the input of AIM2, i.e., pt
- t_i: the output of $\mathsf{Mer}[e_i]^{-1}$ for $i = 1, \ldots, \ell$
- z: the output of Lin

From $\mathsf{Mer}[e_i]^{-1}(x+c_i) = t_i$, we obtain $3n$ quadratic equations on x and t_i induced by the following relations.

$$\begin{cases} t_i(x + c_i) = t_i^{2^{e_i}}, \\ t_i(x + c_i)^2 = t_i^{2^{e_i}}(x + c_i), \\ t_i^2(x + c_i) = t_i^{2^{e_i}+1}. \end{cases}$$

When x and t_i are of higher degrees with respect to other variables, then the above two relations result in $2n$ equations of degree $\deg x + \deg t_i$ while the last one results in n equations of degree $\max(\deg x + \deg t_i, 2 \deg t_i)$. There are also n quadratic equations on t_i and t_j induced by the following.

$$(c_i + c_j)t_i t_j = t_i^{2^{e_i}} t_j + t_i t_j^{2^{e_j}}.$$

We note that z has the same relation with t_i with respect to x as $z = \mathsf{Mer}[e_*]^{-1}(x + \mathsf{ct})$. Using the brute-force search of quadratic equations on toy parameters, described in the later part of this section, we find that these are all possible (linearly independent) quadratic equations on AIM2. Hence, S_{quad} consists of $3(\ell + 1)n + \binom{\ell+1}{2}n$ quadratic equations.

With fewer variables, the resulting systems would have higher degrees. For example, $\mathsf{Mer}[e_i]^{-1}$ implicitly determines $3n$ quadratic equations in x and t_i as the above, while t_i (resp. x) can be explicitly represented by a polynomial in x (resp. t_i). We can also explicitly represent t_i using t_j for $j \neq i$ or z as follows.

$$\begin{aligned} t_i &= \mathsf{Mer}[e_i]^{-1}\left(\mathsf{Mer}[e_j](t_j) \oplus c_i \oplus c_j\right), \\ &= \mathsf{Mer}[e_i]^{-1}\left(\mathsf{Mer}[e_*](z) \oplus \mathsf{ct}\right). \end{aligned}$$

The degree of t_i with respect to t_j (resp. z) might be greater than the degree of $\mathsf{Mer}[e_i]^{-1} \circ \mathsf{Mer}[e_j]$ (resp. $\mathsf{Mer}[e_i]^{-1} \circ \mathsf{Mer}[e_*]$) due to the constant addition, while we estimate the degree of the composition (without constant addition) for simplicity.

Scheme	Name	#Var	Variables	(#Eq, Deg)	Complexity		
					k	d_{reg}	Time (bits)
AIM2-I	S_1	n	x	$(2n, 76)$	-	-	-
	S_2	$2n$	t_1, t_2	$(3n, 2)$	62	15	207.9
	S_{quad}	$3n$	x, t_1, t_2	$(12n, 2)$	0	16	185.3
AIM2-III	S_1	n	x	$(2n, 116)$	-	-	-
	S_2	$2n$	t_1, t_2	$(3n, 2)$	100	20	301.9
	S_{quad}	$3n$	x, t_1, t_2	$(12n, 2)$	0	22	262.4
AIM2-V	S_1	n	x	$(2n, 172)$	-	-	-
	S_2	$2n$	t_2, z	$(n, 2) + (2n, 38)$	253	30	513.5
	S_3	$3n$	t_1, t_2, t_3	$(6n, 2)$	2	47	503.7
	S_{quad}	$4n$	x, t_1, t_2, t_3	$(18n, 2)$	9	32	411.4

Table 4: Optimal systems of equations and their security against algebraic attacks. $(\#\text{Eq}, \text{Deg}) = (a, b)$ means that the system contains a equations of degree b. All the complexities are measured by (5). k is the number of guessed bits and d_{reg} is the degree of regularity.

Table 4 summarizes a system of equations of the lowest degree for each type, where such systems are denoted $S_1, S_2, \ldots, S_{\mathsf{quad}}$, respectively, according to the number of variables. The complexities are measured by (5). In the case of S_1 systems of n variables, we did not compute the complexities since the degree greater than $n/2$ makes the XL algorithm to consider $\approx 2^n$ monomials. This requires at least time complexity $O(2^{2n})$, which is infeasible.

