STS2005 (Analytic Geometry and Calculus I)
Midterm Exam Solutions

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1. (5 pts) Prove the limit $\lim_{x\to 4}(2x-5)=3$ using ε, δ definition of a limit.

Solution. We first let $\varepsilon > 0$ be given. We will find $\delta > 0$ such that $|x-4| < \delta$ implies $|(2x-5)-3| < \varepsilon$.

Since $|(2x-5)-3| = |2x-8| = 2|x-4| < \varepsilon$, we can let $\delta = \dfrac{\varepsilon}{2}$. Thus we have found a δ such that $|x-4| < \delta$ implies $|(2x-5)-3| < \varepsilon$, therefore we have shown that $\lim_{x\to 4}(2x-5)=3$. □

2. (5 pts each) Find the derivatives of the following:

(a) $f(x) = \dfrac{1}{\sqrt[3]{x^2-1}}$

(b) $y = \tan\left(e^t\right) + e^{\tan t}$

Solution.

(a)

$$\begin{aligned} f(x) &= \frac{1}{\sqrt[3]{x^2-1}} \\ \frac{d}{dx} f(x) &= \frac{d}{dx}\frac{1}{\sqrt[3]{x^2-1}} \\ &= \frac{d}{dx}\left(x^2-1\right)^{-\frac{1}{3}} \\ &= -\frac{1}{3}\left[\frac{d}{dx}\left(x^2-1\right)\right]\left(x^2-1\right)^{-\frac{4}{3}} \\ &= -\frac{2}{3}x\left(x^2-1\right)^{-\frac{4}{3}} \end{aligned}$$

(b)

$$
\begin{aligned}
y =& \tan\left(e^t\right) + e^{\tan t} \\
\frac{dy}{dt} =& \frac{d}{dt}\left[\tan\left(e^t\right) + e^{\tan t}\right] \\
=& \frac{d}{dt}\tan\left(e^t\right) + \frac{d}{dt}e^{\tan t} \\
=& \left(\frac{d}{dt}e^t\right)\sec^2\left(e^t\right) + \left(\frac{d}{dt}\tan t\right)e^{\tan t} \\
=& e^t\sec^2\left(e^t\right) + e^{\tan t}\sec^2 t
\end{aligned}
$$

3. (5 pts each) Prove the following:

(1) $\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$

(2) $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), -1 < x < 1$

(3) $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}, x > 1$

Solution.

(1)

$$
\begin{aligned}
& \sinh(x+y) \\
=& \frac{e^{x+y} - e^{-x-y}}{2} \\
=& \frac{2e^{x+y} - 2e^{-x-y}}{4} \\
=& \frac{2e^{x+y} + (e^{x-y} - e^{y-x}) - (e^{x-y} - e^{y-x}) - 2e^{-x-y}}{4} \\
=& \frac{e^{x+y} + e^{x-y} - e^{-x-y} - e^{y-x}}{4} + \frac{e^{x+y} - e^{x-y} + e^{y-x} - e^{-x-y}}{4} \\
=& \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\
=& \sinh x\cosh y + \cosh x\sinh y
\end{aligned}
$$

(2) Let $x = \tanh y$. Then

$$\begin{aligned}
x =& \frac{\sinh y}{\cosh y} \\
=& \frac{e^y - e^{-y}}{e^y + e^{-y}} \\
=& \frac{e^{2y} - 1}{e^{2y} + 1} \\
\Rightarrow \left(e^{2y} + 1\right) x =& e^{2y} - 1 \\
\Rightarrow xe^{2y} + x =& e^{2y} - 1 \\
\Rightarrow (x-1) e^{2y} =& -(x+1) \\
\Rightarrow e^{2y} =& -\frac{x+1}{x-1} \\
\Rightarrow y =& \frac{1}{2} \ln \left(\frac{x+1}{x-1}\right)
\end{aligned}$$

Since $x = \tanh y$, $y = \tanh^{-1} x = \dfrac{1}{2} \ln \left(\dfrac{1+x}{1-x}\right)$ where $-1 < x < 1$.

