Propositional Logic

A proposition is a declarative sentence that is either T (1)/F (0)

- Negation ¬
- Conjunction ∧
- Disjunction ∨
- Implication, if-then →

T for everything else except for $T \rightarrow F = F$

p	q	p→q	¬p ∨ q
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Different Ways of Expressing $p \rightarrow q$

if p, then q	p implies q
if p, q	p only if q
q unless $\neg p$	q when p
q if p	
q whenever p	p is sufficient for q
q follows from p	q is necessary for p

a necessary condition for p is q

a sufficient condition for q is p

- Biconditional, if-and-only-if ↔

T when both $p \leftrightarrow q$ has the same value (T/F)

Tautology: a proposition that is always true
Contradiction: a proposition that is always false

- **Contrapositive** of $p \rightarrow q$ is for $\neg q \rightarrow \neg p$
  $p \rightarrow q \equiv \neg q \rightarrow \neg p$
- **Converse** of $p \rightarrow q$ is for $q \rightarrow p$
  $p \rightarrow q$ is NOT LOGICALLY EQUIVALENT to $q \rightarrow p$
- **Inverse** of $p \rightarrow q$ is for $\neg p \rightarrow \neg q$
  $p \rightarrow q$ is NOT LOGICALLY EQUIVALENT to $\neg p \rightarrow \neg q$

De Morgan's Law

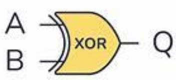
$i. \neg(p \wedge q) \equiv \neg p \vee \neg q$ $\qquad ii. \neg(p \vee q) \equiv \neg p \wedge \neg q$

Predicates & Quantifiers

- Predicate: A statement whose truth value depends on one or more variables.
- **Universal Quantifier, ∀:** "For all x, P(x) is true" → $\forall x, P(x)$
- **Existential Quantifier**, ∃: "There exist some x, where P(x) is true" → $\exists x, P(x)$
- **Domain of discourse:** Object/Category/Item being considered

Proof

1. Direct proof, $p \rightarrow q$
   If p = T, then q = T
2. Contrapositive, $\neg q \rightarrow \neg p$
3. Vacuous proof, $p \rightarrow q$
   If p = F, then q can be T or F to be T.
4. Trivial proof, $p \rightarrow q$
   If q = T, then p can be T or F to be T.
5. Proof by Equivalence, $p \leftrightarrow q$
   Show that if $p \rightarrow q$ & $q \rightarrow p$



A	B	Q
0	0	0
0	1	1
1	0	1
1	1	0

6. Proof by Contradiction: Negate the current statement and assume that the negated statement is T until proven F.
7. Proof by Counterexample: Show that at least in one condition, the statement is F.
8. Constructive proof: For proving $\exists x, P(x)$. Proof P(x) = T for 1 element in the domain discourse.
9. Proof by Cases: Identify all possible sub-cases and prove.

- For proving an "if-then-", consider using direct proofs, proofs by contraposition, or contradiction.
- For proving an "A if and only if B", think of proving both "A implies B" and "B implies A".
- For proving "There exists - such that -", think of a constructive proof.
- For disproving a statement, consider showing a counterexample.
- Use rules of inference wherever you can.

Rules of Inference

Rule of Inference	*Tautology*	*Name*
p $p \rightarrow q$ $\therefore q$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus ponens
$\neg q$ $p \rightarrow q$ $\therefore \neg p$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus tollens
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \rightarrow q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \rightarrow (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \rightarrow p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$	Resolution

Premises (assume true)

Conclusion (true)

Set theory

- The set of integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- The set of nonnegative integers: $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$
- The set of positive integers: $\mathbb{Z}^+ = \{1, 2, \ldots\}$

Using predicates, we can also write $\mathbb{Z}^+$ as $\mathbb{Z}^+ = \{x | (x \in Z) \wedge (x > 0)\}$

- The set of real numbers: $\mathbb{R} = \{x | x \text{ is real}\}$
- The set of positive real numbers: $\mathbb{R}^+ = \{x | x \text{ is real} \wedge x > 0\}$
- The set of rational numbers: $\mathbb{Q} = \{x | x = \frac{a}{b}, a, b \in \mathbb{Z}, b \neq 0\}$
- The empty set (or the *null* set) $\phi = \{\}$

