# Homework 3

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1. **(10 points)** Determine whether or not the following statements are true or false. If you think a statement is true, give your reasoning. If you think it is false, provide a counterexample.

   (a) For all sets $A$ and $B$, if $B \subseteq \bar{A}$, then $A \cap B = \emptyset$.

   (b) For all sets $A, B, C$, if $B \subseteq C$ and $A \cap C = \emptyset$, then $A \cap B = \emptyset$.

**a) For all sets A and B, if $B \subseteq \bar{A}$, then $A \cap B = \varnothing$.**

**True:** If $B \subseteq \bar{A}$, it means that every element in set B is in complement A, or can also be said that every element in set B is not in A. This means that if we take the intersection of set A and set B, we will get null ($\varnothing$) as there is no elements that are both in set A and set B.

**b) For all sets A, B, C, if $B \subseteq C$ and $A \cap C = \varnothing$, then A** $\cap$ B = $\varnothing$.

**True**: If $B \subseteq C$ and $A \cap C = \varnothing$, it means that every element in set B is also in set C and that the intersection of set A and set C results to empty ($\varnothing$). This means that there is no elements that are both in set A and set C. Since every element of set B is also in set C, it is safe to conclude that the intersection of set A and set B must be empty ($\varnothing$).

2. **(10 points)** Indicate which of the following relationships are true and which are false, together with a brief explanation in words why you think that is the case:

(a) $Z^+ \subseteq Q$.
(b) $Q \subseteq Z$.

(c) $Q \cap R = Q$.
(d) $Z^+ \cap R = Z^+$.
(e) $\emptyset \subset \mathbb{N}$.

**a)** $Z^+ \subseteq Q$ : The set of positive integers is a subset of the set of rational numbers.

**True**: Every positive integer is also a rational number since every positive integer can be expressed as a fraction with a denominator of 1. Hence, this relationship is true.

**b)** $Q \subseteq Z$ : The set of rational numbers is a subset of the set of integers.

**False**: The set of rational numbers includes both integers and fractions. Not all rational numbers are integers. Hence, this relationship is false.

**c) Q ∩ R = Q** : The intersection of set of rational numbers and set of real numbers are set of rational numbers

**True**: Rational numbers are real numbers. Hence, this statement is true.

**d)** $Z^+$ ∩ **R =** $Z^+$: The intersection of set of positive integers and set of real numbers are set of positive integers.

**True:** Every positive integers are also real number. Hence, this statement is true.

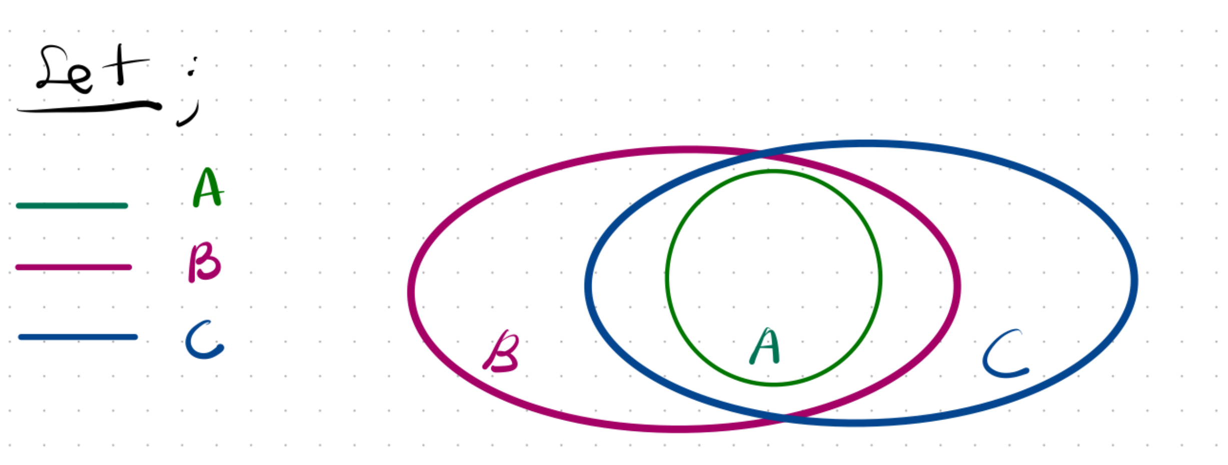
**e) ø ⊂ N**: Null is a subset of nonnegative integers.

