CS-GY 6763: Lecture 3
Finish Chebyshev's, Exponential Concentration Inequalities

NYU, Prof. Ainesh Bakshi

DISTINCT ELEMENTS PROBLEM

Input: $d_1, \ldots, d_n \in \mathcal{U}$ where $\mathcal{U}$ is a huge universe of items.

Output: Number of distinct inputs, D.

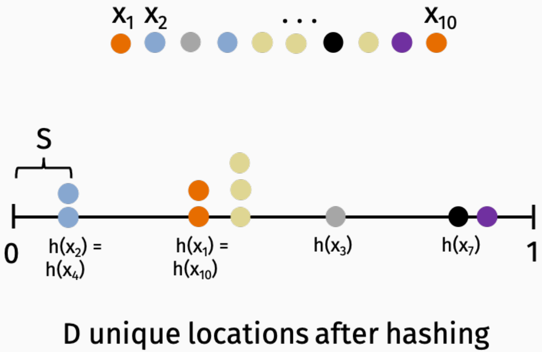
Example: $f(1, 10, 2, 4, 9, 2, 10, 4) \rightarrow D = 5$

Flajolet–Martin (simplified):

- Choose random hash function $h : \mathcal{U} \rightarrow [0, 1]$.
- $S = 1$
- For $i = 1, \ldots, n$
    - $S \leftarrow \min(S, h(x_i))$
- Return: $\frac{1}{S} - 1$

FM ANALYSIS

Let D equal the number of distinct elements in our stream.



Intuition: When D is larger, S will be smaller. Makes sense to return the estimate $\tilde{D} = \frac{1}{S} - 1$.

What is $\mathbb{E}S$?

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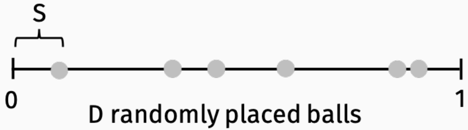

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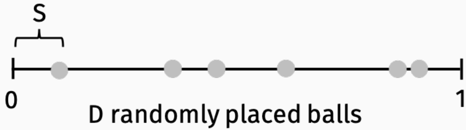

Let D equal the number of distinct elements in our stream.

Lemma

$\mathbb{E}S = \frac{1}{D+1}$.

THE CALCULUS PROOF

Proof: $\mathrm{E[S]} = \int_0^1 \Pr[S \geq \lambda] d\lambda$Exercise: Why?
$= \int_0^1 (1-\lambda)^D d\lambda$
$= \text{-}(1\text{-}\lambda)^{D+1} \frac{}{D+1\Big|_{\lambda=0}^{1}}$
$= 1 \frac{}{D+1}$

Hint: For a non-negative random variable $X = \int_0^\infty \mathbf{1}(X \geq t) dt$. Then,

$$\mathbb{E}[X] = \mathbb{E}[\int_0^\infty \mathbf{1}(X \geq t) dt] = \int_0^\infty \Pr[X \geq t] dt.$$

VISUALIZATION

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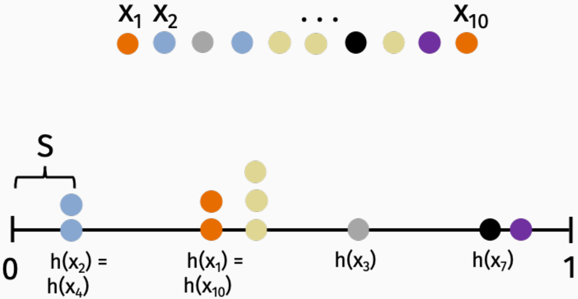

PROVING CONCENTRATION

$\mathbb{E}S = \frac{1}{D+1}$. **Estimate:** $\tilde{D} = \frac{1}{S} - 1$. We have for $\epsilon < \frac{1}{4}$:

$$\text{If } (1-\epsilon)\mathbb{E}S \leq S \leq (1+\epsilon)\mathbb{E}S, \text{ then:}$$

$$(1-4\epsilon)D \leq \tilde{D} \leq (1+4\epsilon)D.$$

Proof.

