Homework 2

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1. **(10 points) Translate the following statements into English, where C(x) is "x is a comedian" and F(x) is "x is funny" and the domain consist of all people.**

b. $\forall x(C(x) \rightarrow F(x))$: If everyone is a comedian, then everyone is funny.

c. $\forall x(C(x) \land F(x))$: Everyone is a comedian and everyone is funny.

d. $\exists x(C(x) \rightarrow F(x))$: If some people are a comedian, then they are funny.

e. $\exists x(C(x) \land F(x))$: Some people are a comedian and some people are funny.

2. **(10 points) Let :-**

- **P (x) be the statement "x can speak Russian"**
- **Q(x) be the statement "x knows the computer language C++"**
- **the domain is all students at ISU.**

Express the following statements:

a. There is a student at your school who can speak Russian and who knows C++.

Answer: $\exists x(P(x) \land Q(x))$

b. There is a student at your school who can speak Russian but who doesn't know C++.

Answer: $\exists x(P(x) \land \neg Q(x))$

c. Every student at your school either can speak Russian or knows C++.

Answer: $\forall x(Q(x) \lor P(x))$

d. No student at your school can speak Russian or knows C++

Answer: $\forall x(\neg P(x) \land \neg Q(x))$

3. **(10 points) Determine the truth value of the statement ∃x∀yP (x ≤ y^2) if the domain of the variables consists of**

a. The positive reals.

The statement ∃x∀yP (x ≤ y^2) is true for positive reals. For any x positive real number, there exists a positive real number y such that x ≤ y^2.

b. The integers.

The statement ∃x∀yP (x ≤ y^2) is true for the integers.

- If we take (x = 0), then y can be any integer: positive, zero, or negative integer, and the condition x ≤ y^2 will still be satisfied.
- Therefore, for any integer x, there exist an integer y such that x ≤ y^2.

c. The non-zero real numbers.

The statement ∃x∀yP (x ≤ y^2) is true for the non-zero real numbers.

- If we take (x = -1/2), then then y^2 can be any non-negative zero real number greater than or equal to x.
- Therefore, for any non-zero real numbers, there exist an integer y such that x≤ y^2.

4. **(10 points) Determine whether each of these are valid arguments. If an argument is correct, what rule of inference is being used? If not, what is the logical error? You need to do this by writing down appropriate propositions and proceeding like we did in class**

a. If x is a positive real number, then x^2 is a positive real number. Therefore, if a^2 is positive, where a is a real number, then a is a positive real number.

Answer:

- This statement is not a valid argument.
- This is a fallacy that resembles the modus ponens.
- The reason why the statement is not a valid argument is because the exponent 2 of any real number, positive and negative, will result in a positive real number.

- Hence, it is wrong to conclude if a^2, then a is a positive real number as a could also be a negative real number.

b. If x^2≠0, where x is a real number, then x≠0. Let a be a real number with a^2 ≠0. Then a≠0.

Answer:

- This statement is a valid argument.
- Based on the rule of inference, this is a modus ponens.
- Given that any number to the power of 2 is not equal to zero, then it is impossible that the number is zero.
- Hence, it is true to conclude that if a^2≠0, then a≠0.

5. **(10 points) Use the rules of inference to deduce the conclusion from the premises.**

a.

$$
\begin{array}{l}
p \lor q \\
q \to r \\
p \land s \to t \\
\neg r \\
\neg q \to u \land s \\
\hline
\therefore\ t.
\end{array}
$$

- For q →r, since ¬r, by Modus tollens, ¬q
- For p∨q, since ¬q, by Disjunctive syllogism, p
- For ¬q → u ∧ s, by Modus ponens, u ∧ s is true. Therefore both u and s must be true.
- For p ∧ s → t, since p ∧ s is true by Conjunction, t.
- Hence, the conclusion that **t is valid**.

b.

$$
\begin{aligned}
&p \to q \\
&r \lor s \\
&\neg s \to \neg t \\
&\neg q \lor s \\
&\neg s \\
&\neg p \land r \to u \\
&w \lor t \\
&\text{-------------------------} \\
&\therefore \; u \land w.
\end{aligned}
$$

- For r ∨ s, since ¬s, by Disjunctive syllogism, r
- For ¬s → ¬t, by Modus ponens, ¬t
- For ¬q ∨ s, since ¬s, by Disjunctive syllogism, ¬q
- For p → q, since ¬q, by Modus tollens, ¬p
- For ¬p ∧ r → u, since ¬p ∧ r evaluates to false, then by Modus ponens, u
- For w ∨ t, since ¬t ,by Disjunctive syllogism, w
- Hence, the conclusion that **u ∧ w is valid.**

6. **(10 points) Recall that a number is called rational if it can be written as the ratio of two integers. Numbers that are not rational are called irrational. Prove by contraposition the following statement:**

"If r is irrational, then $\sqrt{r}$ is irrational."