**True:** The empty set is a proper subset of any nonempty set. Hence, this relationship is true.

3. **(10 points)** Prove the *distributive law* for any three sets $A, B, C$:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

You can use any method of proof. For example, for a formal logic proof you might want to consider an element $x \in A \cup (B \cap C)$ and construct a chain of logical deductions to show that $x$ also belongs to $(A \cup B) \cap (A \cup C)$. Or you could use Venn diagrams.




4. **(10 points)** Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Determine $A \times B$ and $B \times A$. Are they equal?

- The number of possible ordered pairs (a, b) = $4 \times 2 = 8$
- $A \times B =$ {(a, y), (a, z), (b, y), (b, z), (c, y), (c, z), (d, y), (d, z)}
- $B \times A =$ {(y, a), (y, b), (y, c), (y, d), (z, a), (z, b), (z, c), (z, d)}
- Hence, $A \times B \neq B \times A$.

5. **(10 points)** Prove or disprove that for all sets $A, B$ and $C$, we have (i) $A \times (B - C) = (A \times B) - (A \times C)$. (ii) $\overline{A} \times \overline{(B \cup C)} = \overline{A \times (B \cup C)}$.

**i) A × (B − C) = (A × B) − (A × C)**

- Pick arbitrary element $(x, y) \in A \times (B - C)$.
- The $x$ is in A while the $y$ is in the intersection of B and the complement of C.
- Therefore, $(x, y)$ is in the Cartesian product of A and B and not in the Cartesian product of A and C, $(x, y) \in A \times B - A \times C$
- This prove that $A \times (B - C) \subseteq A \times B - A \times C$ as $(x, y)$ is arbitrary element in $A \times (B - C)$.

- Next, prove that $A \times (B - C) \supseteq A \times B - A \times C$
- Pick arbitrary element $(x, y) \in A \times B - A \times C$.
- The (x, y) is in the Cartesian product of A and B and not in Cartesian product of A and C.
- Therefore, $x \in A, y \in B$, and $y \notin C$.
- This implies that y is in B-C.
- Therefore, we conclude that $(x, y) \in A \times (B - C)$.

**ii)** $\bar{A} \times (B \cup C)^c = (A \times (B \cup C))^c$

- Let (x, y) be arbitrary element in $(A^c \times (B \cup C)^c), (x, y) \in (A^c \times (B \cup C)^c)$
  - This means that the $x \notin A, y \notin B \cup C$
- Let (x, y) be arbitrary element in $(A \times (B \cup C))^c, (x, y) \in (A \times (B \cup C))^c$
  - This means that $(x, y) \notin A \times (B \cup C)$
- Since $x \notin A$, we have $(x, y) \notin (A \times (B \cup C))$
- Therefore, $((x, y) \in A \times (B \cup C))^c$

6. **(10 points)** Let $E$ be the set of even integers and $O$ be the set of odd integers. Define a function:

$$f : E \times O \to \mathbb{Z}$$

such that $f(x, y) = xy$. Is $f$ one-to-one? Is $f$ onto? For either question, if your answer is yes, then prove it; if not, then provide a counterexample.