Inverting the inequalities,

$$\frac{1}{(1+\epsilon)\mathbb{E}S} \leq \frac{1}{S} \leq \frac{1}{(1-\epsilon)\mathbb{E}S}$$

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Using $\mathbb{E}S = \frac{1}{D+1}$,

$$\frac{D+1}{1+\epsilon} \leq \frac{1}{S} \leq \frac{D+1}{1-\epsilon}$$

$$\implies (1-\epsilon)D + (1-\epsilon) - 1 \leq \frac{1}{S} - 1 \leq (1+2\epsilon)D + (1+2\epsilon) - 1$$

$$\implies (1-\epsilon)D - \epsilon \leq \tilde{D} \leq (1+2\epsilon)D + 2\epsilon$$

□

CALCULUS PROOF

Lemma

$\mathrm{Var}[S] = \mathbb{E}[S^2] - \mathbb{E}[S]^2 = \frac{2}{(D+1)(D+2)} - \frac{1}{(D+1)^2} \leq \frac{1}{(D+1)^2}.$

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Proof:

$$\begin{aligned} \mathbb{E}[S^2] &= \int_0^1 \Pr[S^2 \geq \lambda] d\lambda \\ &= \int_0^1 \Pr[S \geq \sqrt{\lambda}] d\lambda \\ &= \int_0^1 (1 - \sqrt{\lambda})^D d\lambda \\ &= \frac{2}{(D+1)(D+2)} \end{aligned}$$

www.wolframalpha.com/input?i=antiderivative+of+%281-sqrt%28x%29%29%5ED

FM ANALYSIS

Recall we want to show that, with high probability, $(1-\epsilon)\mathbb{E}[S] \leq S \leq (1-\epsilon)\mathbb{E}[S]$.

- $\mathbb{E}[S] = \frac{1}{D+1} = \mu$.
- $\mathrm{Var}[S] \leq \frac{1}{(D+1)^2} = \mu^2$. Standard deviation: $\sigma \leq \mu$.
- Want to bound $\Pr[|S-\mu| \geq \epsilon\mu] \leq \delta$.

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Chebyshev's: $\Pr[|S-\mu| \geq \epsilon\mu] = \Pr[|S-\mu| \geq \epsilon\sigma] \leq \frac{1}{\epsilon^2}$.

Vacuous bound. Our variance is way too high!

VARIANCE REDUCTION

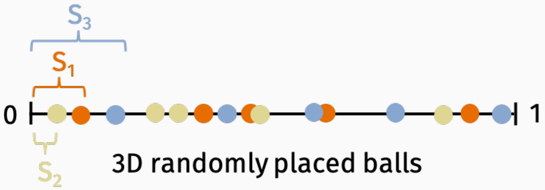
Trick of the trade: Repeat many independent trials and take the mean to get a better estimator.

Given i.i.d. (independent, identically distributed) random variables $X_1, \ldots, X_n$ with mean μ and variance σ^2, what is:

- $\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \mu$
- $\text{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2} \cdot n \cdot \sigma^2$

FM ANALYSIS

Using independent hash functions, maintain k independent sketches $S_1, \ldots, S_k$.



Flajolet–Martin:

- Choose k random hash function $h_1, \ldots, h_k : \mathcal{U} \to [0, 1]$.
- $S_1 = 1, \ldots, S_k = 1$
- For $i = 1, \ldots, n$
  - $S_j \leftarrow \min(S_j, h_j(x_i))$ for all $j \in 1, \ldots, k$.
- $S = (S_1 + \ldots + S_k)/k$
- Return: $\frac{1}{S} - 1$

FM ANALYSIS

1 estimator:

- $\mathbb{E}[S] = \frac{1}{D+1} = \mu$.
- $\mathrm{Var}[S] = \mu^2$

k **estimators**:

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- By Chebyshev, $\Pr[|S - \mathbb{E}S| \geq c\mu/\sqrt{k}] \leq \frac{1}{c^2}$.

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Setting $c = 1/\sqrt{\delta}$ and $k = \frac{1}{\epsilon^2\delta}$ gives:

$$\Pr[|S - \mu| \geq \epsilon\mu] \leq \delta.$$

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$$\Pr[|S - \mu| \geq \epsilon\mu] \leq \delta.$$

Total space complexity: $O\left(\frac{1}{\epsilon^2\delta}\right)$ to estimate distinct elements up to error ϵ with success probability $1 - \delta$.

