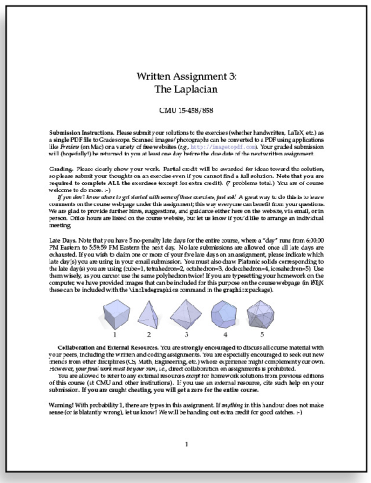
Discrete Differential Geometry
离散差分几何

CARNEGIE MELLON UNIVERSITY 15-458/768 | SPRING 2025 | TUE/THU, TIME TBD | ROOM TBD

Assignment 3: The Laplacian
作业 3：拉普拉斯人

Written 写

These exercises will lead you through two different derivations for the *cotan-Laplace operator*. As we' ll discuss in class, this operator is basically the "Swiss army knife" of discrete differential geometry and digital geometry processing, opening the door to a huge number of interesting algorithms and applications. Note that this time the exercises all come from the course notes—you will need to read the accompanying notes in order to familiarize yourself with the necessary material (though actually we' ve covered much of this stuff in class already!)



Written Assignment 3:
The Laplacian

CMU 15-458/858

Submission Instructions. Please submit your solutions to the exercises (whether handwritten, LaTeX, etc.) as a single PDF file to Gradescope. Scanned images/photographs can be converted to a PDF using applications like *Preview* (on Mac) or a variety of free websites (*e.g.*, http://imagetopdf.com). Your graded submission will (hopefully!) be returned to you at least one day before the due date of the next written assignment.

Grading. Please clearly show your work. Partial credit will be awarded for ideas toward the solution, so please submit your thoughts on an exercise even if you cannot find a full solution. **Note that you are required to complete ALL the exercises (except for extra credit).** (7 problems total.) You are of course welcome to do more. :-)

If you don't know where to get started with some of these exercises, just ask! A great way to do this is to leave comments on the course webpage under this assignment; this way everyone can benefit from your questions. We are glad to provide further hints, suggestions, and guidance either here on the website, via email, or in person. Office hours are listed on the course website, but let us know if you'd like to arrange an individual meeting.

Late Days. Note that you have 5 no-penalty late days for the entire course, where a "day" runs from 6:00:00 PM Eastern to 5:59:59 PM Eastern the next day. No late submissions are allowed once all late days are exhausted. If you wish to claim one or more of your five late days on an assignment, please indicate which late day(s) you are using in your email submission. You must also draw Platonic solids corresponding to the late day(s) you are using (cube=1, tetrahedron=2, octahedron=3, dodecahedron=4, icosahedron=5). Use them wisely, as you cannot use the same polyhedron twice! If you are typesetting your homework on the computer, we have provided images that can be included for this purpose on the course webpage (in LaTeX these can be included with the `\includegraphics` command in the graphicx package).

1 2 3 4 5

Collaboration and External Resources. You are strongly encouraged to discuss all course material with your peers, including the written and coding assignments. You are especially encouraged to seek out new friends from other disciplines (CS, Math, Engineering, etc.) whose experience might complement your own. However, *your final work must be your own, i.e.*, direct collaboration on assignments is prohibited.

You are allowed to refer to any external resources *except* for homework solutions from previous editions of this course (at CMU and other institutions). If you use an external resource, cite such help on your submission. **If you are caught cheating, you will get a zero for the entire course.**

Warning! With probability 1, there are typos in this assignment. If *anything* in this handout does not make sense (or is blatantly wrong), let us know! We will be handing out extra credit for good catches. :-)

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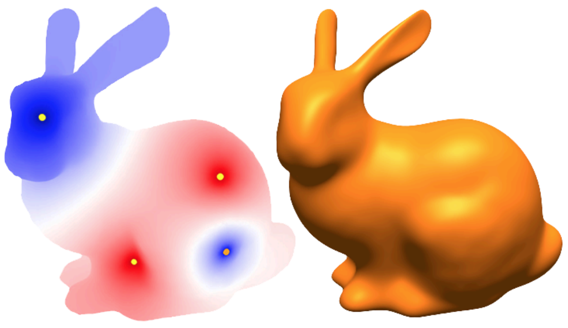
这些练习将引导您了解 cotan-Laplace 算子的两种不同推导。正如我们将在课堂上讨论的那样，这个算子基本上是离散微分几何和数字几何处理的“瑞士军刀”，为大量有趣的算法和应用打开了大门。请注意，这次的练习都来自课程笔记——你需要阅读随附的笔记以熟悉必要的材料（尽管实际上我们已经在课堂上涵盖了大部分这些内容！

Coding 编码

For the coding portion of this assignment, you will build the so-called *"cotan-Laplace"* matrix and start to see how it can be used for a broad range of surface processing tasks, including the *Poisson equation* and two kinds of *curvature flow*.