- $f$ is a one-to-one.
  - Consider two distinct pairs $(x_1, y_1)$ and $(x_2, y_2)$ in $E \times O$ such that $f(x_1, y_1) = f(x_2, y_2)$.
  - We need to show that $(x_1, y_1) = (x_2, y_2)$
  - Given that $x \neq 0$ and that $f(x_1, y_1) = xy_1$ and $f(x_2, y_2) = xy_2$.
  - Since $y_1$ and $y_2$ are odd integers, they cannot be zero and we can cancel $x$ from both sides, leading to $y_1 = y_2$.
  - If $x = 0$, then $y_1$ and $y_2$ can be any odd integers, and $x = 0$ would not affect the injectivity.
  - Therefore, the function is one-to-one.
- $f$ is a an onto function

- For any $z \in Z$, we need to find $(x, y) \in E \times O$ such that $f(x, y) = xy = z$.
- If $z$ is even, we can choose $x = \frac{z}{2}$ and $y = 1$ (any odd integer)
- If $z$ is odd, we can choose $x = 1$ and $y = z$.
- Therefore, the function is onto.

- In conclusion, the function $f : E \times O \longrightarrow Z$ defined as $f(x, y) = xy$ is both one-to-one and onto.

7. **(10 points)** Let $f : A \to B$ and $g : B \to C$ be functions. Let $h : A \to C$ be their composition, i.e., $h(a) = g(f(a))$.
   (a) Prove that if $f$ and $g$ are surjections, then so is $h$.
   (b) Prove that if $f$ and $g$ are bijections then so is $h$.

a) **Prove that if $f$ and $g$ are surjections, then so is $h$.**

- Given an arbitrary element $c$, where $c \in C$. Since $g$ is a surjection, there exist $b$ where $b \in B$ and that $g(b) = c$.
- Since f is also a surjection, there exist $a$ where $a \in A$ and that $f(a) = b$.
- As $h(a) = g(f(a))$, then $h(a) = g(f(a)) = g(b) = c$.
- This shows that for any $c \in C$, there exists an $a \in A$ where $h(a) = c$.

b) **Prove that if $f$ and $g$ are bijections, then so is $h$.**

- Proof that $f$ and $g$ are surjections, then so is $h$.
  - Given an arbitrary element $c$, where $c \in C$. Since $g$ is a surjection, there exist $b$ where $b \in B$ and that $g(b) = c$.
  - Since f is also a surjection, there exist $a$ where $a \in A$ and that $f(a) = b$.

- As $h(a) = g(f(a))$, then $h(a) = g(f(a)) = g(b) = c$.
- This shows that for any $c \in C$, there exists an $a \in A$ where $h(a) = c$.

- Proof that $f$ and $g$ are injections, then so is $h$
  - Let $h(a_1) = h(a_2)$ for some $a_1$ and $a_2$ in A.
  - This implies that $g(f(a_1)) = g(f(a_2))$.
  - Since $g$ is injective, $f(a_1) = f(a_2)$.
  - And since $f$ is injective, $a_1 = a_2$.
  - Therefore, $h$ is injective.
- Since it is proven that both $f$ and $g$ are surjective and injective , then so is $h$, therefore, $h$ is a surjective and injective, making it a bijection.

8. **(10 points)** Determine if the following are functions. The domain is $\mathbb{R}$ and the co-domain is $\mathbb{R}$. $(i) f(x) = 1/x, (ii) f(x) = \sqrt{x}, (iii) f(x) = \pm\sqrt{x^2 + 1}$.

i) $f(x) = 1/x$

- This is a function.
- But this function is not defined for x = 0 as division by zero is undefined.
- Therefore $f(x) = 1/x$ is not a function for the entire domain R since it excludes x = 0.

ii) $f(x) = \sqrt{x}$

- This is a function.
- For every real number $x \geq 0$, there is a unique non-negative real number $\sqrt{x}$.
- However, for $x < 0$, $\sqrt{x}$ is not a real number.
- Hence the domain of this function should be $x \geq 0$.

iii) $f(x) = \pm\sqrt{x^2+1}$

- This is not a function.
- The $\pm$ makes the co-domain value be both negative and positive real numbers.
- This means that there are two possible values of $f(x)$ due to the $\pm$.
- Specifically, for each $x$, there are two distinct values: one positive and one negative.
- This violates the definition of a function, where each input is related to exactly one output.

