Homework 4

Name: Aina Azman

Net_ID: qistina

1. **Problem 9.1.3**

3. For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
c) $\{(2, 4), (4, 2)\}$
d) $\{(1, 2), (2, 3), (3, 4)\}$
e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
f) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

a)

- Reflexive: False, as (1,1) and (4,4) are missing.
- Symmetric: False, as (2, 4) are present but not (4, 2) and (3, 4) are present but not (4, 3).
- Antisymmetric: False, as (2, 3) and (3,2) are both present and that 3 ≠ 2.
- Transitive: True. All pairs are transitive.

b)

- Reflexive: True, as all elements have self loop.
- Symmetric: True. All pairs are symmetric.
- Antisymmetric: False, as (1, 2) and (2,1) are both present and that 1 ≠ 2.

- Transitive: True. All pairs are transitive.

c)

- Reflexive: False, as all (1,1), (2, 2), (3, 3) and (4, 4) are missing.
- Symmetric: True. All pairs are symmetric.
- Antisymmetric: False, as (2, 4) and (4, 2) are both present and that $4 \neq 2$.
- Transitive: False.

d)

- Reflexive: False, as all (1,1), (2, 2), (3, 3) and (4, 4) are missing.
- Symmetric: False, as missing (2, 1), (3, 2) and (4, 3).
- Antisymmetric: True. All pair are antisymmetric.
- Transitive: False.

e)

- Reflexive: True, as all elements have self loop.
- Symmetric: True. All pairs are symmetric.
- Antisymmetric: True. All pair are antisymmetric.
- Transitive: True.

f)

- Reflexive: False, as all (1,1), (2, 2), (3, 3) and (4, 4) are missing.
- Symmetric: False, as there is (1, 4) but not (4, 1).
- Antisymmetric: False, as there exist (1, 3) and (3, 1) and $3 \neq 1$.
- Transitive: False.

2. **Problem 9.1.6**

6. Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

a) $x + y = 0$.
b) $x = \pm y$.
c) $x - y$ is a rational number.
d) $x = 2y$.
e) $xy \geq 0$.
f) $xy = 0$.
g) $x = 1$.
h) $x = 1$ or $y = 1$.

a)

- Reflexive: False. $x + x = 2x$ and not 0, unless if $x = 0$.
- Symmetric: True. If $x + y = 0$, then $y + x = 0$.
- Antisymmetric: False. If $x + y = 0$ and $y + x = 0$, it doesn't imply that $x = y$. For example, $x = 1$ and $y = -1$.
- Transitive: False. If $x + y = 0$ and $y + z = 0$, then $x + y = y + z$, we will get $x = z, x - z = 0$ and not $x + z = 0$.

b)

- Reflexive: True. $x = \pm x$.
- Symmetric: True. If $x = \pm y$, then $y = \pm x$.
- Antisymmetric: False. If $x = \pm y$ and $y = \pm x$, it is possible that $x \neq y$.
- Transitive: True. If $x = \pm y$ and $y = \pm z$, then $x = \pm z$.

c)

- Reflexive: True. For $x - x$ the answer will always be 0 and 0 is rational number.
- Symmetric: True. If $x - y =$ rational number, then $y - x =$ rational number.

- Antisymmetric: False. If $x - y =$ rational number and $y - x =$ rational number, it is possible that $x \neq y$.
- Transitive: True. If $x - y =$ rational number and $y - z =$ rational number, then $x - z =$ rational number.

d)

- Reflexive: False. $x = 2x$ unless x = 0
- Symmetric: False. If $x = 2y$, it implies that $y = x \div 2$, unless x = y = 0.
- Antisymmetric: True. If $x = 2y$ and $y = 2x$, it implies that $x = y = 0$.
- Transitive: False. If $x = 2y$ and $y = 2z$, then $x = 4z$ and not $x = 2z$.

e)

- Reflexive: True. $xx \geq 0$.
- Symmetric: True. If $xy \geq 0$, then $yx \geq 0$.
- Antisymmetric: False. It is possible that for $xy \geq 0$, and $yx \geq 0$ and $x \neq y$.
- Transitive: True. If $xy \geq 0$, and $yz \geq 0$, then $xz \geq$ 0.

f)

- Reflexive: False. $x \times x = x^2 \neq 0$, unless x = 0.
- Symmetric: True. If $xy = 0$, then $yx = 0$.
- Antisymmetric: False. If $xy = 0$, then $x \neq y$.
- Transitive: False. If $xy = 0$ and $yz = 0$, it doesn’t mean that $xz = 0$.

g)

- Reflexive: False. This condition satisfies only if x = 1.
- Symmetric: False. If $(x, y) \in R$, it implies that x = 1, but it doesn’t guarantee that $(y, x) \in R$ since y can be any real number.

