RC Circuit Transient Analysis using Laplace Transform for both DC and AC Voltage Source

Aimilios's Circuits

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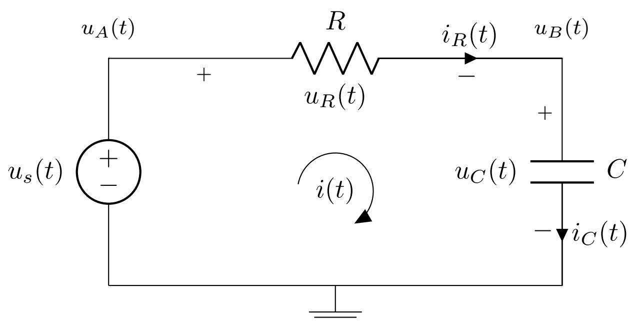
1 RC Circuit



1.1 V-I Relations

$u_R(t) = R \cdot i_R(t)$ $\qquad$ $i_R(t) = \frac{1}{R} \cdot u_R(t)$

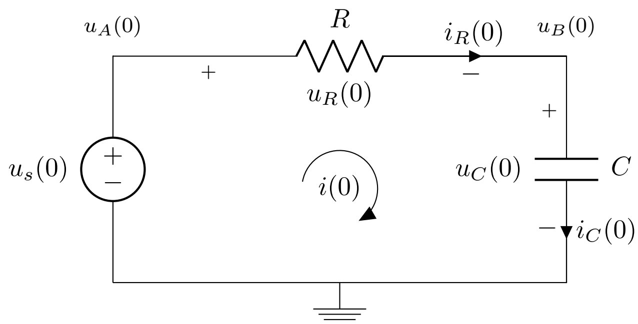
$u_C(t) = \frac{1}{C} \cdot \int_0^t i_C(t)dt + u_C(0)$ $\qquad$ $i_C(t) = C \cdot \frac{du_C(t)}{dt}$

1.2 V-I Relations to Mesh Current and Nodal Voltage

$u_R(t) = u_A(t) - u_B(t)$ $\qquad$ $i_R(t) = i(t)$

$u_C(t) = u_B(t)$ $\qquad$ $i_C(t) = i(t)$

2 Initial Conditions



For $t = 0$:
Initial Capacitor Voltage: $u_c(0)$, so from the circuit:

$$\begin{aligned} u_B(0) - 0 &= u_c(0) \\ u_B(0) &= u_c(0) \end{aligned} \tag{1}$$

Kirchoffs Current Law in node A:

$$\begin{aligned} u_s(0) &= u_A(0) - 0 \\ u_A(0) &= u_s(0) \end{aligned} \tag{2}$$

Kirchoffs Current Law in node B:

$$\begin{aligned} i_R(0) &= i_C(0) \\ \frac{u_A(0) - u_B(0)}{R} &= c\frac{du_B(t)}{dt}|_{t=0} \\ u'_B(0) &= \frac{1}{RC}(u_A(0) - u_B(0)) \\ u'_B(0) &= \frac{1}{RC}(u_s(0) - u_c(0)) \end{aligned} \tag{3}$$

Kirchoffs Voltage Law in loop:

$$\begin{aligned} -u_s(0) + u_R(0) + u_C(0) &= 0 \\ -u_s(0) + R * i_R(0) + u_C(0) &= 0 \\ -u_s(0) + R * i(0) + u_C(0) &= 0 \\ i(0) &= \frac{u_s(0) - u_C(0)}{R} \end{aligned} \tag{4}$$

All initial Conditions:

$$
\begin{aligned}
u_A(0) &= u_s(0) \\
u_B(0) &= u_c(0) \\
u'_B(0) &= \frac{1}{RC}(u_s(0) - u_c(0)) \\
i(0) &= \frac{u_s(0) - u_C(0)}{R}
\end{aligned}
$$

3 Mesh Analysis

3.1 I Formula

$$
\begin{aligned}
-u_s(t) + u_R(t) + u_C(t) &= 0 \\
-u_s(t) + R \cdot i_R(t) + \frac{1}{C} \cdot \int_0^t i_C(t)dt + u_C(0) &= 0 \\
-u_s(t) + R \cdot i(t) + \frac{1}{C} \cdot \int_0^t i(t)dt + u_C(0) &= 0 \\
\text{Laplace Transform:} - V_s(s) + R \cdot I(s) + \frac{1}{C}\frac{I(s)}{s} + \frac{u_C(0)}{s} &= 0 \\
I(s)(s + \frac{1}{RC}) &= \frac{1}{R}(sV_s(s) - u_C(0)) \\
I(s) &= \frac{1}{R}(sV_s(s) - u_C(0))\frac{1}{(s + \frac{1}{RC})}
\end{aligned}
$$

