Historical Note

The methods of analysis of variance were developed by R. A. Fisher in the early 1920s.

Introduction

The F test, used to compare two variances as shown in Chapter 9, can also be used to compare three or more means. This technique is called *analysis of variance,* or *ANOVA*. It is used to test claims involving three or more means. (*Note:* The F test can also be used to test the equality of two means. But since it is equivalent to the t test in this case, the t test is usually used instead of the F test when there are only two means.) For example, suppose a researcher wishes to see whether the means of the time it takes three groups of students to solve a computer problem using HTML, Java, and PHP are different. The researcher will use the ANOVA technique for this test. The z and t tests should not be used when three or more means are compared, for reasons given later in this chapter.

For three groups, the F test can show only whether a difference exists among the three means. It cannot reveal where the difference lies—that is, between $\overline{X}_1$ and $\overline{X}_2$, or $\overline{X}_1$ and $\overline{X}_3$, or $\overline{X}_2$ and $\overline{X}_3$. If the F test indicates that there is a difference among the means, other statistical tests are used to find where the difference exists. The most commonly used tests are the Scheffé test and the Tukey test, which are also explained in this chapter.

The analysis of variance that is used to compare three or more means is called a *one-way analysis of variance* since it contains only one variable. In the previous example, the variable is the type of computer language used. The analysis of variance can be extended to studies involving two variables, such as type of computer language used and mathematical background of the students. These studies involve a *two-way analysis of variance.* Section 12–3 explains the two-way analysis of variance.

12–1 One-Way Analysis of Variance

OBJECTIVE 1

Use the one-way ANOVA technique to determine if there is a significant difference among three or more means.

When an F test is used to test a hypothesis concerning the means of three or more populations, the technique is called **analysis of variance** (commonly abbreviated as **ANOVA**).

The **one-way analysis of variance** test is used to test the equality of three or more means using sample variances.

The procedure used in this section is called the **one-way analysis of variance** because there is only one independent variable that distinguishes between the different populations in the study. The independent variable is also called a *factor*.

At first glance, you might think that to compare the means of three or more samples, you can use the t test, comparing two means at a time. But there are several reasons why the t test should not be done.

First, when you are comparing two means at a time, the rest of the means under study are ignored. With the F test, all the means are compared simultaneously. Second, when you are comparing two means at a time and making all pairwise comparisons, the probability of rejecting the null hypothesis when it is true is increased, since the more t tests that are conducted, the greater is the likelihood of getting significant differences by chance alone. Third, the more means there are to compare, the more t tests are needed. For example, for the comparison of 3 means two at a time, 3 t tests are required. For the comparison of 5 means two at a time, 10 tests are required. And for the comparison of 10 means two at a time, 45 tests are required.

As the number of populations to be compared increases, the probability of making a type I error using multiple t tests for a given level of significance α also increases. To address this problem, the technique of analysis of variance is used. This technique involves a comparison of two estimates of the same population variance.

Recall that the characteristics of the F distribution are as follows:

1. The values of F cannot be negative, because variances are always positive or zero.
2. The distribution is positively skewed.

3. The mean value of F is approximately equal to 1.
4. The F distribution is a family of curves based on the degrees of freedom of the variance of the numerator and the degrees of freedom of the variance of the denominator.

Even though you are comparing three or more means in this use of the F test, *variances* are used in the test instead of means.

With the F test, two different estimates of the population variance are made. The first estimate is called the **between-group variance,** and it involves finding the variance of the means. The second estimate, the **within-group variance,** is made by computing the variance using all the data and is not affected by differences in the means. If there is no difference in the means, the between-group variance estimate will be approximately equal to the within-group variance estimate, and the F test value will be approximately equal to one. The null hypothesis will not be rejected. However, when the means differ significantly, the between-group variance will be much larger than the within-group variance; the F test value will be significantly greater than one; and the null hypothesis will be rejected. Since variances are compared, this procedure is called *analysis of variance* (ANOVA).

The formula for the F test is

$$F = \frac{\text{variance between groups}}{\text{variance within groups}}$$

The variance between groups measures the differences in the means that result from the different treatments given to each group. To calculate this value, it is necessary to find the *grand mean* $\overline{X}_{\text{GM}}$, which is the mean of all the values in all of the samples. The formula for the grand mean is

$$\overline{X}_{\text{GM}} = \frac{\Sigma X}{N}$$

This value is used to find the between-group variance s_B^2. This is the variance among the means using the sample sizes as weights.

