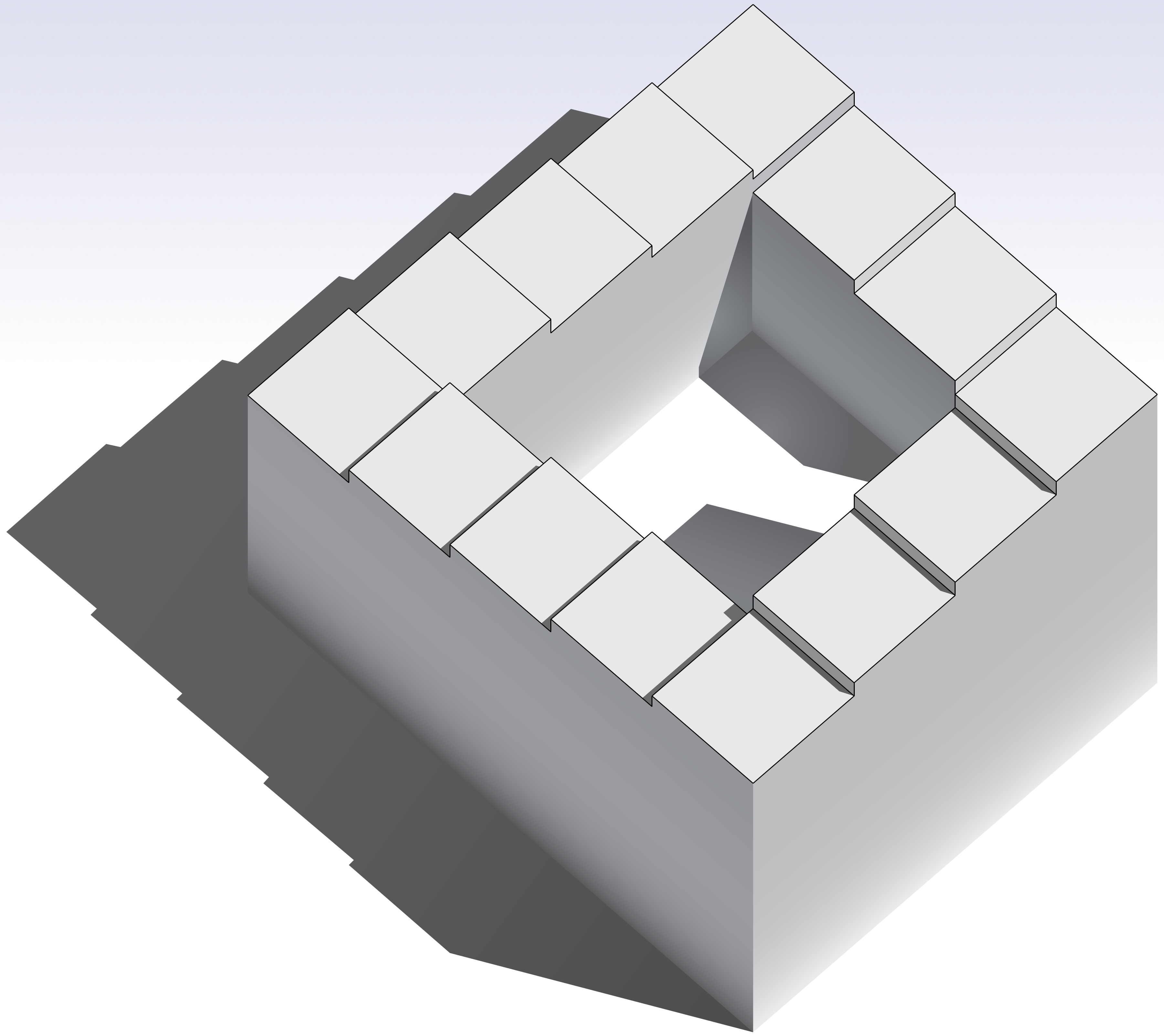
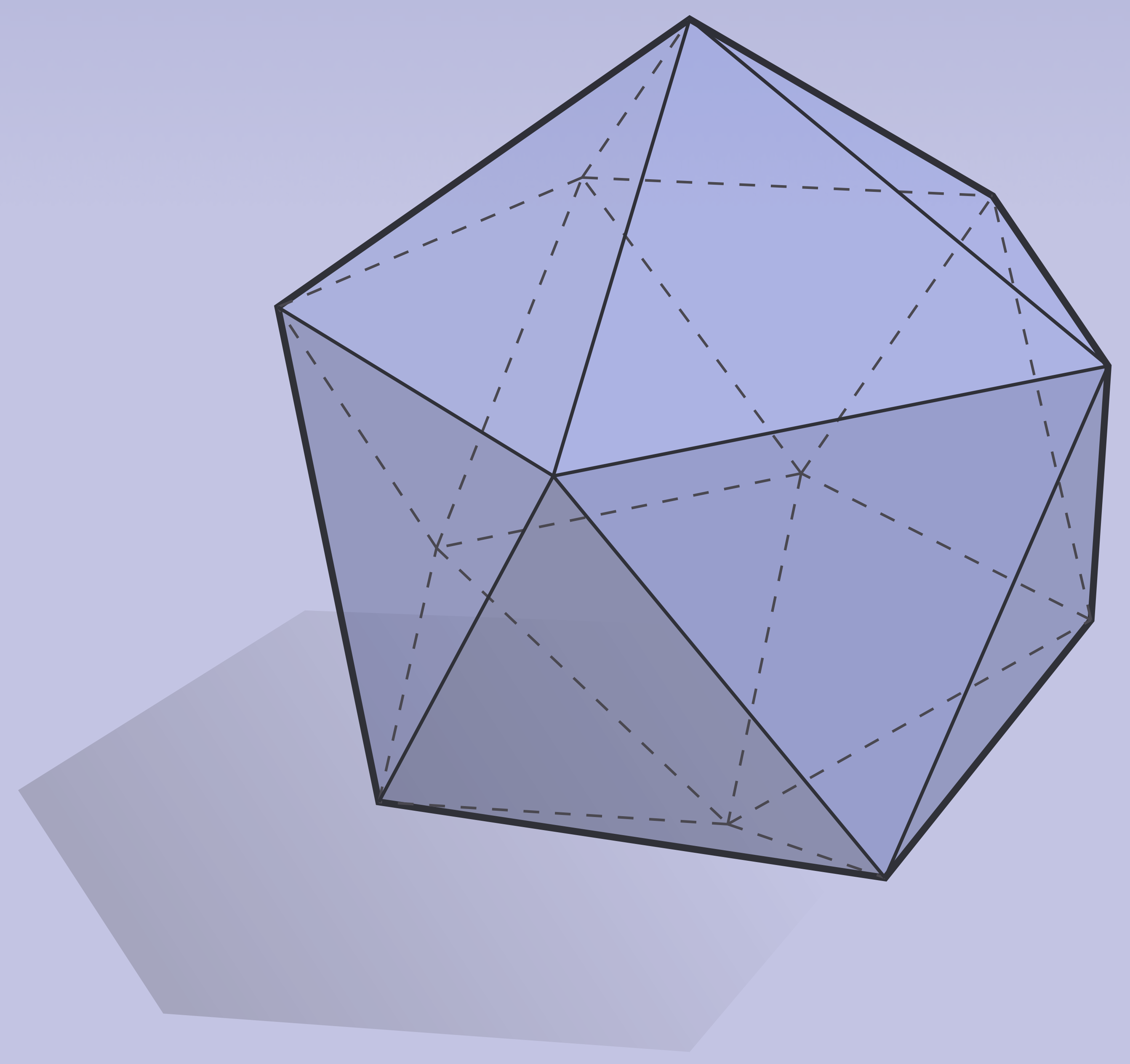


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

Discrete Curvature I



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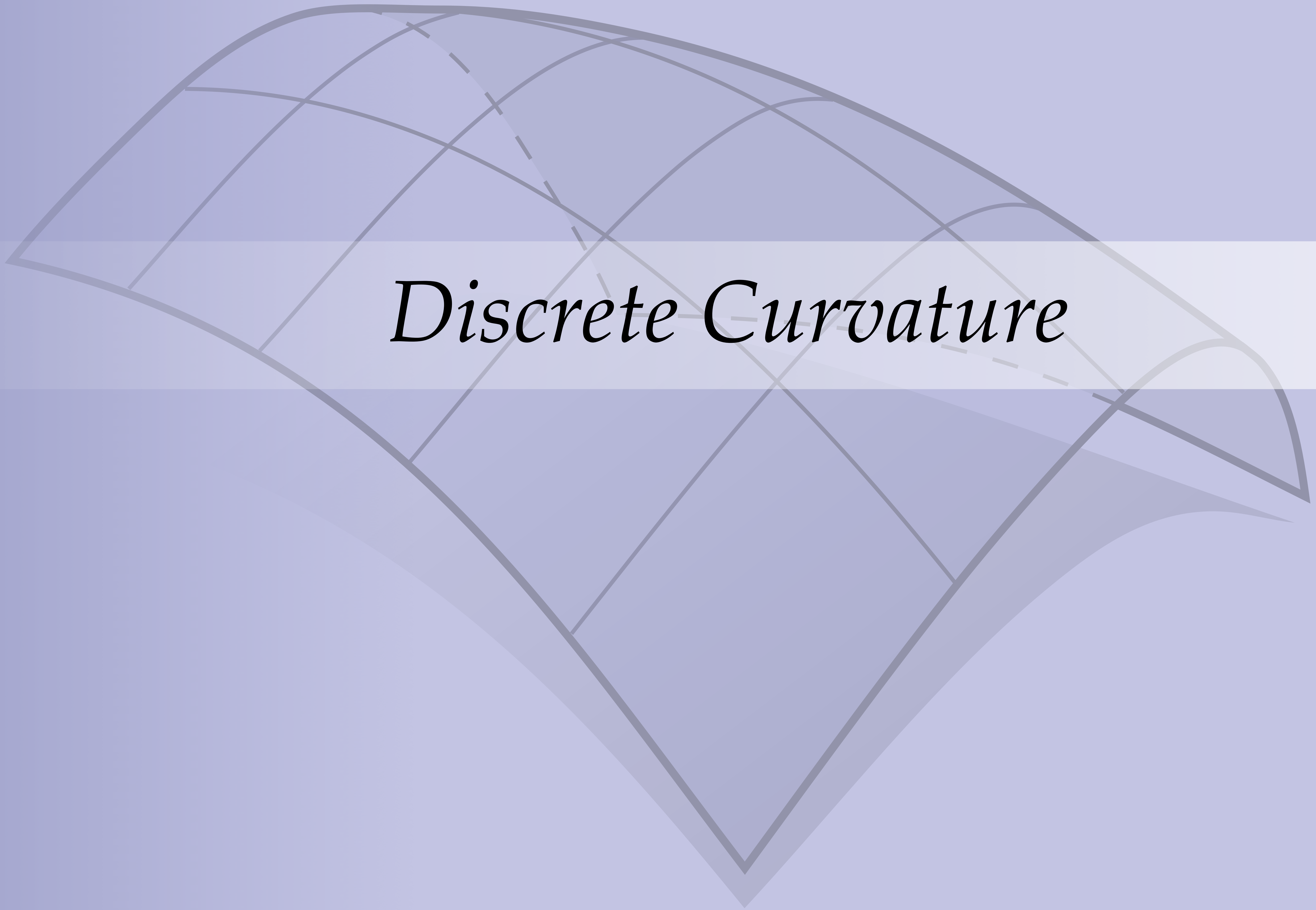
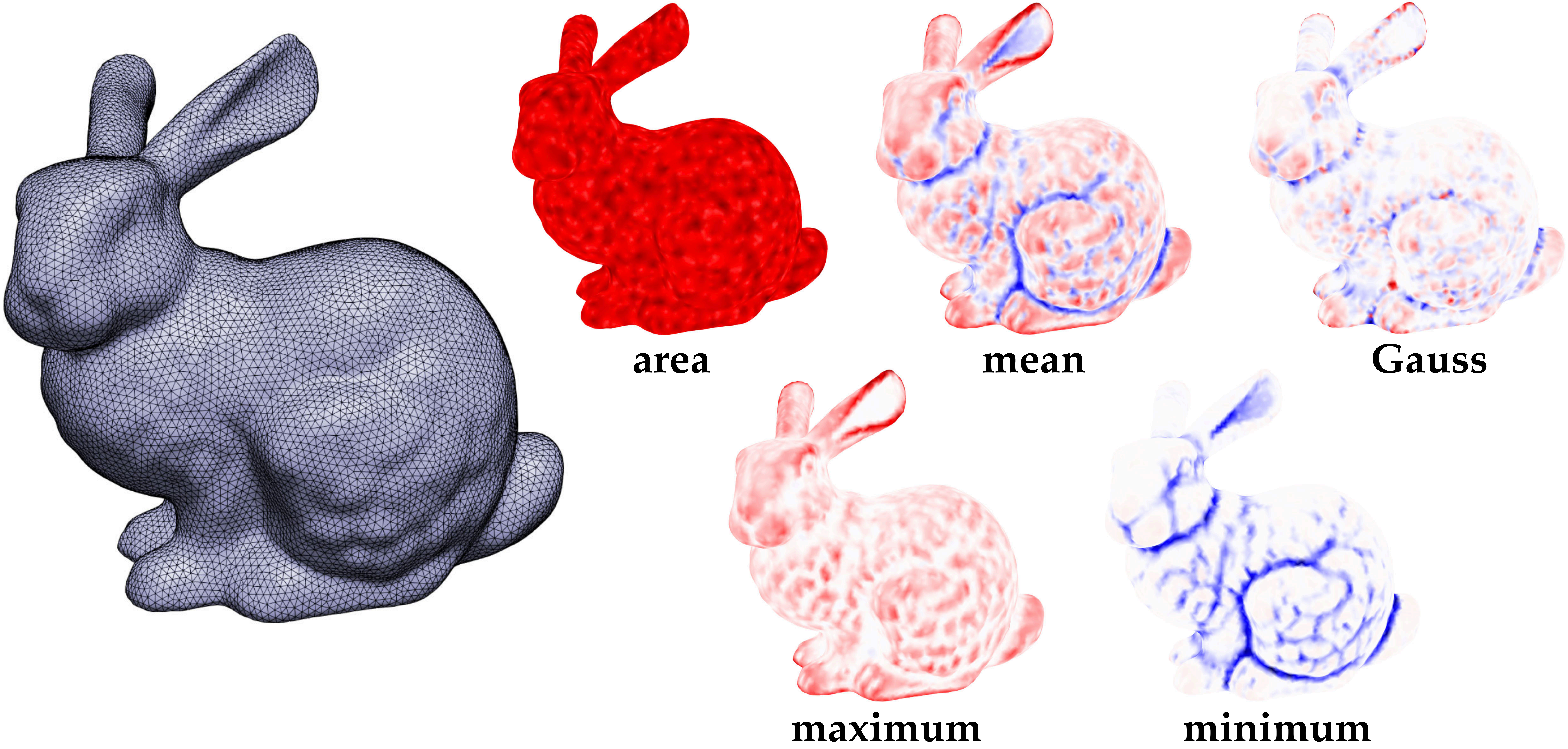
Discrete Curvature

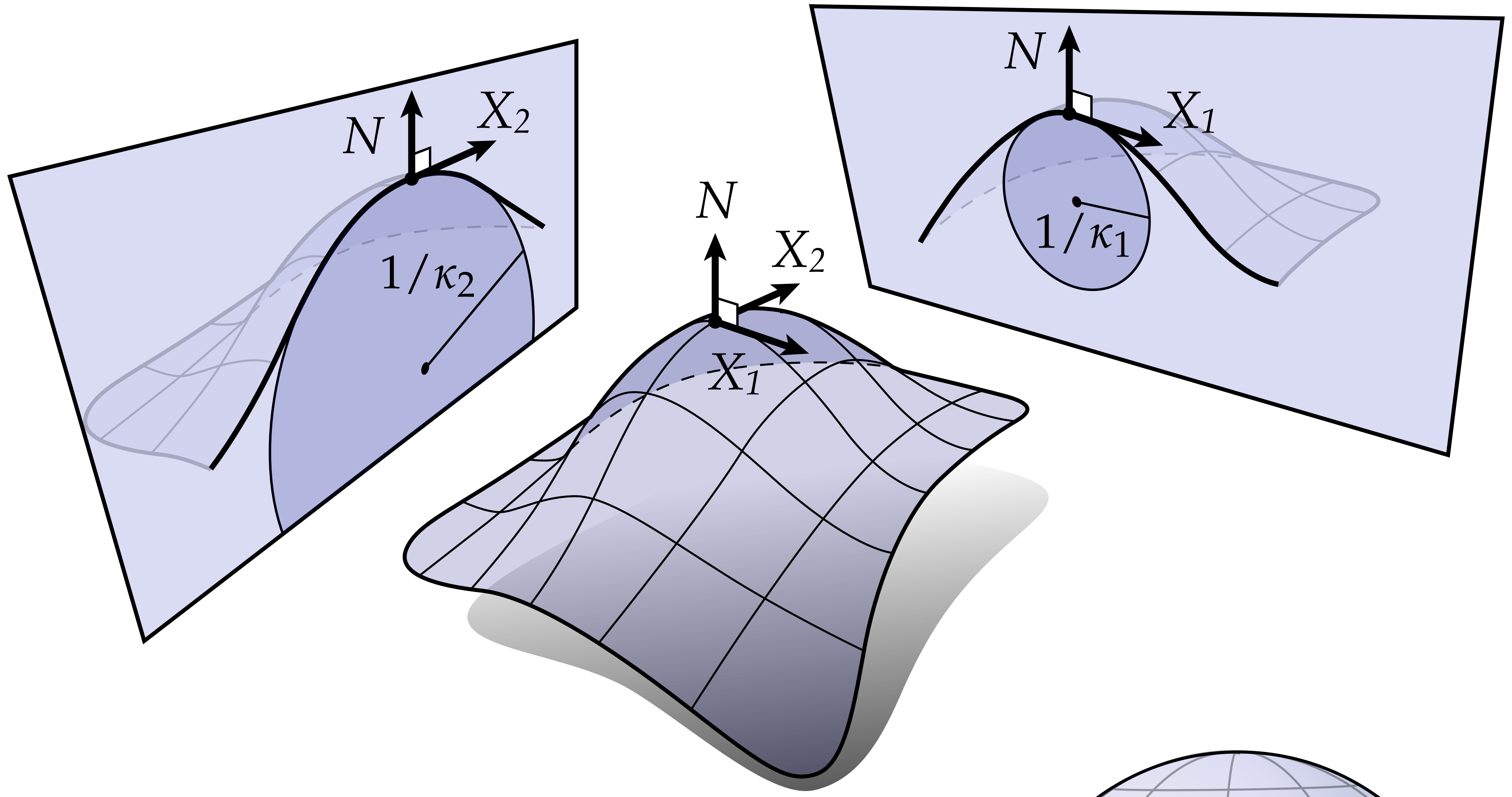


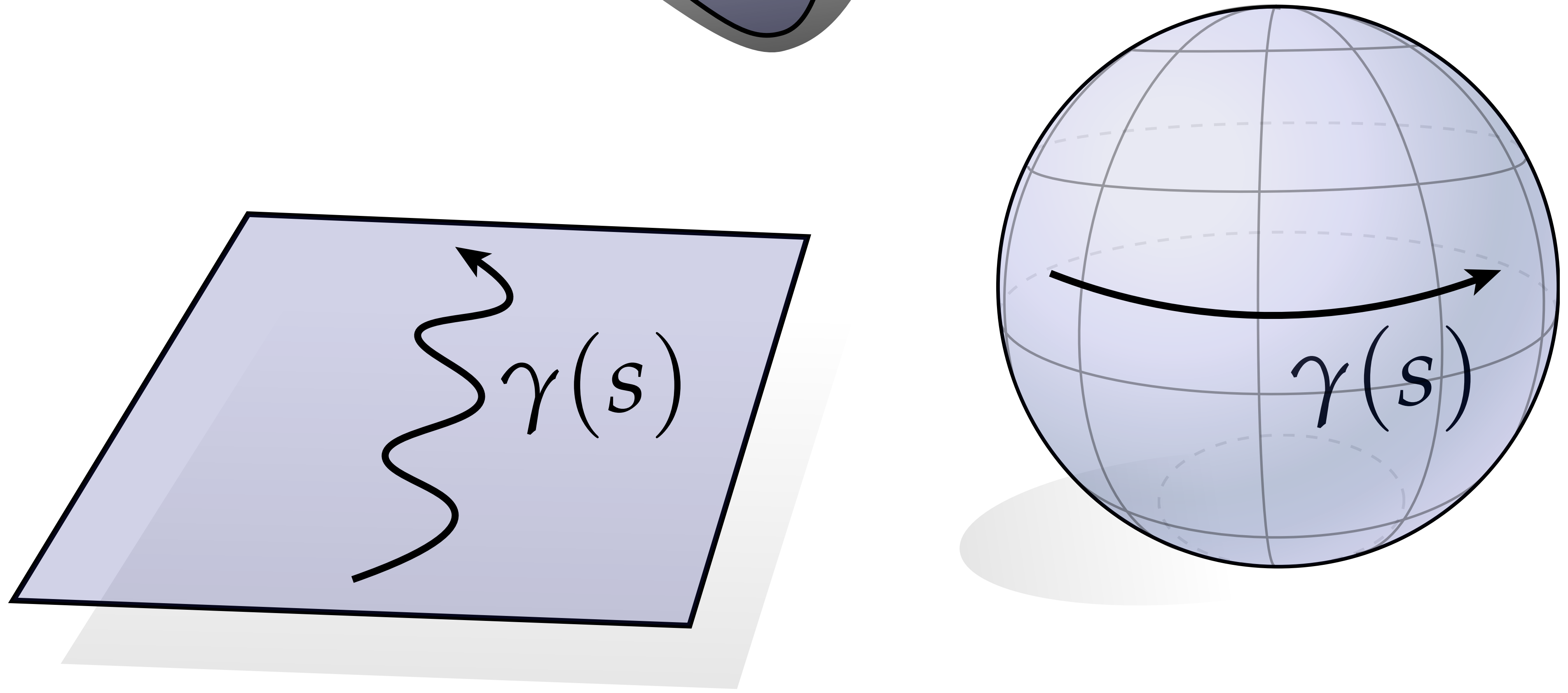
Discrete Curvature—Visualized



Curvature of Surfaces

- In smooth setting, had many different curvatures (normal, principal, Gauss, mean, geodesic, …)
- In discrete setting, appear to be many disconnected ways to discretize curvatures
- Actually, there is a unified viewpoint that helps explain many common choices…



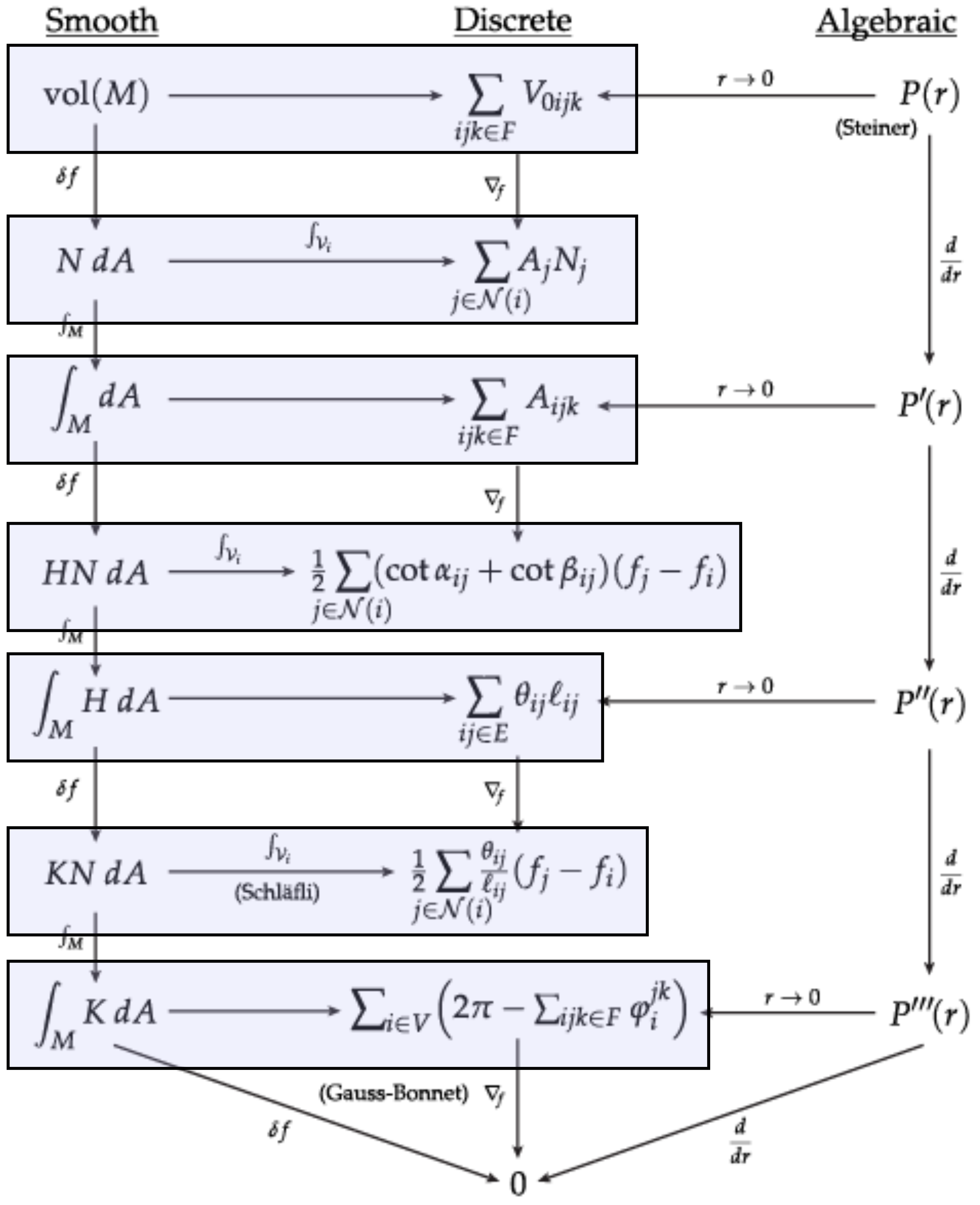


A Unified Picture of Discrete Curvature

- By making some connections between smooth and discrete surfaces, we get a unified picture of many different discrete curvatures scattered throughout the literature