1. **A set** $\{a_1, a_2, \ldots, a_n\}$

- Repetitions removed, ex: if A = {1, 1, 2, 3}, write as A = {1, 2, 3}
- Order of elements in a set doesn't matter, ex {1, 2, 3} = {2, 3, 1}
- Is a subset of itself, $A \subseteq A$
- The empty set is a subset of every set, ∅

Operation on sets:

a. Union: $A \cup B = \{x | (x \in A) \vee (x \in B)\}$

b. Intersection: $A \cap B = \{x | (x \in A) \wedge (x \in B)\}$

c. Complement: $\bar{A}\ or\ A^c = \{x | x \notin A\}$

d. Set difference: $A - B = \{x | (x \in A) \wedge (x \notin B)\}$

Subsets vs set membership vs equality

a. $3 \in \{1, 2, 3\}$ - TRUE

b. $\{3\} \subset \{1, 2, 3\}$ – TRUE

c. $\{3\} \subseteq \{1, 2, 3\}$ – TRUE

d. $\{3\} \in \{\{1\}, \{2\}, \{3\}\}$ – TRUE, sets of "sets"

e. $\{2\} \in \{1, 2, 3\}$ – WRONG

f. $2 \subseteq \{1, 2, 3\}$ – WRONG

2. **Ordered pair, tuple (a_1, a_2)**

Order matters, $(a_1, a_2) \neq (a_2, a_1)$

Cartesian Product

- Cartesian product of set A and B is the set of all ORDERED PAIRS (a, b), $a \in A \;\&\; b \in B$
- $A \times B \neq B \times A$

Cardinality of sets (Size)

- Cardinality of $A \times B = |A||B|$
- **Power set** of A is the set of all subsets of A, if A = {1, 2,3}, then the power set:
  - $2^3 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, (2,3), \{1,2,3\}\}$
- $|A \cap B| \leq |A| \;\&\; |A \cap B| \leq |B|$
- $|A \cup B| \geq |A| \;\&\; |A \cup B| \geq |B|$
- $|A \cup B| \leq |A| + |B|$

Functions $f: X \rightarrow Y$

- X – domain of f, Y – co-domain of f.
- $\boldsymbol{f(x) = y}$**,** x pre-image of y, y is the image of x under f.
- Not every element of Y necessarily gets a pre-image.
- If $y \epsilon Y$ are images of x under f, it is called the range of f, Range(f)

a. **Injective Function** One-to-one

➢ **Each element in the co-domain Y** has **AT MOST 1** incoming arrow.

✓ To prove: Do **proof-by-contradiction**. Assume that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$. Deduce that $x_1 = x_2$, thus leading to a contradiction.

- To disapprove: Do **proof-by-counterexample.** Find x_1, x_2 such that $f(x_1) = f(x_2)$. For onto, find

b. **Surjective Function** Onto

➢ **Each element in the co-domain Y** has **AT LEAST 1** incoming arrow.

✓ To prove: Do **direct proof.** Assume some generic y in co-domain and prove that $y = f(x)$ for some x in domain.

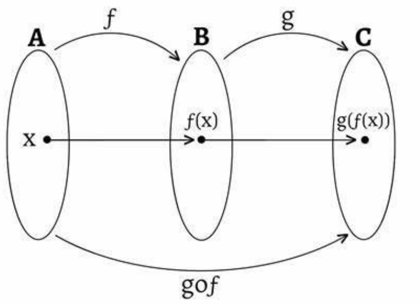
- To disapprove: Do **counterexample**. Find y that has no inverse image in X.

c. **Bijection Function** 1-to-1 and onto

➢ **One arrow out of every element in domain X**

➢ **One arrow into every element in co-domain Y**

d. **Composition of function**



Relations

- $R \subseteq A \times B \;\rightarrow\; aRb\ or\ (a, b) \in R$, where R is a relation that makes $a \in A\ related\ to\ b \in B$
- Every function is a relation BUT NOT vice versa.

a. **Properties of Relation**

1. **Reflexivity:** R is reflexive if aRa
2. **Symmetry:** R is symmetric if aRb iff bRa
3. **Transitivity:** R is transitive if for any triple $a, b, c\ \epsilon A$, aRb and bRc, then aRc.
4. **Irreflexive:** R is irreflexive if for all element a of A is not related to itself.
5. **Antisymmetric:** R is antisymmetric if I. If aRb and bRa, then a = b II. Exist only aRb, no bRa

b. **"Equivalence" Relation:**

- Reflexive
- Transitive
- Symmetric

If R is an equivalence relation: if aRb, then a is equivalent to b.

c. **Equivalence Classes**

▶ Let $A = \mathbb{Z}$ and $R = \{(a, b) | a - b \text{ is even}\}$.