- Contrapositive statement: "If $\sqrt{r}$ is rational, then r is rational."
- Assume that $\sqrt{r}$ is rational, This means that r can expressed as the ratio of two integers, a and b and b≠0, $\sqrt{r}$ = (a/b).

- $r = (a/b)^2$
- Now r is expressed as the ratio a^2 / b^2.
- Since r is expressed as the ratio of two integers, r is rational.
- As a result. the original statement “If r is irrational, then $\sqrt{r}$ is irrational.” is proven by contraposition.

7. **(10 points) An integer n is called *frumpy* if $n^2 + 2n$ is an odd number. Prove that all *frumpy* numbers are themselves odd numbers. (Clearly state your method of proof in the beginning.**

Proof by Contradiction:

- Assume that there exist a frumpy number n that is not odd.
- This means that n is even, therefore $n = 2k$ for some integer k.
- Hence, $n^2 + 2n$ will be $(2k)^2 + 2(2k) = 4k^2 + 4k = 4k(k+1)$.
- Since k and k+1 is a consecutive integer, then one of them must be an odd number.
- The product of an even number ($4k$) and an odd number ($k+1$) will always be even.
- This contradicts our initial assumption that n is a $frumpy$ number as frumpy numbers are defined to be odd.
- Therefore, our assumption that there exist a frumpy number n that is not odd must be false, and we conclude that all frumpy numbers are odd.

8. **(10 points) Given n arbitrary real numbers $a_1, a_2, ..., a_n$, prove that at least one of these numbers is greater than or equal to their average. (Clearly state your method of proof in the beginning).**

Proof by Contradiction:

- Given n arbitrary real numbers $a_1, a_2, ..., a_n$. Assume that none of these numbers is greater than or equal to their average.

- This means that each number must be less than the average.
- $A = (a_1 + a_2 + ... + a_n) \div n$
- We assume that for each i, a_i, is less than the average A; $a_i < A$ for $i = 1, 2, 3, \ldots, n$.
- Summing up the inequalities for all i ;$a_1 + a_2 + ... + a_n < nA$
- However, this contradicts the definition of the average. Since the sum of the a_i is equal to n times A, there must be at least one value a_i that is greater than or equal to the average, A.
- Therefore, at least one of the arbitrary real numbers $a_1, a_2, ..., a_n$ must be greater or equal to their average.

9. **(10 points) Prove that if n is a positive integer, then n is an odd number if and only if 5n + 6 is an odd number.**

p = n is an odd number

q = 5n + 6 is an odd number

<u>Proof by Equivalence:</u>

a. **p → q**

- Assume n is an odd number, so $n = 2k + 1$ for some positive integer k.
- $5n + 6 = 5(2k + 1) + 6 = 10k + 11 = 2(5k + 5) + 1$. The final equation resembles the form of $n = 2k + 1$.
- Therefore, $5n + 6$ results in an odd positive integer.
- Therefore, p → q.

b. **q → p**

- Assume that $5n + 6$ is an odd number, so $5n + 6 = 20k + 1$.
- $5n = 20k - 5$. Therefore, $n = 4k - 1$.
- $n = 4k - 1$ will result in an odd positive integer.

- Therefore, q → p.

Hence, it is proven that n is an odd number if and only if 5n+6 is an odd number given that n is a positive integer.

10. **(10 points) We call any group of people who have all shook hands with each other at some point in the past a *clique*. We call any group of people who have never met each other a *cabal*.**

Prove using cases that any given collection of six (6) CPRE 310 students includes either a clique of three (3) students, or a cabal of three (3) students

<u>Proof by contradiction:</u>

- Assume that there is a group of six CPRE310 students where no three students form a clique and no three students form a cabal.

Case 1: No clique of three students

- Consider any student in the group and assume that they are not part of a clique of three.
- Therefore, this student must be a part of the a cabal of three.

Case 2: No cabal of three

- Consider any student in the group and assume that they are not part of a cabal of three.
- Therefore, this student must be a part of the clique of three.

In either case, we found a group of three students if a clique and a cabal. Thus, the assumption that there are no group of three of a cabal and a clique is false. Therefore, it is proven that any given collection of six CPRE310 students includes either a cliques of three students or a cabal of three students.

11. **(10 points) Prove that there are no solutions in integers x and y to the equation** $2x^2 + 5y^2 = 14$

Proof by contradiction:

- Given that $2x^2 + 5y^2 = 14$. Assume that there exists at least a pair of integers x and y to the equation.
- $5y^2 = 14 - 2x^2$
- $5y^2 = 2(7 - x^2)$
- $5y^2/2 = (x - \sqrt{7})(x + \sqrt{7})$
- Based on the above statement, x is equal to $\sqrt{7} or - \sqrt{7}$, which are not integers.
- Since x itself is not integer, the product $(x - \sqrt{7})(x + \sqrt{7})$ cannot be integer either, contradicting the assumption that y can be an integer.
- This contradiction implies that the assumption in integers x and y of the existence of a pair (x, y) is false.
- Therefore, there are no solutions in integers x and y to the equation $2x^2 + 5y^2 = 14$.