BRUTE-FORCE SEARCH OF QUADRATIC EQUATIONS. For a given overdetermined quadratic system, algebraic attacks tend to solve the system faster when the system has more linearly independent equations. To lower bound the complexity of the algebraic attacks, we have to find all linearly independent equations. To brute-force search all the equations, we did a experiment as follows.

1. Set variables as follows.
   - x: the input of AIM2, i.e., pt
   - t_i: the output of $\mathsf{Mer}[e_i]^{-1}$ for $i = 1, \ldots, \ell$
   - z: the output of $\mathsf{Mer}[e_*]^{-1}(x + \mathsf{ct})$
2. Make a generic quadratic equation with indeterminate coefficients $a_{\alpha,\beta,\gamma} \in \mathbb{F}_2$;

$$\sum_{\substack{\alpha,\gamma\in\mathbb{F}_2^n,\beta\in\mathbb{F}_2^{\ell n} \\ \mathsf{hw}(\alpha)+\mathsf{hw}(\beta)+\mathsf{hw}(\gamma)\le 2}} a_{\alpha,\beta,\gamma} x^\alpha t_i^{\beta_i} z^\gamma = 0 \tag{9}$$

   where $\beta = (\beta_1, \ldots, \beta_\ell)$.
3. Randomly sample $x \in \mathbb{F}_{2^n}$, and compute corresponding t_i and z. Substitute those values to (9).
4. Repeat the previous step for $O\left(\binom{(\ell+2)n}{2}\right)$ times.
5. Solve the linear system with respect to $a_{\alpha,\beta,\gamma}$. The quadratic equations for the target system can be computed by substituting such $a_{\alpha,\beta,\gamma}$ to (9).

For S_{quad}, this experiment found that $12n$ quadratic equations for AIM2-I, III and $18n$ quadratic equations for AIM2-V. For S_2 of AIM2-I, III, it found that $3n$ quadratic equations. For S_3 of AIM2-V, it found that $6n$ quadratic equations. We remark that this experiment does not consider the affine layer by introducing redundant variable z. Although this may find more equations than the real count, we checked that all the equations obtained from the experiment are linearly independent.

This experiment can be easily generalized for a general degree d. However, the generalized experiment will include *all* equations of degree d expanded from quadratic equations. By this reason, we rather found equations of higher degree by hand than running the generalized experiment.

RESISTANCE TO FAST EXHAUSTIVE SEARCH. The fast exhaustive search attacks in [BCC+10, Bou22] are infeasible if target polynomial system is of high degree. Although the nominal time complexity of the fast exhaustive search is $4d\log(n)2^n$, there is a hidden preprocessing cost

$$T = \sum_{k=0}^{d-1} k\binom{n}{k}\binom{k}{\downarrow \min(d-k,k)} \geq \frac{2d}{3}2^{2d/3}\binom{n}{\lfloor 2d/3 \rfloor}$$

in binary operations where $\binom{n}{\downarrow k} = \sum_{i=0}^{k}\binom{n}{i}$. One can see that $T \gg d2^n$ if $d \geq 0.341n$. Furthermore, if $d \geq n/2$ then the memory complexity should be higher than 2^n bits.

ON NEW VARIABLES OTHER THAN S-BOX OUTPUTS. In Section 4.2, Liu increases the number of quadratic equations by introducing variables ($w = \mathsf{pt}^{-1}$) other than the inputs and outputs of the S-boxes without significantly increasing the degree of the overall system. We address the security of AIM2 against this attack on AIM for general exponents.

For simplicity, we denote $t_{\ell+1} = z$ and $c_{\ell+1} = \mathsf{ct}$. To make a successful attack by introducing new variables $w_i = (\mathsf{pt} + c_i)^a$ instead of t_i for all but one $i = 1, \ldots, \ell+1$, the following two conditions should hold.

1. The number of quadratic equations between x and the chosen w_i's should be greater than the number of quadratic equations in S_{quad}.
2. The degree $\deg t_i$ of t_i with respect to x and w_i's should not be too large for the chosen i's.

To the best of our knowledge, computing the number of quadratic equations induced by the power mapping of the form $y = x^a$ for $x, y \in \mathbb{F}_{2^n}$ requires $O(n^2)$ time [NGG09]. However, if the exponent a satisfies $\mathsf{hw}(a) \leq 2$ or $\mathsf{hw}(a+1) \leq 2$ (including the inverse exponent $a = 2^n - 2$), it is easy to find some quadratic equations directly from $y = x^a$ or $xy = x^{a+1}$. For these special cases of a, we find a lower bound for $\deg t_i$ with respect to x and w_i. Using the lower bound, we can show that introducing these kinds of new variables does not work well on the parameter sets for AIM2.