9. **(20 points)** Determine if these functions from $\mathbb{Z}$ (domain) to $\mathbb{Z}$ (co-domain) are one-to-one (injective). $(i) f(n) = n - 1, (ii) f(n) = n^2 + 1, (iii) f(n) = n^3, (iv) f(n) = \lceil n/2 \rceil$.

i) $f(n) = n - 1$

- This function is injective.
    - Assume $f(n_1) = f(n_2)$.
    - Then $n_1 - 1 = n_2 - 1$.
    - Solving for $n_1$ and $n_2$, we get $n_1 = n_2$
    - Therefore, $f(n) = n - 1$ is an injective function.

ii) $f(n) = n^2 + 1$

- This is function not injective.
    - Assume $f(n_1) = f(n_2)$.
    - Then $n_1^2 + 1 = n_2^2 + 1$.
    - Solving for $n_1$ and $n_2$, we get $n_1 = n_2$ and $n_1 = -n_2$

- Therefore, $f(n) = n^2 + 1$ is not injective  as there are two domain values that map to the same co-domain value.

iii) $f(n) = n^3$

- This function is injective.
  - Assume $f(n_1) = f(n_2)$
  - Then $n_1^3 = n_2^3$
  - Solving for $n_1$ and $n_2$, we get $n_1 = n_2$ after taking the cube root on both sides.
  - Therefore, $f(n) = n^3$ is an injective function.

iv) $f(n) = \lceil n/2 \rceil$

- This function is not injective.
  - Assume $f(n_1) = f(n_2)$
  - Given that $n_1 = 3$ and $n_2 = 4$
  - Then $\lceil 3/2 \rceil = 2$ and $\lceil 4/2 \rceil = 2$.
  - Therefore, $f(n) = \lceil n/2 \rceil$ is not an injective function as distinct values of domain $n$ should not map to the same co-domain value.

10. **(20 points)** Determine if these functions from $\mathbb{Z} \times \mathbb{Z}$ (domain) to $\mathbb{Z}$ (co-domain) are onto (surjective). $(i) f(m, n) = m + n, (ii) f(m, n) = m^2 - n^2, (iii) f(m, n) = m, , (iv) f(m, n) = |m| - |n|$.

i) $f(m, n) = m + n$

- This function is a surjective.

- For any integer $z$, we can find $m = z$ and $n = 0$ or vice versa such that $f(m, n) = m + n = z$.
- Therefore, $f(m, n) = m + n$ is onto.

ii) $f(m, n) = m^2 - n^2$

- This function is not a surjective.
  - Consider $z = -1$.
  - There is no pair integers $m, n$ such that $m^2 - n^2 = -1$.
  - Therefore, $f(m, n) = m^2 - n^2$ is not onto.

iii) $f(m, n) = m$

- This function is a surjective.
  - For any integer $z$, we can find $m = z$ and $n = 0$ such that $f(m, n) = m$ = z.
  - Therefore, $f(m, n) = m$ is onto.

iv) $f(m, n) = |m| - |n|$

- This function is a surjective.
  - For any integer $z$, we can find $m = z$ and $n = 0$ and vice versa such that $f(m, n) = |m| - |n| = z$.
  - Therefore, $f(m, n) = |m| - |n|$ is onto.

11. **(10 points)** Find $f \circ g$ and $g \circ f$ if $f(x) = x^2 + 1$ and $g(x) = x + 2$ if the functions are from $\mathbb{R}$ (domain) to $\mathbb{R}$ (co-domain).

- $f \circ g$
    - $(f \circ g)(x) = f(g(x)) = f(x+2) = (x+2)^2 + 1 = x^2 + 4x + 5$
    - Hence, $(f \circ g) = x^2 + 4x + 5$

- $g \circ f$
    - $(g \circ f) = g(x^2 + 1) = x^2 + 3$
    - Hence, $(g \circ f) = x^2 + 3$