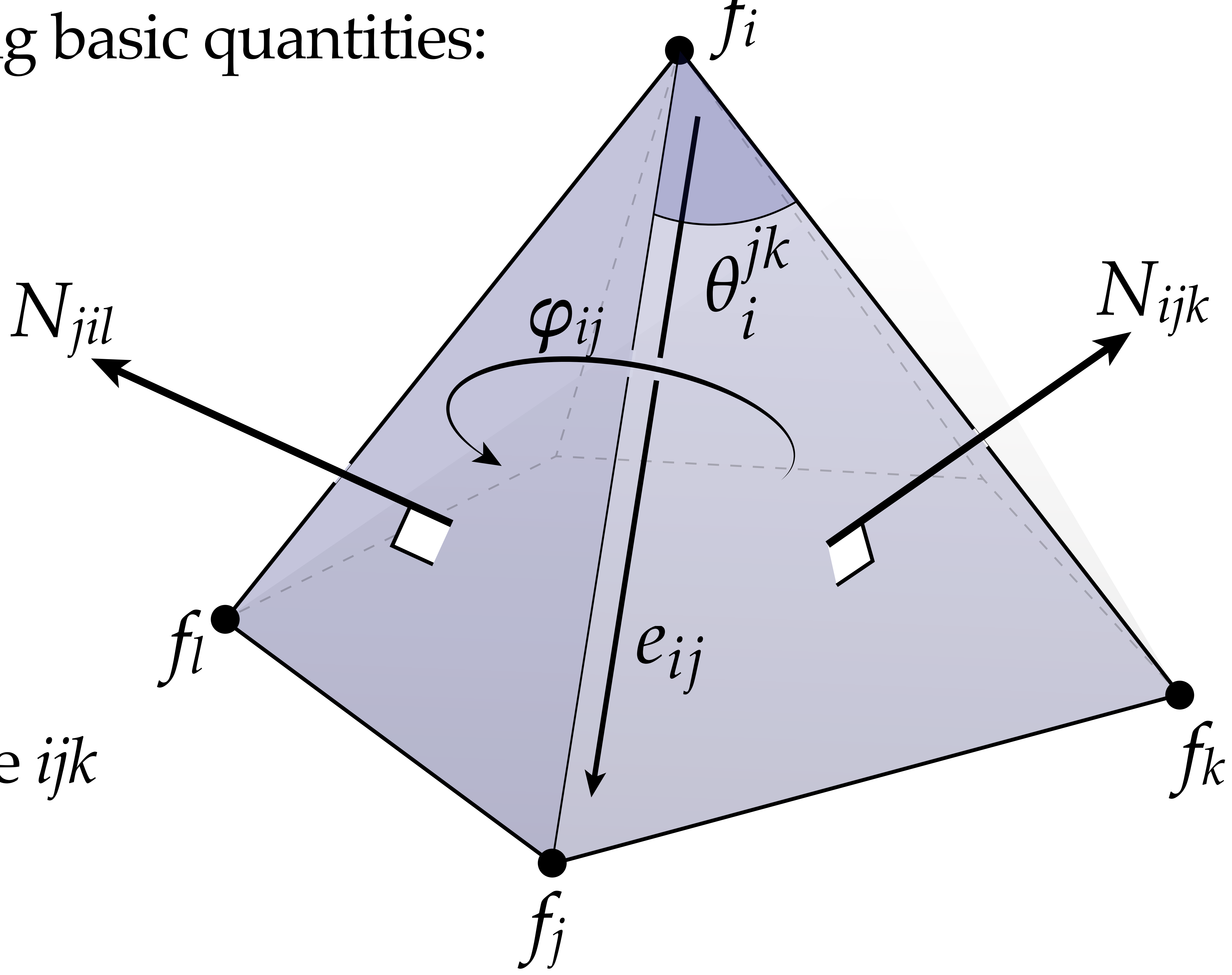
- To tell the full story we'll need a few pieces:
  - **geometric derivatives**
  - **Steiner polynomials**
  - **sequence of curvature variations**
  - **basic theorems** (Gauss-Bonnet, Schläfli, $\Delta f = 2HN$)

- Start with *integral* <u>viewpoint (1st lecture)</u>, then cover *variational* viewpoint (2nd lecture).



Quantities & Conventions

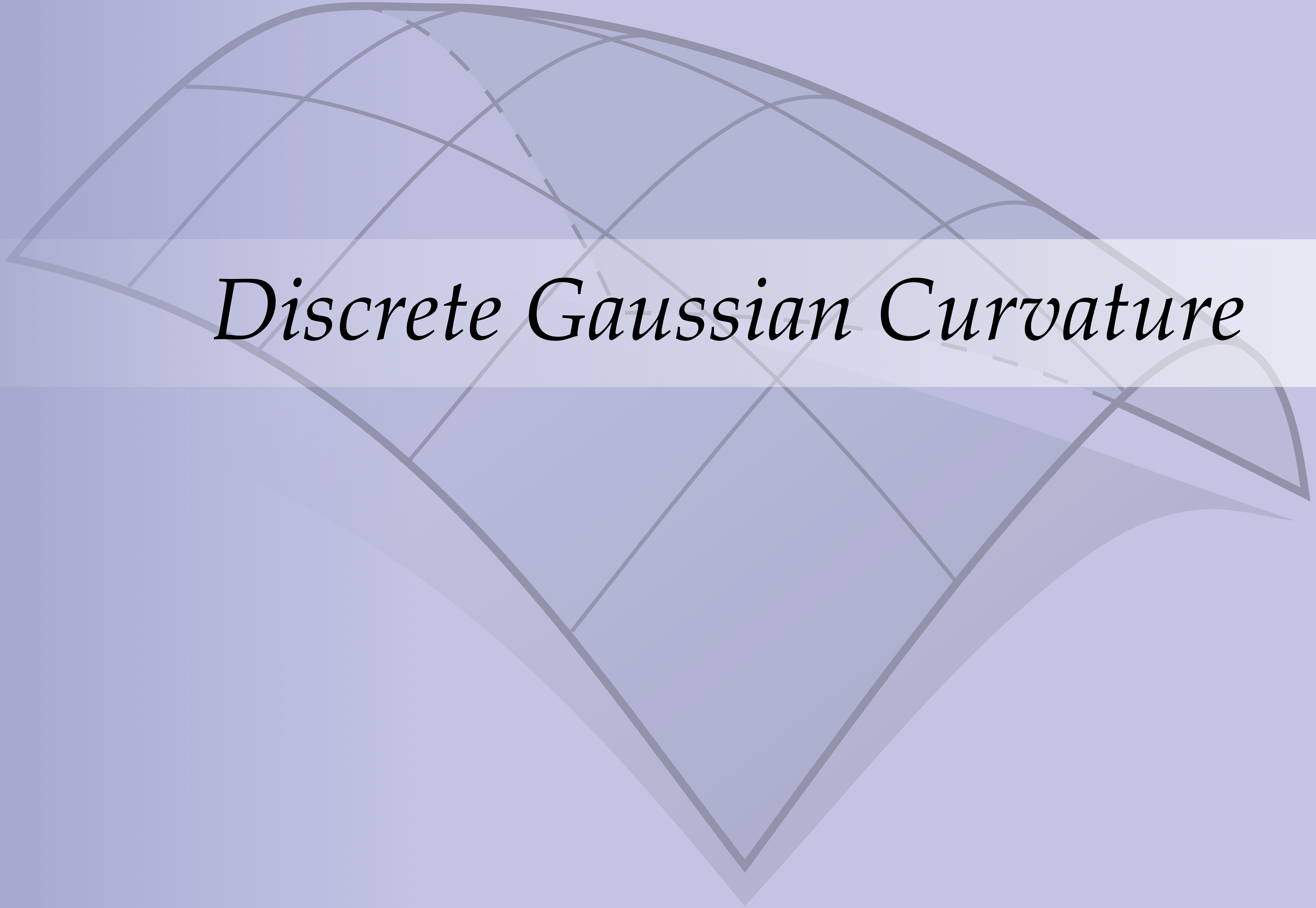
- Throughout we will consider the following basic quantities:
  - f_i — position of vertex i
  - e_{ij} — vector from i to j
  - ℓ_{ij} — length of edge ij
  - A_{ijk} — area of triangle ijk
  - N_{ijk} — unit normal of triangle ijk
  - θ_i^{jk} — interior angle at vertex i of triangle ijk
  - φ_{ij} — dihedral angle at oriented edge ij



$$\varphi_{ij} := \text{atan2}(\hat{e}_{ij} \cdot N_{ijk} \times N_{jil}, N_{ijk} \cdot N_{jil}), \qquad \hat{e}_{ij} := e_{ij}/\ell_{ij}$$

Q: Which of these quantities are discrete differential forms? (And what kind?)

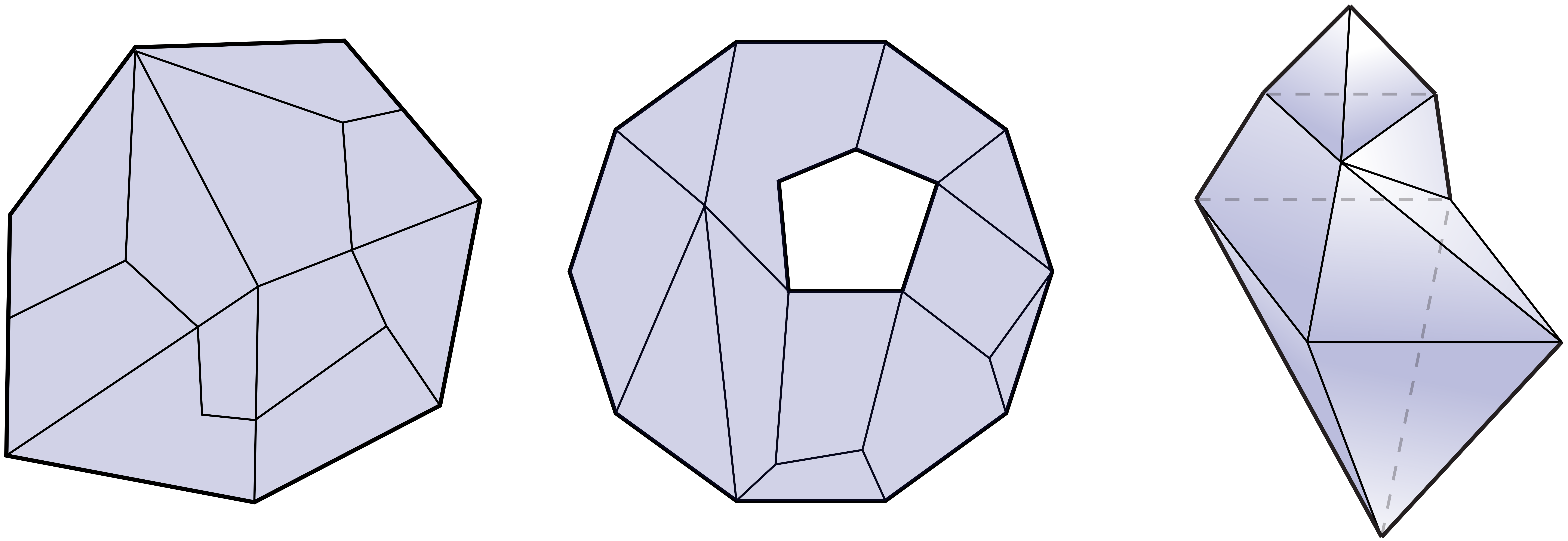
Discrete Gaussian Curvature



Euler Characteristic

The **Euler characteristic** of a polyhedral surface is the constant

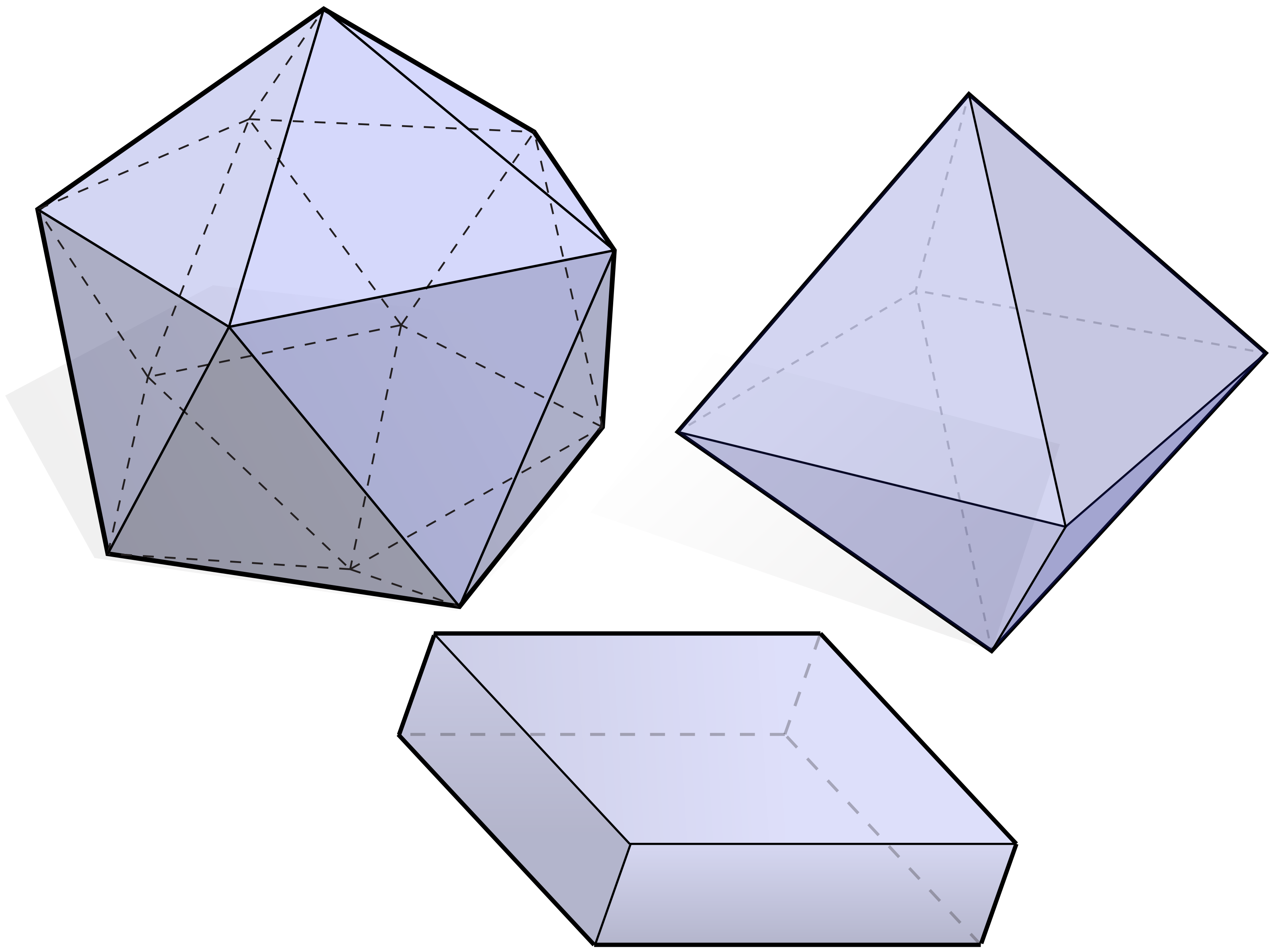
$$\boxed{\chi := V - E + F}$$



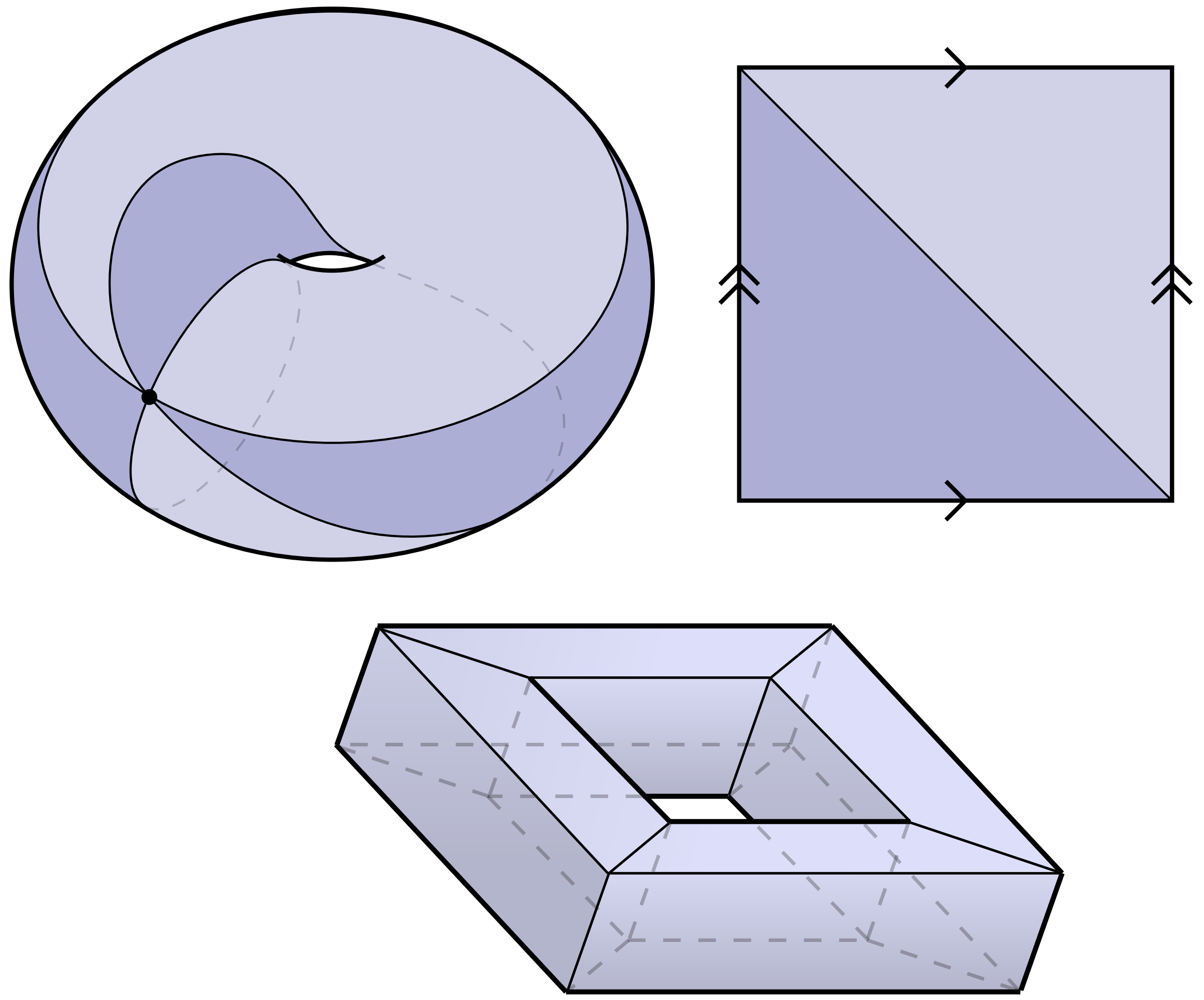
$\chi = 1$ $\chi = 0$ $\chi = 2$

Topological Invariance of the Euler Characteristic

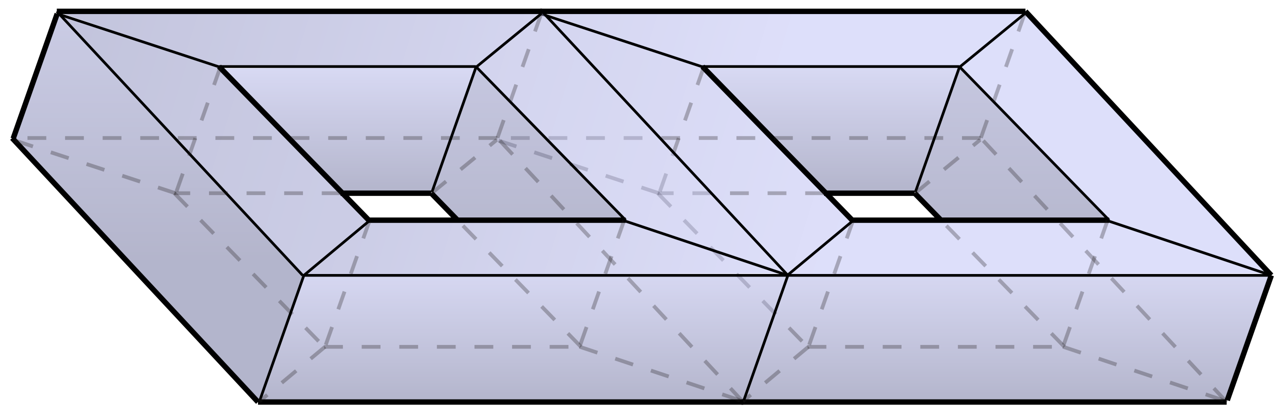
Fact. (L'Huilier) The Euler characteristic is a *topological invariant* of a polyhedral surface, i.e., it does not depend on the vertex positions or choice of tessellation. *E.g.*, for a torus of genus g, $\chi = 2\text{-}2g$:



$g = 0$



$g = 1$

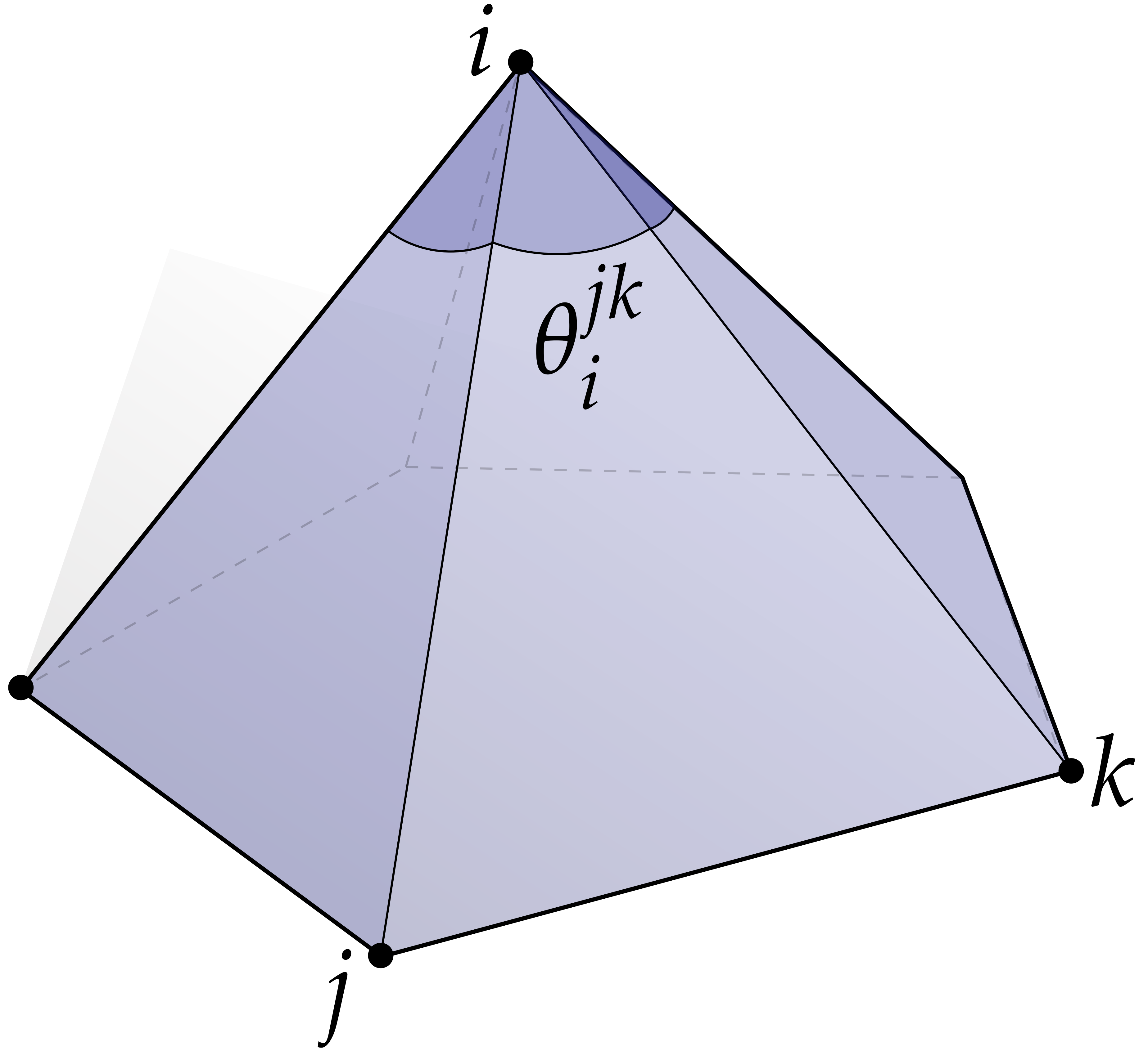


$g = 2$

Angle Defect

- The **angle defect** at a vertex i is the deviation of the sum of interior angles from the Euclidean angle sum of 2π:

$$\Omega_i := 2\pi - \sum_{ijk} \theta_i^{jk}$$



Intuition: how "flat" is the vertex?

