Your model may not always fit the experimental situation perfectly, but you should try to choose a model that *best fits* the population relative frequency histogram. The better the model approximates reality, the better your inferences will be. Fortunately, many continuous random variables have mound-shaped frequency distributions, such as the data in Figure 6.1(d). The **normal probability distribution** provides a good model for describing this type of data.

6.2 THE NORMAL PROBABILITY DISTRIBUTION

Continuous probability distributions can assume a variety of shapes. However, a large number of random variables observed in nature possess a frequency distribution that is approximately mound-shaped or, as the statistician would say, is approximately a normal probability distribution. The formula that generates this distribution is shown next.

NORMAL PROBABILITY DISTRIBUTION

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \qquad -\infty < x < \infty$$

The symbols e and π are mathematical constants given approximately by 2.7183 and 3.1416, respectively; μ and σ ($\sigma > 0$) are parameters that represent the population mean and standard deviation, respectively.

The graph of a normal probability distribution with mean μ and standard deviation σ is shown in Figure 6.5. The mean μ locates the *center* of the distribution, and the distribution is *symmetric* about its mean μ. Since the total area under the normal probability distribution is equal to 1, the symmetry implies that the area to the right of μ is .5 and the area to the left of μ is also .5. The *shape* of the distribution is determined by σ, the population standard deviation. As you can see in Figure 6.6, large values of σ reduce the height of the curve and increase the spread; small values of σ increase the height of the curve and reduce the spread. Figure 6.6 shows three normal probability distributions with different means and standard deviations. Notice the differences in shape and location.

FIGURE 6.5
Normal probability distribution

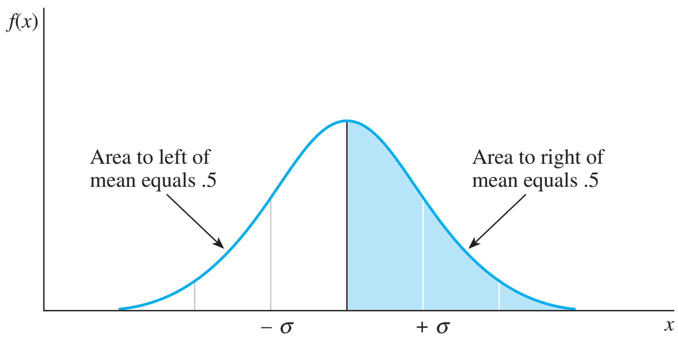
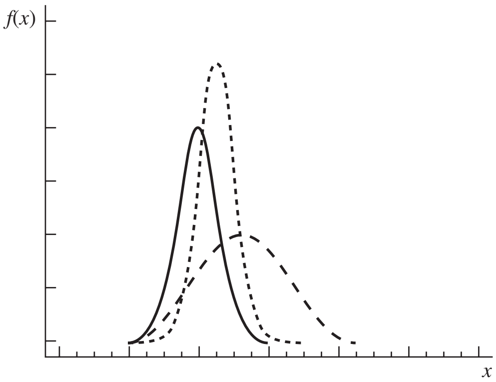


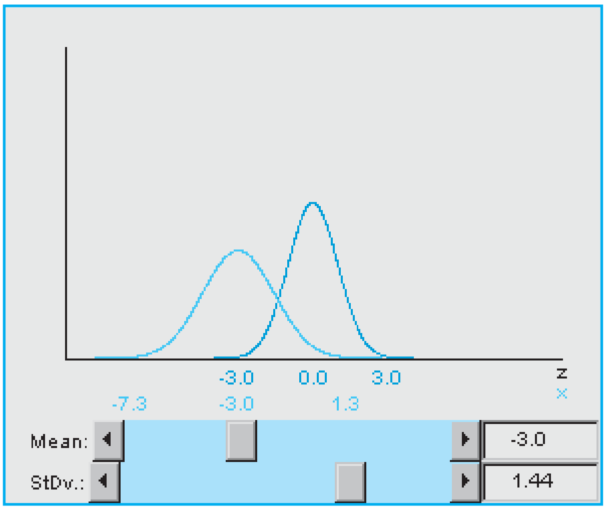
FIGURE 6.6
Normal probability distributions with differing values of μ and σ



MY APPLET

The Java applet called **Visualizing Normal Curves** gives a visual display of the normal distribution for values of μ between -10 and $+8$ and for values of σ between .5 and 1.8. The dark blue curve is the standard normal z with mean 0 and standard deviation 1. You can use this applet to compare its shape to the shape of other normal curves (the red curve on your monitor, light blue in Figure 6.7) by moving the sliders to change the mean and standard deviation. What happens when you change the mean? When you change the standard deviation?

FIGURE 6.7
Visualizing Normal Curves applet



You rarely find a variable with values that are infinitely small ($-\infty$) or infinitely large ($+\infty$). Even so, many *positive* random variables (such as heights, weights, and times) have distributions that are well approximated by a normal distribution. According to the Empirical Rule, almost all values of a normal random variable lie in the interval $\mu \pm 3\sigma$. As long as the values within three standard deviations of the mean are *positive,* the normal distribution provides a good model to describe the data.