NOTE ON FAILURE PROBABILITY

$O\left(\frac{1}{\epsilon^2\delta}\right)$ space is an impressive bound:

- $1/\epsilon^2$ dependence cannot be improved.

[1] Technically, if we account for the bit complexity of storing $S_1, \ldots, S_k$ and the hash functions $h_1, \ldots, h_k$, the space complexity is $O\left(\frac{\log D}{\epsilon^2\delta}\right)$.

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- $1/\epsilon^2$ dependence cannot be improved.
- No linear dependence on number of distinct elements D.[1]
- But... $1/\delta$ dependence is not ideal. For 95% success rate, pay a $\frac{1}{5\%} = 20$ factor overhead in space.

We can get a better bound depending on $O(\log(1/\delta))$ using exponential tail bounds. We will see next.

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DISTINCT ELEMENTS IN PRACTICE

In practice, we cannot hash to real numbers on $[0,1]$. Could use a finite grid, but more popular choice is to hash to integers (bit vectors).

Real Flajolet-Martin / HyperLogLog:

$h(x_1)$	1010010
$h(x_2)$	1001100
$h(x_3)$	1001110
⋮	
$h(x_n)$	1011000

- Estimate # distinct elements based on maximum number of trailing zeros **m**.
- The more distinct hashes we see, the higher we expect this maximum to be.

LOGLOG SPACE

Total Space: $O\left(\frac{\log\log D}{\epsilon^2} + \log D\right)$ for an ϵ approximate count.

"Using an auxiliary memory smaller than the size of this abstract, the LogLog algorithm makes it possible to estimate in a single pass and within a few percents the number of different words in the whole of Shakespeare's works." – Flajolet, Durand.

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Using HyperLogLog to count 1 billion distinct items with 2% accuracy:

$$\begin{aligned}\text{space used } &= O\left(\frac{\log\log D}{\epsilon^2} + \log D\right)\\ &= \frac{1.04 \cdot \lceil \log_2 \log_2 D \rceil}{\epsilon^2} + \lceil \log_2 D \rceil \text{ bits}\end{aligned}$$

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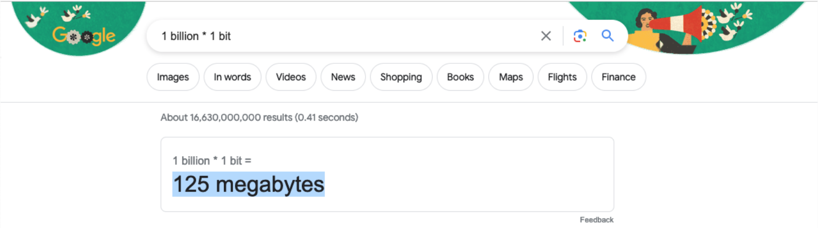
HYPERLOGLOG IN PRACTICE

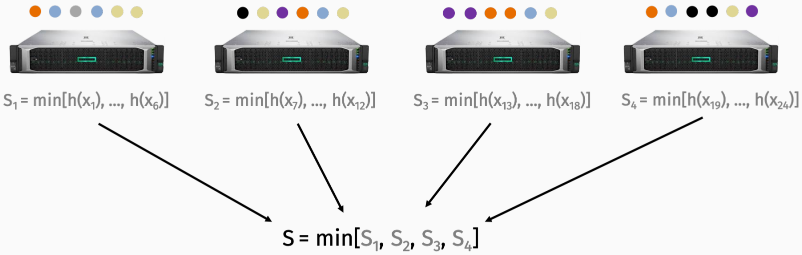
Although, to be fair, storing a dictionary with 1 billion bits only takes 125 megabytes. Not tiny, but not unreasonable.



These estimators become more important when you want to count many different things (e.g., a software company tracking clicks on 100s of UI elements).

DISTRIBUTED DISTINCT ELEMENTS

Also very important in distributed settings.



Distinct elements summaries are "mergeable". No need to share lists of distinct elements if those elements are stored on different machines. Just share minimum hash value.