- Antisymmetric: False. If $(x, y) \in R$, it implies that x = 1, but it doesn't guarantee that $(y, x) \in R$ since y can be any real number. Hence, it is possible that $x \neq y$.
- Transitive: Yes. If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$ holds as long as x = 1.

h)

- Reflexive: False. This condition satisfies only if x = 1.
- Symmetric: False. If $(x, y) \in R$, it implies that either x = 1 and y = 1, but it doesn't guarantee that $(y, x) \in R$ since the condition may not hold for the other variable.
- Antisymmetric: False. If $(x, y) \in R$, it implies that either x = 1 and y = 1, but it doesn't guarantee that $(y, x) \in R$ since the condition may not hold for the other variable. Hence, it is possible that $x \neq y$.
- Transitive: Yes. If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$ holds.

3. **Problem 9.5.15**

15. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that R is an equivalence relation.

- A relation on a set is called equivalence relation if it is reflexive, symmetric, and transitive.
- Reflexive:
  - $((a, b), (c, d)) \in R$ as $a + b = b + a$.
- Symmetric:
  - $((a, b), (c, d)) \in R$ as if $a + d = b + c$, then $c + b = d + a$.

- Transitive:
    - If $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$, then $a+d=b+c$ and $c+e=d+f$, so $a+d+c+e=b+c+d+f$ and $a+e=b+f$, so $((a,b),(e,f)) \in R$.
- Hence, the set R is an equivalence relation.

4. **<u>Problem 9.5.41</u>**

41. Which of these collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$?

a) $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$

b) $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$

c) $\{2, 4, 6\}, \{1, 3, 5\}$ **d)** $\{1, 4, 5\}, \{2, 6\}$

- A partition is a grouping of elements of a given set into disjoint subsets given that it satisfies the two conditions:
    - The union of subsets gives the whole set.
    - The subsets are disjoint.

a) False. The subsets are not disjoint.

b) True.

c) True.

d) False. There are no element 3 in any of the subsets.

5. **(20 points)** Consider an n-player round robin tournament where every pair of players play each other exactly once. Assume that there are no ties and every game has a winner. Then, the tournament can be represented via a directed graph with n nodes where the edge $x \to y$ means that x *has beaten* y in their game.

   (a) Explain why the tournament graph does not have cycles (loops) of size 1 or 2.

   (b) We can interpret this graph in terms of a relation where the domain of discourse is the set of n players. Explain whether the "beats" relation for any given tournament is always/sometimes/never:
   (i) asymmetric
   (ii) reflexive
   (iii) irreflexive
   (iv) transitive.

a) The tournament graph does not have cycles of size 1 or 2 as:

- Cycle of size 1 (self -loops) would imply that a player beat themselves, which is not possible in a tournament scenario where every game has a distinct winner.
- Cycle of size 2 would imply that player A beat player B and player B beat player A. However, this contradicts the assumption that there are no ties and every game has a winner.

b)

i. Asymmetric: Always. The "beats" relation is always asymmetric. If player A has beaten player B, it implies that player B cannot beat player A. In other words, if there is a directed edge from node A to node B in the graph, there cannot be a directed edge from node B to node A.

ii. Reflexive: Never. The "beats" relation is never reflexive because no player can beat themselves in a tournament.

iii. Irreflexive: Always. The "beats" relation is always irreflexive because no player can beat themselves in a tournament.

iv. Transitive: Sometimes. The "beats" relation is sometimes transitive. If player A beats player B, and player B beats player C, it does not necessarily imply that A beat C directly. However, it is possible that there is an indirect transitive where A indirectly beat C through B. Hence, the transitivity of this relation depends on the specific outcomes of the games.

6. **(20 points)** Let W be the set of all words in the sentence, "The sky above the port was the color of television, tuned to a dead channel." Define a relation R on W as follows: for any words $w_1, w_2 \in W$, $w_1 R w_2$ if the first letter of w_1 is the same as the first letter of w_2 without regard to upper or lower cases.

   (a) Prove that R is an equivalence relation.