(3) Let $x = \cosh y$. Then

$$\begin{aligned}
\frac{d}{dx} x =& \frac{d}{dx} \cosh y \\
\Rightarrow 1 =& \frac{dy}{dx} \sinh y \\
\Rightarrow \frac{dy}{dx} =& \frac{1}{\sinh y}
\end{aligned}$$

Since $\cosh^2 y - \sinh^2 y = 1$, $\sinh y = \pm\sqrt{\cosh^2 y - 1}$. Given $x > 1$, $\frac{dy}{dx} > 0$, thus

$$\begin{aligned}
\frac{dy}{dx} =& \frac{1}{\sqrt{\cosh^2 y - 1}} \\
=& \frac{1}{\sqrt{x^2 - 1}}
\end{aligned}$$

and since $x = \cosh y$, $y = \cosh^{-1} x$, therefore $\dfrac{d}{dx} \cosh^{-1} x = \dfrac{1}{\sqrt{x^2 - 1}}$ where $x > 1$.

4. (5 pts) Find the approximation of $\sqrt{3.96}$ by using the linear approximation or differential.

Solution. We let $f(x) = \sqrt{x}$. then $f'(x) = \dfrac{1}{2\sqrt{x}}$. Then the linear approximation of $f(x)$ at $x = 4$ is given by

$$\begin{aligned} L(x) &= f'(4)(x-4) + f(4) \\ &= \frac{x-4}{4} + 2 \\ &= \frac{x}{4} + 1 \end{aligned}$$

Thus we can approximate $\sqrt{3.96} = f(3.96)$ as $L(3.96) = 1.99$.

5. (5 pts) Use the mean value theorem to prove the inequality

$$|\sin x - \sin b| \leq |a-b| \text{ for all } a \text{ and } b.$$

Solution. By the mean value theorem, there exists $c \in (a,b)$ such that

$$\left| \frac{\sin b - \sin a}{b-a} \right| = |\cos c|$$

Since $|\cos c| \leq 1$, $\left| \frac{\sin b - \sin a}{b-a} \right| \leq 1$, which gives $|\sin x - \sin b| \leq |a-b|$ for all $a \neq b$. If $a = b$, $0 = |\sin x - \sin b| \leq |a-b| = 0$, thus the statement holds. □

6. (5 pts each) Evaluate the following:

(1) $\lim_{x \to 0} \frac{\tan 3x}{\sin 2x}$
(2) $\lim_{x \to 1^+} \left[\ln\left(x^7 - 1\right) - \ln\left(x^5 - 1\right) \right]$

Solution.

(1) By L'Hospital's rule,

$$\begin{aligned} &\lim_{x \to 0} \frac{\tan 3x}{\sin 2x} \\ =&\lim_{x \to 0} \frac{d/dx \tan 3x}{d/dx \sin 2x} \\ =&\lim_{x \to 0} \frac{3\sec^2 3x}{2\cos 2x} \\ =&\frac{3}{2} \lim_{x \to 0} \frac{1}{\cos 2x \cos^2 3x} \\ =&\frac{3}{2} \end{aligned}$$

(2)

$$
\begin{aligned}
&\lim_{x\to 1^+}\left[\ln\left(x^7-1\right)-\ln\left(x^5-1\right)\right]\\
=&\lim_{x\to 1^+}\ln\frac{x^7-1}{x^5-1}\\
=&\lim_{x\to 1^+}\ln\frac{(x-1)\left(x^6+x^5+x^4+x^3+x^2+x+1\right)}{(x-1)\left(x^4+x^3+x^2+x+1\right)}\\
=&\ln\lim_{x\to 1^+}\frac{(x-1)\left(x^6+x^5+x^4+x^3+x^2+x+1\right)}{(x-1)\left(x^4+x^3+x^2+x+1\right)}\\
=&\ln\lim_{x\to 1^+}\frac{x^6+x^5+x^4+x^3+x^2+x+1}{x^4+x^3+x^2+x+1}\\
=&\ln\frac{7}{5}=\ln 7-\ln 5
\end{aligned}
$$

7. (5 pts each) Find the following of the curve $y=(1-x)\,e^x$:

(1) Asymptotes.
(2) Intervals of increase or decrease.
(3) Concavity and points of inflection.

