Now let's consider a different type of question. Suppose that two different reaction temperatures t_1 and t_2 can be used in a chemical process. The engineer conjectures that t_1 will result in higher yields than t_2. If the engineers can demonstrate that t_1 results in higher yields, then a process change can probably be justified. Statistical hypothesis testing is the framework for solving problems of this type. In this example, the engineer would be interested in formulating hypotheses that allow him or her to demonstrate that the mean yield using t_1 is higher than the mean yield using t_2. Notice that there is no emphasis on estimating yields; instead, the focus is on drawing conclusions about a hypothesis that is relevant to the engineering decision.

This chapter and Chapter 8 discuss parameter estimation. Chapters 9 and 10 focus on hypothesis testing.

Learning Objectives

After careful study of this chapter, you should be able to do the following:

1. Explain the general concepts of estimating the parameters of a population or a probability distribution
2. Explain the important role of the normal distribution as a sampling distribution
3. Understand the central limit theorem
4. Explain important properties of point estimators, including bias, variance, and mean square error
5. Know how to construct point estimators using the method of moments and the method of maximum likelihood
6. Know how to compute and explain the precision with which a parameter is estimated
7. Know how to construct a point estimator using the Bayesian approach

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7-1 Point Estimation

Statistical inference always focuses on drawing conclusions about one or more parameters of a population. An important part of this process is obtaining estimates of the parameters. Suppose that we want to obtain a **point estimate** (a reasonable value) of a population parameter. We know that before the data are collected, the observations are considered to be random variables, say, $X_1, X_2, \ldots, X_n$. Therefore, any function of the observation, or any **statistic**, is also a random variable. For example, the sample mean $\overline{X}$ and the sample variance S^2 are statistics and random variables.

Another way to visualize this is as follows. Suppose we take a sample of $n = 10$ observations from a population and compute the sample average, getting the result $\overline{x} = 10.2$. Now we repeat this process, taking a second sample of $n = 10$ observations from the same population and the resulting sample average is 10.4. The sample average depends on the observations in the sample, which differ from sample to sample because they are random variables. Consequently, the sample average (or any other function of the sample data) is a random variable.

Because a statistic is a random variable, it has a probability distribution. We call the probability distribution of a statistic a **sampling distribution**. The notion of a sampling distribution is very important and will be discussed and illustrated later in the chapter.

When discussing inference problems, it is convenient to have a general symbol to represent the parameter of interest. We will use the Greek symbol θ (theta) to represent the parameter. The symbol θ can represent the mean μ, the variance σ^2, or any parameter of interest to us. The objective of point estimation is to select a single number based on sample data that is the most plausible value for θ. A numerical value of a sample statistic will be used as the point estimate.

In general, if X is a random variable with probability distribution $f(x)$, characterized by the unknown parameter θ, and if $X_1, X_2, \ldots, X_n$ is a random sample of size n from X, the statistic

$\hat{\Theta} = h(X_1, X_2, \ldots, X_n)$ is called a **point estimator** of θ. Note that $\hat{\Theta}$ is a random variable because it is a function of random variables. After the sample has been selected, $\hat{\Theta}$ takes on a particular numerical value $\hat{\theta}$ called the **point estimate** of θ.

Point Estimator

A **point estimate** of some population parameter θ is a single numerical value $\hat{\theta}$ of a statistic $\hat{\Theta}$. The statistic $\hat{\Theta}$ is called the **point estimator**.

As an example, suppose that the random variable X is normally distributed with an unknown mean μ. The sample mean is a point estimator of the unknown population mean μ. That is, $\hat{\mu} = \overline{X}$. After the sample has been selected, the numerical value $\overline{x}$ is the point estimate of μ. Thus, if $x_1 = 25$, $x_2 = 30$, $x_3 = 29$, and $x_4 = 31$, the point estimate of μ is

$$\overline{x} = \frac{25 + 30 + 29 + 31}{4} = 28.75$$

Similarly, if the population variance σ^2 is also unknown, a point estimator for σ^2 is the sample variance S^2, and the numerical value $s^2 = 6.9$ calculated from the sample data is called the *point estimate of* σ^2.