   (b) Enumerate all possible equivalence classes in R. (As per lecture, any equivalence class is the set of all elements in W that are related to each other via R.)

a)

- Reflexive: For any word w in W, wRw holds as the first letter of w will always be the same of the first letter of w. Hence, relation R is reflexive.
- Symmetric: For any words w_1, w_2 in W, if $w_1 R w_2$ holds, it means that the first letter of w_1 and w_2 are the same. This means that $w_2 R w_1$ will also holds. Hence, relation R is symmetric.
- Transitive: For any words, w_1, w_2 and w_3 in W, if $w_1 R w_2$ and $w_2 R w_3$ holds, it means that the first letter of w_1 and w_2 is the same and w_2 and w_3 is the same. This means that $w_1 R w_3$ will also holds. Hence, relation R is transitive.
- Since relation R are reflexive, symmetric and transitive, it is proven that relation R is an equivalence relation.

b)

1. Set of words starting with 'T': {"The", "the", "television", "tuned", "to"}
2. Set of words starting with 'S': {"sky"}
3. Set of words starting with 'A': {"above", "a"}
4. Set of words starting with 'P': {"port"}
5. Set of words starting with 'W': {"was"}
6. Set of words starting with 'C': {"color", "channel"}

7. Set of words starting with ‘O’: {"of"}
8. Set of words starting with ‘D’: {"dead"}

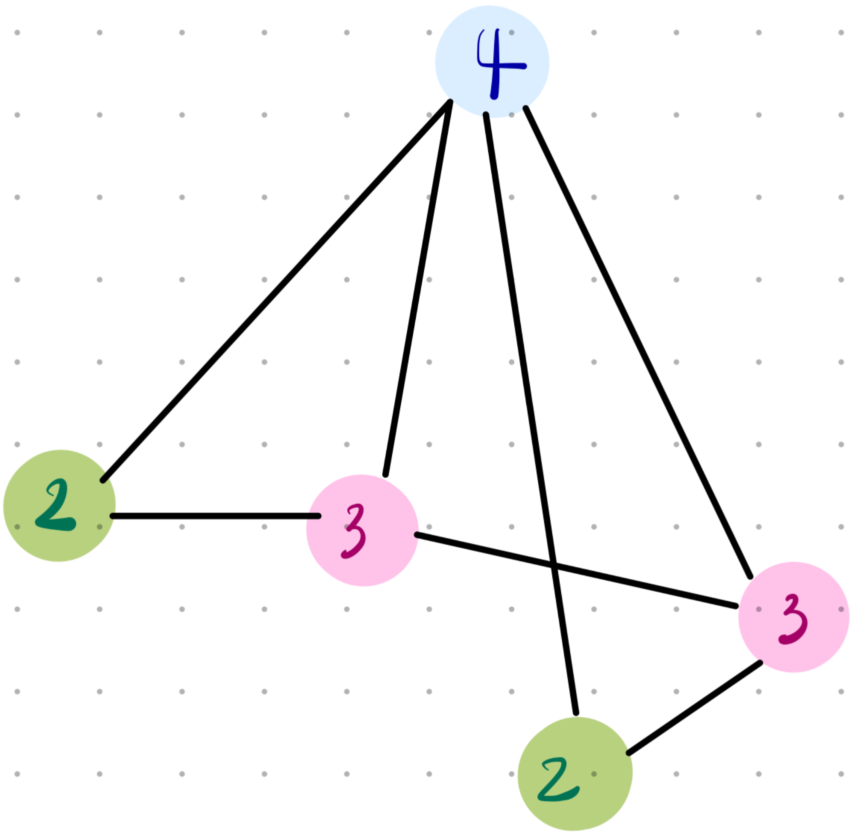
7. **(20 points)** How many edges does a simple undirected graph have if its degree sequence is (i) $4, 3, 3, 2, 2$ $(ii) 5, 2, 2, 2, 2, 1$? In both cases draw the corresponding graphs.