Gaussian Curvature as Ratio of Ball Areas

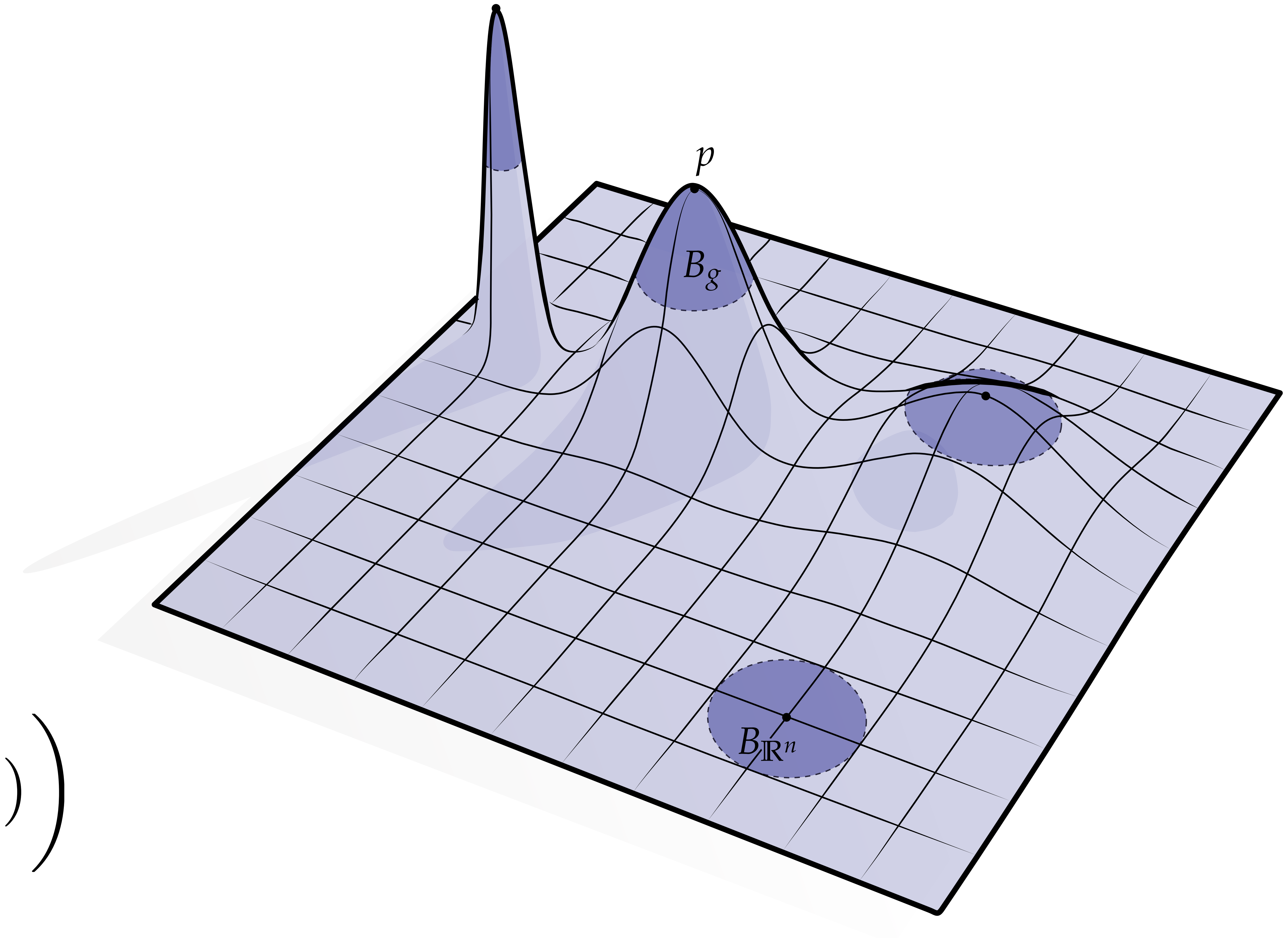
- Recall that Gaussian curvature captures deviation of the area of a small ball on a surface from a ball of equal radius in the plane

Roughly speaking,

$$K \;\propto\; 1 - \frac{|B_g|}{|B_{\mathbb{R}^2}|}$$

More precisely:

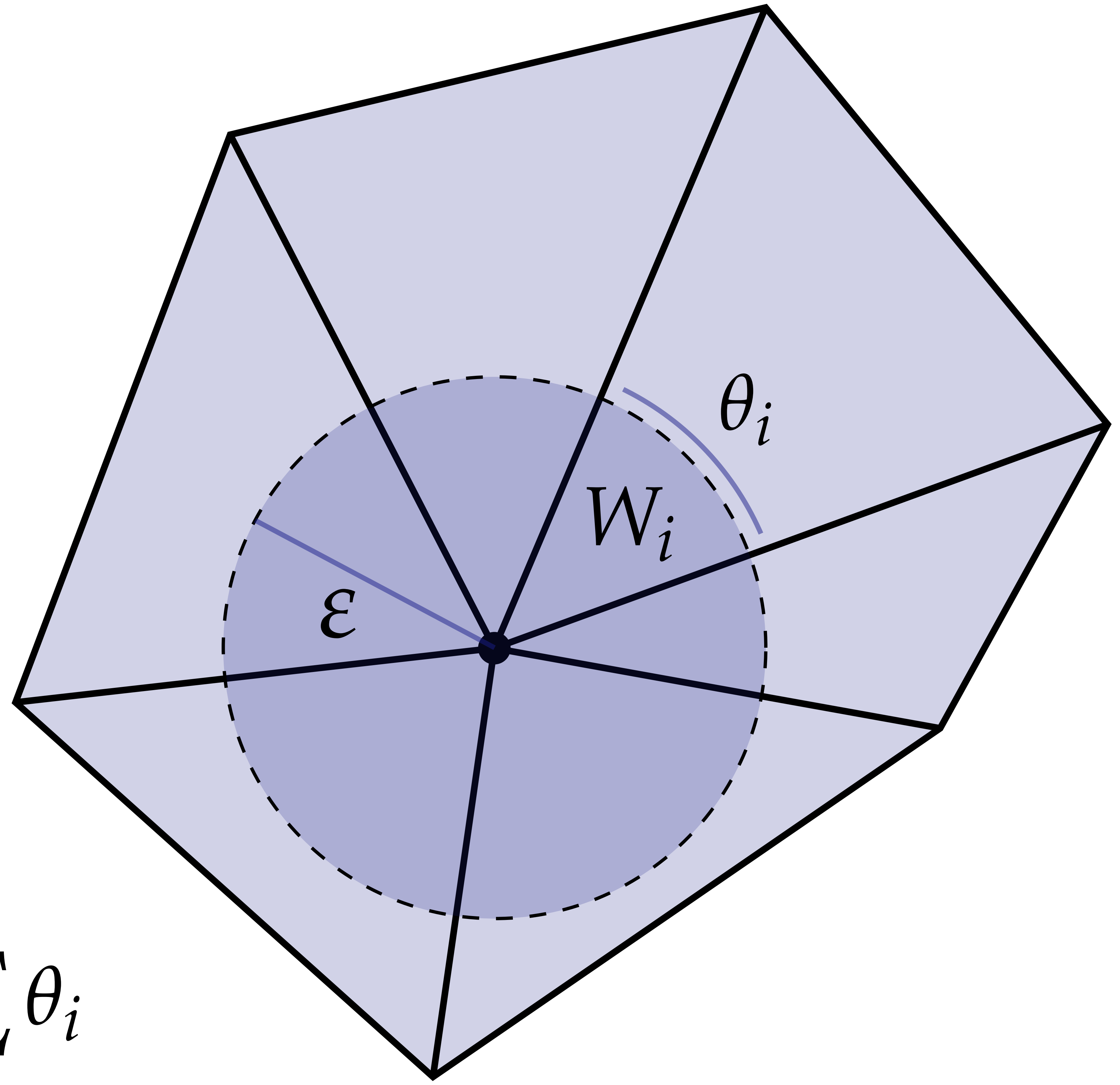
$$|B_g(p,\varepsilon)| = |B_{\mathbb{R}^2}(p,\varepsilon)| \left(1 - \frac{K}{12}\varepsilon^2 + O(\varepsilon^3)\right)$$



Discrete Gaussian Curvature—Intrinsic

- For small radii ε, we have

$$\frac{\varepsilon^2}{12}K \approx 1 - \frac{|B_g(\varepsilon)|}{|B_{\mathbb{R}^2}(\varepsilon)|}$$



Discrete case:

area of Euclidean ball $\quad |B_{\mathbb{R}^2}(\varepsilon)| = \pi\varepsilon^2$

area of geodesic "wedge" $\quad W_i(\varepsilon) = \frac{\theta_i}{2\pi}|B_{\mathbb{R}^2}| = \frac{1}{2}\varepsilon^2\theta_i$

area of geodesic ball $\quad |B_g(\varepsilon)| = \sum_i W_i(\varepsilon) = \frac{\varepsilon^2}{2}\sum_i \theta_i$

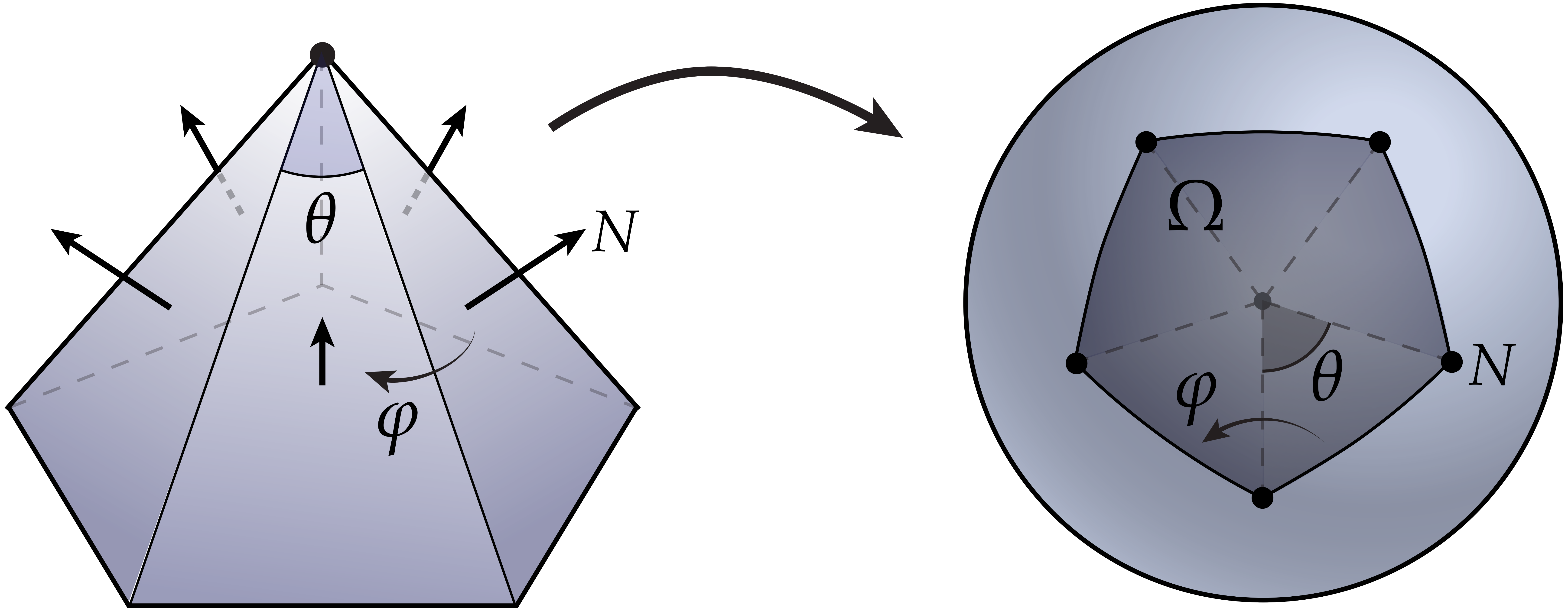
Then

$$\frac{\varepsilon^2}{12}K = 1 - \frac{1}{2\pi}\sum_i \theta_i \quad \Longleftrightarrow \quad \boxed{2\pi - \sum_i \theta_i = \frac{1}{6}\pi\varepsilon^2 K}$$

(can think of angle defect as *integrated* curvature)

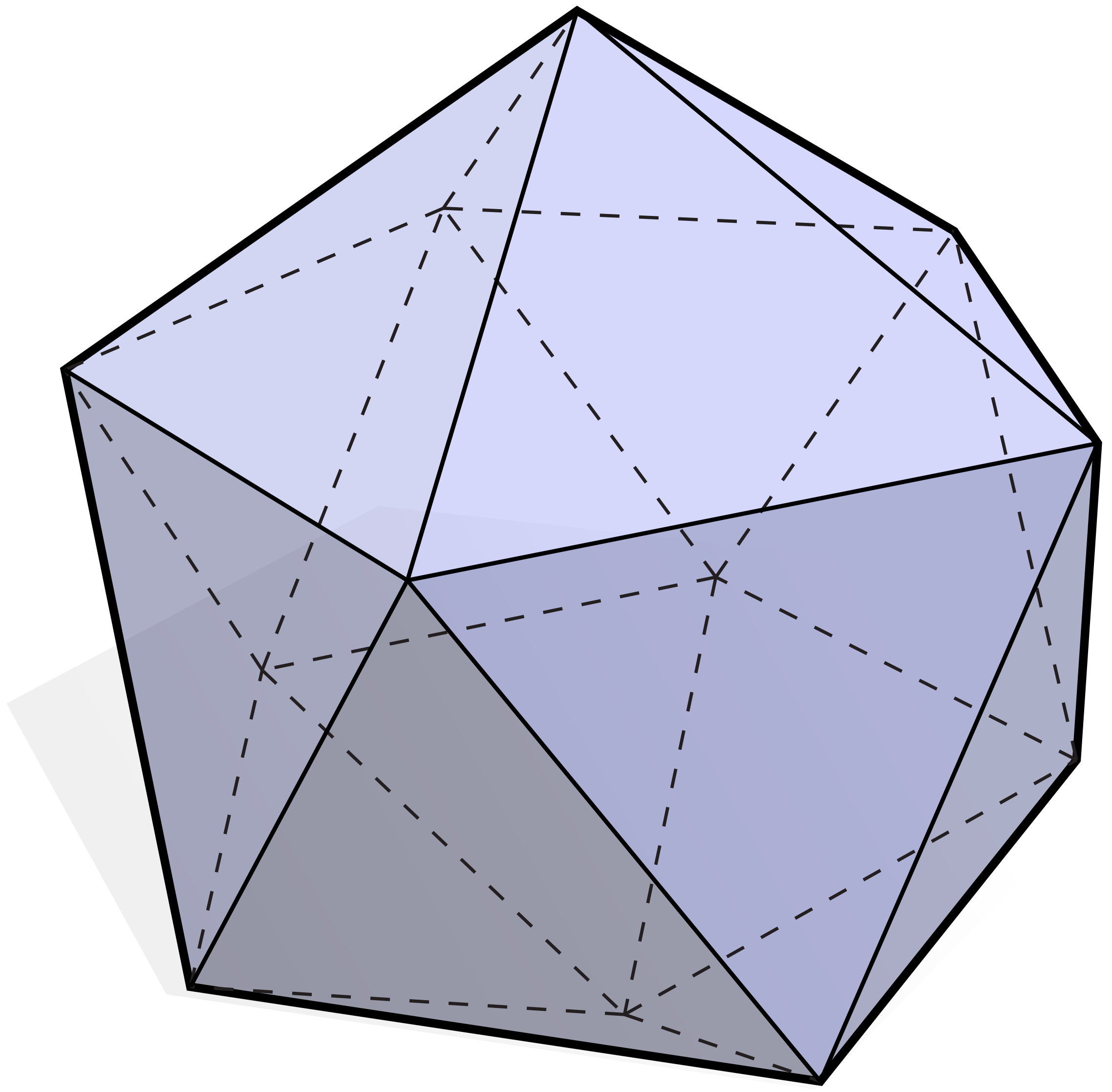
Discrete Gaussian Curvature—Extrinsic

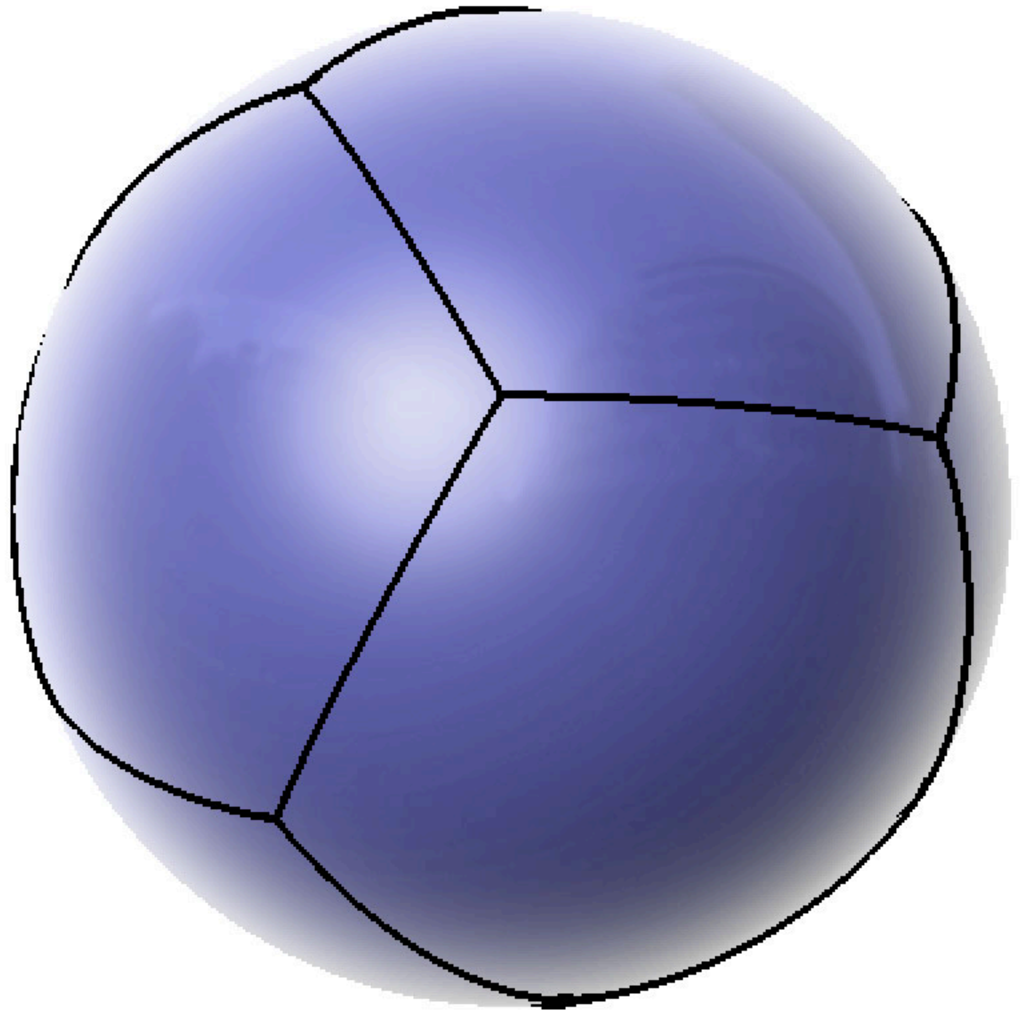
- Consider the discrete Gauss map:
  - unit normals on surface become points on the sphere
  - connecting these points to the sphere center makes a new vertex star
  - dihedral angles on surface become interior angles on sphere and vice versa
  - **angle defect on surface becomes area on the sphere**



Total Angle Defect Theorem

- Consider a closed convex polyhedron in R^3
- **Q:** Given each angle defect is an area on the sphere, what might we guess about <u>total</u> angle defect?
- **A:** Equal to area of unit sphere! 4π
- Can in fact argue that total angle defect is equal to 4π for *any* polyhedron with spherical topology
- **Theorem.** For any polyhedron of genus g, total angle defect is equal to $2\pi(2\text{-}2g)$.
- Does this theorem remind you of anything?



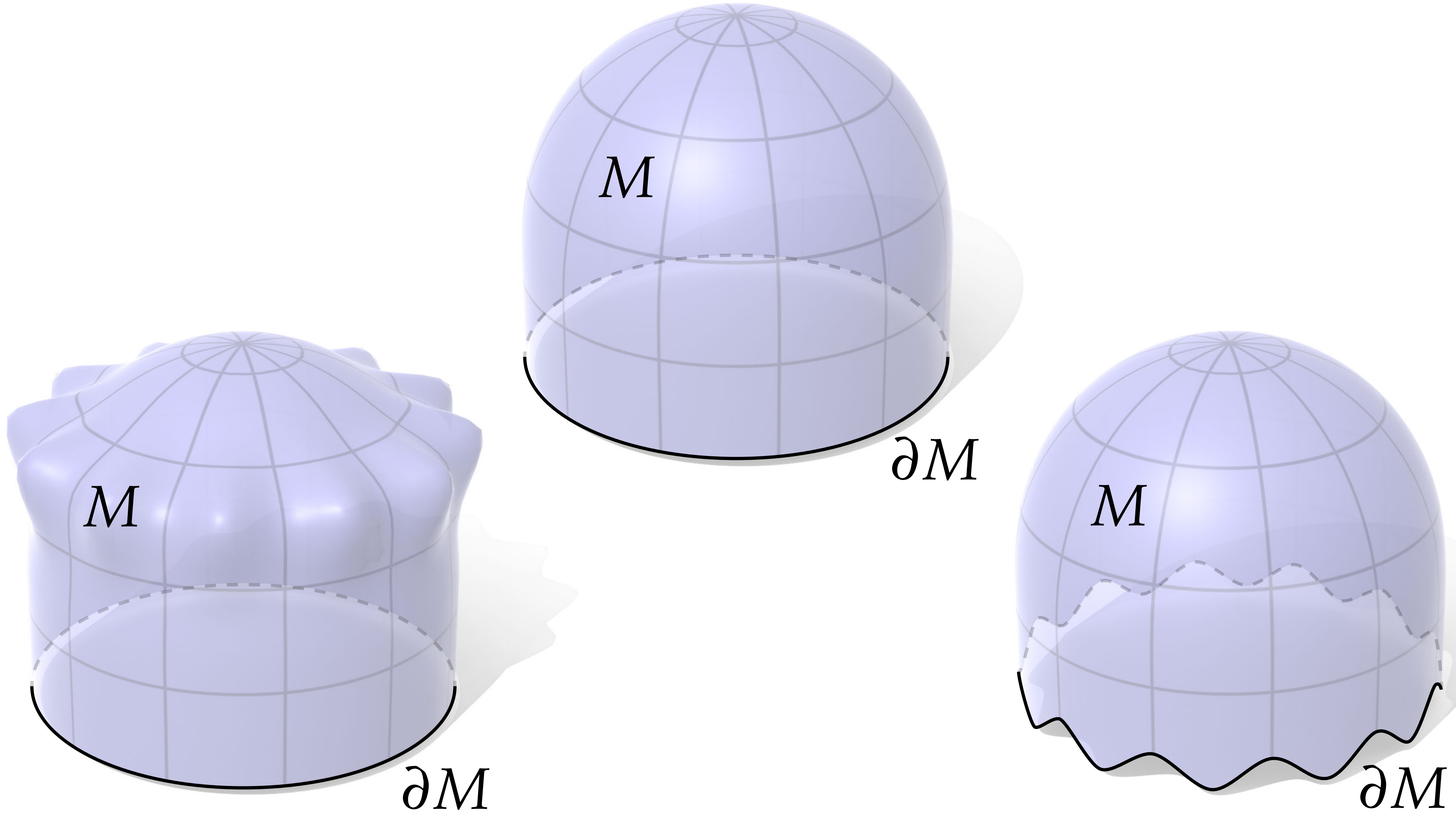


Review: Gauss-Bonnet Theorem

- Gauss-Bonnet theorem says total Gaussian curvature plus geodesic curvature along the boundary is always equal to 2π times *Euler characteristic* χ
- "Total angle defect theorem" is really a discrete analogue of Gauss-Bonnet
- **Q:** How do we generalize discrete theorem to surfaces with boundary?

