Sampler for Different Ranking Models

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Simulating farmers' preference for different varieties requires sampling from different ranking models. Here I code samplers for sampling from Thurstone Model [8], Bradley-Terry Model [1], Plackett-Luce Model [7][5], and Mallows Model [6]. The samplers all sample a complete ranking from one farmer.

1 Problem Statement

We are given a set of $|V|$ *varieties* $V = \{v_1, \dots, v_{|V|}\}$ and a set of $|G|$ *farmers* $G = \{g_1, \dots, g_{|G|}\}$. Each farmer g_i is provided with $k < |V|$ varieties and returns a *ranking* σ_{g_i} as an ordered list of the form $\{v_1^{(g_i)}, \succ \cdots \succ v_k^{(g_i)}\}$. The data we get is a collection of $|G|$ rankings $\Sigma = \{\sigma_{g_1}, \dots, \sigma_{g_{|G|}}\}$. Let $v_i \succ_\sigma v_j$ denote that v_i ranks higher than v_j according to the pairwise preferences expressed by σ, and let $\sigma_{g_i} \subset \sigma_{g_j}$ denote that all varieties ranked by σ_{g_i} are ranked by σ_{g_j} and their relative orders are preserved in σ_{g_j}. Furthermore we let $r_{v_i}^{(g)}$ denote the rank of v_i in σ_g.

A ranking model $\mathcal{P}$ is a probability distribution that assigns different probabilities to all possible rankings of V. And we want to sample from $\mathcal{P}$ given the parameters of $\mathcal{P}$.

2 Thurstone Model

Thurstone proposed a ranking process [8] where the ranking σ of $|V|$ varieties (full ranking) given by a certain farmer is determined by the ordering of a set of latent values $\mathbf{y} = (y_{v_1}, \dots, y_{v_{|V|}})$ and

$$\mathbf{y} \sim \mathcal{N}(S, \Gamma).$$

Here $S = \{s_{v_1}, \dots, s_{v_{|V|}}\}$ is called the *score* parameter. For simplicity we assume Γ is diagonal, i.e. elements in $\mathbf{y}$ are independent with each other, since farmers won't have prior knowledge on the kinship between varieties and they draw each y_{v_i} independently (however elements in S may have dependence between each other). Algorithm 1 gives a detailed description of the sampling process.

Algorithm 1 Sampler for Thurstone Model

```
procedure rThurstone(S, var)
    y ← independent normal variables according to S, var
    Sort y in the decreasing order
    return indexes of sorted y
```

Now consider all pairwise comparisons. Following independence, we have

$$y_{v_i} - y_{v_j} \sim \mathcal{N}(s_{v_i} - s_{v_j}, \Gamma_{ii} + \Gamma_{jj}).$$

Hence the ranking model is given by

$$\mathcal{P}(\sigma|S, \Gamma) = \prod_{v_i \succ_\sigma v_j} \Phi\left(\frac{s_{v_i} - s_{v_j}}{\sqrt{\Gamma_{ii} + \Gamma_{jj}}}\right), \tag{1}$$

where Φ is the C.D.F. of the standard normal distribution.

3 Bradley-Terry Model

This model was proposed by Bradley and Terry [1] and has various extensions. This model is also parameterized by the *score* parameter $S = (s_{v_1}, \ldots, s_{v_{|V|}})$ with the constaint $s_{v_1} = 0$ in order to be identifiable. It also focuses on pairwise comparisons, namely we have

$$\mathbb{P}(v_i \succ_\sigma v_j) = \frac{1}{1 + \exp\{-(s_{v_i} - s_{v_j})\}}. \tag{2}$$

Assuming independence among pairwise comparisons, the ranking model is given by

$$\mathcal{P}(\sigma|S) = \prod_{v_i \succ_\sigma v_j} \frac{1}{1 + \exp\{-(s_{v_i} - s_{v_j})\}}. \tag{3}$$

Algorithm 2 provides a detailed description of the sampling process.