Solution. Let $y=f(x)$. Since $f(x)=(1-x)\,e^x$, we can derive $f'(x)=-xe^x$, $f''(x)=-(x+1)\,e^x$. Both $f(x)$ and $f'(x)$ is continuous and differentiable on $\mathbb{R}$.

(1)

$$
\begin{aligned}
&\lim_{x\to-\infty}f(x)\\
=&\lim_{x\to-\infty}(1-x)\,e^x\\
=&0
\end{aligned}
$$

Thus $y=f(x)$ has an asymptote of $y=0$. No other linear asymptotes can be found; $\lim_{x\to\infty}f(x)$ gives $-\infty$.

(2) If $f'(x)>0$, f will increase, and if $f'(x)<0$, f will decrease.
Since $f'(x)=-xe^x=0\Rightarrow x=0$, $f'(x)<0$ where $x<0$, $f'(x)>0$ where $x>0$. Therefore f increases on $(-\infty,0)$ and decreases on $(0,\infty)$.

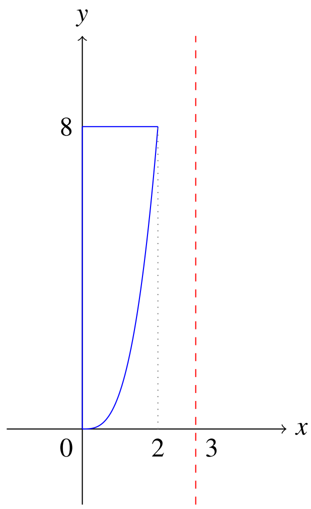
(3) $f''(x)=-(x+1)\,e^x$, therefore

$$
\begin{aligned}
x=-1&\Rightarrow f''(x)=0\\
x<-1&\Rightarrow f''(x)>0\\
x>-1&\Rightarrow f''(x)<0
\end{aligned}
$$

Thus f is concave on $(-\infty,-1)$ and is convex on $(-1,\infty)$. The point of inflection resides at $\left(-1,2e^{-1}\right)$.

8. (5 pts) Use the method of cylindrical shells to find the volume generated by rotating bounded by curves $y = x^3$, $y = 8$, $x = 0$ about the line $x = 3$.

Solution. The plot of the region bounded by $y = x^3$, $y = 8$, $x = 0$ is:



Therefore the volume V is given by

$$\begin{aligned}
V &= \int_0^2 h \times 2\pi r \, dx \\
&= 2\pi \int_0^2 \left(8 - x^3\right)(3 - x)\, dx \\
&= 2\pi \int_0^2 x^4 - 3x^3 - 8x + 24 \, dx \\
&= 2\pi \left[\frac{1}{5}x^5 - \frac{3}{4}x^4 - 4x^2 + 24x\right]_0^2 \\
&= \frac{264}{5}\pi = 52.8\pi
\end{aligned}$$

9. (5 pts each) Evaluate the integral or determine whether each improper integral is convergent or divergent. If the improper integral converges, evaluate the integral.

(1) $\int \tan^2 x \sec x \, dx$

(2) $\int_0^{\frac{\pi}{2}} \cos 5x \cos 10x \, dx$

(3) $\displaystyle\int \frac{dx}{\cos x - 1}$

(4) $\displaystyle\int \frac{x^4+9x^2+x+2}{x^2+9}\,dx$

(5) $\displaystyle\int_1^{\infty} \frac{e^{\frac{1}{x}}}{x^2}\,dx$

(6) $\displaystyle\int_0^4 \frac{dx}{x^2-x-2}$

Solution.