Let $c \in \mathbb{F}_{2^n}$ and $\bar{e} = (2^e - 1)^{-1} \bmod (2^n - 1)$ for some $e \in \{e_1, \ldots, e_\ell, e_*\}$ be given. For $w = (\mathsf{pt} + c)^a$ and $t = (\mathsf{pt} + c)^{\bar{e}}$, we want to show that t should be of at least certain degree with respect to pt and w when a is one of the following types:

- $a = -1$;
- $a = 2^p + 1$, where $p \in \{2, ..., n-1\}$;
- $a = 2^p - 1$, where $p \in \{2, ..., n-1\}$;
- $a = 2^p + 2^q - 1$, where $p, q \in \{2, ..., n-1\}$, $p \neq q$.

We note that $w = (x+c)^a$ and $w = (x+c)^{2^i a}$ for some positive integer i result in equivalent systems.

Define

$$D_{\min,a} := \min_u \left\{\mathsf{hw}_n(u) + \mathsf{hw}_n(\bar{e} - a \cdot u)\right\}$$

and

$$D_{\min} := \min_a \{D_{\min,a}\}.$$

$D_{\min}$ is the lower bound of the degree of t with respect to w and pt by

$$t = w^u \cdot (\mathsf{pt} + c)^{\bar{e} - a \cdot u}.$$

At first, suppose $a = 2^p + 2^q - 1$ for some $p, q \in \{2, ..., n-1\}$ where $p \neq q$. By the definition, we have

$$D_{\min,2^p+2^q-1} = \min_u \left\{\mathsf{hw}_n(u) + \mathsf{hw}_n(\bar{e} - (2^p + 2^q - 1) \cdot u)\right\}.$$

By using the fact $\mathsf{hw}_n(x) + \mathsf{hw}_n(y) \geq \mathsf{hw}_n(x+y)$, we have

$$\begin{aligned}
&2 \cdot \mathsf{hw}_n(u) + \mathsf{hw}_n(\bar{e} - (2^p + 2^q - 1) \cdot u) \\
&= \mathsf{hw}_n(2^p \cdot u) + \mathsf{hw}_n(2^q \cdot u) + \mathsf{hw}_n(\bar{e} - (2^p + 2^q - 1) \cdot u) \\
&\geq \mathsf{hw}_n(\bar{e} + u),
\end{aligned}$$

and it implies that

$$D_{\min, 2^p+2^q-1} \geq \min_u \left\{ \max \left\{ \mathsf{hw}_n(u), \mathsf{hw}_n(\bar{e}+u) - \mathsf{hw}_n(u) \right\} \right\}. \tag{10}$$

Now we want to lower bound $\mathsf{hw}(\bar{e}+u)$ for arbitrary u. For an integer x, define

$$\mathsf{NumSeg}_n(x) := \left| \left\{ i \in \{0, ..., n-1\} : 2 \mid (2^i \cdot x \bmod (2^n - 1)), 4 \nmid (2^i \cdot x \bmod (2^n - 1)) \right\} \right|$$

which counts the number of connected "1"s in the n-bit binary representation of x allowing bitwise rotation. Then, for an integer x and $h \in \{0, \ldots, n-1\}$,

$$\mathsf{NumSeg}_n(x + 2^h) \geq \mathsf{NumSeg}_n(x) - 1$$

so we get

$$\mathsf{hw}_n(\bar{e}+u) \geq \mathsf{NumSeg}_n(\bar{e}+u) \geq \mathsf{NumSeg}_n(\bar{e}) - \mathsf{hw}_n(u),$$

Together with (10), we have

$$D_{\min, 2^p+2^q-1} \geq \min_u \left\{ \max \left\{ \mathsf{hw}_n(u), \mathsf{NumSeg}_n(\bar{e}) - 2 \cdot \mathsf{hw}_n(u) \right\} \right\} \geq \lceil \mathsf{NumSeg}_n(\bar{e})/3 \rceil$$