Gauss-Bonnet

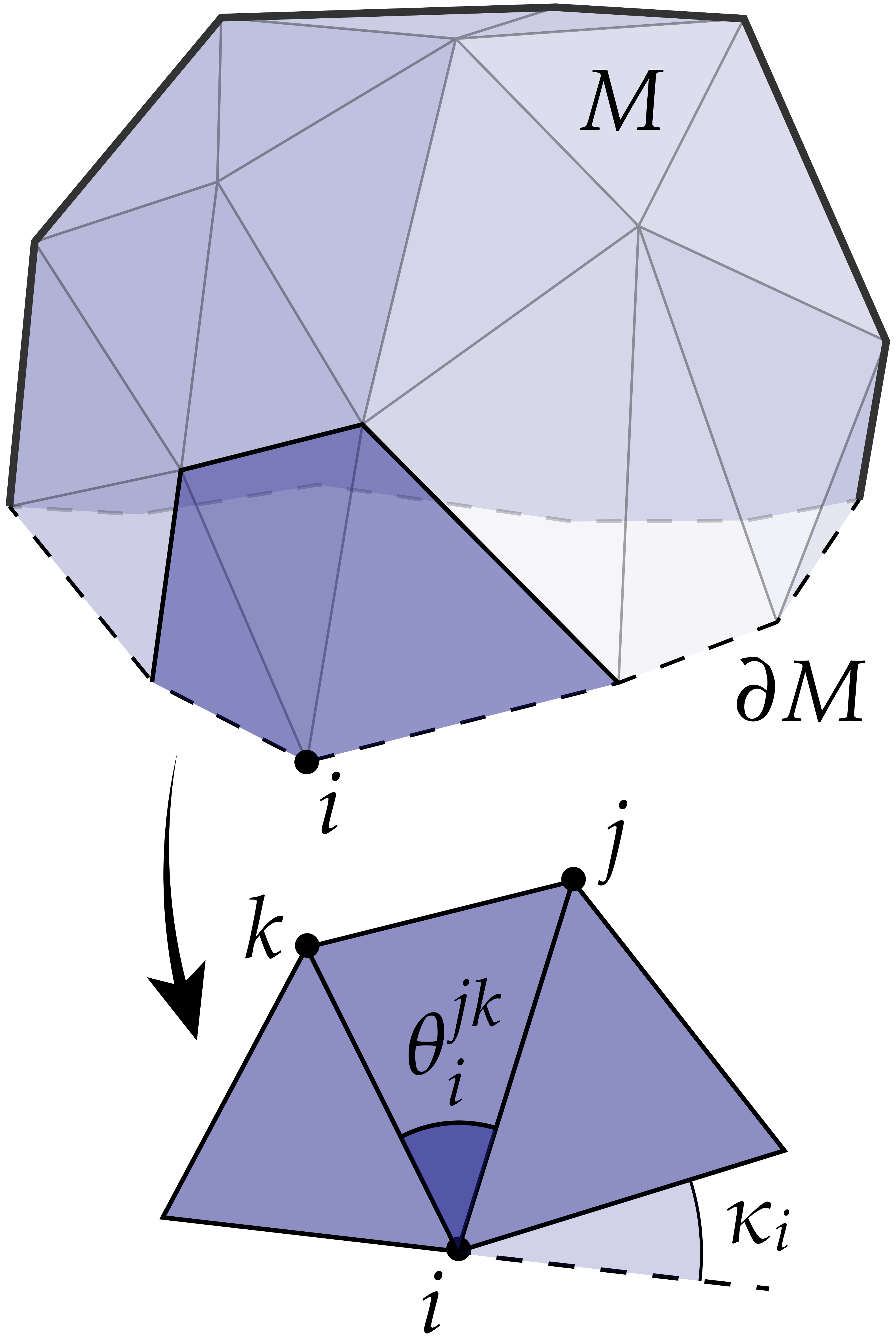
$$\int_M K\,dA + \int_{\partial M} \kappa_g\,ds = 2\pi\chi$$



Discrete Boundary Curvature

- Angle defect Ω_i provides discrete analogue of Gaussian curvature K
- Intuitively: captures failure of vertex star to be "flattenable"
- Since every boundary vertex star can be flattened without stretching, *boundary vertices have zero Gaussian curvature*
- But can still measure how straight boundary itself is, via discrete geodesic curvature:

$$\kappa_i := \pi - \sum_{ijk} \theta_i^{jk}$$



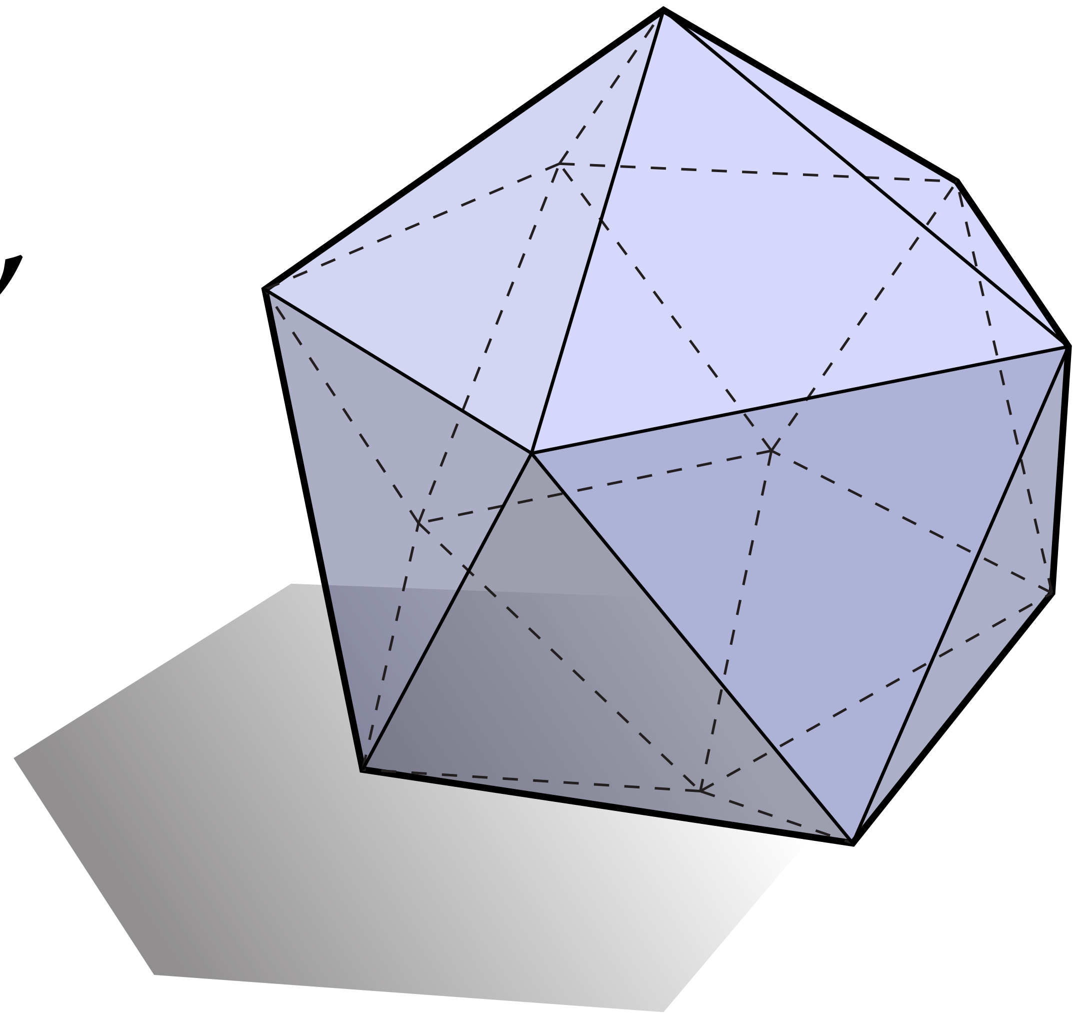
Discrete Gauss Bonnet Theorem

Theorem. For a smooth surface M with Gauss curvature K and geodesic curvature κ_g,

$$\int_M K\,dA + \int_{\partial M} \kappa_g\,ds = 2\pi\chi$$

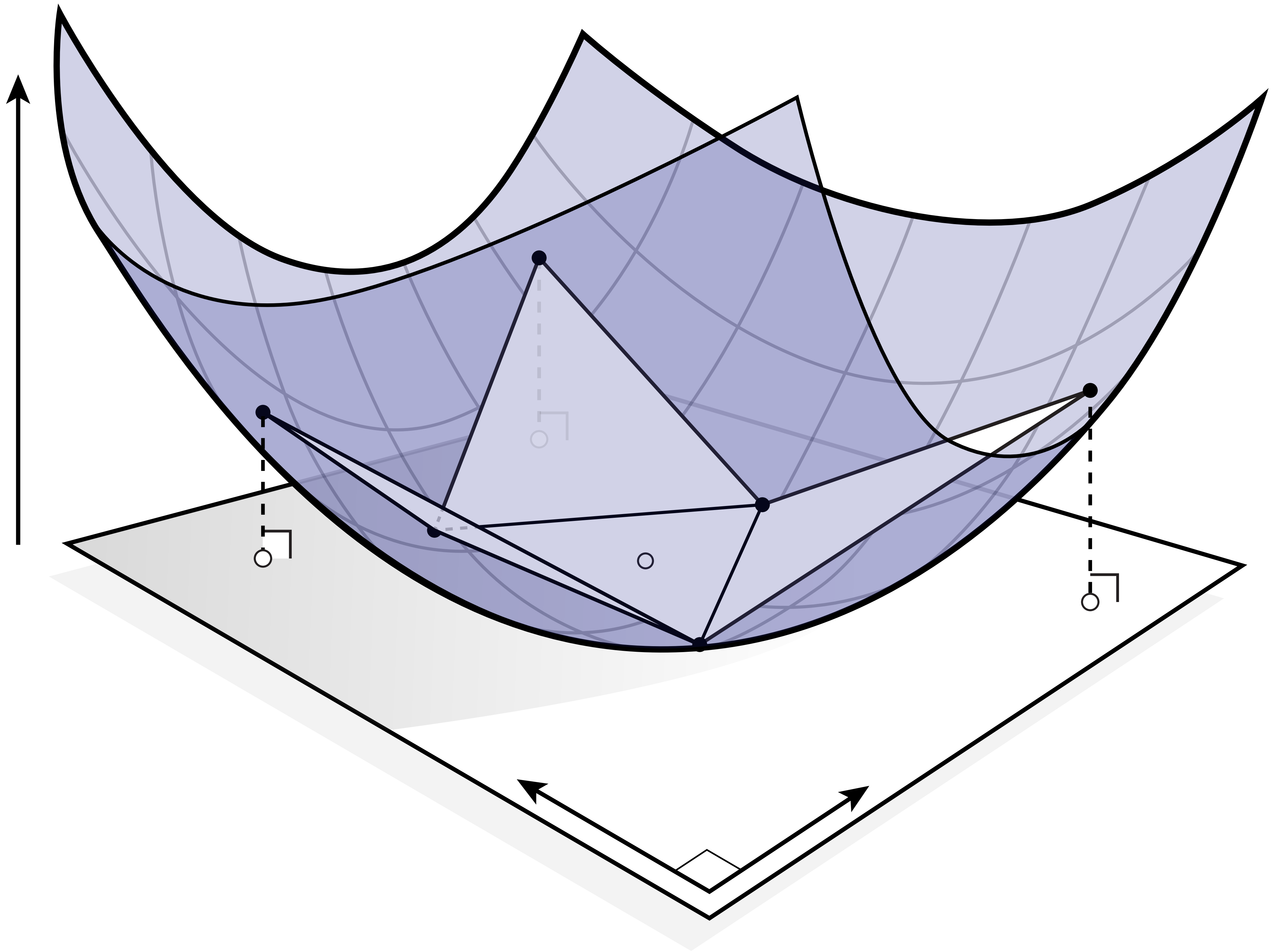
Theorem. For a simplicial surface $K = (V,E,F)$ with interior angle defects Ω_i, boundary angle defects κ_i,

$$\sum_{i \in \mathrm{int} V} \Omega_i + \sum_{i \in \partial V} \kappa_i = 2\pi\chi$$



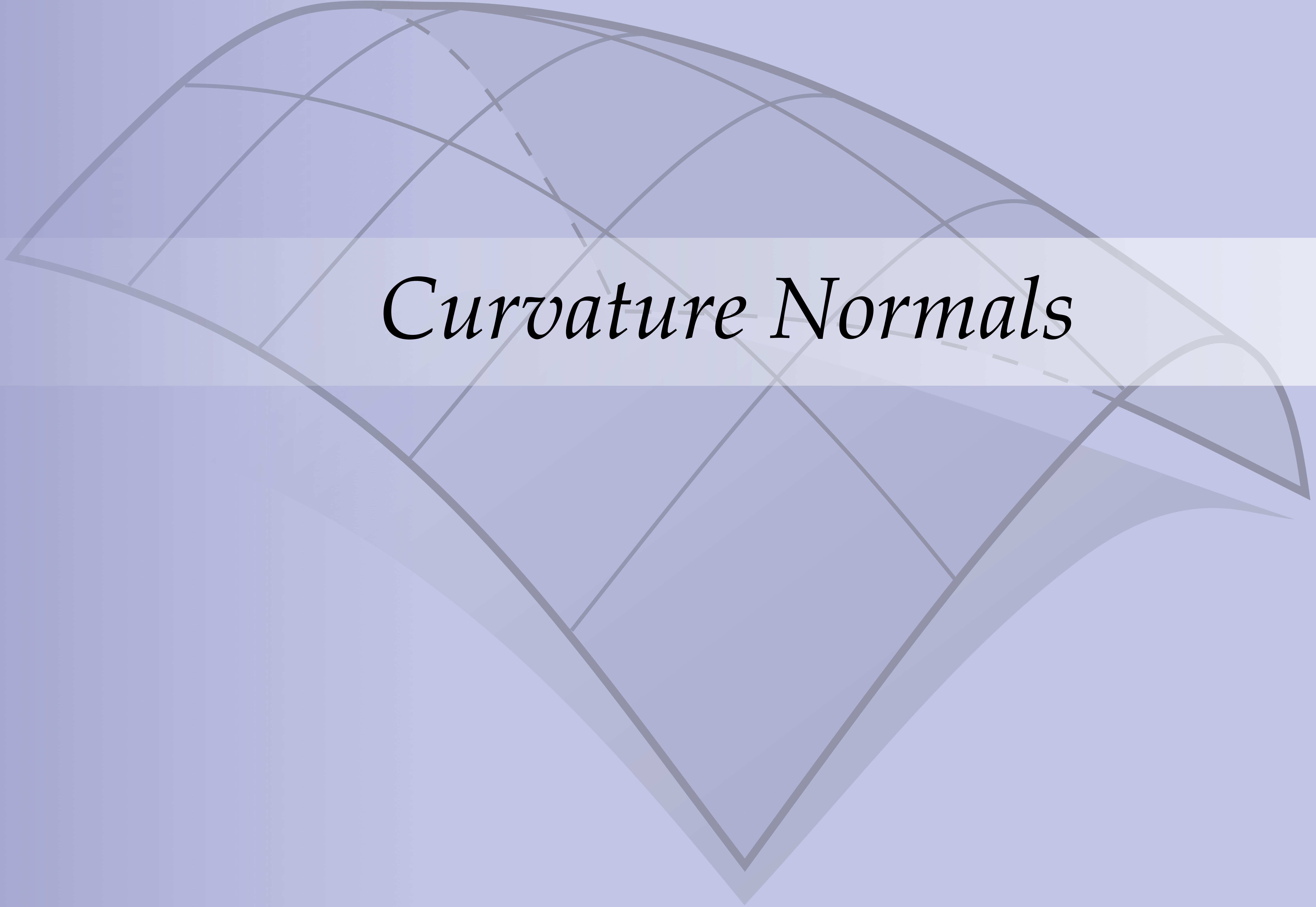
Approximating Gaussian Curvature

- Many other ways to <u>approximate</u> Gaussian curvature
- *E.g.*, locally fit quadratic functions, compute smooth Gaussian curvature
- Which way is "best"?
  - values from quadratic fit won't satisfy Gauss-Bonnet
  - angle defects won't converge[1] unless vertex valence is 4 or 6
- In general, no best way; each choice has its own pros & cons

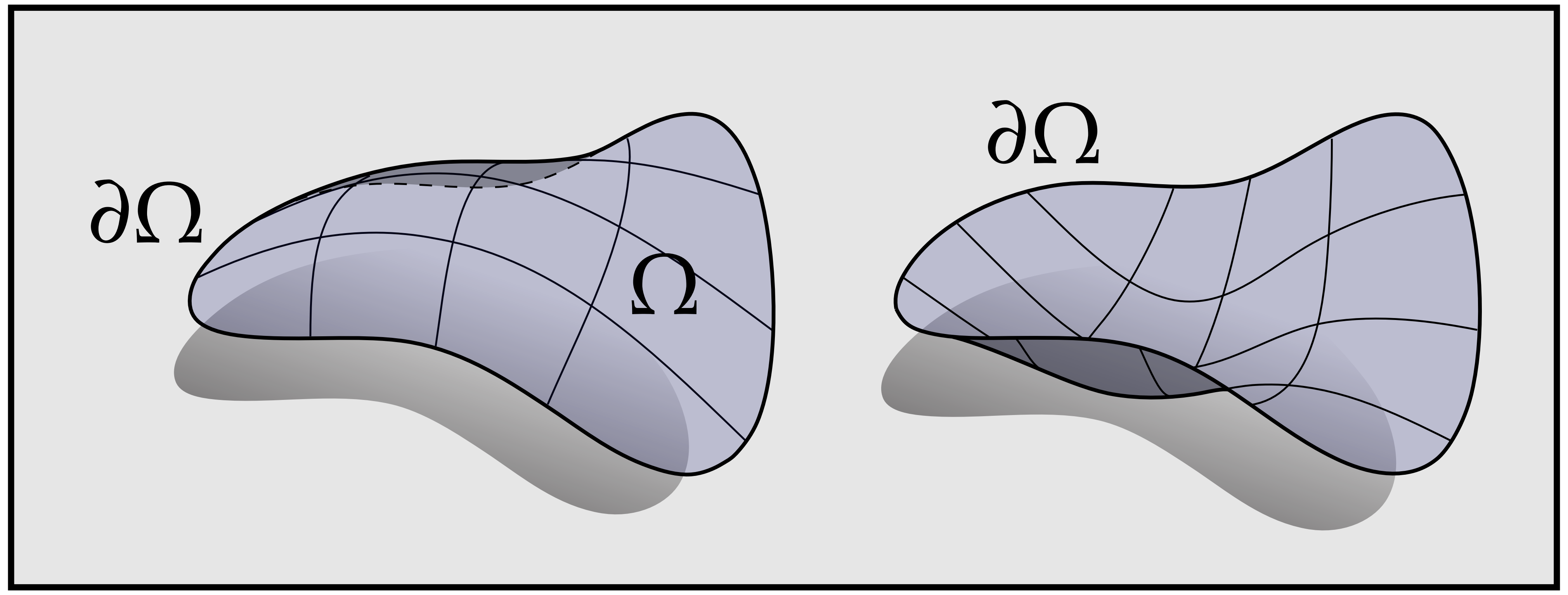


[1]Borrelli, Cazals, Morvan, *"On the angular defect of triangulations and the pointwise approximation of curvatures"*

Curvature Normals



Curvature Normals



- Earlier we saw vector area, which was the integral of the 2-form NdA
- This 2-form is one of three quantities we can naturally associate with a surface:

$$\frac{1}{2}\, df \wedge df \quad = \quad NdA \qquad \textbf{(area normal)}$$

$$\frac{1}{2}\, df \wedge dN \quad = \quad HNdA \qquad \textbf{(mean curvature normal)}$$

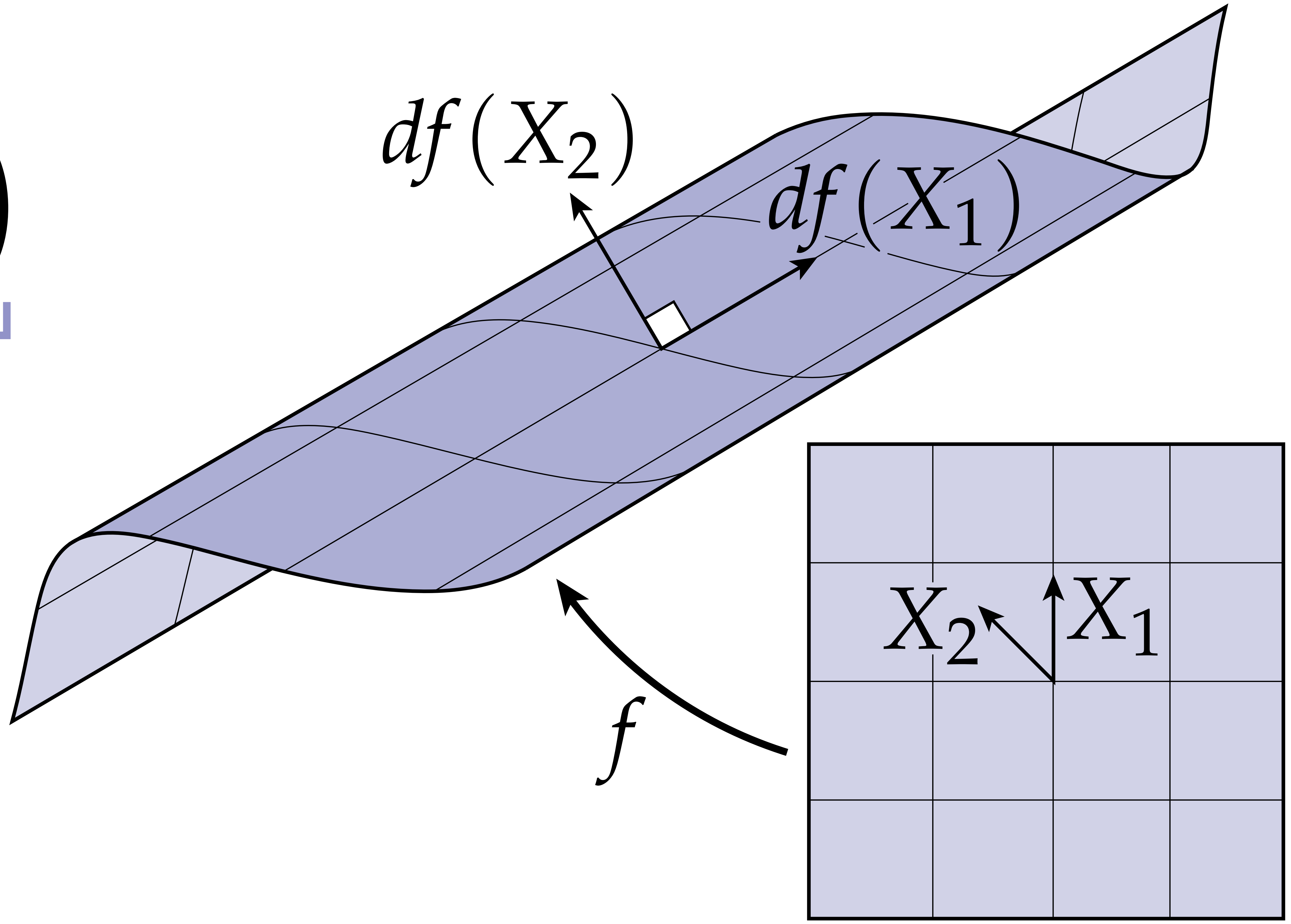
$$\frac{1}{2}\, dN \wedge dN \quad = \quad KNdA \qquad \textbf{(Gauss curvature normal)}$$

- Effectively *"mixed areas"* of ordinary surface area and area on sphere

Review: Principal Curvature

- **Principal directions** X_1, X_2 describe directions of min/max bending
- Key relationship that is helpful in <u>many</u> derivations:

$$\overbrace{dN}^{\text{change in normal}}\underbrace{(X_i)}_{\text{ith principal direction}} = \overbrace{\kappa_i}^{\text{ith principal curvature}}\underbrace{df(X_i)}_{\text{ith principal direction (R^3)}}$$



Curvature Normals—Derivation

For any surface f with normals N, we have:

$$\begin{aligned} df \wedge df(X_1, X_2) = {} & df(X_1) \times df(X_2) - df(X_2) \times df(X_1) = \\ & 2df(X_1) \times df(X_2) = \boxed{2NdA(X_1, X_2)} \end{aligned}$$

$$\begin{aligned} df \wedge dN(X_1, X_2) = {} & df(X_1) \times dN(X_2) - df(X_2) \times dN(X_1) = \\ & \kappa_1 df(X_1) \times df(X_2) - \kappa_2 df(X_2) \times df(X_1) = \\ & (\kappa_1 + \kappa_2) df(X_1) \times df(X_2) = \boxed{2HNdA(X_1, X_2)} \end{aligned}$$

$$\begin{aligned} dN \wedge dN(X_1, X_2) = {} & dN(X_1) \times dN(X_2) - dN(X_2) \times dN(X_1) = \\ & \kappa_1 \kappa_2 df(X_1) \times df(X_2) - \kappa_2 \kappa_1 df(X_2) \times df(X_1) = \\ & 2K df(X_1) \times df(X_2) = \boxed{2KNdA(X_1, X_2)} \end{aligned}$$