▶ **Definition:** For a given a let $[a] = \{b | b \in A,\ (b, a) \in R\}$, i.e., $[a]$ is the set of elements that are related to a.

▶ What is $[0]$ and what is $[1]$. Do you see a pattern emerging?

▶ $\{\ldots, -4, -2, 0, 2, \ldots\}$ and $\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$

▶ There are exactly two "equivalence classes".

d. **Partitions**

- Grouping of elements of a given set **into DISJOINT SUBSETS.**
- The union of subsets gives us the WHOLE set.
- The subsets are DISJOINT for ALL PAIR of subsets.

e. **Order relations** (can use Hasse Diagram)

Partial order	Total order
○ Reflexive ○ Transitive ○ Antisymmetric *Not every pair of elements need to be related	○ For $a_1 \in A,\ a_2 \in A$, either $a_1 R a_2\ or\ a_2 R a_1$. ○ Antisymmetric ○ Transitive *Any total order is also a partial order.

Graph

Directed Graph

- Vertex degree: in-degree and out-degree
- E = (u, v), u – start vertex, v – end vertex

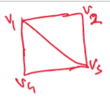
Undirected Graph

- The relation is symmetric for the pairs.
- Undirected edge between nodes a and b, {a, b} since order doesn't matter
- Vertex degree: number of edges touching that node.
- Vertices and b are ADJACENT if they are connected by an edge.
- Given E = (a, b), then E is said to be an incident on a and b.

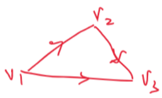
Simple Graph:

- Undirected graph
- No self-loops
- No multiple edges between nodes. At most one edge between distinct vertices.

Representing graphs:



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}_{4\times 4}$$



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}_{3\times 3}$$

Adjacency matrix is symmetric for undirected graphs. $A = A^T$

Degree theorem, for undirected graph:

$$\sum_{v \in V} deg(v) = 2|E|$$

Handshaking Lemma:

$$\sum_{v \in V_{odd}} deg(v) + \sum_{v \in V_{even}} deg(v) = 2|E|$$

Degree theorem, for directed graph:

$$\sum_{v \in V} deg^+(v) + \sum_{v \in V} deg^-(v) = |E|$$

Graph Traversal

- Walk – Both edges and nodes can be visited more than once.
- Path – any walk that does not contain repeated edges. However, nodes can be repeated.
- Simple path – any path that does not contain repeated vertices.

Bipartite Graphs, $K_{m,n}$

- m vertices on one side and n vertices on another. Edges = m * n
- **Hall's Marriage Theorem: A** bipartite graph, G = $(V_1 \cup V_2, E)$ has a complete matching from $V_1 to V_2$ iff if $|N(A)| \geq |A|$ for all subsets $A \subseteq V_1$.

Special Graphs

- Line graph. [$v_1 - v_2 - v_3 - v_n$](n nodes, n-1 edges)
- Cycle, C_n [Closed loop] $(n \geq 3)$
- Tree, [Connected graph containing no cycles]
- Star graph, [central node connected to the other outer nodes] (n nodes, n-1 edges)
- Wheel, [star graph but the outer is connected] (n nodes, 2(n-1) edges)
- Complete graph, K_n, [connecting each of the n nodes with every other node] (n nodes, $\frac{n(n-1)}{2}$edges)

Connectivity

- **Strongly connected graph:** It is possible to reach every vertex from every other vertex for all cases.
- **Weakly connected graph:** The underlying graph is fully connected.

Mathematical Induction (Base case & Inductive Step)

- Provide: Predicate, Base case, Inductive step

Inductive Step

- **General form of Mathematical Induction: Assume P(k) is true and prove P(k+1) is true.**
  - Suppose P(k) is true for [ex: k > 0, equation]
  - $P(k) \rightarrow P(k+1)$, show that P(k+1) is true by making use of P(k) since P(k) is supposed to be true.