HYPERLOGLOG IN PRACTICE

Implementations: Google PowerDrill, Facebook Presto, Twitter Algebird, Amazon Redshift.

Use Case: Exploratory SQL-like queries on tables with 100's of billions of rows.

- Count number of distinct users in Germany that made at least one search containing the word 'auto' in the last month.
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Answering a query requires a (distributed) linear scan over the database: 2 seconds in Google's distributed implementation.

Google Paper: "Processing a Trillion Cells per Mouse Click"

BEYOND CHEBYSHEV

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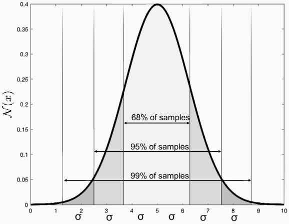

68-95-99 rule for Gaussian bell-curve. $\mathbf{X} \sim \mathbf{N}(\mathbf{0}, \sigma^2)$

Chebyshev's Inequality:

$$\Pr(|X - \mathbb{E}[X]| \geq 1\sigma) \leq 100\%$$
$$\Pr(|X - \mathbb{E}[X]| \geq 2\sigma) \leq 25\%$$
$$\Pr(|X - \mathbb{E}[X]| \geq 3\sigma) \leq 11\%$$
$$\Pr(|X - \mathbb{E}[X]| \geq 4\sigma) \leq 6\%.$$

Truth:

$$\Pr(|X - \mathbb{E}[X]| \geq 1\sigma) \approx 32\%$$
$$\Pr(|X - \mathbb{E}[X]| \geq 2\sigma) \approx 5\%$$
$$\Pr(|X - \mathbb{E}[X]| \geq 3\sigma) \approx 1\%$$
$$\Pr(|X - \mathbb{E}[X]| \geq 4\sigma) \approx .01\%$$

GAUSSIAN CONCENTRATION

$X \sim \mathcal{N}(\mu, \sigma^2)$ has probability density function (PDF) p with:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \mathbf{e}^{-(\mathbf{x}-\mu)^2/2\sigma^2}$$

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Lemma (Gaussian Tail Bound)

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Compare this to:

Lemma (Chebyshev's Inequality)

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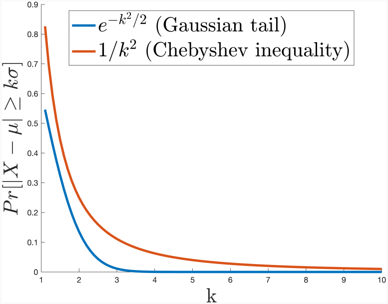
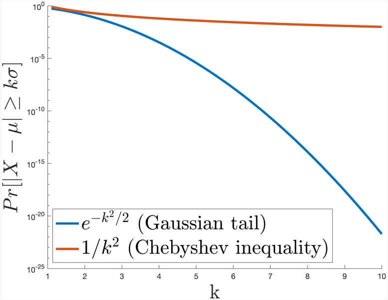


Standard y-scale.



Logarithmic y-scale.

Takeaway: Gaussian random variables concentrate much tighter around their expectation than variance alone predicts (i.e., than Chebyshevs's inequality predicts).

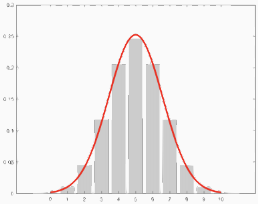
Why does this matter for algorithm design?

CENTRAL LIMIT THEOREM

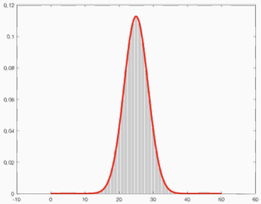
Theorem (CLT – Informal)

Any sum of mutually independent, (identically distributed) r.v.'s $X_1, \ldots, X_n$ with mean μ and finite variance σ^2 converges to a Gaussian r.v. with mean $n \cdot \mu$ and variance $n \cdot \sigma^2$, as $n \to \infty$.

$$S = \sum_{i=1}^{n} X_i \Longrightarrow \mathcal{N}(n \cdot \mu, n \cdot \sigma^2).$$



(a) Distribution of # of heads after 10 coin flips, compared to a Gaussian.