Estimation problems occur frequently in engineering. We often need to estimate

- The mean μ of a single population
- The variance σ^2 (or standard deviation σ) of a single population
- The proportion p of items in a population that belong to a class of interest
- The difference in means of two populations, $\mu_1 - \mu_2$
- The difference in two population proportions, $p_1 - p_2$

Reasonable point estimates of these parameters are as follows:

- For μ, the estimate is $\hat{\mu} = \overline{x}$, the sample mean.
- For σ^2, the estimate is $\hat{\sigma}^2 = s^2$, the sample variance.
- For p, the estimate is $\hat{p} = x/n$, the sample proportion, where x is the number of items in a random sample of size n that belong to the class of interest.
- For $\mu_1 - \mu_2$, the estimate is $\hat{\mu}_1 - \hat{\mu}_2 = \overline{x}_1 - \overline{x}_2$, the difference between the sample means of two independent random samples.
- For $p_1 - p_2$, the estimate is $\hat{p}_1 - \hat{p}_2$, the difference between two sample proportions computed from two independent random samples.

We may have several different choices for the point estimator of a parameter. For example, if we wish to estimate the mean of a population, we might consider the sample mean, the sample median, or perhaps the average of the smallest and largest observations in the sample as point estimators. To decide which point estimator of a particular parameter is the best one to use, we need to examine their statistical properties and develop some criteria for comparing estimators.

7-2 Sampling Distributions and the Central Limit Theorem

Statistical inference is concerned with making **decisions** about a population based on the information contained in a random sample from that population. For instance, we may be interested in the mean fill volume of a container of soft drink. The mean fill volume in the population is required to be 300 milliliters. An engineer takes a random sample of 25 containers and computes the sample average fill volume to be $\bar{x} = 298.8$ milliliters. The engineer will probably decide that the population mean is $\mu = 300$ milliliters even though the sample mean was 298.8 milliliters because he or she knows that the sample mean is a reasonable estimate of μ and that a sample mean of 298.8 milliliters is very likely to occur even if the true population mean is $\mu = 300$ milliliters. In fact, if the true mean is 300 milliliters, tests of 25 containers made repeatedly, perhaps every five minutes, would produce values of $\bar{x}$ that vary both above and below $\mu = 300$ milliliters.

The link between the probability models in the earlier chapters and the data is made as follows. Each numerical value in the data is the observed value of a random variable. Furthermore, the random variables are usually assumed to be independent and identically distributed. These random variables are known as a *random sample*.

Random Sample

The random variables $X_1, X_2, \ldots, X_n$ are a **random sample** of size n if (a) the X_i's are independent random variables and (b) every X_i has the same probability distribution.

The observed data are also referred to as a *random sample*, but the use of the same phrase should not cause any confusion.

The assumption of a random sample is extremely important. If the sample is not random and is based on judgment or is flawed in some other way, statistical methods will not work properly and will lead to incorrect decisions.

The primary purpose in taking a random sample is to obtain information about the unknown population parameters. Suppose, for example, that we wish to reach a conclusion about the proportion of people in the United States who prefer a particular brand of soft drink. Let p represent the unknown value of this proportion. It is impractical to question every individual in the population to determine the true value of p. To make an inference regarding the true proportion p, a more reasonable procedure would be to select a random sample (of an appropriate size) and use the observed proportion $\hat{p}$ of people in this sample favoring the brand of soft drink.

The sample proportion, $\hat{p}$, is computed by dividing the number of individuals in the sample who prefer the brand of soft drink by the total sample size n. Thus, $\hat{p}$ is a function of the observed values in the random sample. Because many random samples are possible from a population, the value of $\hat{p}$ will vary from sample to sample. That is, $\hat{p}$ is a random variable. Such a random variable is called a **statistic**.

Statistic

A **statistic** is any function of the observations in a random sample.

We have encountered statistics before. For example, if $X, X_2, \ldots, X_n$ is a random sample of size n, the **sample mean** $\bar{X}$, the **sample variance** S^2, and the **sample standard deviation** S are statistics. Because a statistic is a random variable, it has a probability distribution.

Sampling Distribution

The probability distribution of a statistic is called a **sampling distribution**.