(1)

$$
\begin{aligned}
&\int \tan^2 x \sec x\,dx \\
=&\int \left(\sec^2 x - 1\right)\sec x\,dx \\
=&\int \sec^3 x\,dx - \int \sec x\,dx \\
=&\frac{1}{2}\tan x\sec x - \frac{1}{2}\int \sec x\,dx \qquad \because \int \sec^m x\,dx = \frac{\sin x\sec^{m-1} x}{m-1} + \frac{m-2}{m-1}\int \sec^{m-2} x\,dx \\
=&\frac{1}{2}\tan x\sec x - \frac{1}{2}\int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx \\
=&\frac{1}{2}\tan x\sec x - \frac{1}{2}\ln(\sec x + \tan x)
\end{aligned}
$$

(2)

$$
\begin{aligned}
&\int_0^{\frac{\pi}{2}} \cos 5x \cos 10x\,dx \\
=&\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 5x + \cos 15x\,dx \qquad \because \cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha-\beta) + \cos(\alpha+\beta)\right) \\
=&\frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \cos 5x\,dx + \int_0^{\frac{\pi}{2}} \cos 15x\,dx\right] \\
=&\frac{1}{2}\left[\frac{1}{5}\sin 5x\Big|_0^{\frac{\pi}{2}} + \frac{1}{15}\sin 15x\Big|_0^{\frac{\pi}{2}}\right] \\
=&\frac{1}{2}\left(\frac{1}{5} - \frac{1}{15}\right) = \frac{1}{15}
\end{aligned}
$$

(3)

$$\begin{aligned}
&\int \frac{dx}{\cos x - 1} \\
=&\int \frac{2\,du}{(u^2+1)\left(\frac{1-u^2}{u^2+1}-1\right)} \qquad u \triangleq \tan\frac{x}{2} \qquad \cos x = \frac{1-u^2}{u^2+1} \qquad du = \frac{1}{2}\,dx\sec^2\frac{x}{2} \Rightarrow dx = \frac{2\,du}{u^2+1} \\
=&-\int \frac{1}{u^2}\,du \\
=&\frac{1}{u}+C \\
=&\cot\frac{x}{2}+C
\end{aligned}$$

(4)

$$\begin{aligned}
&\int \frac{x^4+9x^2+x+2}{x^2+9}\,dx \\
=&\int x^2+\frac{x+2}{x^2+9}\,dx \\
=&\int x^2\,dx+\int \frac{x}{x^2+9}\,dx+2\int \frac{1}{x^2+9}\,dx \\
=&\frac{1}{3}x^3+\frac{1}{2}\int \frac{2x}{x^2+9}\,dx+2\int \frac{1}{x^2+9}\,dx \\
=&\frac{1}{3}x^3+\frac{1}{2}\ln\left(x^2+9\right)+\frac{2}{3}\tan^{-1}\frac{x}{3}+C
\end{aligned}$$

(5)

$$\begin{aligned}
&\int_1^\infty \frac{e^{\frac{1}{x}}}{x^2}\,dx \\
=&\lim_{t\to\infty}\int_1^t \frac{e^{\frac{1}{x}}}{x^2}\,dx \\
=&-\lim_{t\to\infty}\int_1^t e^u\,du \qquad u=\frac{1}{x} \qquad du=-\frac{1}{x^2} \\
=&-\lim_{t\to\infty} e^t+e=-\infty
\end{aligned}$$

Thus the improper integral diverges.

(6)

$$\begin{aligned}
&\int_0^4 \frac{dx}{x^2-x-2} \\
=&\lim_{t\to 2^-}\int_0^t \frac{dx}{x^2-x-2}+\lim_{t\to 2^+}\int_t^4 \frac{dx}{x^2-x-2} \\
=&\frac{1}{3}\lim_{t\to 2^-}\int_0^t\left(\frac{1}{x-2}-\frac{1}{x+1}\right)dx+\frac{1}{3}\lim_{t\to 2^+}\int_t^4\left(\frac{1}{x-2}-\frac{1}{x+1}\right)dx
\end{aligned}$$

Thus the improper integral diverges.