(b) Distribution of # of heads after 50 coin flips, compared to a Gaussian.

Recall:

Definition (Mutual Independence)

Random variables $X_1, \ldots, X_n$ are mutually independent if, for all possible values $v_1, \ldots, v_n$,

$$\Pr[X_1 = v_1, \ldots, X_n = v_n] = \Pr[X_1 = v_1] \cdot \ldots \cdot \Pr[X_n = v_n]$$

Strictly stronger than pairwise independence.

EXERCISE

If I flip a fair coin 100 times, lower bound the chance I get between 30 and 70 heads?

Let's approximate the probability by assuming the limit of the CLT holds exactly – i.e., that this sum looks exactly like a Gaussian random variable.

Lemma (Gaussian Tail Bound)

For $X \sim \mathcal{N}(\mu, \sigma^2)$:

$$\Pr[|X - \mathbb{E}X| \geq k \cdot \sigma] \leq 2e^{-k^2/2}.$$

Recall, $\mathbb{E}[X] = n \cdot 0.5 = 50$ and $\sigma(X) = \sqrt{n \cdot 0.5 \cdot 0.5} = 5$. Setting $k = 4$ gives:

$$\Pr[|X - 50| \geq 20] \leq 2e^{-8}.$$

$2e^{-8} = .06\%$. Chebyshev's inequality gave a bound of 6.25%.

QUANTITATIVE VERSIONS OF THE CLT

These back-of-the-envelop calculations can be made rigorous! Lots of different "versions" of bound which do so.

- Chernoff bound
- Bernstein bound
- Hoeffding bound
- ...

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Different assumptions on random varibles (e.g. binary vs. bounded), different forms (additive vs. multiplicative error), etc.

Wikipedia is your friend.

Theorem (Chernoff Bound)

Let $X_1, X_2, \ldots, X_n$ be independent $\{0,1\}$-valued random variables and let $p_i = \mathbb{E}[X_i]$, where $0 < p_i < 1$. Then the sum $S = \sum_{i=1}^n \mathbb{E}[X_i] = \sum_{i=1}^n X_i$, which has mean $\mu = \sum_{i=1}^n p_i$, satisfies

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$$\Pr[S \geq (1+\epsilon)\mu] \leq e^{\frac{-\epsilon^2 \mu}{2+\epsilon}}.$$

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$$\Pr[S \geq (1+\epsilon)\mu] \leq e^{\frac{-\epsilon^2 \mu}{2+\epsilon}}.$$

and for $0 < \epsilon < 1$

$$\Pr[S \leq (1-\epsilon)\mu] \leq e^{\frac{-\epsilon^2 \mu}{2}}.$$

CHERNOFF BOUND

Theorem (Chernoff Bound Corollary)

Let $X_1, \ldots, X_n$ be independent $\{0,1\}$-valued r.v.s with $p_i = \mathbb{E}[X_i]$. Let $S = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[S]$. For $\epsilon \in (0,1)$,

$$\Pr[|S - \mu| \geq \epsilon\mu] \leq 2e^{-\epsilon^2\mu/3}.$$

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Connection to Gaussian tail bound $\Pr[|S - \mu| \geq k\sigma] \lesssim 2e^{-k^2/2}$:

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Connection to Gaussian tail bound $\Pr[|S - \mu| \geq k\sigma] \lesssim 2e^{-k^2/2}$:

$$\mathrm{Var}[S] = \sum_{i=1}^n \mathrm{Var}[X_i] = \sum_{i=1}^n (p_i - p_i^2) \leq \sum_{i=1}^n p_i = \mu \implies \sigma \leq \sqrt{\mu}.$$

CHERNOFF BOUND

Theorem (Chernoff Bound Corollary)

Let $X_1, \ldots, X_n$ be independent $\{0,1\}$-valued r.v.s with $p_i = \mathbb{E}[X_i]$. Let $S = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[S]$. For $\epsilon \in (0,1)$,

$$\Pr[|S - \mu| \geq \epsilon\mu] \leq 2e^{-\epsilon^2\mu/3}.$$

Connection to Gaussian tail bound $\Pr[|S - \mu| \geq k\sigma] \lesssim 2e^{-k^2/2}$:

$$\mathrm{Var}[S] = \sum_{i=1}^n \mathrm{Var}[X_i] = \sum_{i=1}^n (p_i - p_i^2) \leq \sum_{i=1}^n p_i = \mu \implies \sigma \leq \sqrt{\mu}.$$