Algorithm 2 Sampler for Bradley-Terry Model

procedure RBT(S)
    Sample count matrix C according to pairwise probabilities
    Calculate the number of wins $\mathbf{w}$ for each variety
    Sort $\mathbf{w}$ in the decreasing order
    return indexes of sorted $\mathbf{w}$

4 Plackett-Luce Model

This model was proposed independently by Plackett [7] and Luce [5]. It is the generalization of Bradley-Terry model, allowing the comparison of more than 2 varieties concurrently, and it is also parameterized by the score $S = (s_{v_1}, \dots, s_{v_{|V|}})$ with the constraint $s_{v_1} = 0$. The ranking model is given by

$$\mathcal{P}(\sigma|S) = \prod_{v_i \in V} \frac{\exp\{s_{v_i}\}}{\exp\{s_{v_i}\} + \sum_{v_i \succ_\sigma v_j} \exp\{s_{v_j}\}} \tag{4}$$

This model has a Thurstonian interpretation, according to [9]. When each farmer draw latent values $\mathbf{y}$ independently from

$$y_{v_i} \sim \text{Gumbel}(\mu_{v_i}, \beta),$$

where $\beta > 0$ is an arbitrary fixed scale parameter, the ranking induced is identical to the Plackett-Luce model with parameterization:

$$s_{v_i} = \frac{\mu_{v_i}}{\beta} + \ln\Big(\sum_{v_j \in V} \exp\{s_{v_j}\}\Big). \tag{5}$$

This interpretation gives rise to algorithm 3.

Algorithm 3 Sampler for Plackett-Luce Model

procedure RPL(S, β)
    Calculate $\boldsymbol{\mu}$ by (5)
    $\mathbf{y} \sim \text{Gumbel}(\boldsymbol{\mu}, \beta)$
    Sort $\mathbf{y}$ in the decreasing order
    return indexes of sorted $\mathbf{y}$

5 Mallows Model

This model was proposed by Mallow [6] and is parameterized by a reference ranking $\bar{\sigma}$ and a dispersion parameter $\phi \in (0, 1]$. Intuitively it describes a distribution over rankings with respect to a distance measure $\delta(\sigma, \bar{\sigma})$. The ranking model is given by

$$p(\sigma|\bar{\sigma}, \phi) = \frac{1}{Z}\phi^{\delta(\sigma,\bar{\sigma})}, \tag{6}$$

where $Z = \sum_{\sigma'} \phi^{\delta(\sigma',\bar{\sigma})}$. Here we follow the path of [4] and use Kendall's tau distance [3], which is given by

$$\delta(\sigma, \bar{\sigma}) = \sum_{v_i \succ_\sigma v_j} \mathbb{1}\{v_j \succ_{\bar{\sigma}} v_i\} \tag{7}$$

Doignon el al. introduces the Repeated Insersion Model (RIM) which allows us to sample efficiently from a Mallows model [2]. Let $\bar{\sigma} = \bar{\sigma}_1\bar{\sigma}_2 \cdots \bar{\sigma}_{|V|}$ be the reference ranking and let an *insersion vector* $\mathbf{j} = (j_1, \ldots, j_{|V|})$ be any positive integer vector satisfying $j_i \leq i, \forall i \leq |V|$. Then an insersion vector has the one-to-one correspondence to a ranking by placing each $\bar{\sigma}_i$ into rank j_i for all $i \leq |V|$. Each $j_i \leq i$ is drawn with probability p_{ij_i}, which is given by

$$p_{ij} = \frac{\phi^{i-j}}{1 + \phi + \ldots, +\phi^{i-1}} \tag{8}$$

for $j \leq i \leq |V|$. The details are shown in Algorithm 4.

Algorithm 4 Sampler for Mallows Model

```
procedure rMallow(σ̄, φ)
    r ← σ̄_1                                        ▷ Initialize
    for i = 2, ..., |V| do
        Calculate insertion probabilities p by (8)
        x ~ Multinomial(1, p)
        t ← the index of 1 in x
        Insert σ̄_i into rank t
        r ← the new ranking
    return r
```

References

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