Getting Started

Please implement the following routines in



对于本作业的编码部分，您将构建所谓的“cotan-Laplace”矩阵，并开始了解如何将其用于广泛的表面处理任务，包括泊松方程和两种曲率流。

开始

请在

1. `core/geometry.js`:
   - `laplaceMatrix`
   - `massMatrix`
2. `projects/poisson-problem/scalar-poisson-problem.js`:
   - `constructor`
   - `solve`
3. `projects/geometric-flow/mean-curvature-flow.js`:
   - `buildFlowOperator`
   - `integrate`
4. `projects/geometric-flow/modified-mean-curvature-flow.js`:
   - `constructor`
   - `buildFlowOperator`

Notes

笔记

- Sections 6.4-6 of the course notes describe how to build the cotan-Laplace matrix and mass matrices, and outline how they can be used to solve equations on a mesh. In these applications you will be required to setup and solve a linear system of equations $Ax = b$ where A is the Laplace matrix, or some slight modification thereof. Highly efficient numerical methods such as Cholesky Factorization can be used to solve such systems, but require A to be symmetric positive definite. Notice that the cotan-Laplace matrix described in the notes is **negative semi-definite**. To make it compatible for use with numerical methods like the Cholesky Factorization, your implementation of `laplaceMatrix` should instead produce a **positive definite matrix**, *i.e.*, it should represent the expression

$$(\Delta u)_i = \frac{1}{2}\sum_{ij}(\cot\alpha_{ij}+\cot\beta_{ij})(u_i – u_j).$$

(Note that $u_i - u_j$ is reversed relative to the course notes.) To make this matrix strictly positive definite (rather than semidefinite), you should also add a small offset such as 10^{-8} to the diagonal of the matrix (which can be expressed in code as a floating point value `1e-8`). This technique is known as Tikhonov regularization.

课程笔记的第 6.4-6 节描述了如何构建 cotan-Laplace 矩阵和质量矩阵，并概述了如何使用它们来求解网格上的方程。在这些应用程序中，您将需要设置和求解一个线性方程组 $Ax = b$ ，其中 A 是拉普拉斯矩阵，或对其进行一些轻微的修改。高效的数值方法（如 Cholesky Factorization）可用于求解此类系统，但要求 A 是对称的正定。请注意，注释中描述的 cotan-Laplace 矩阵是负半定矩阵。为了使其与 Cholesky Factorization 等数值方法兼容，您的 implementation `laplaceMatrix` 应该生成一个正定矩阵，即，它应该表示表达式

$$(\Delta u)_i = \frac{1}{2}\sum_{ij}(\cot\alpha_{ij}+\cot\beta_{ij})(u_i – u_j).$$

（请注意，相对于 $u_i - u_j$ 课程笔记是相反的）。要使此矩阵严格为正定（而不是半定），您还应该添加一个小偏移量，例如 10^{-8} 矩阵的对角线（可以在代码中表示为 floating point value `1e-8` ）。这种技术被称为 Tikhonov 正则化。

- The mass matrix is a diagonal matrix containing the barycentric dual area of each vertex.

  质量矩阵是一个对角矩阵，其中包含每个顶点的重心对偶区域。

- In the scalar Poisson problem, your goal is to discretize and solve the equation $\Delta\phi = \rho$ where ρ is a scalar density on vertices and Δ is the Laplace operator. Be careful about appropriately incorporating dual areas into the discretization of this equation (i.e., think about where and how the mass matrix should appear); also remember that the right-hand side cannot have a constant component (since then there is no solution).

  在标量泊松问题中，您的目标是离散化并求解方程 $\Delta\phi = \rho$ ，其中 ρ 是顶点上的标量密度， Δ 是拉普拉斯算子。小心地将对偶区域适当地合并到该方程的离散化中（即，考虑质量矩阵应该出现在何处以及如何出现）;还要记住，右侧不能有 constant 分量（因为那时没有解）。

- In your implementation of the implicit mean curvature flow algorithm, you can encode the surface $f : M \to \mathbb{R}^3$ as a single DenseMatrix of size $|V| \times 3$, and solve with the same (scalar) cotan-Laplace matrix used for the previous part. When constructing the flow operator, you should follow section 6.6 of the notes. But be careful - when we discretize equations of the form $\Delta f = \rho$, we create systems of the form $Af = M\rho$. So you'll need to add in the mass matrix somewhere. Also, recall that our discrete Laplace matrix is the negative of the actual Laplacian.

在隐式均值曲率流算法的实现中，可以将表面 $f: M \to \mathbb{R}^3$ 编码为大小 $|V| \times 3$ 为的单个 DenseMatrix，并使用与上一部分相同的（标量）cotan-Laplace 矩阵进行求解。在构造 flow 运算符时，您应该遵循注释的第 6.6 节。但要小心 - 当我们离散 形式的 $\Delta f = \rho$ 方程 时，我们创建的方程形式 $Af = M\rho$ 为 。所以你需要在某处添加质量矩阵。另外，回想一下，我们的离散拉普拉斯矩阵是实际拉普拉斯算子的负数。

- The modified mean curvature flow is nearly identical to standard mean curvature flow. The one and only difference is that you should **not** update the cotan-Laplace matrix each time step, i.e., you should always be using the cotans from the original input mesh. The mass matrix, however, must change on each iteration.

  修正后的平均曲率流与标准平均曲率流几乎相同。唯一的区别是，您不应该在每个时间步长都更新 cotan-Laplace 矩阵，即，您应该始终使用来自原始输入网格的 cotans。但是，质量矩阵必须在每次迭代时更改。

Submission Instructions

Please only turn in the source files `geometry.[cpp|js]`, `scalar-poisson-problem.[cpp|js]`, `mean-curvature-flow.[cpp|js]` and `modified-mean-curvature-flow.[cpp|js]`.

Handin instructions can be found in the Assignments section of the main page.

提交说明

请只提交源文件 `geometry.[cpp|js]` 、 `scalar-poisson-problem.[cpp|js]` 和 `mean-curvature-flow.[cpp|js]` `modified-mean-curvature-flow.[cpp|js]` 。

提交说明可以在主页的 Assignments 部分找到。

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