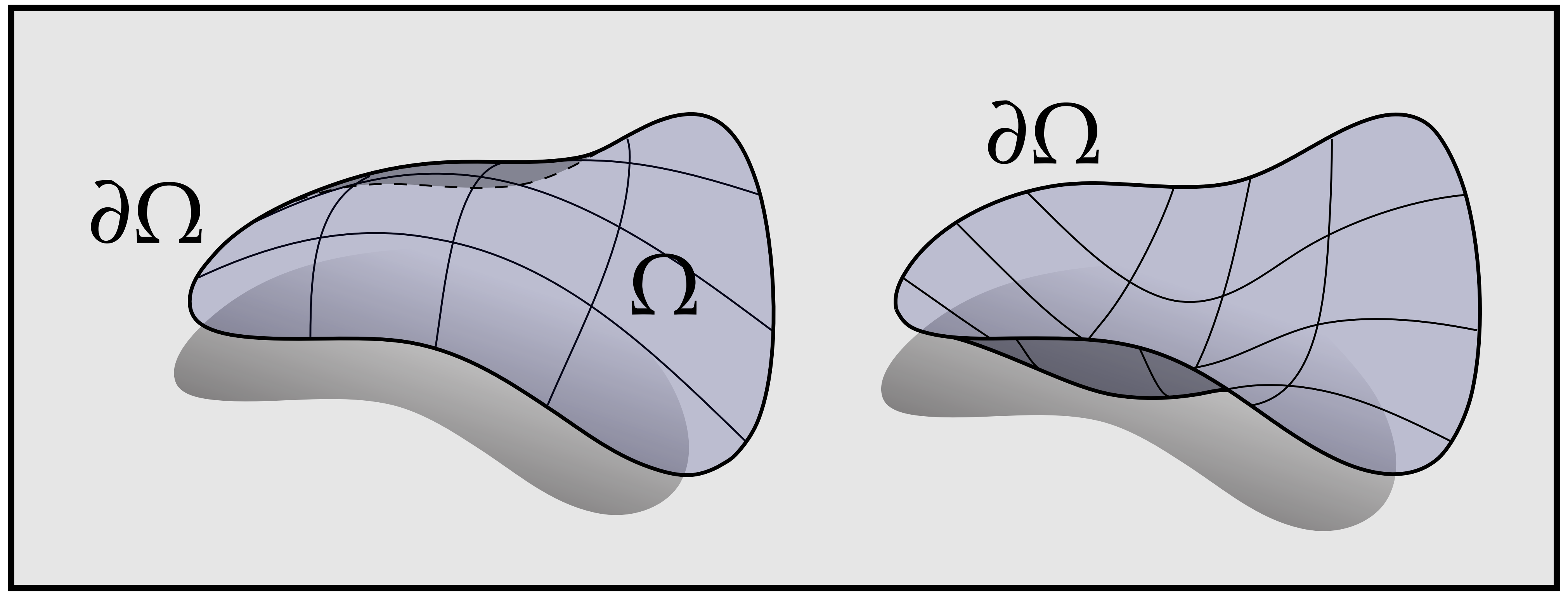
Discrete Vector Area

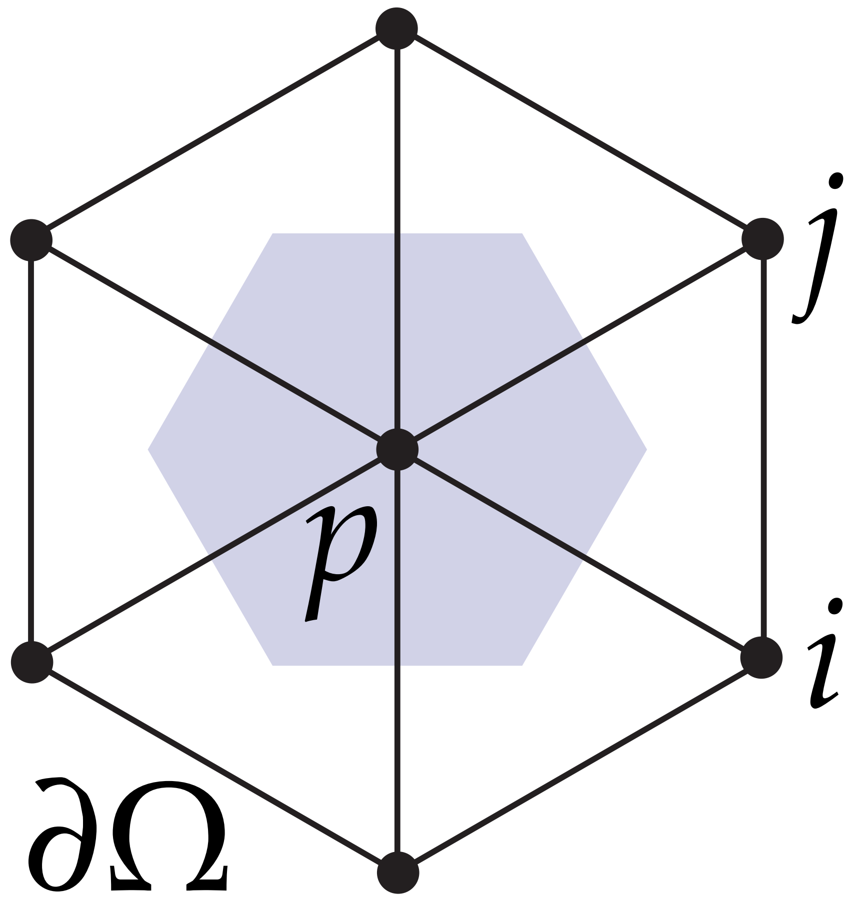
- Recall smooth vector area: $\int_\Omega N\, dA = \frac{1}{2}\int_\Omega df \wedge df = \frac{1}{2}\int_{\partial\Omega} f \times df$
- **Idea:** Integrate NdA over dual cell to get normal at vertex p

$$\frac{1}{3}\int_\Omega N\, dA = \frac{1}{6}\int_{\partial\Omega} f \times df =$$

$$\frac{1}{6}\sum_{ij\in\partial\Omega}\int_{e_{ij}} f \times df =$$

$$\frac{1}{6}\sum_{ij\in\partial\Omega}\frac{f_i+f_j}{2} \times (f_j - f_i) = \boxed{\frac{1}{6}\sum_{ij\in\partial\Omega} f_i \times f_j}$$





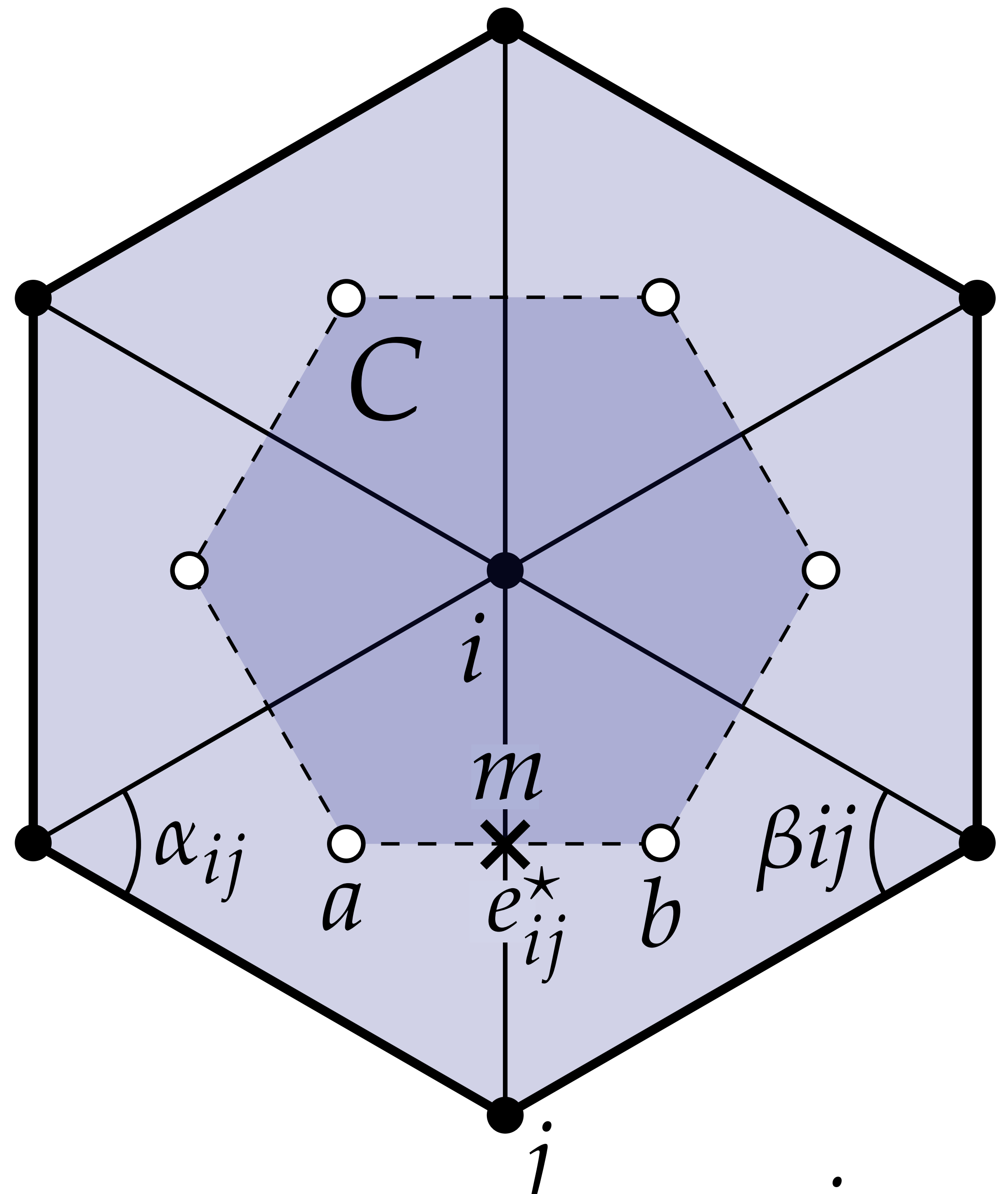
Q: What kind of quantity is the final expression?

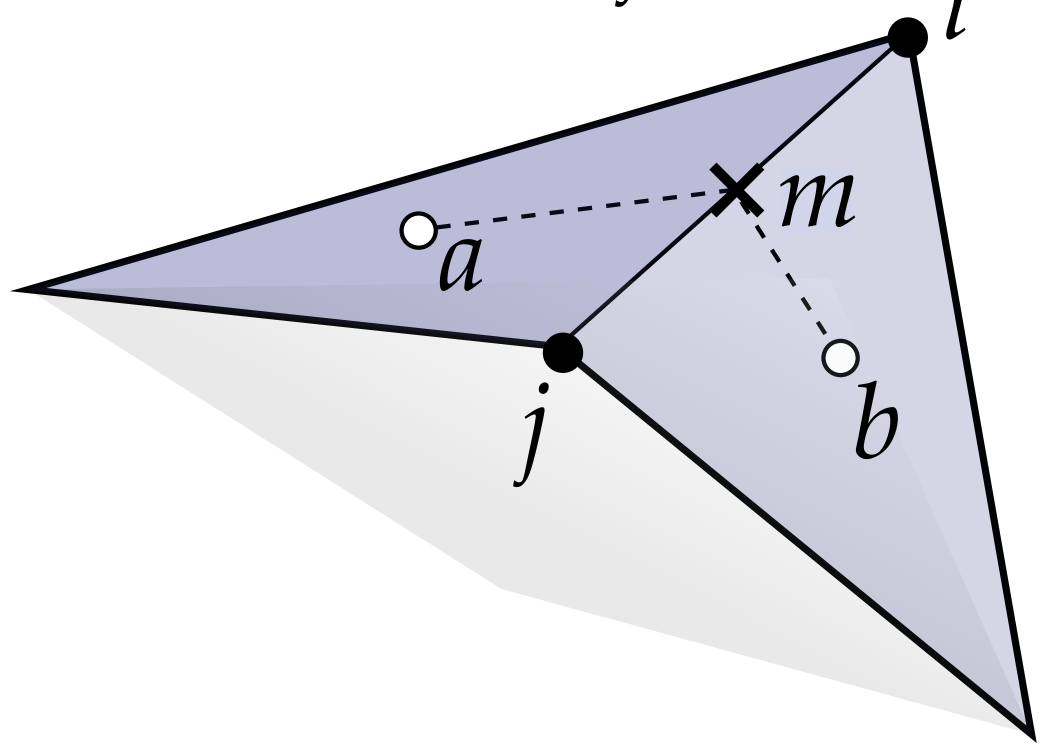
Discrete Mean Curvature Normal

Similarly, integrating HN over a circumcentric dual cell C yields

$$\int_C HN\,dA = \int_C df \wedge dN = \int_C dN \wedge df = \int_C d(N \wedge df) =$$

$$\int_{\partial C} N \wedge df = \sum_j \int_{e^\star_{ij}} N \wedge df = \sum_j N_a \times (m-a) + N_b \times (b-m)$$





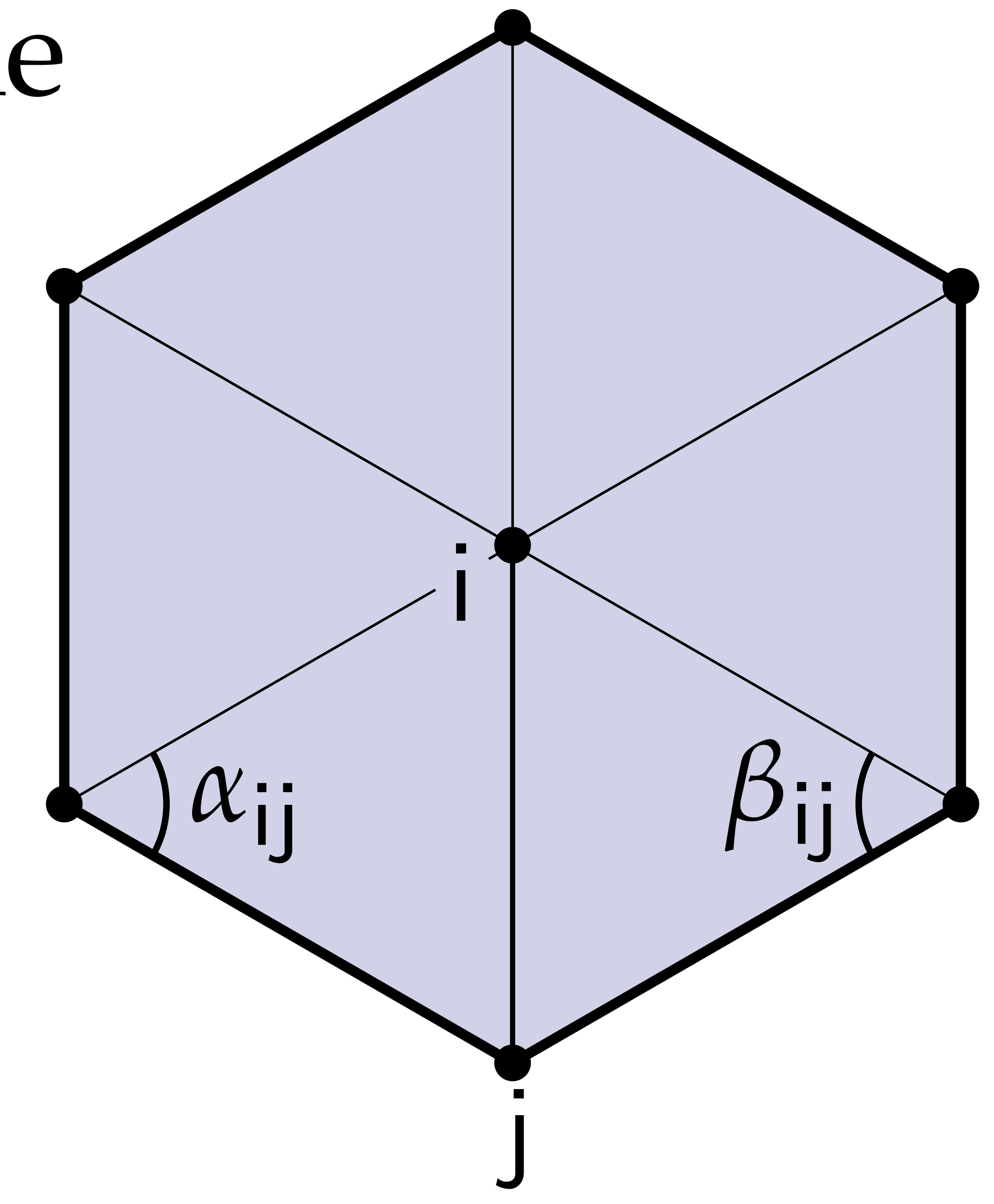
- Since N × is an in-plane 90-degree rotation, both terms in the summand are parallel to the edge vector e_{ij}
- The length of the sum equals the length $\ell^\star_{ij}$ of the dual edge
- Ratio of dual/primal length is given by cotan formula, yielding

$$\boxed{(HN)_i := \tfrac{1}{2} \sum_{ij \in E} (\cot \alpha_{ij} + \cot \beta_{ij})(f_i - f_j)}$$

Mean Curvature Normal via Laplace-Beltrami

- Another well-known fact: the mean curvature normal can be expressed via the *Laplace-Beltrami operator** Δ
- **Fact.** For any smooth immersed surface f, $\Delta f = 2HN$.
- Discretizing Δ via the *cotangent formula* yields the same expression for the discrete mean curvature normal:

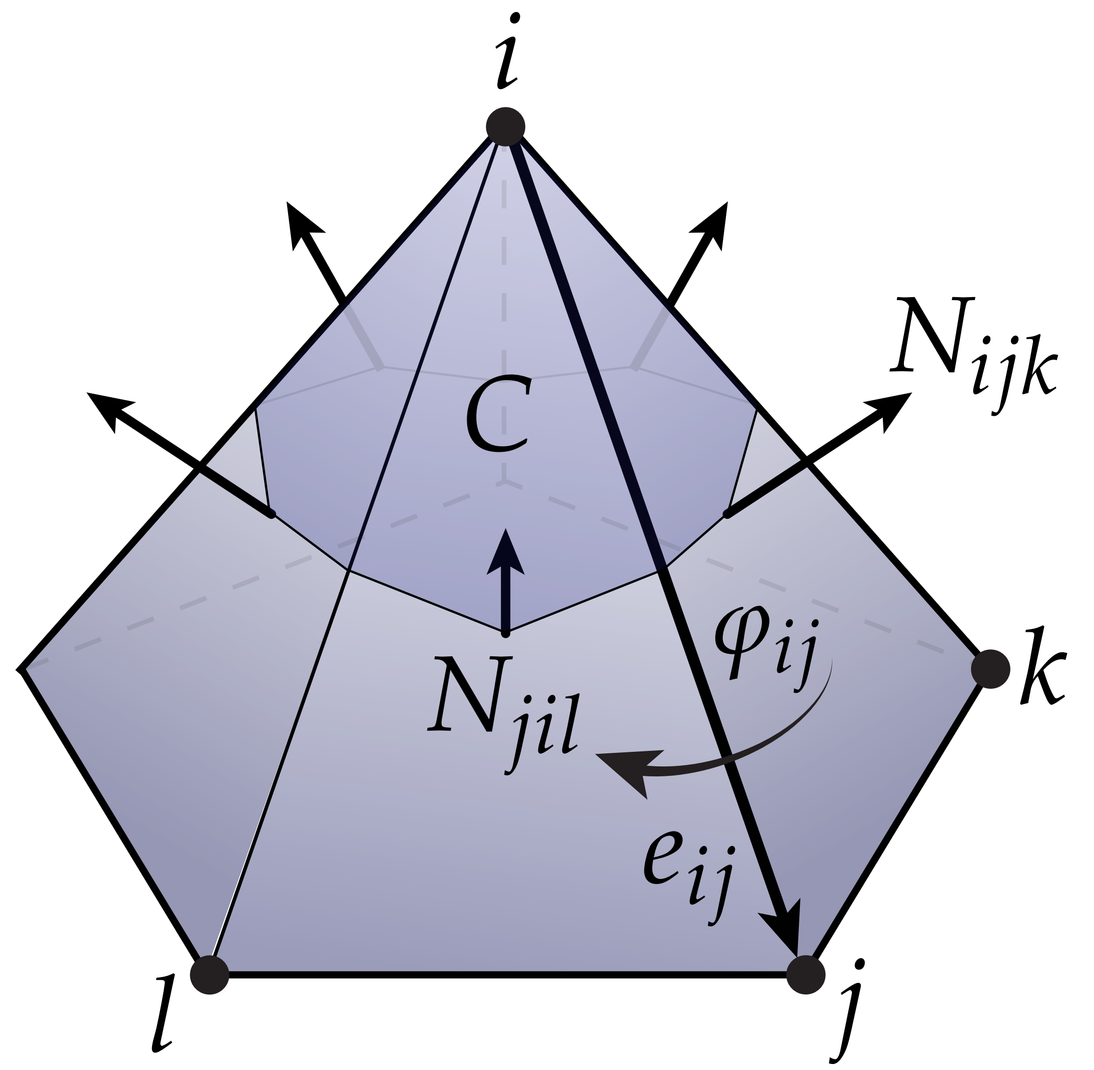
$$(\Delta f)_i = \frac{1}{2} \sum_{ij \in E} (\cot \alpha_{ij} + \cot \beta_{ij})(f_j - f_i)$$



*Will see *much* more of the Laplacian in upcoming lectures!

Discrete Gauss Curvature Normal

- A similar calculation leads to an expression for the (discrete) Gauss curvature normal
- One key difference: rather than viewing N as linear along edges, we imagine it makes an arc on the unit sphere

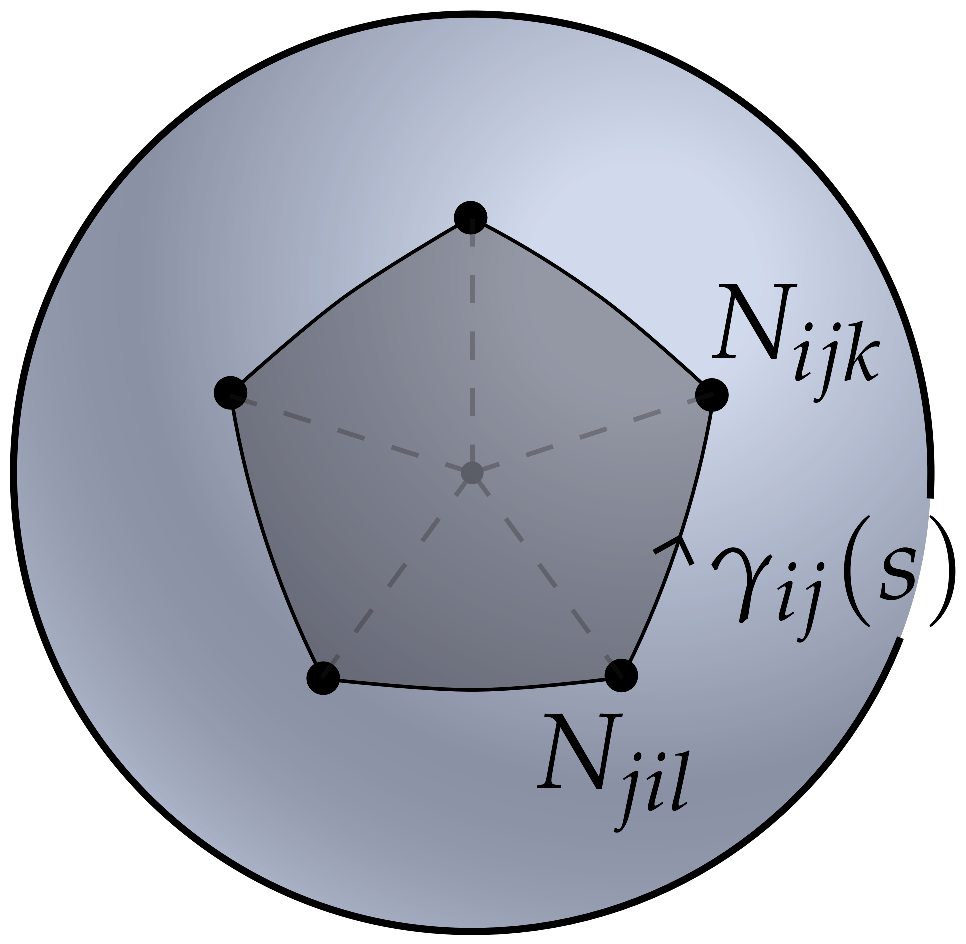


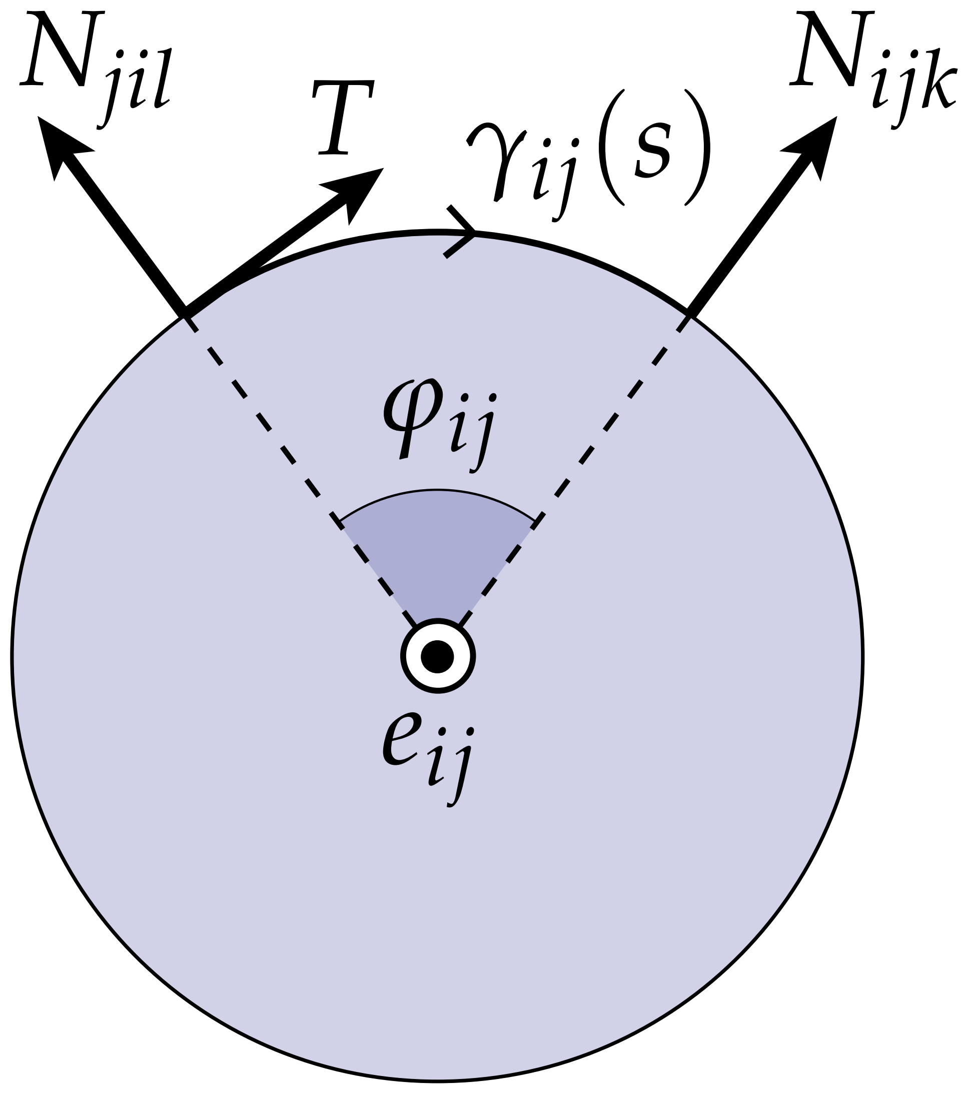
$$2\int_C KN\,dA = \int_C dN\wedge dN = \int_C d(N\wedge dN) =$$

$$\int_{\partial C} N\wedge dN = \int_{\partial C} N\times dN(\gamma')\,ds =$$

$$\int_{\partial C} N\times T\,ds = \sum_j \int_{\partial C}\frac{e_{ij}}{|e_{ij}|}\,ds = \sum_j \frac{e_{ij}}{\ell_{ij}}\varphi_{ij}$$

$$\boxed{(KN)_i := \frac{1}{2}\sum_{ij\in E}\frac{\varphi_{ij}}{\ell_{ij}}(f_j - f_i)}$$