$$\text{Total degrees} = 2 \times edges$$

i) Number of edges: 7

$$4 + 3 + 3 + 2 + 2 = 2 \times edges$$

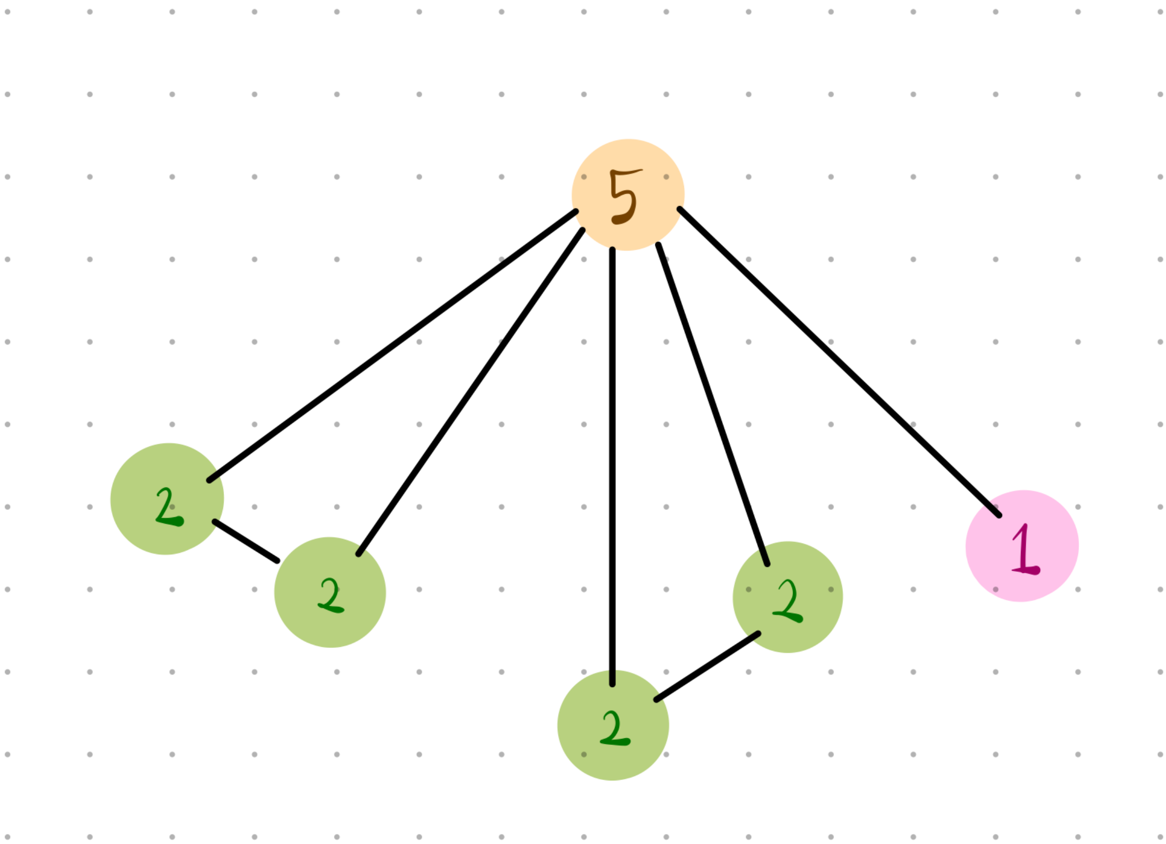
$$edges = 7$$



ii) Number of edges: 7

$$5 + 2 + 2 + 2 + 2 + 1 = 2 \times edges$$

$$edges = 7$$



8. **(20 points)** Let G be a simple undirected graph with n vertices and m edges. Let c_1 and c_2 be the minimum and maximum vertex degrees in G. Show that $c_2 \geq 2m/n \geq c_1$.

- Based on the handshaking lemma or the handshaking theorem:

$$\sum_{1}^{i} deg(v_i) = 2m, \forall i = 1, 2, ...n$$

where the total sum of degrees in a graph G is twice as the number of edges since each edge contributes to the degree of two vertices.

- Given that:

$$nc_2 \geq \sum_{1}^{i} deg(v_i) = 2m \geq nc_1$$

- Then it is proven that:

$$c_2 \geq \frac{2m}{n} \geq c_1$$

9. **(10 points)** A simple undirected graph is called regular if every vertex has the same degree. How many vertices does a regular graph of degree four with 10 edges have?

- Each vertex is connected to exactly four other vertices.
- Since the edges of the graph is 10, then the total degrees would be:

$$degrees = 2 \times edges = 2 \times 10 = 20$$

- Each vertex has the degree of 4. Hence:

$$4n = 20$$

- Solving for n:

$$n = 5$$

- Hence, a regular graph of degree four with 10 edges has 5 vertices.

10. **(20 points)** Let G be a simple undirected graph. $\bar{G}$ represents the complementary graph of G. This is a graph that has the same vertex set as G and is obtained as follows. If (u, v) is an edge in G then (u, v) is not an edge in $\bar{G}$. Conversely if (u, v) is not an edge in G, then (u, v) is an edge in $\bar{G}$.

- Suppose that G has 15 edges and $\bar{G}$ has 13 edges. Then, how many vertices does G have?

- To obtain the maximum number of edges with n vertices, we use the formula:

$$\text{Total edges} = \frac{n(n-1)}{2}$$

- Since G and G^{-} share the same vertex set, the total number of edges in G and G^{-} is the same:

$$\text{Total edges in G} + \text{Total edges in } G^{-} = \frac{n(n-1)}{2}$$

- Given that G has 15 edges and its complementary graph has 13 edges.

$$2(13 + 15) = n^2 - n$$

$$56 = n^2 - n$$

$$n^2 - n - 56 = 0, n = 8, -7$$

- Since it is impossible to have negative vertices, we discard n = -7.
- Hence, we have n = 8. G has 8 vertices.