Similarly, we have

$$\begin{aligned}
D_{\min, 2^p-1} &\geq \min_u \left\{ \max \left\{ \mathsf{hw}_n(u), \mathsf{hw}_n(\bar{e}+u) \right\} \right\} \geq \lceil \mathsf{NumSeg}_n(\bar{e})/2 \rceil, \\
D_{\min, 2^p+1} &\geq \min_u \left\{ \max \left\{ \mathsf{hw}_n(u), \mathsf{hw}_n(\bar{e}) - \mathsf{hw}_n(u) \right\} \right\} \geq \lceil \mathsf{hw}_n(\bar{e})/2 \rceil, \\
D_{\min, -1} &\geq \min_u \left\{ \mathsf{hw}_n(u) + \mathsf{hw}_n(\bar{e}+u) \right\} \geq \lceil \mathsf{NumSeg}_n(\bar{e}) \rceil,
\end{aligned}$$

and overall, we get following lower bound:

$$D_{\min} \geq \lceil \mathsf{NumSeg}_n(\bar{e})/3 \rceil.$$

Since the largest degree reaching while running a Gröbner basis computation algorithm or the XL algorithm (also known as *solving degree* [DS13]) should be larger than or equal to the degree of the system, we can lower bound the security of AIM2 against Liu's attack. Let us denote following variables.

- x: the input of AIM2, i.e., $x = \mathsf{pt}$.
- w_i: $(\mathsf{pt} + c_i)^{a_i}$ for some fixed a_i, for $i \in [\ell + 1]$, while define $c_{\ell+1} = \mathsf{ct}$.
- t_i: $(\mathsf{pt} + c_i)^{(2^{e_i}-1)^{-1} \bmod (2^n-1)}$, for $i \in [\ell + 1]$, while define $c_{\ell+1} = \mathsf{ct}$ and $e_{\ell+1} = e_*$.

One can construct system by replacing some of t-variables with w-variables. Table 5 summarizes the lower bound of time complexity (from (5)) of these systems and the bound of $D_{\min}$ for each exponents. We only considered replacing some of t-variables in S_{quad} to w-variables, since otherwise we would get a system with a lot higher degree.

Scheme	$(e_1, D_{\min})$	$(e_2, D_{\min})$	$(e_3, D_{\min})$	$(e_*, D_{\min})$	Complexity		
					k	sd	Time (bits)
AIM2-I	$(49, 16)$	$(99, 18)$	-	$(3, 15)$	0	≥ 15	176.2
AIM2-III	$(17, 27)$	$(37, 28)$	-	$(5, 26)$	0	≥ 26	298.4
AIM2-V	$(191, 21)$	$(219, 28)$	$(7, 25)$	$(3, 29)$	0	≥ 21	288.5

Table 5: Lower bounds of the degrees of the system when introducing variables other than the S-box outputs. $(e_i, D_{\min}) = (e, d)$ means that there is no such f with $\deg(f) < d$ where $t_i = \mathsf{Mer}[e_i]^{-1}(\mathsf{pt}) = f(\mathsf{pt}, w)$, while there exists degree 2 polynomial $g(\mathsf{pt}, w) = 0$. All the complexities are measured by (3). k is the number of guessed bits and sd is the solving degree, which is larger than at least one of $D_{\min}$.

5.3 Other Attacks on AIM2

For larger exponents, it will take slightly more time to compute the (inverse) Mersenne S-boxes. This leads to a slightly larger complexity of the brute-force attack and the Grover's algorithm. The complexities of quantum algebraic attacks will be changed not critically as new quadratic systems are found for AIM2. For `QuantumBooleanSolve` [FHK+17], the complexity becomes $O(2^{0.462 \cdot \ell n})$ since there are quadratic systems in ℓn Boolean variables for all the instances of AIM2. The complexity of GroverXL [BY18] is $2^{(1.1061+o(1))n}$ for AIM2-I, III and $2^{(1.3568+o(1))n}$ for AIM2-V. We remark that these attacks are not better than the Grover's algorithm.

As differential probability and linear probability of an S-box is the same as its inverse, most of the analysis on statistical attacks will remain unchanged except the weight of a correlation trail. Since e_1 becomes larger than $n/2$, the weight is lower bounded by $n - 2e_*$ (with the previous bound being $2(n - e_1 - e_*)$). We note that it does not imply that linear cryptanalysis is feasible since an adversary is not given a large enough number of plaintext-ciphertext pairs to mount this analysis.

5.4 Effect on Efficiency

The main feature of AIM is to fully utilize the repeated multipliers in BN++ when proving an AIM instance. Although the S-boxes in the first round are replaced by inverse Mersenne S-boxes, the structure of AIM2 still remains unchanged, so the signature size will be unchanged as well.

In AIMer, for every input share $[\![x]\!]$ of an S-box, the prover and the verifier should compute $[\![x]\!]^{2^e}$. For a larger exponent e, this computation will take more time. From our experiment, signing and verification of the new AIMer is expected to be about 10% slower.

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