The formula for the between-group variance, denoted by s_B^2, is

$$s_B^2 = \frac{\Sigma n_i(\overline{X}_i - \overline{X}_{\text{GM}})}{k - 1}$$

where k = number of groups
n_i = sample size
$\overline{X}_i$ = sample mean

This formula can be written out as

$$s_B^2 = \frac{n_1(\overline{X}_1 - \overline{X}_{\text{GM}})^2 + n_2(\overline{X}_2 - \overline{X}_{\text{GM}})^2 + \cdots + n_k(\overline{X}_k - \overline{X}_{\text{GM}})^2}{k - 1}$$

Next find the within group variance, denoted by s_W^2. The formula finds the overall variance by calculating a weighted average of the individual variances. It does not involve using differences of means. The formula for the within-group variance is

$$s_W^2 = \frac{\Sigma(n_i - 1)s_i^2}{\Sigma(n_i - 1)}$$

where n_i = sample size
s_i^2 = variance of sample

This formula can be written out as

$$s_W^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 + \cdots + (n_k - 1)s_k^2}{(n_1 - 1) + (n_2 - 2) + \cdots + (n_k - 1)}$$

Finally, the F test value is computed. The formula can now be written using the symbols s_B^2 and s_W^2.

The formula for the F test for one-way analysis of variance is

$$F = \frac{s_B^2}{s_W^2}$$

where s_B^2 = between-group variance
s_W^2 = within-group variance

As stated previously, a significant test value means that there is a high probability that this difference in means is not due to chance, but it does not indicate where the difference lies.

The degrees of freedom for this F test are d.f.N. $= k - 1$, where k is the number of groups, and d.f.D. $= N - k$, where N is the sum of the sample sizes of the groups $N = n_1 + n_2 + \cdots + n_k$. The sample sizes need not be equal. The F test to compare means is always right-tailed.

The results of the one-way analysis of variance can be summarized by placing them in an **ANOVA summary table.** The numerator of the fraction of the s_B^2 term is called the **sum of squares between groups,** denoted by SS_B. The numerator of the s_W^2 term is called the **sum of squares within groups,** denoted by SS_W. This statistic is also called the *sum of squares for the error.* SS_B is divided by d.f.N. to obtain the between-group variance. SS_W is divided by $N - k$ to obtain the within-group or error variance. These two variances are sometimes called **mean squares,** denoted by MS_B and MS_W. These terms are used to summarize the analysis of variance and are placed in a summary table, as shown in Table 12–1.

TABLE 12–1 Analysis of Variance Summary Table

Source	Sum of squares	d.f.	Mean square	*F*
Between	SS_B	$k - 1$	MS_B	
Within (error)	SS_W	$N - k$	MS_W	
Total				

Unusual Stat

The *Journal of the American College of Nutrition* reports that a study found no correlation between body weight and the percentage of calories eaten after 5:00 P.M.

In the table,

$$SS_B = \text{sum of squares between groups}$$
$$SS_W = \text{sum of squares within groups}$$
$$k = \text{number of groups}$$
$$N = n_1 + n_2 + \cdots + n_k = \text{sum of sample sizes for groups}$$
$$MS_B = \frac{SS_B}{k - 1}$$
$$MS_W = \frac{SS_W}{N - k}$$
$$F = \frac{MS_B}{MS_W}$$

To use the F test to compare two or more means, the following assumptions must be met.

Assumptions for the *F* Test for Comparing Three or More Means

1. The populations from which the samples were obtained must be normally or approximately normally distributed.
2. The samples must be independent of one another.
3. The variances of the populations must be equal.
4. The samples must be simple random samples, one from each of the populations.

In this book, the assumptions will be stated in the exercises; however, when encountering statistics in other situations, you must check to see that these assumptions have been met before proceeding.

The steps for computing the F test value for the ANOVA are summarized in this Procedure Table.

Procedure Table

Finding the *F* Test Value for the Analysis of Variance

Step 1 Find the mean and variance of each sample.

$$(\overline{X}_1, s_1^2), (\overline{X}_2, s_2^2), \ldots, (\overline{X}_k, s_k^2)$$

Step 2 Find the grand mean.

$$\overline{X}_{\text{GM}} = \frac{\Sigma X}{N}$$

Step 3 Find the between-group variance.

$$s_B^2 = \frac{\Sigma n_i(\overline{X}_i - \overline{X}_{\text{GM}})^2}{k - 1}$$

Step 4 Find the within-group variance.

$$s_W^2 = \frac{\Sigma(n_i - 1)s_i^2}{\Sigma(n_i - 1)}$$

Step 5 Find the F test value.

$$F = \frac{s_B^2}{s_W^2}$$

The degrees of freedom are

$$\text{d.f.N.} = k - 1$$

where k is the number of groups, and

$$\text{d.f.D.} = N - k$$

where N is the sum of the sample sizes of the groups

$$N = n_1 + n_2 + \cdots + n_k$$

The one-way analysis of variance follows the regular five-step hypothesis-testing procedure.