3.2 Solution According to Source Voltage Signal

3.2.1 For DC Signal

$$
\begin{aligned}
u_s(t) &= Us \cdot u(t) \\
V_s(s) &= \mathcal{L}\big\{u_s(t)\big\} = \frac{U_s}{s} \\
I(s) &= \frac{1}{R}(U_s - u_C(0))\frac{1}{(s + \frac{1}{RC})}
\end{aligned}
$$

3.2.2 For AC Cosine Signal

$$
\begin{aligned}
u_s(t) &= Us \cdot cos(\omega t) \\
V_s(s) &= \mathcal{L}\big\{u_s(t)\big\} = U_s\frac{s}{s^2 + \omega^2} \\
I(s) &= \frac{1}{R}(Us\frac{s^2}{s^2 + \omega^2} - u_C(0))\frac{1}{(s + \frac{1}{RC})}
\end{aligned}
$$

3.3 Solution

3.3.1 DC Solution

$$i(t) = \frac{Us - u_C(0)}{R} e^(\frac{-t}{RC})$$

3.3.2 AC Cosine Solution

$$i(t) = \frac{Us \cdot C \cdot \omega \cdot sin(\omega t) - Us \cdot C^2 \cdot R \cdot \omega^2 \cdot cos(\omega t)}{R^2 \cdot C^2 \cdot \omega^2 + 1} - e^(\frac{-t}{RC}) \cdot \frac{u_C(0) \cdot R^2 \cdot C^2 \cdot \omega^2 - Us + u_C(0)}{R \cdot (R^2 \cdot C^2 \cdot \omega^2 + 1)}$$

4 Nodal Analysis

4.1 V Formula

Kirchoffs Current Law in node A:

$$\begin{aligned} u_s(t) &= u_A(t) - 0 \\ u_A(t) &= u_s(t) \end{aligned} \tag{5}$$

Kirchoffs Current Law in node B:

$$\begin{aligned} i_R(t) &= i_C(t) \\ \frac{u_A(t) - u_B(t)}{R} &= c\frac{du_B(t)}{dt} \\ \frac{1}{RC} \cdot (u_A(t) - u_B(t)) - u_B'(t) &= 0 \\ \text{Laplace Transform:} \frac{1}{RC} \cdot V_A(s) - \frac{1}{RC} \cdot V_B(s) - (s \cdot V_B(s) - u_B(0)) &= 0 \\ \frac{1}{RC} \cdot V_A(s) - \frac{1}{RC} \cdot V_B(s) - s \cdot V_B(s) + u_B(0) &= 0 \\ \text{From (1),(5):} \frac{1}{RC} \cdot V_s(s) - \frac{1}{RC} \cdot V_B(s) - s \cdot V_B(s) + u_C(0) &= 0 \\ V_B(s) \cdot (s + \frac{1}{RC}) &= \frac{1}{RC} \cdot V_s(s) + u_C(0) \\ V_B(s) &= \frac{1}{RC} \cdot (V_s(s) + u_C(0)) \cdot \frac{1}{(s + \frac{1}{RC})} \end{aligned}$$

4.2 Solution According to Source Voltage Signal

4.2.1 For DC Signal

$$u_s(t) = Us \cdot u(t)$$
$$V_s(s) = \mathcal{L}\{u_s(t)\} = \frac{U_s}{s}$$
$$V_B(s) = \frac{1}{RC} \cdot (U_s + u_C(0)) \cdot \frac{1}{(s + \frac{1}{RC})}$$

4.2.2 For AC Cosine Signal

$$u_s(t) = Us \cdot cos(\omega t)$$
$$V_s(s) = \mathcal{L}\{u_s(t)\} = U_s \frac{s}{s^2 + \omega^2}$$
$$V_B(s) = \frac{1}{RC} \cdot (Us \frac{s^2}{s^2 + \omega^2} + u_C(0)) \cdot \frac{1}{(s + \frac{1}{RC})}$$

4.3 Solution

4.3.1 DC Solution

$$u_B(t) = Us + (u_C(0) - Us) \cdot e^(\frac{-t}{RC})$$

4.3.2 AC Cosine Solution

$$U_B(t) = \frac{Us \cdot cos(\omega t) + Us \cdot C \cdot R \cdot sin(\omega t)}{R^2 \cdot C^2 \cdot \omega^2 + 1} + e^(\frac{-t}{RC}) \cdot \frac{u_C(0) \cdot R^2 \cdot C^2 \cdot \omega^2 \cdot + u_C(0) - Us}{R^2 \cdot C^2 \cdot \omega^2 + 1}$$