For example, the probability distribution of $\bar{X}$ is called the **sampling distribution of the mean**. The sampling distribution of a statistic depends on the distribution of the population, the size of the sample, and the method of sample selection. We now present perhaps the most important sampling distribution. Other sampling distributions and their applications will be illustrated extensively in the following two chapters.

Consider determining the sampling distribution of the sample mean $\bar{X}$. Suppose that a random sample of size n is taken from a normal population with mean μ and variance σ^2. Now each observation in this sample, say, $X_1, X_2, \ldots, X_n$, is a normally and independently distributed random variable with mean μ and variance σ^2. Then because linear functions of independent, normally distributed random variables are also normally distributed (Chapter 5), we conclude that the sample mean

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

has a normal distribution with mean

$$\mu_{\bar{X}} = \frac{\mu + \mu + \cdots + \mu}{n} = \mu$$

and variance

$$\sigma^2_{\bar{X}} = \frac{\sigma^2 + \sigma^2 + \cdots + \sigma^2}{n^2} = \frac{\sigma^2}{n}$$

If we are sampling from a population that has an unknown probability distribution, the sampling distribution of the sample mean will still be approximately normal with mean μ and variance σ^2/n if the sample size n is large. This is one of the most useful theorems in statistics, called the **central limit theorem**. The statement is as follows:

Central Limit Theorem

If $X_1, X_2, \ldots, X_n$ is a random sample of size n taken from a population (either finite or infinite) with mean μ and finite variance σ^2 and if $\bar{X}$ is the sample mean, the limiting form of the distribution of

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \quad (7\text{-}1)$$

as $n \to \infty$, is the standard normal distribution.

It is easy to demonstrate the central limit theorem with a **computer simulation experiment**. Consider the lognormal distribution in Fig. 7-1. This distribution has parameters $\theta = 2$ (called the **location** parameter) and $\omega = 0.75$ (called the **scale** parameter), resulting in mean $\mu = 9.79$ and standard deviation $\sigma = 8.51$. Notice that this lognormal distribution does not look very much like the normal distribution; it is defined only for positive values of the random variable X and is skewed considerably to the right. We used computer software to draw 20 samples at random from this distribution, each of size $n = 10$. The data from this sampling experiment are shown in Table 7-1. The last row in this table is the average of each sample $\bar{x}$.

The first thing that we notice in looking at the values of $\bar{x}$ is that they are not all the same. This is a clear demonstration of the point made previously that any statistic is a random

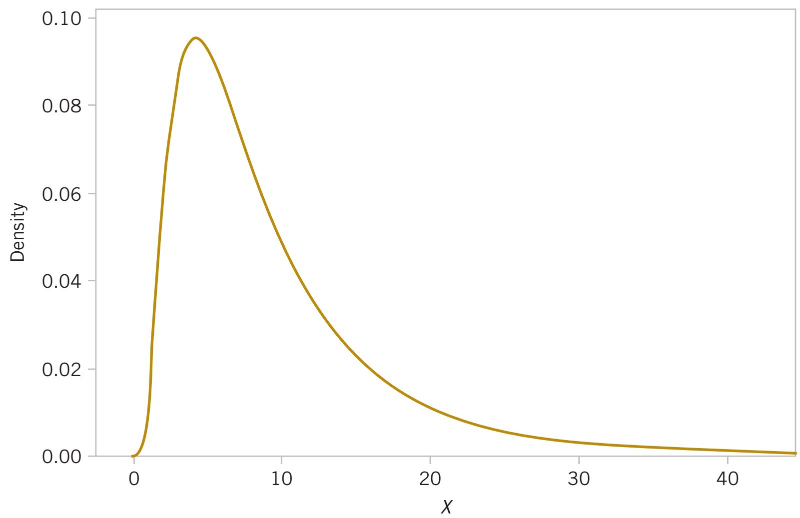

FIGURE 7-1 A lognormal distribution with $\theta = 2$ and $\omega = 0.75$.

variable. If we had calculated any sample statistic (s, the sample median, the upper or lower quartile, or a percentile), they would also have varied from sample to sample because they are random variables. Try it and see for yourself.