Step 1 State the hypotheses.
Step 2 Find the critical values.
Step 3 Compute the test value.
Step 4 Make the decision.
Step 5 Summarize the results.

Examples 12–1 and 12–2 illustrate the computational procedure for the ANOVA technique for comparing three or more means, and the steps are summarized in the Procedure Table.

EXAMPLE 12–1 Miles per Gallon

A researcher wishes to see if there is a difference in the fuel economy for city driving for three different types of automobiles: small automobiles, sedans, and luxury automobiles. He randomly samples four small automobiles, five sedans, and three luxury automobiles. The miles per gallon for each is shown. At $\alpha = 0.05$, test the claim that there is no difference among the means. The data are shown.

Small	Sedans	Luxury
36	43	29
44	35	25
34	30	24
35	29	
	40	

Source: U.S. Environmental Protection Agency.

Step 1 State the hypotheses and identify the claim.

H_0: $\mu_1 = \mu_2 = \mu_3$ (claim)
H_1: At least one mean is different from the others

Step 2 Find the critical value.

$$N = 12 \qquad k = 3$$
$$\text{d.f.N.} = k - 1 = 3 - 1 = 2$$
$$\text{d.f.D.} = N - k = 12 - 3 = 9$$

The critical value from Table H in Appendix A with $\alpha = 0.05$ is 4.26.

Step 3 Compute the test value.

a. Find the mean and variance for each sample. (Use the formulas in Chapter 3.)

For the small cars: $\overline{X} = 37.25$ $\quad s^2 = 20.917$
For the sedans: $\overline{X} = 35.4$ $\quad s^2 = 37.3$
For the luxury cars: $\overline{X} = 26$ $\quad s^2 = 7$

b. Find the grand mean.

$$\overline{X}_{\text{GM}} = \frac{\Sigma X}{N} = \frac{36 + 44 + 34 + \cdots + 24}{12} = \frac{404}{12} = 33.667$$

c. Find the between-group variance.

$$s_B^2 = \frac{\Sigma n(\overline{X}_i - \overline{X}_{\text{GM}})^2}{k - 1}$$
$$= \frac{4(37.25 - 33.667)^2 + 5(35.4 - 33.667)^2 + 3(26 - 33.667)^2}{3 - 1}$$
$$= \frac{242.717}{2} = 121.359$$

d. Find the within-group variance.

$$s_W^2 = \frac{\Sigma(n_i - 1)s_i^2}{\Sigma(n_i - 1)} = \frac{(4 - 1)(20.917) + (5 - 1)(37.3) + (3 - 1)7}{(4 - 1) + (5 - 1) + (3 - 1)}$$
$$= \frac{225.951}{9} = 25.106$$

e. Find the F test value.

$$F = \frac{s_B^2}{s_W^2} = \frac{121.359}{25.106} = 4.83$$

Step 4 Make the decision. The test value $4.83 > 4.26$, so the decision is to reject the null hypothesis. See Figure 12–1.

FIGURE 12–1 Critical Value and Test Value for Example 12–1

Step 5 Summarize the results. There is enough evidence to conclude that at least one mean is different from the others.

The **ANOVA summary table** is shown in Table 12–2.

TABLE 12–2 Analysis of Variance Summary Table for Example 12–1

Source	Sum of squares	d.f.	Mean square	F
Between	242.717	2	121.359	4.83
Within (error)	225.954	9	25.106	
Total	468.671	11		

The P-values for ANOVA are found by using the same procedure shown in Section 9–5. For Example 12–1, the F test value is 4.83. In Table H with d.f.N. $= 2$ and d.f.D. $= 9$, the F test value falls between $\alpha = 0.025$ with an F value of 5.71 and $\alpha = 0.05$ with an F value of 4.26. Hence, $0.025 < P\text{-value} < 0.05$. In this case, the null hypothesis is rejected at $\alpha = 0.05$ since the $P\text{-value} < 0.05$. The TI-84 P-value is 0.0375.

EXAMPLE 12–2 Employees at Toll Road Interchanges

A state employee wishes to see if there is a significant difference in the number of employees at the interchanges of three state toll roads. The data are shown. At $\alpha = 0.05$, can it be concluded that there is a significant difference in the average number of employees at each interchange?

Pennsylvania Turnpike	Greensburg Bypass/ Mon-Fayette Expressway	Beaver Valley Expressway
7	10	1
14	1	12
32	1	1
19	0	9
10	11	1
11	1	11

Source: Pennsylvania Turnpike Commission.

SOLUTION

Step 1 State the hypotheses and identify the claim.

H_0: $\mu_1 = \mu_2 = \mu_3$

H_1: At least one mean is different from the others (claim).

Step 2 Find the critical value. Since $k = 3$, $N = 18$, and $\alpha = 0.05$,

d.f.N. $= k - 1 = 3 - 1 = 2$

d.f.D. $= N - k = 18 - 3 = 15$

The critical value is 3.68.