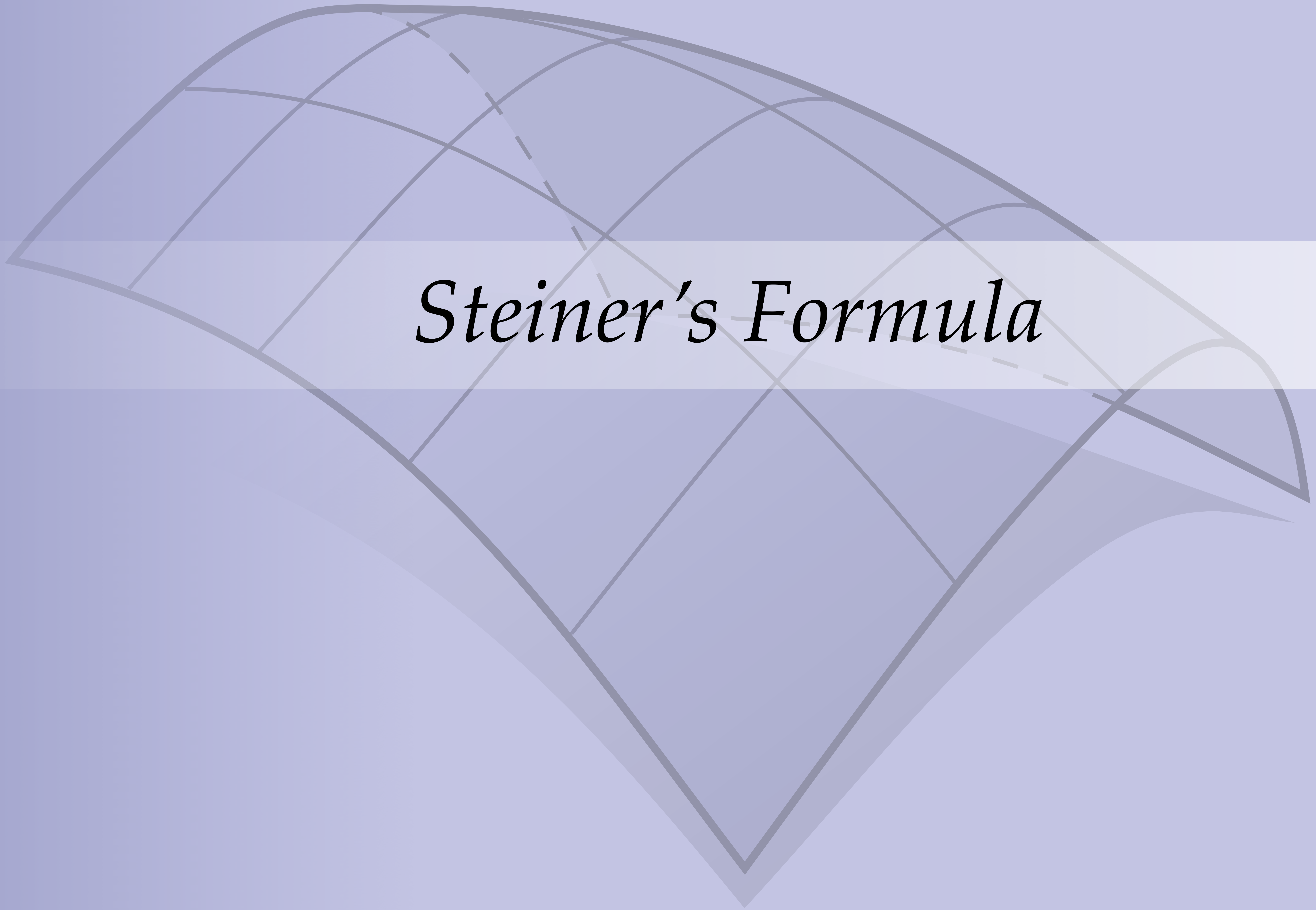




(view along e_{ij})

Discrete Curvature Normals — Summary

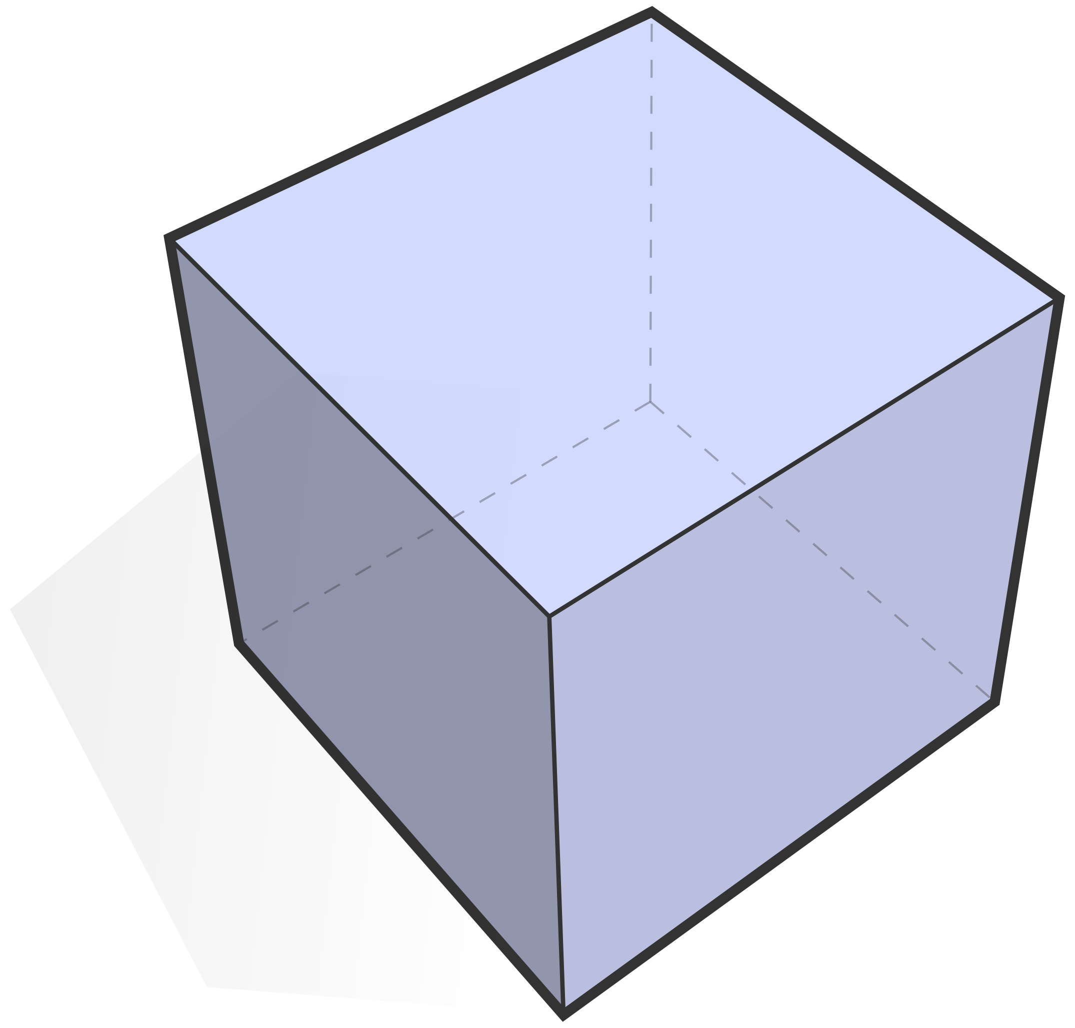
	area (NdA)	**mean** $(HNdA)$	**Gauss** $(KNdA)$
smooth	$\frac{1}{2}\, df \wedge df$	$\frac{1}{2}\, df \wedge dN$	$\frac{1}{2}\, dN \wedge dN$
discrete	$\frac{1}{6} \sum_{ijk \in \mathrm{St}(i)} f_j \times f_k$	$\frac{1}{2} \sum_{ij \in \mathrm{St}(i)} (\cot \alpha_{ij} + \cot \beta_{ij})(f_i - f_j)$	$\frac{1}{2} \sum_{ij \in \mathrm{St}(i)} \frac{\varphi_{ij}}{\ell_{ij}} (f_j - f_i)$



Steiner's Formula

Steiner Approach to Curvature

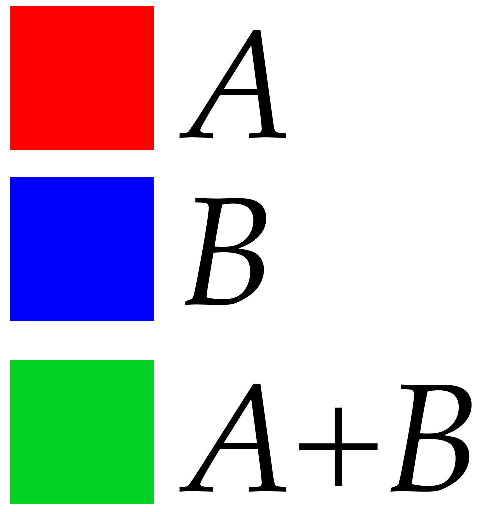
- What's the curvature of a discrete surface (polyhedron)?
- Simply taking derivatives of the surface gives a useless answer: zero except at vertices/edges, where derivative is ill-defined ("infinite")
- **Steiner approach**: "smooth out" the surface; define discrete curvature in terms of this *mollified* surface

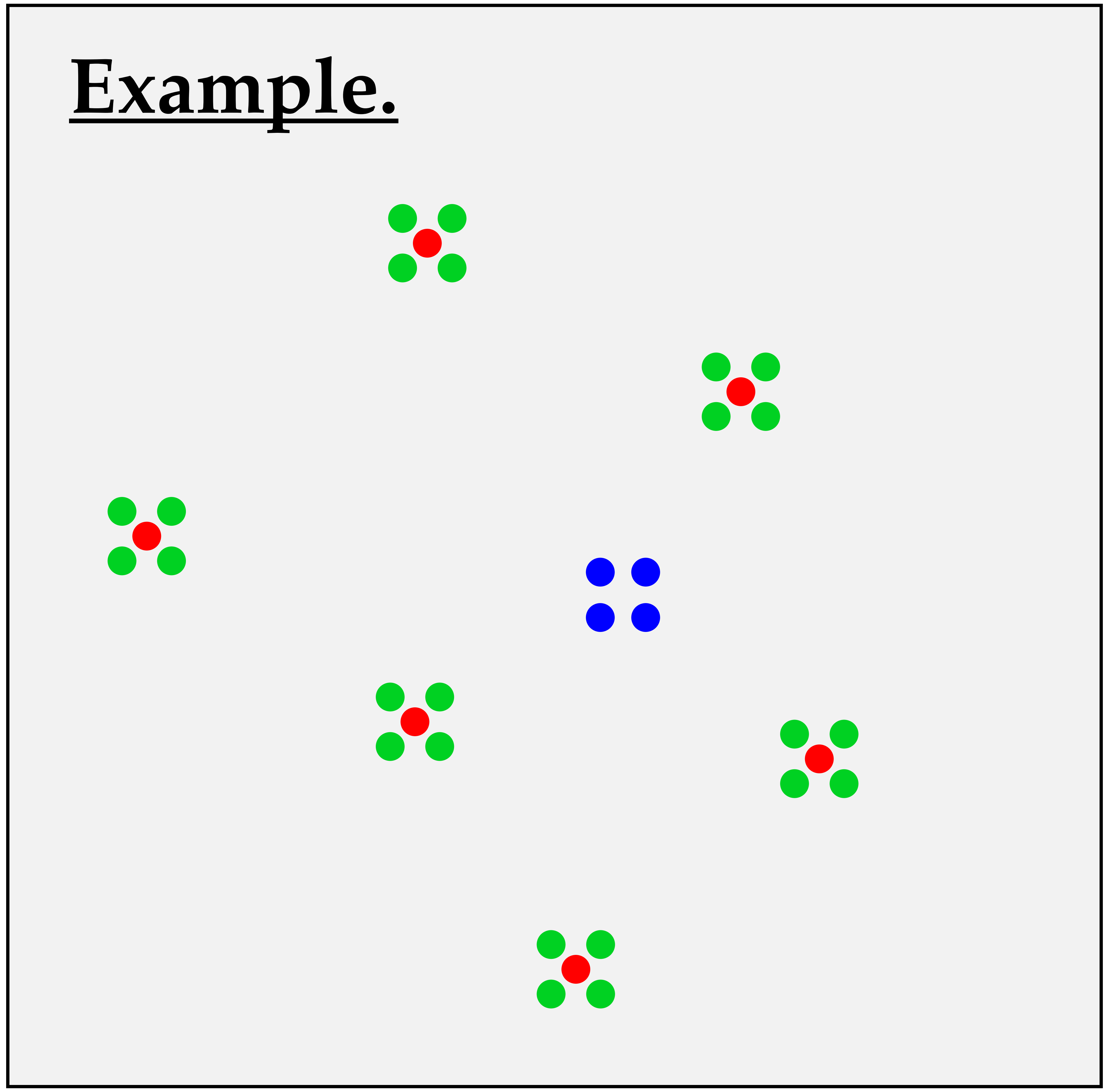


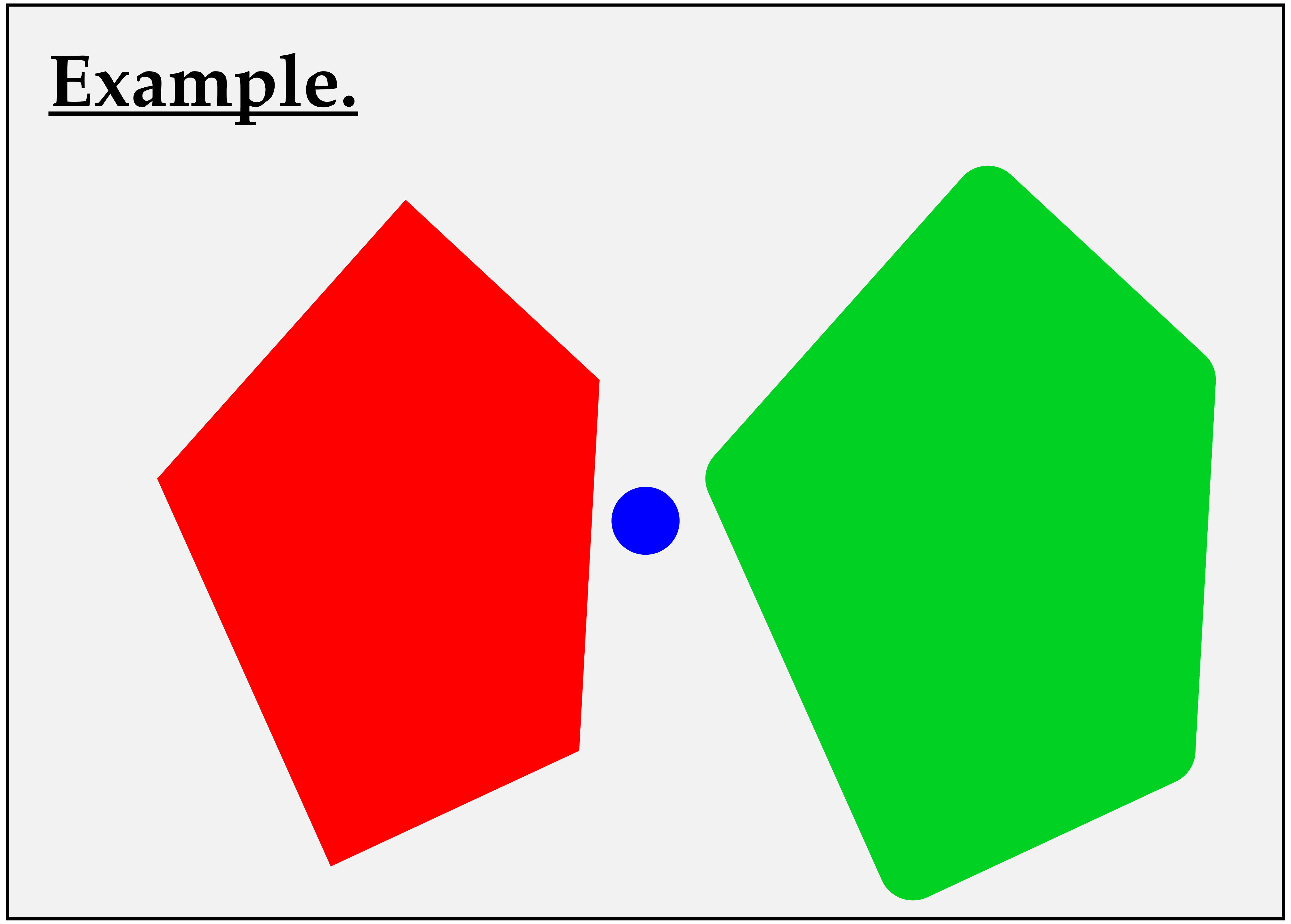
Minkowski Sum

- Given two sets A, B in R^n, their *Minkowski sum* is the set of points

$$A + B := \{a + b \mid a \in A, b \in B\}$$







Q: Does translation of A, B matter?

Mollification of Polyhedral Surfaces

- **Steiner approach:** smooth out or “mollify” polyhedral surface by taking Minkowski sum with ball of radius $\varepsilon > 0$
- Measure curvature, take limit as ε goes to zero to get discrete definition
- Have to be careful about nonconvex polyhedra… same formulas hold

Steiner Formula

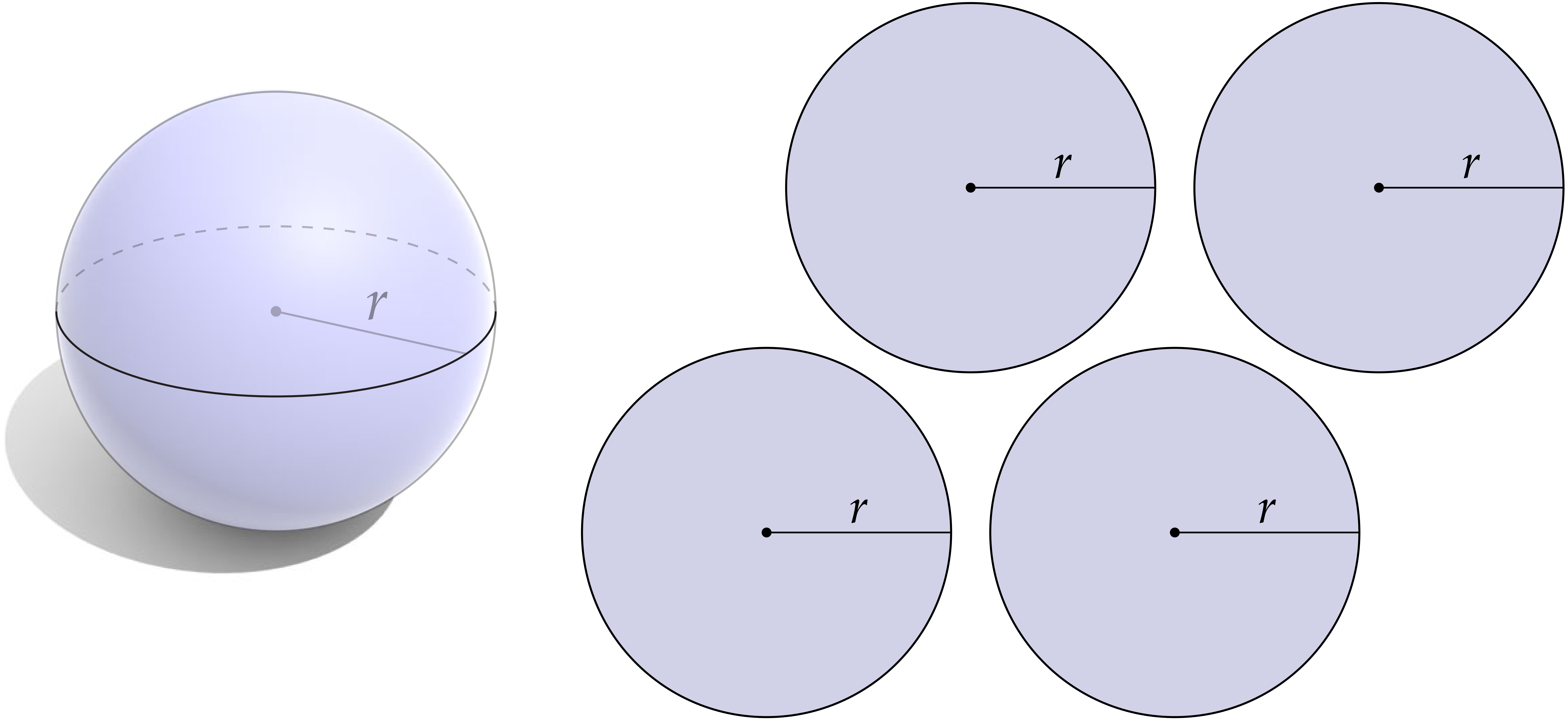
- **Theorem.** (Steiner) Let A be any convex body in R^n, and let B_ε be a ball of radius ε. Then the volume of the Minkowski sum $A+B_\varepsilon$ can be expressed as a polynomial in ε:

$$\mathrm{volume}(A + B_\varepsilon) = \mathrm{volume}(A) + \sum_{k=1}^{n} \Phi_k(A)\varepsilon^k$$

- Coefficients Φ_k are called *quermassintegrals* ("cross-dimension integrals")
  - describe how quickly the volume grows
- This volume growth is related to (discrete) curvature…