- **Strong Induction: Assume that all of P(1), P(2),..,P(k) is true and use any combination of these k predicates to prove P(k+1) is true.**
- Use when the truth of P(k+1) requires that P(I) is true for ALL integers $I \leq k$.
- Necessary to prove multiple base cases

Arithmetic Progressions

- Sequences of numbers (integers or reals) defined via a recursion of the form:

$$a_0 = b,$$
$$a_n = a_{n-1} + d.$$

  are called *arithmetic progressions*.
- An arithmetic progression is any sequence of regularly-spaced numbers of the form $(b, b+d, b+2d, b+3d, \ldots)$.

Geometric Progressions

$$a_0 = b,$$
$$a_n = ra_{n-1}.$$

- What sequence does this generate?

$b, br, br^2, br^3, \ldots$

Summations and Recursions

$$S_0 = a_0$$
$$S_n = S_{n-1} + a_n.$$

- Here are a few examples of summations. You should try to express these recursively.

$$\sum_{i=0}^{n} i = 1 + 2 + \ldots + n.$$ $S_{n+1} = S_n + (n+1)$

$$\sum_{i=0}^{4} i^2 = 0 + 1 + 4 + 9 + 16 = 30.$$ $S_{n+1} = S_n + (n+1)^2$

$$\sum_{i=0}^{3} (-1)^i = 1 + (-1) + 1 + (-1) = 0.$$

$$\sum_{i=3}^{73} i^3 = 3^3 + 4^3 + 5^3 + 6^3 + \ldots 73^3.$$

Time Complexity

Algorithm	Best Time Complexity	Average Time Complexity	Worst Time Complexity
Linear Search	O(1)	O(n)	O(n)
Binary Search	O(1)	O(log n)	O(log n)
Bubble Sort	O(n)	O(n^2)	O(n^2)
Selection Sort	O(n^2)	O(n^2)	O(n^2)
Insertion Sort	O(n)	O(n^2)	O(n^2)
Merge Sort	O(nlogn)	O(nlogn)	O(nlogn)
Quick Sort	O(nlogn)	O(nlogn)	O(n^2)
Heap Sort	O(nlogn)	O(nlogn)	O(nlogn)

Counting

1. **Way to choose an element**
   a. **Addition Rule**
- Disjoint sets:
  $A \cap B = \emptyset$ then $|A| + |B|$ (cardinality/size)
- Non-disjoint sets, PIE: $|A \cup B| = |A| + |B| - |A \cap B|$

   b. **Product Rule**
- $A \times B$ is the Cartesian product of A and B, containing all ordered pairs (a, b) where $a \in A \ \& \ b \in B$.
- Number of possibilities (a, b): $A \times B = |A| \times |B|$
- If there are different conditions (ex: password length 7, 8, 9) **I. Find the possibilities of 7, 8, 9 II. Sum it all.**

2. **Permutations and Arrangement Principle** (order of selection matters)
- The number of ways to arrange r distinct elements drawn from a set of n elements:
  $$P(n, r) = n \cdot (n-1) \cdots (n-r+1)$$
  n = total num. of elements, r = num. of place (_ _ _)
- The number of ways to arrange all n elements of the set into a sequence:
  $$P(n, n) = n!\,, 0! = 1 \ \& \ 1! = 1$$
- Hence, to find the number of possible ways with different arrangements:
  $$P(n, r) = \frac{n!}{(n-r)!}$$

3. **Combinations and Selection Principle** (the order of selection doesn't matter, no repeats)
   c. **Division Rule**
- Used when we want permutation without repeats. Ex: the knight example = (a, b, c), (b, c, a), (c, a, b) – DON'T WANT REPEATS
- "n choose r":
  $$\binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}$$

4. **Counting with Repetitions** (Ball & Sticks argument)
- Donut examples: 17 donut, 4 varieties
- n objects/item from r varieties:
  $$\binom{n+r-1}{r-1} = \binom{n+r-1}{n}$$

5. **Pigeon Hole**

If N pigeons are placed in K holes, then there is at least A HOLE that holds at least $ceil\left(\frac{N}{K}\right)$

- ✓ How many different functions are there from a set with 10 elements to sets with 2 elements? Answer: 2^{10}

How many different functions are there from a set with 10 elements to sets with 5 elements? Answer: 5^{10}