6.3 TABULATED AREAS OF THE NORMAL PROBABILITY DISTRIBUTION

To find the probability that a normal random variable x lies in the interval from a to b, we need to find the area under the normal curve between the points a and b (see Figure 6.2). However (see Figure 6.6), there are an infinitely large number of normal distributions—one for each different mean and standard deviation. A separate table of areas for each of these curves is obviously impractical. Instead, we use a standardization procedure that allows us to use the same table for all normal distributions.

The Standard Normal Random Variable

A normal random variable x is **standardized** by expressing its value as the number of standard deviations (σ) it lies to the left or right of its mean μ. This is really just a change in the units of measure that we use, as if we were measuring in inches rather than in feet! The standardized normal random variable, z, is defined as

$$z = \frac{x - \mu}{\sigma}$$

MY TIP

Area under the z-curve equals 1.

or equivalently,

$$x = \mu + z\sigma$$

From the formula for z, we can draw these conclusions:

- When x is less than the mean μ, the value of z is negative.
- When x is greater than the mean μ, the value of z is positive.
- When $x = \mu$, the value of $z = 0$.

The probability distribution for z, shown in Figure 6.8, is called the **standardized normal distribution** because its mean is 0 and its standard deviation is 1. Values of z on the left side of the curve are negative, while values on the right side are positive. The area under the standard normal curve to the left of a specified value of z—say, z_0—is the probability $P(z \le z_0)$. This **cumulative area** is recorded in Table 3 of Appendix I and is shown as the shaded area in Figure 6.8. An abbreviated version of Table 3 is given in Table 6.1. Notice that the table contains both positive and negative values of z. The left-hand column of the table gives the value of z correct to the tenth place; the second decimal place for z, corresponding to hundredths, is given across the top row.

FIGURE 6.8
Standardized normal distribution

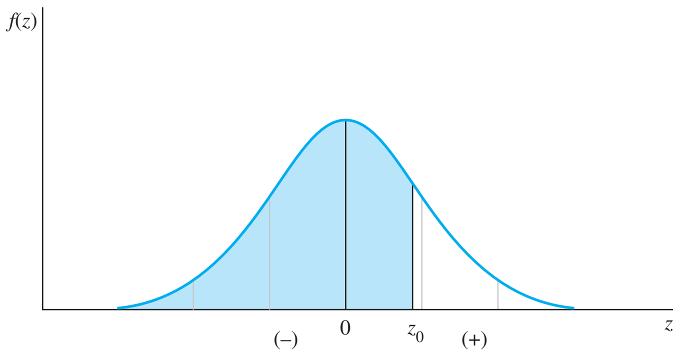


TABLE 6.1

Abbreviated Version of Table 3 in Appendix I

Table 3. Areas Under the Normal Curve

z	.00	.01	.02	.03	. . .	.09
−3.4	.0003	.0003	.0003	.0003		
−3.3	.0005	.0005	.0005	.0004		
−3.2	.0007	.0007	.0006	.0006		
−3.1	.0010	.0009	.0009	.0009		
−3.0	.0013	.0013	.0013	.0012	. . .	.0010
−2.9	.0019	.	.	.		
−2.8	.0026	.	.	.		
−2.7	.0035	.	.	.		
−2.6	.0047					
−2.5	.0062					
.	.					
.	.					
.	.					
−2.0	.0228					
.	.					
.	.					
.	.					

Table 3. Areas Under the Normal Curve (*continued*)

z	.00	.01	.02	.03	.04	. . .	.09
0.0	.5000	.5040	.5080	.5120	.5160		
0.1	.5398	.5438	.5478	.5517	.5557		
0.2	.5793	.5832	.5871	.5910	.5948		
0.3	.6179	.6217	.6255	.6293	.6331		
0.4	.6554	.6591	.6628	.6664	.6700	. . .	.6879
0.5	.6915	.	.	.			
0.6	.7257	.	.	.			
0.7	.7580	.	.	.			
0.8	.7881						
0.9	.8159						
.	.						
.	.						
.	.						
2.0	.9772						

EXAMPLE 6.3

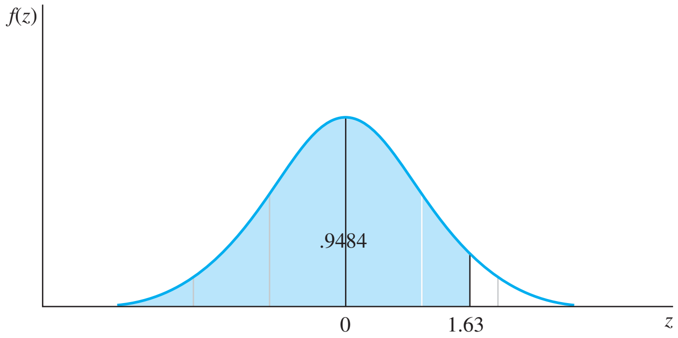
Find $P(z \leq 1.63)$. This probability corresponds to the area to the left of a point $z = 1.63$ standard deviations to the right of the mean (see Figure 6.9).