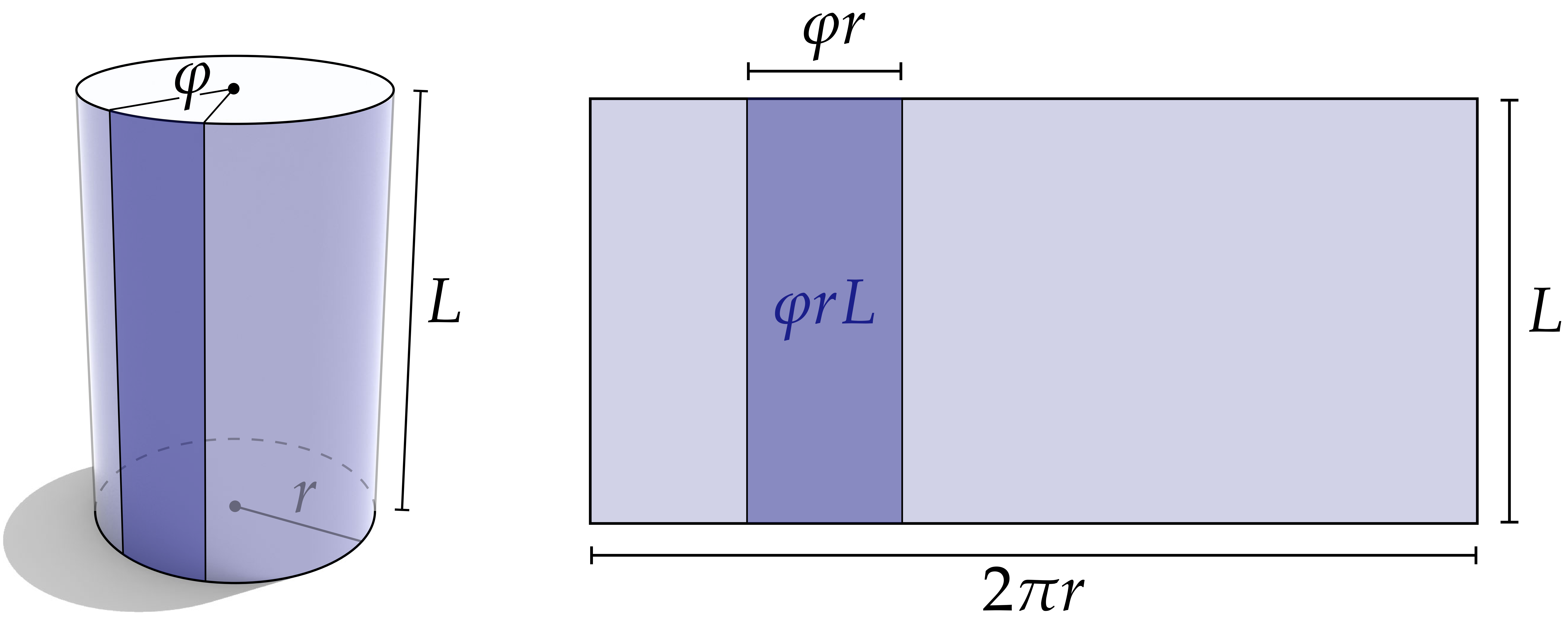
Surface Area of a Sphere

Area of a sphere of radius r is $4\pi r^2$



Surface Area of a Cylinder

Area of a cylinder of radius r and length L is $2\pi rL$ (omitting end caps)



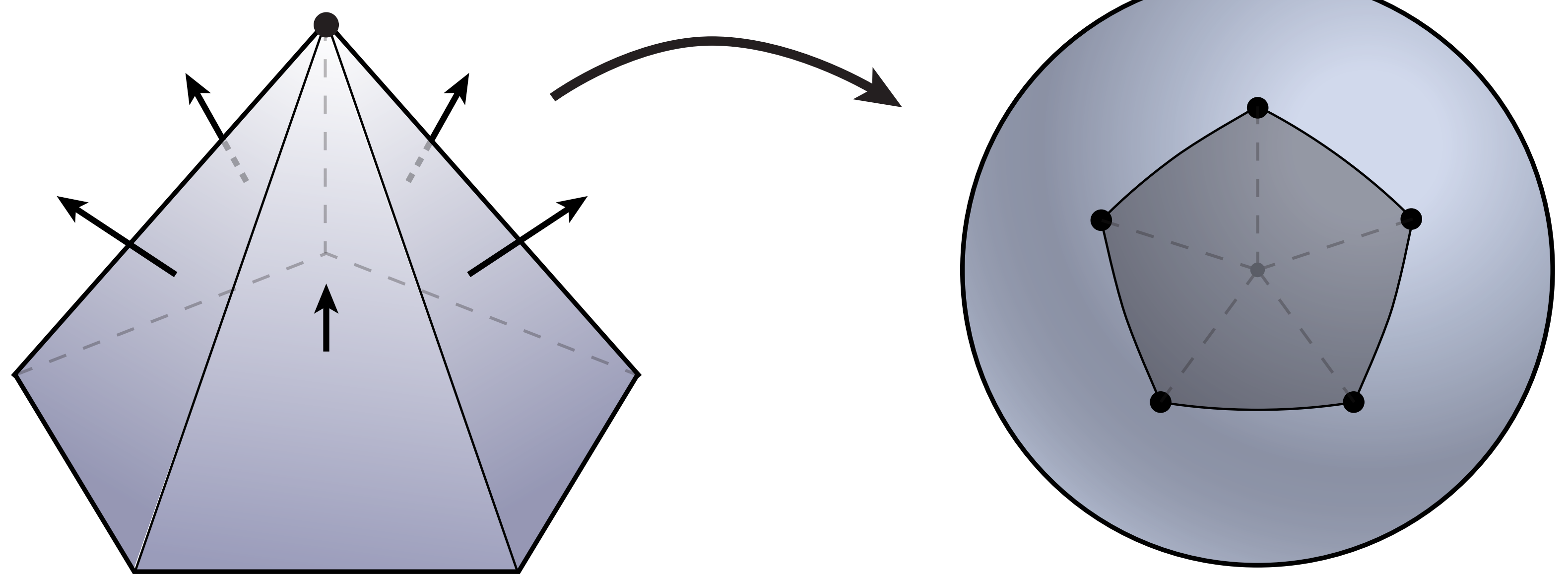
More generally, area of a cylindrical arc of angle φ is equal to φrL

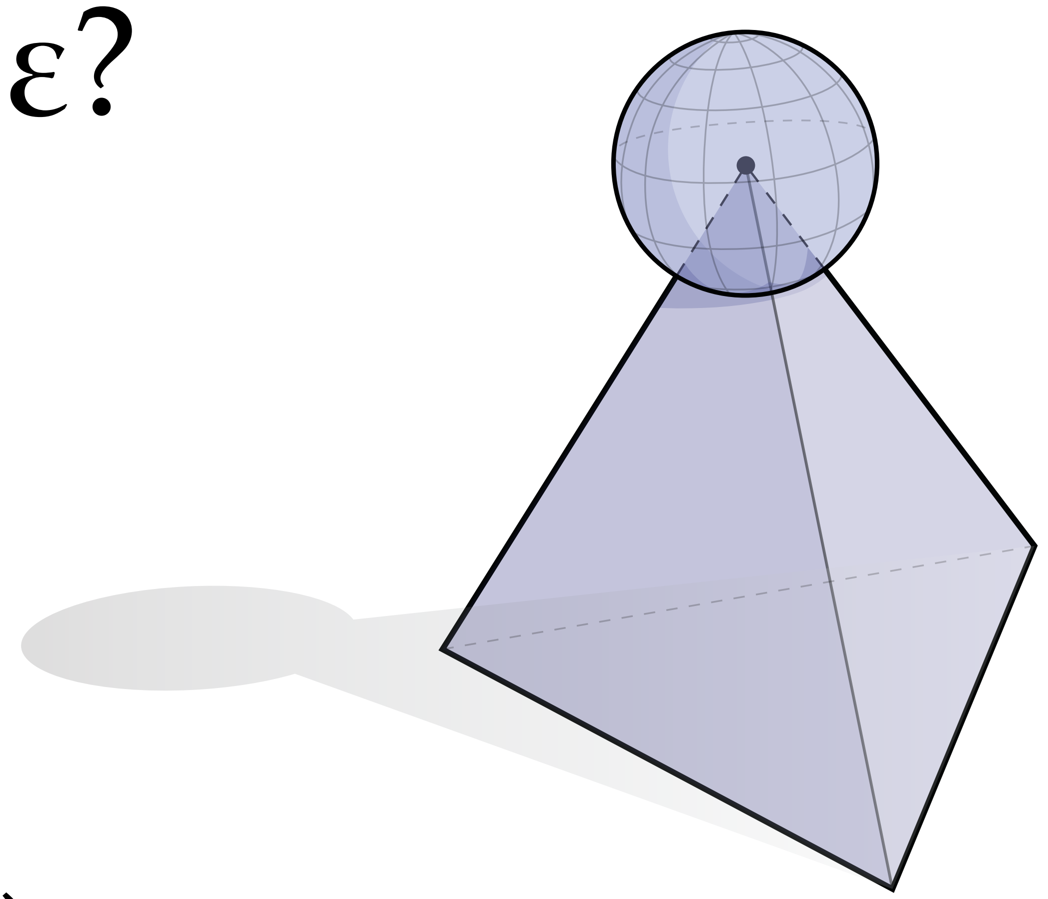
Gaussian Curvature of Mollified Surface

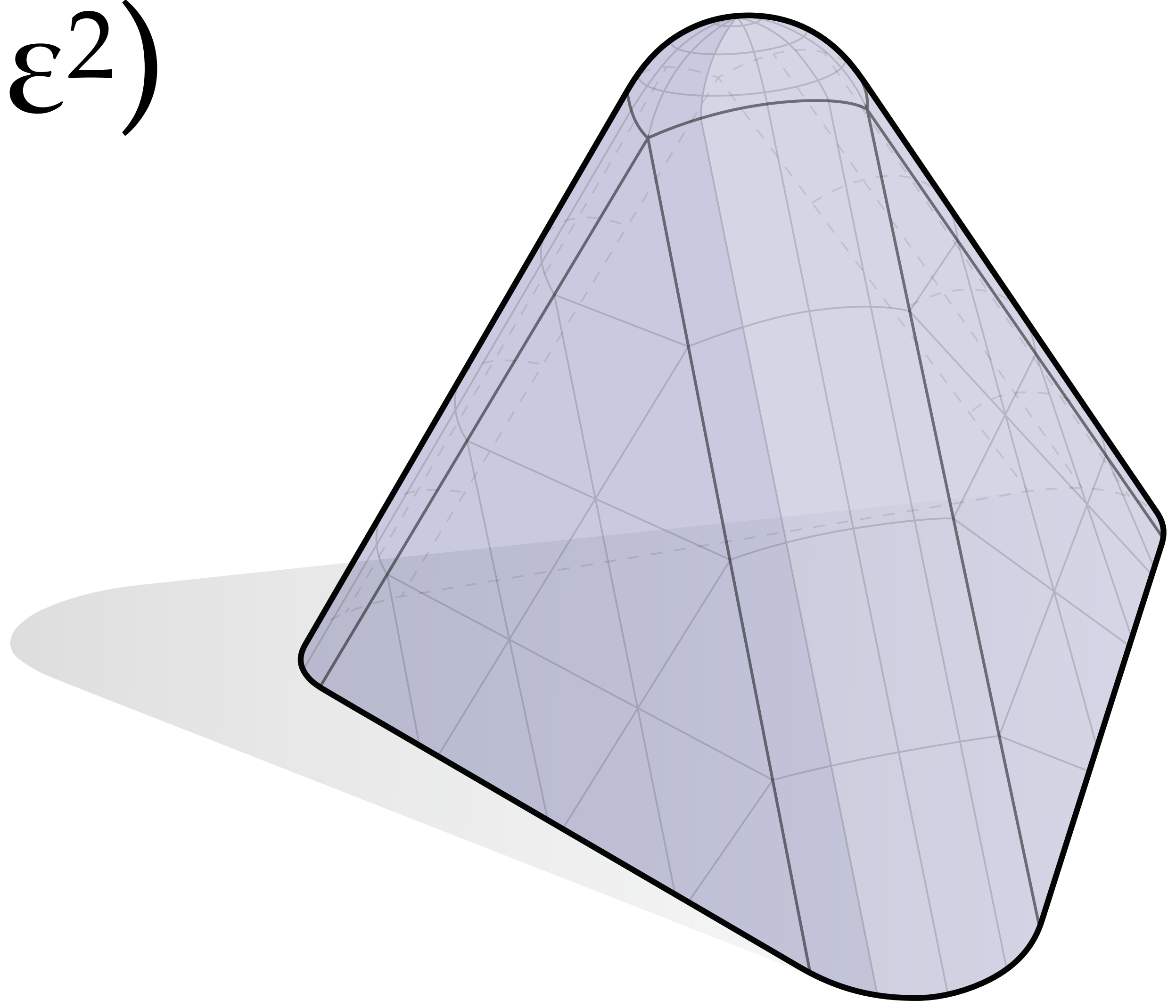
- **Q:** Consider a *closed, convex* polyhedron in R^3; what's the Gaussian curvature K of the mollified surface for a ball of radius ε?
- **Triangles:** $K = 0$
- **Edges:** $K = 0$
- **Vertices?**
  - each contributes a piece of sphere of radius ε (K=1/ε^2)
  - recall (unit) spherical area is equal to *angle defect* Ω_i
  - *total* curvature associated with vertex *i* is hence

$$A_i K_i = \left(\frac{\Omega_i}{4\pi} 4\pi\varepsilon^2\right) \tfrac{1}{\varepsilon^2} = \Omega_i$$

- For whole surface, have $\mathrm{Gauss}(\varepsilon) = \sum_{i\in V} \Omega_i$

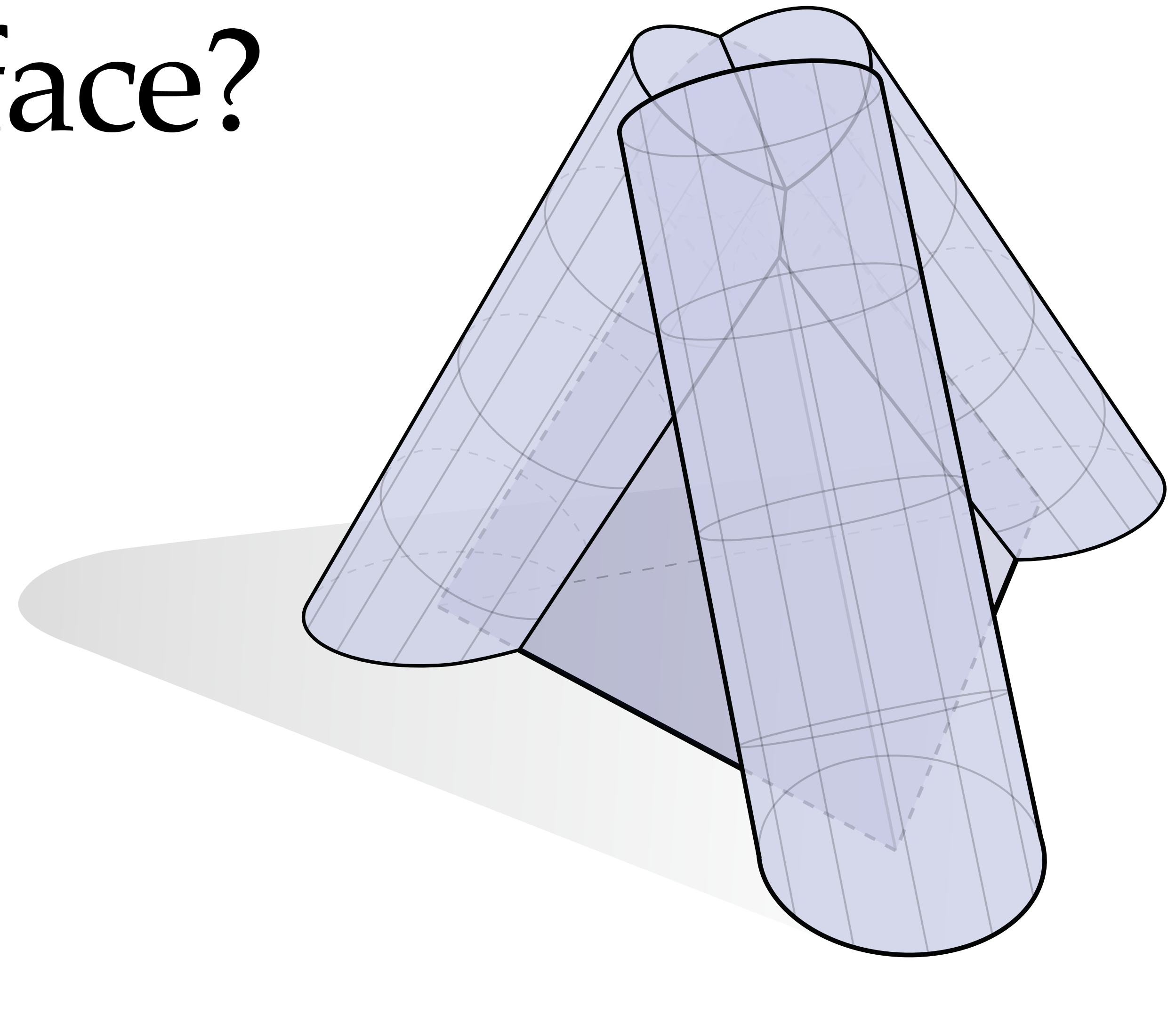


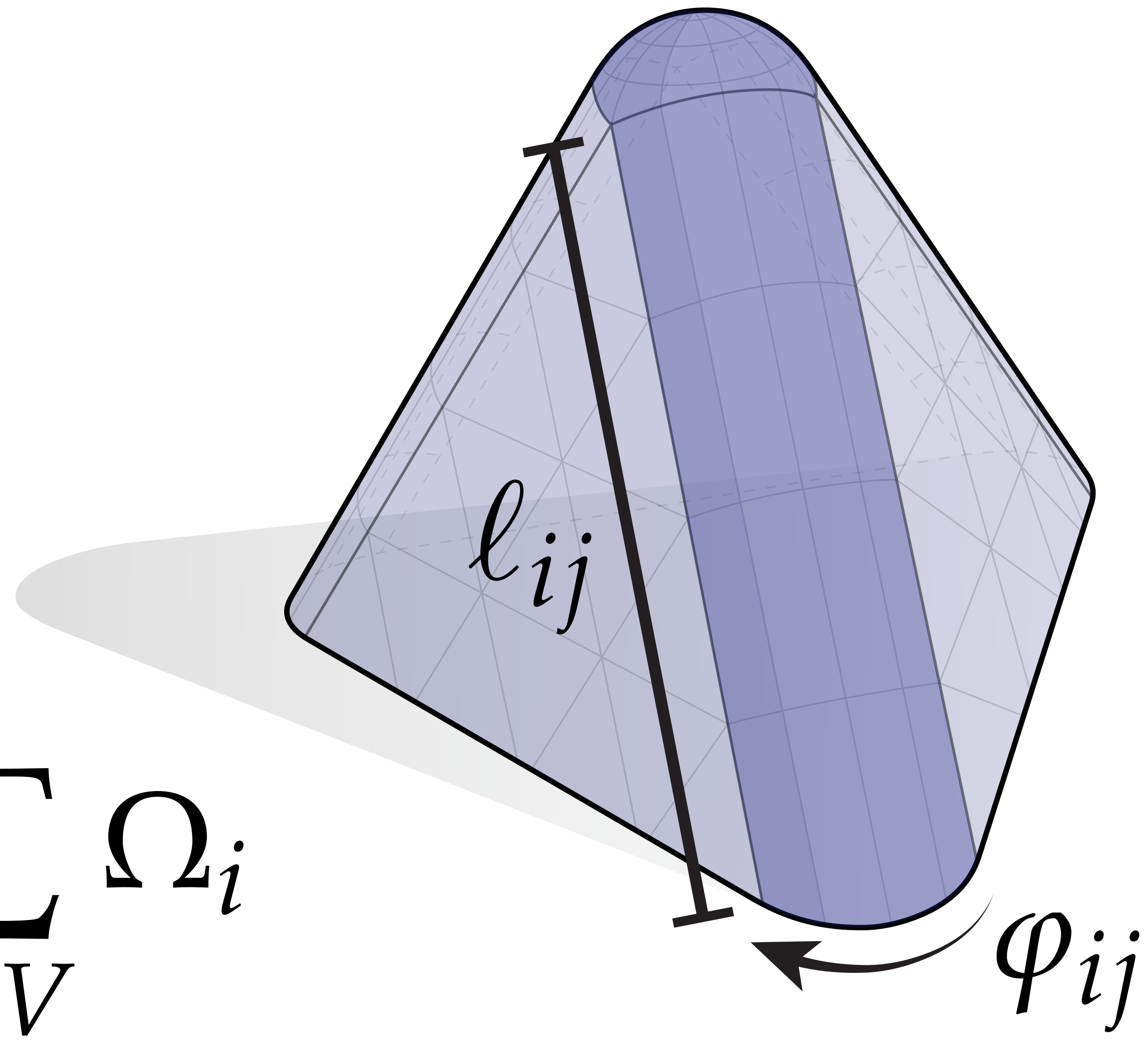




Mean Curvature of Mollified Surface

- **Q:** What's the mean curvature H of the mollified surface?
- **Faces:** $H = 0$
- **Edges?**
  - each contributes a piece of a cylinder (H=1/2ε)
  - area of cylindrical piece is $\ell_{ij}\varphi_{ij}\varepsilon$
  - total mean curvature for edge is hence $H_{ij} = \frac{1}{2}\ell_{ij}\varphi_{ij}$
- **Vertices?**
  - each contributes a piece of the sphere (H=1/ε)
  - area is $(\Omega_i/4\pi)\,4\pi\varepsilon^2 = \Omega_i\varepsilon^2$, hence $H_i = \Omega_i\varepsilon$
- For whole surface, have $\text{mean}(\varepsilon) = \frac{1}{2}\sum_{ij\in E}\ell_{ij}\varphi_{ij} + \varepsilon\sum_{i\in V}\Omega_i$



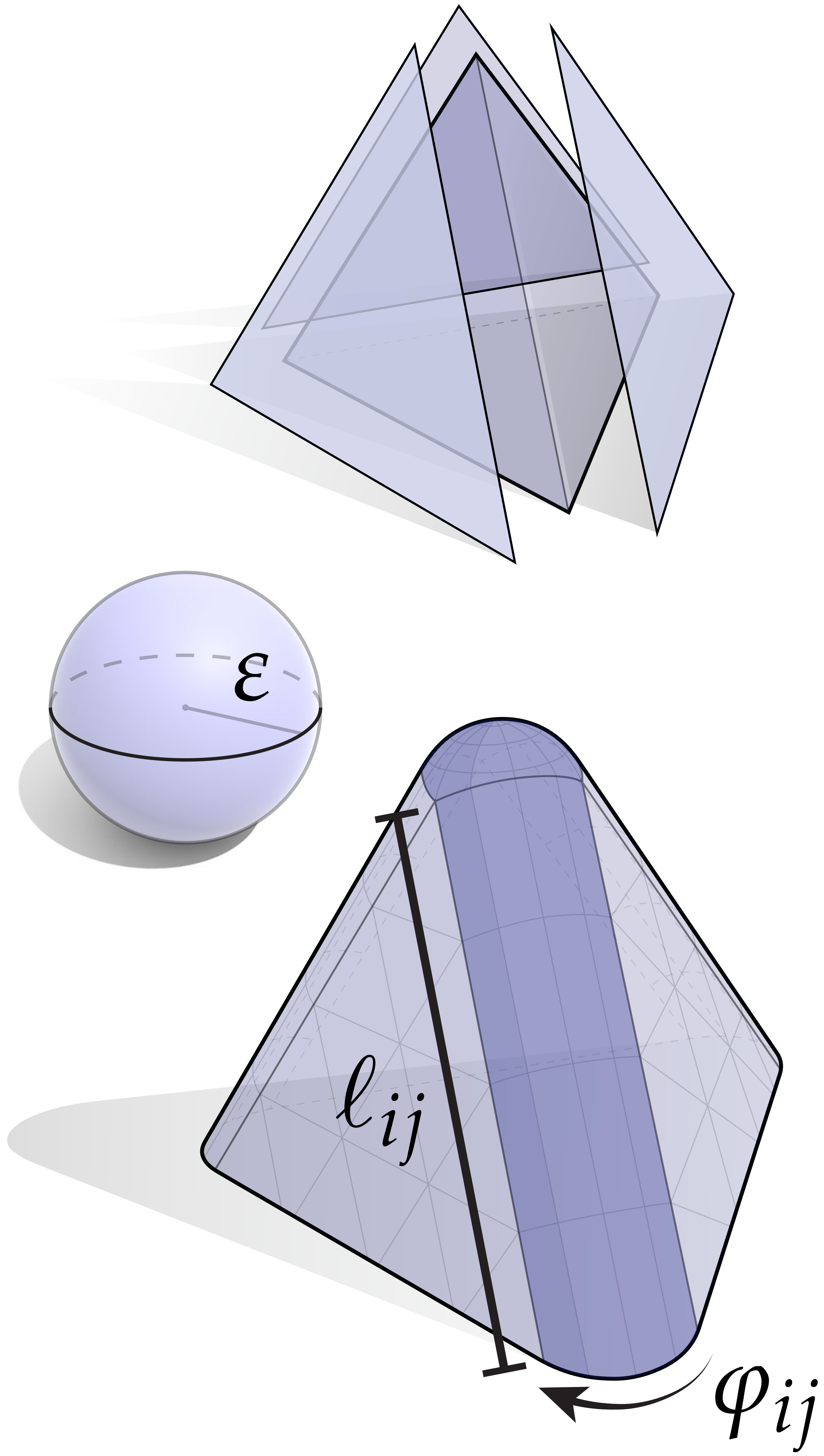


Area of a Mollified Surface

- **Q:** What's the area of the mollified surface?
  - **Faces:** just the original area A_{ijk}
  - **Edges:** $\ell_{ij}\varphi_{ij}\varepsilon$
  - **Vertices:** $\Omega_i\varepsilon^2$
- Total area of the whole surface is then