Step 3 Compute the test value.

a. Find the mean and variance of each sample. The mean and variance for each sample are

Turnpike	$\bar{X}_1 = 15.5$	$s_1^2 = 81.9$
Mon-Fayette	$\bar{X}_2 = 4.0$	$s_2^2 = 25.6$
Beaver Valley	$\bar{X}_3 = 5.8$	$s_3^2 = 29.0$

b. Find the grand mean.

$$\bar{X}_{GM} = \frac{\Sigma X}{N} = \frac{7 + 14 + 32 + \cdots + 11}{18} = \frac{152}{18} = 8.44$$

c. Find the between-group variance.

$$s_B^2 = \frac{\Sigma n_i(\bar{X}_i - \bar{X}_{GM})^2}{k - 1}$$

$$= \frac{6(15.5 - 8.44)^2 + 6(4 - 8.44)^2 + 6(5.8 - 8.44)^2}{3 - 1}$$

$$= \frac{459.16}{2} = 229.58$$

d. Find the within-group variance.

$$s_W^2 = \frac{\Sigma(n_i - 1)s_i^2}{\Sigma(n_i - 1)}$$

$$= \frac{(6 - 1)(81.9) + (6 - 1)(25.6) + (6 - 1)(29.0)}{(6 - 1) + (6 - 1) + (6 - 1)}$$

$$= \frac{682.50}{15} = 45.50$$

e. Find the F test value.

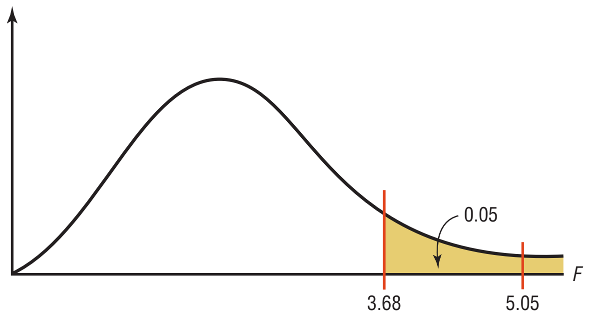
$$F = \frac{s_B^2}{s_W^2} = \frac{229.58}{45.50} = 5.05$$

Step 4 Make the decision. Since $5.05 > 3.68$, the decision is to reject the null hypothesis. See Figure 12–2.

Interesting Facts

The weight of 1 cubic foot of wet snow is about 10 pounds while the weight of 1 cubic foot of dry snow is about 3 pounds.

FIGURE 12–2 Critical Value and Test Value for Example 12–2



Step 5 Summarize the results. There is enough evidence to support the claim that there is a difference among the means. The ANOVA summary table for this example is shown in Table 12–3.

TABLE 12–3 Analysis of Variance Summary Table for Example 12–2

Source	Sum of squares	d.f.	Mean square	*F*
Between	459.16	2	229.58	5.05
Within	682.50	15	45.50	
Total	1141.66	17		

The *P*-values for ANOVA are found by using the procedure shown in Section 9–2. For Example 12–2, find the two α values in the tables for the F distribution (Table H), using d.f.N. $= 2$ and d.f.D. $= 15$, where $F = 5.05$ falls between. In this case, 5.05 falls between 4.77 and 6.36, corresponding, respectively, to $\alpha = 0.025$ and $\alpha = 0.01$; hence, $0.01 < P\text{-value} < 0.025$. Since the *P*-value is between 0.01 and 0.025 and since $P\text{-value} < 0.05$ (the originally chosen value for α), the decision is to reject the null hypothesis. (The *P*-value obtained from a calculator is 0.021.)

When the null hypothesis is rejected in ANOVA, it only means that at least one mean is different from the others. To locate the difference or differences among the means, it is necessary to use other tests such as the Tukey or the Scheffé test.

Applying the Concepts 12–1

Colors That Make You Smarter

The following set of data values was obtained from a study of people's perceptions on whether the color of a person's clothing is related to how intelligent the person looks. The subjects rated the person's intelligence on a scale of 1 to 10. Randomly selected group 1 subjects were shown people with clothing in shades of blue and gray. Randomly selected group 2 subjects were shown people with clothing in shades of brown and yellow. Randomly selected group 3 subjects were shown people with clothing in shades of pink and orange. The results follow.

Group 1	Group 2	Group 3
8	7	4
7	8	9
7	7	6
7	7	7
8	5	9
8	8	8
6	5	5
8	8	8
8	7	7
7	6	5
7	6	4
8	6	5
8	6	4

1. Use ANOVA to test for any significant differences between the means.
2. What is the purpose of this study?
3. Explain why separate *t* tests are not accepted in this situation.

See page 686 for the answers.