MY TIP

$P(z \leq 1.63) = P(z < 1.63)$

Solution The area is shaded in Figure 6.9. Since Table 3 in Appendix I gives areas under the normal curve to the left of a specified value of z, you simply need to find the tabled value for $z = 1.63$. Proceed down the left-hand column of the table to $z = 1.6$ and across the top of the table to the column marked .03. The intersection of this row and column combination gives the area .9484, which is $P(z \leq 1.63)$.

FIGURE 6.9
Area under the standard normal curve for Example 6.3

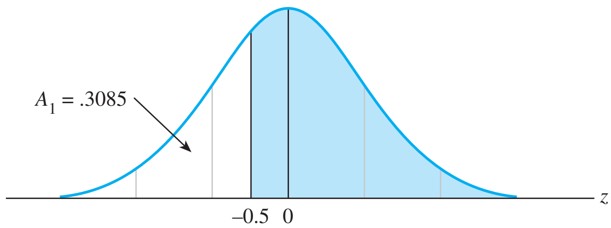


Areas to the left of $z = 0$ are found using negative values of z.

EXAMPLE 6.4

Find $P(z \geq -.5)$. This probability corresponds to the area to the *right* of a point $z = -.5$ standard deviation to the left of the mean (see Figure 6.10).

FIGURE 6.10
Area under the standard normal curve for Example 6.4



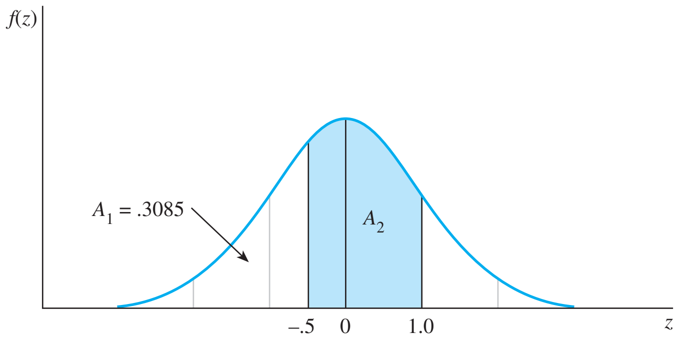
Solution The area given in Table 3 is the area to the left of a specified value of z. Indexing $z = -.5$ in Table 3, we can find the area A_1 to the *left* of $-.5$ to be .3085. Since the area under the curve is 1, we find

$$P(z \geq -.5) = 1 - A_1 = 1 - .3085 = .6915.$$

EXAMPLE 6.5

Find $P(-.5 \leq z \leq 1.0)$. This probability is the area between $z = -.5$ and $z = 1.0$, as shown in Figure 6.11.

FIGURE 6.11
Area under the standard normal curve for Example 6.5



Solution The area required is the shaded area A_2 in Figure 6.11. From Table 3 in Appendix I, you can find the area to the left of $z = -.5$ ($A_1 = .3085$) and the area to the left of $z = 1.0$ ($A_1 + A_2 = .8413$). To find the area marked A_2, we subtract the two entries:

$$A_2 = (A_1 + A_2) - A_1 = .8413 - .3085 = .5328$$

That is, $P(-.5 \le z \le 1.0) = .5328$.

MY PERSONAL TRAINER

How Do I Use Table 3 to Calculate Probabilities under the Standard Normal Curve?

- To calculate the area to the left of a z-value, find the area directly from Table 3.
- To calculate the area to the right of a z-value, find the area in Table 3, and subtract from 1.
- To calculate the area between two values of z, find the two areas in Table 3, and subtract one area from the other.

Exercise Reps

Consider a standard random variable with mean $\mu = 0$ and standard deviation $\sigma = 1$. Use Table 3 and fill in the probabilities below. The third probability is calculated for you.

The Interval	Write the Probability	Rewrite the Probability (if needed)	Find the Probability
Less than 1.5	$P(z <$ _____ $)$		
Greater than 2	$P(z >$ _____ $)$		
Greater than 2.33	$P(z > \underline{2.33})$	$1 - P(x \le 2.33)$	$1 - .9901 = .0099$
Between -1.96 and 1.96	$P($ _____ $< z <$ _____ $)$		
Between -1.24 and 2.37	$P($ _____ $< z <$ _____ $)$		
Less than or equal to -1	$P(z \le$ _____ $)$		

Progress Report

- Still having trouble? Try again using the Exercise Reps at the end of this section.
- Mastered the z-table? You can skip the Exercises Reps at the end of this section!

Answers are located on the perforated card at the back of this book.

EXAMPLE 6.6

Find the probability that a normally distributed random variable will fall within these ranges:

1. One standard deviation of its mean
2. Two standard deviations of its mean