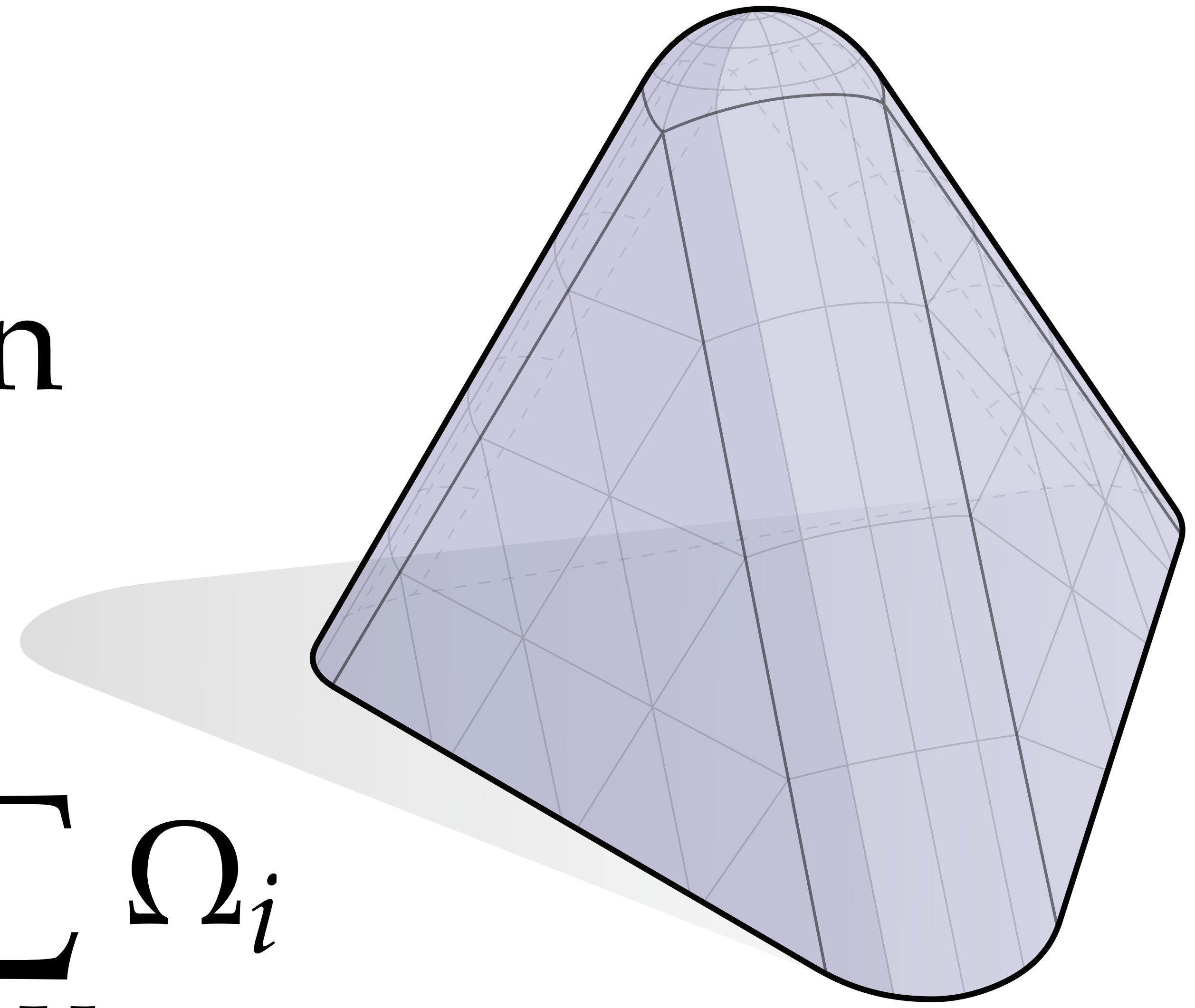
$$\text{area}(\varepsilon) = \sum_{ijk\in F} A_{ijk} + \varepsilon \sum_{ij\in E} \ell_{ij}\varphi_{ij} + \varepsilon^2 \sum_{i\in V} \Omega_i$$

- By (discrete) Gauss-Bonnet, last sum equals $2\pi\chi$



Volume of Mollified Surface

- **Q:** What's the total volume of the mollified surface?
- Starting to see a pattern—if V_0 is original volume, then



$$\text{volume}(\varepsilon) = V_0 + \varepsilon \sum_{ijk \in F} A_{ijk} + \tfrac{1}{2}\varepsilon^2 \sum_{ij \in E} \ell_{ij}\varphi_{ij} + \tfrac{1}{3}\varepsilon^3 \sum_{i \in V} \Omega_i$$

- **Faces** add "slabs" of thickness ε, hence volume εA
- **Edges** add cylindrical wedges of volume $\frac{1}{2}\ell_{ij}\varphi_{ij}\varepsilon^2$ (cylinder: $\pi r^2 L$)
- **Vertices** add spherical cones of volume $\frac{1}{3}\Omega_i\varepsilon^3$ (sphere: $4\pi r^3/3$)

Steiner Polynomial for Polyhedra in R^3

- Volume of mollified polyhedron is a polynomial in radius ε

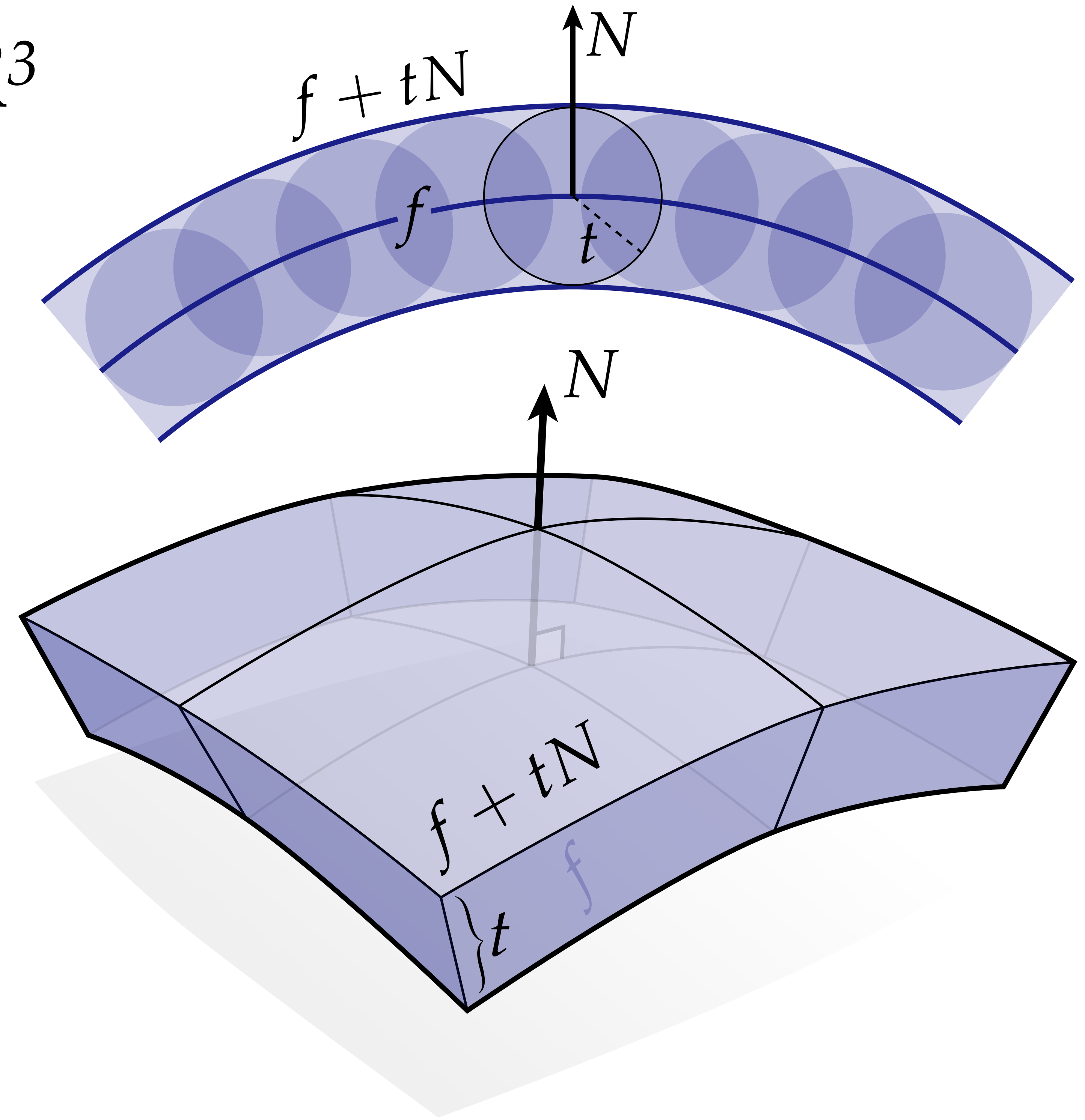
$$\text{volume}(\varepsilon) = V_0 + \varepsilon \sum_{ijk \in F} A_{ijk} + \tfrac{1}{2}\varepsilon^2 \sum_{ij \in E} \ell_{ij}\varphi_{ij} + \tfrac{1}{3}\varepsilon^3 \sum_{i \in V} \Omega_i$$

- Derivatives w.r.t. ε give total area, mean curvature, Gauss curvature:

$\frac{d}{d\varepsilon}\text{volume}_\varepsilon = \text{area}_\varepsilon$ $\quad$ $\frac{d}{d\varepsilon}\text{area}_\varepsilon = 2\text{mean}_\varepsilon$ $\quad$ $\frac{d}{d\varepsilon}\text{mean}_\varepsilon = \text{Gauss}_\varepsilon$ $\quad$ $\frac{d}{d\varepsilon}\text{Gauss}_\varepsilon = 0$

Steiner Polynomial for Surfaces in R^3

- Not surprisingly, there is an analogous formula for surfaces in R^3
- Taking a Minkowski sum w/ a small ball of radius $\varepsilon > 0$ is the same as shifting the surface in the normal direction by ε
- Consider a surface $f : M \longrightarrow R^3$ with Gauss map N; let $f_t := f + tN$
- How is the area of the "smoothed" surface changing?



$$dA_t = \tfrac{1}{2}\langle N, df_t \wedge df_t \rangle$$

$$df_t \wedge df_t =$$

$$(df + tdN) \wedge (df + tdN) =$$

$$df \wedge df + 2t df \wedge dN + t^2 dN \wedge dN =$$

$$(1 + 2tH + t^2 K) df \wedge df$$

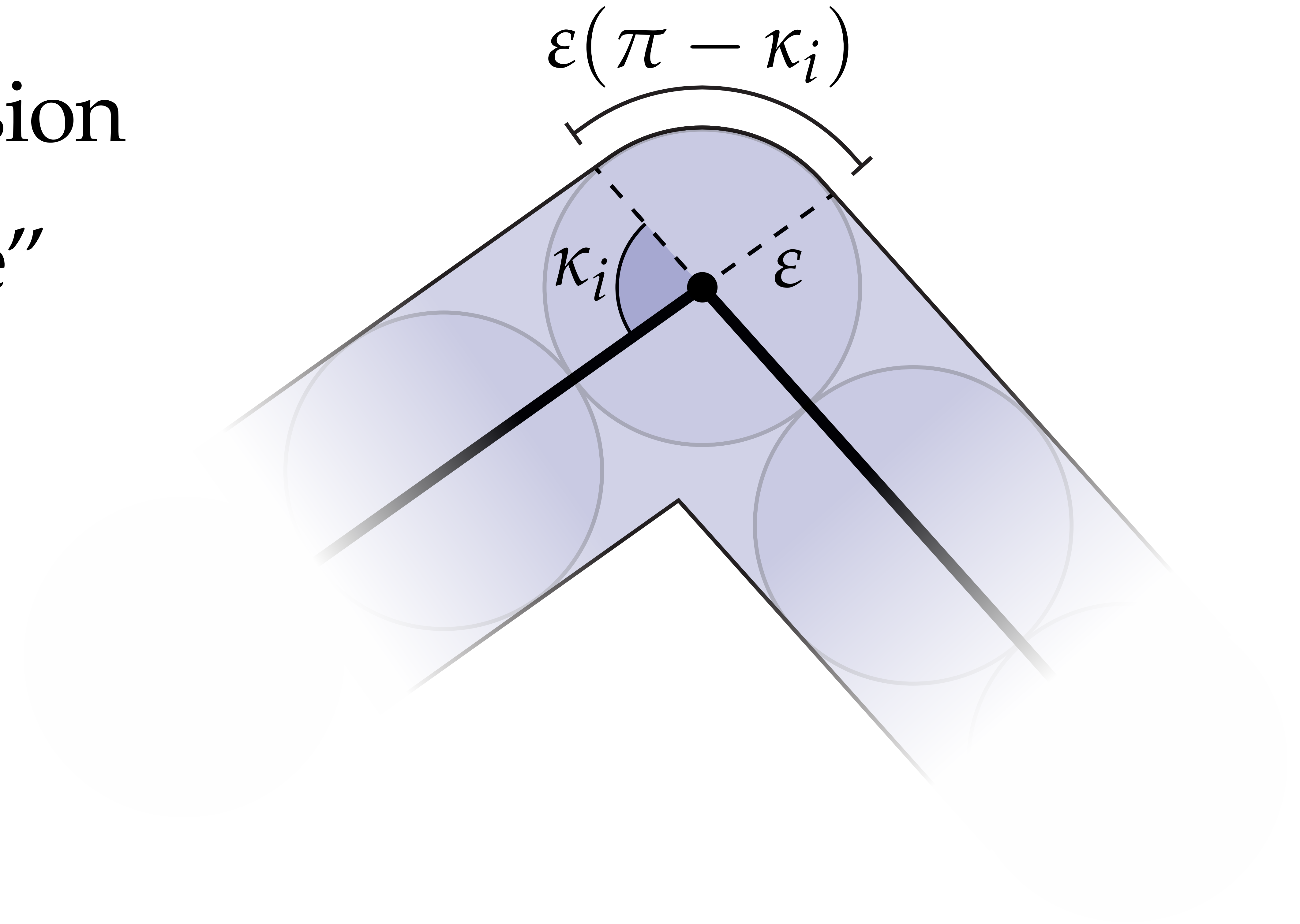
$$\boxed{\implies dA_t = (1 + 2tH + t^2 K) dA_0}$$

Notice:

- surface area given by $df \wedge df$
- spherical area $dN \wedge dN$ gives Gauss curvature
- "mixed area" $df \wedge dN$ gives mean curvature

Discrete Curvature of n-Manifolds

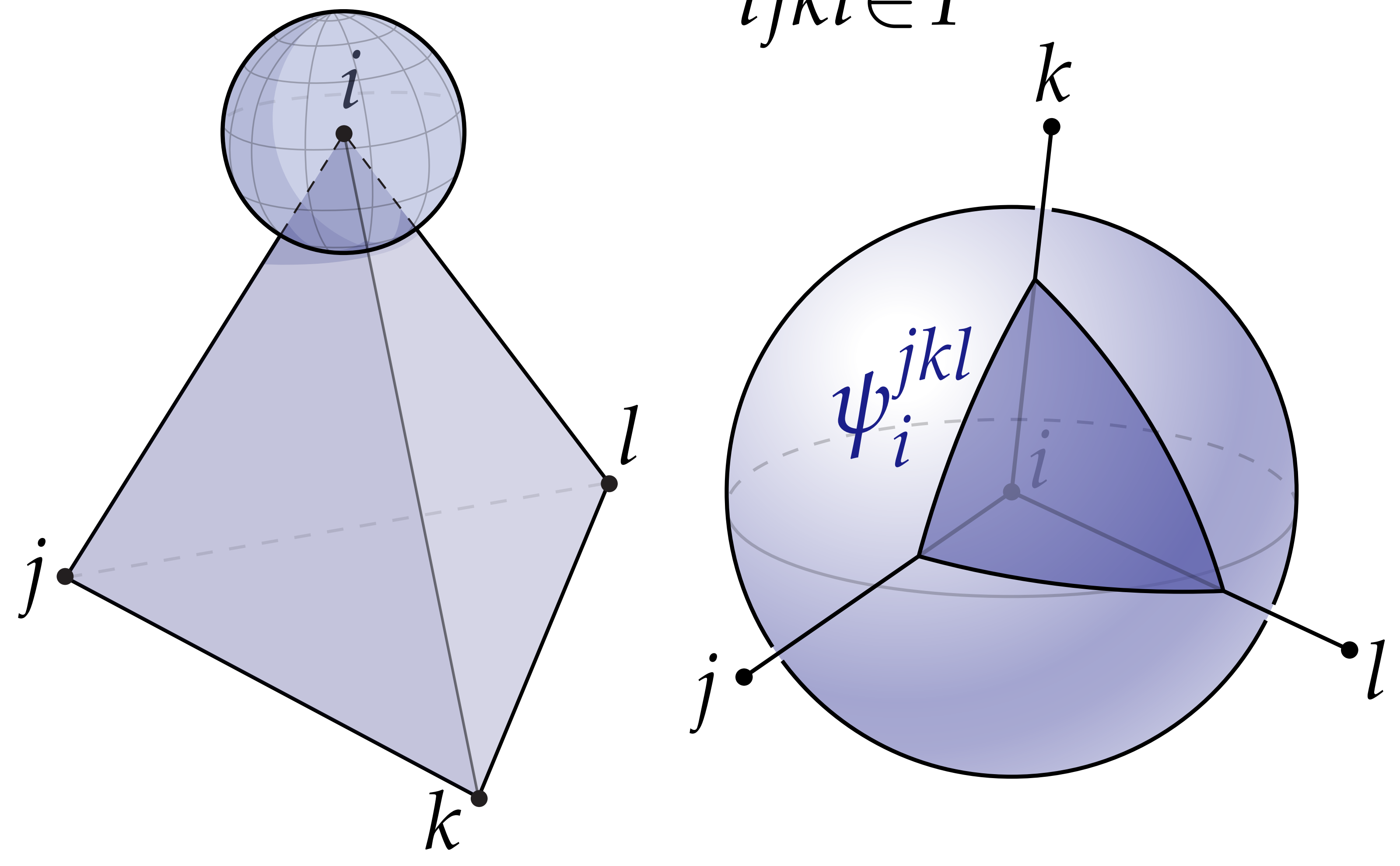
- Can use this same machinery to define/ understand discrete curvature in any dimension
- *E.g.*, for planar curves recover "turning angle" definition of curvature



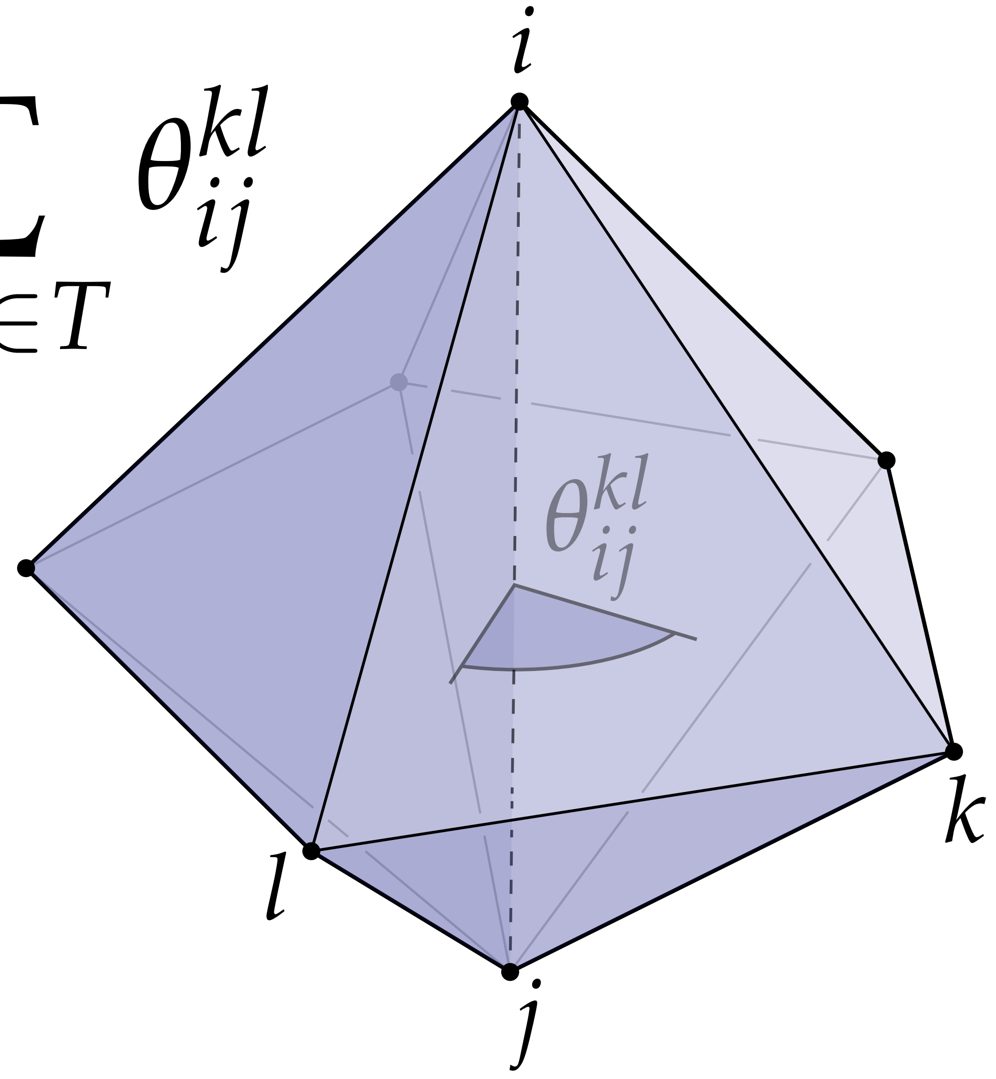
Discrete Curvature of n-Manifolds

- Can use this same machinery to define/understand discrete curvature in any dimension
- *E.g.*, for planar curves recover "turning angle" definition of curvature
- For 3-manifolds:
  - **scalar curvature:** compare total solid angle around vertex to area of Euclidean sphere
  - **Riemann curvature**: compare total dihedral angle around edge to 2π
- In general: consider volume of Minkowski sum with (n+1)-ball of radius ε; derivatives with respect to ε give all the discrete curvatures

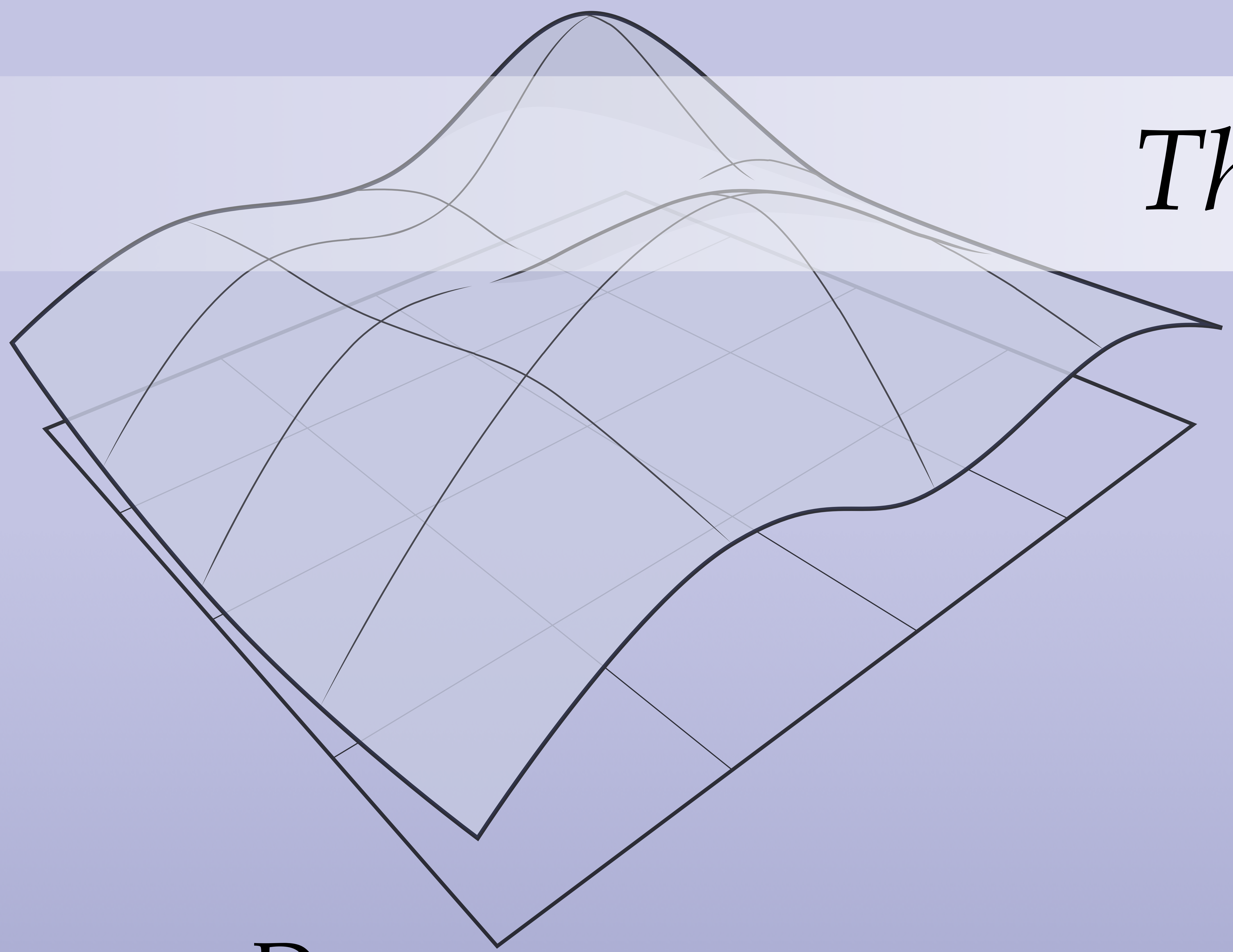
$$S_i = 4\pi - \sum_{ijkl \in T} \psi_i^{jkl}$$



$$R_{ij} = 2\pi - \sum_{ijkl \in T} \theta_{ij}^{kl}$$



Thanks!



DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION

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