According to the central limit theorem, the distribution of the sample average $\bar{x}$ is normal. Figure 7-2 is a normal probability plot of the 20 sample averages $\bar{x}$ from Table 7-1. The observations scatter generally along a straight line, providing evidence that the distribution of the sample mean is normal even though the distribution of the population is very non-normal. This type of sampling experiment can be used to investigate the sampling distribution of any statistic.

The normal approximation for $\bar{X}$ depends on the sample size n. Figure 7-3(a) is the distribution obtained for throws of a single, six-sided true die. The probabilities are equal (1/6) for all the values obtained: 1, 2, 3, 4, 5, or 6. Figure 7-3(b) is the distribution of the average score obtained when tossing two dice, and Fig. 7-3(c), 7-3(d), and 7-3(e) show the distributions of average scores obtained when tossing 3, 5, and 10 dice, respectively. Notice that, although the population (one die) is relatively far from normal, the distribution of averages is approximated reasonably well by the normal distribution for sample sizes as small as five. (The dice throw distributions are discrete, but the normal is continuous.)

The central limit theorem is the underlying reason why many of the random variables encountered in engineering and science are normally distributed. The observed variable of the results from a series of underlying disturbances that act together to create a central limit effect.

TABLE • 7-1 Twenty samples of size $n = 10$ from the lognormal distribution in Figure 7-1.

	Sample									
Obs	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	3.9950	8.2220	4.1893	15.0907	12.8233	15.2285	5.6319	7.5504	2.1503	3.1390
2	7.8452	13.8194	2.6186	4.5107	3.1392	16.3821	3.3469	1.4393	46.3631	1.8314
3	1.8858	4.0513	8.7829	7.1955	7.1819	12.0456	8.1139	6.0995	2.4787	3.7612
4	16.3041	7.5223	2.5766	18.9189	4.2923	13.4837	13.6444	8.0837	19.7610	15.7647
5	9.7061	6.7623	4.4940	11.1338	3.1460	13.7345	9.3532	2.1988	3.8142	3.6519
6	7.6146	5.3355	10.8979	3.6718	21.1501	1.6469	4.9919	13.6334	2.8456	14.5579
7	6.2978	6.7051	6.0570	8.5411	3.9089	11.0555	6.2107	7.9361	11.4422	9.7823
8	19.3613	15.6610	10.9201	5.9469	8.5416	19.7158	11.3562	3.9083	12.8958	2.2788
9	7.2275	3.7706	38.3312	6.0463	10.1081	2.2129	11.2097	3.7184	28.2844	26.0186
10	16.2093	3.4991	6.6584	4.2594	6.1328	9.2619	4.1761	5.2093	10.0632	17.9411
$\bar{x}$	9.6447	7.5348	9.5526	8.5315	8.0424	11.4767	7.8035	5.9777	14.0098	9.8727
Obs	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
1	7.5528	8.4998	2.5299	2.3115	6.1115	3.9102	2.3593	9.6420	5.0707	6.8075
2	4.9644	3.9780	11.0097	18.8265	3.1343	11.0269	7.3140	37.4338	5.5860	8.7372
3	16.7181	6.2696	21.9326	7.9053	2.3187	12.0887	5.1996	3.6109	3.6879	19.2486
4	8.2167	8.1599	15.5126	7.4145	6.7088	8.3312	11.9890	11.0013	5.6657	5.3550
5	9.0399	15.9189	7.9941	22.9887	8.0867	2.7181	5.7980	4.4095	12.1895	16.9185
6	4.0417	2.8099	7.1098	1.4794	14.5747	8.6157	7.8752	7.5667	32.7319	8.2588
7	4.9550	40.1865	5.1538	8.1568	4.8331	14.4199	4.3802	33.0634	11.9011	4.8917
8	7.5029	10.1408	2.6880	1.5977	7.2705	5.8623	2.0234	6.4656	12.8903	3.3929
9	8.4102	6.4106	7.6495	7.2551	3.9539	16.4997	1.8237	8.1360	7.4377	15.2643
10	7.2316	11.5961	4.4851	23.0760	10.3469	9.9330	8.6515	1.6852	3.6678	2.9765
$\bar{x}$	7.8633	11.3970	8.6065	10.1011	6.7339	9.3406	5.7415	12.3014	10.0828	9.1851