Setting $\epsilon\mu = \sigma k = \sqrt{\mu}k$, so $\epsilon = k/\sqrt{\mu}$:

$$\Pr[|S - \mu| \geq \sigma k] \leq 2e^{-k^2/3}$$

Theorem (Bernstein Inequality)

Let $X_1, X_2, \ldots, X_n$ be independent random variables with each $X_i \in [-1, 1]$. Let $\mu_i = \mathbb{E}[X_i]$ and $\sigma_i^2 = \mathrm{Var}[X_i]$. Let $\mu = \sum_{i=1}^n \mu_i$ and $\sigma^2 = \sum_{i=1}^n \sigma_i^2$. Then, for $k \leq \frac{1}{2}\sigma$, $S = \sum_{i=1}^n X_i$ satisfies

$$\Pr[|S - \mu| > k \cdot \sigma] \leq 2e^{-k^2/4}.$$

QUANTITATIVE VERSIONS OF THE CLT

Theorem (Hoeffding Inequality)

Let $X_1, X_2, \ldots, X_n$ be independent random variables with each $X_i \in [a_i, b_i]$. Let $\mu_i = \mathbb{E}[X_i]$ and $\mu = \sum_{i=1}^n \mu_i$. Then, for any $k > 0$, $S = \sum_{i=1}^n X_i$ satisfies:

$$\Pr[|S - \mu| > k] \leq 2e^{\frac{-2k^2}{\sum_{i=1}^n (b_i - a_i)^2}}.$$

HOW ARE THESE BOUNDS PROVEN?

Variance is a natural measure of central tendency, but there are others.

$$q^{\text{th}} \text{ central moment: } \mathbb{E}[(X - \mathbb{E}X)^q]$$

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Idea in brief: Apply Markov's inequality to $\mathbb{E}[(X - \mathbb{E}X)^q]$ for larger q, or more generally to $f(X - \mathbb{E}X)$ for some other non-negative function f. E.g., to $\exp(X - \mathbb{E}X)$. Doing so requires higher-order independence.

EXERCISE

If I flip a fair coin 100 times, lower bound the chance I get between 30 and 70 heads?

Corollary of Chernoff bound: Let $S = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}[S]$. For $0 < \epsilon < 1$,

$$\Pr[|S - \mu| \geq \epsilon\mu] \leq 2e^{-\epsilon^2 \mu/3}$$

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$$\Pr[|S - 50| \geq 20] \leq 2e^{-(2/5)^2 \cdot 50/3} = 2e^{-8/3} \approx 1.4\%.$$

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Chebyshev's inequality gave a bound of 6.25% and Gaussian tail bound gave a bound of 0.06%.

CHERNOFF BOUND APPLICATION

Estimating the bias of a coin: Flip biased coin n times: i.e. the coin is heads with probability b.

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I.e., with probability at least $1 - \delta$, $\frac{\#\text{ heads}}{n} \in [b - \epsilon, b + \epsilon]$.

Proof: Recall, Chernoff bound states

$$\Pr[|S - \mathbb{E}[S]| \geq \alpha \mathbb{E}[S]] \leq 2e^{-\alpha^2 \mathbb{E}[S]/3}.$$

We want $\alpha bn = \epsilon n$, so $\alpha = \epsilon / b$. Then, we have,

$$\Pr[|\#\text{ heads} - b \cdot n| \geq \epsilon n] \leq 2e^{-\epsilon^2 n/3b^2} \leq 2e^{-\epsilon^2 n/3}$$

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Choosing $n = \frac{6\log(1/\delta)}{\epsilon^2}$ implies $2e^{-\epsilon^2 n/3} \leq \delta$.

LOAD BALANCING

Load balancing problem:

Suppose Google answers map search queries using servers $A_1, \ldots, A_q$. Given a query like "new york to rhode island", common practice is to choose a random hash function $h \rightarrow \{1 \ldots, q\}$ and to route this query to server:

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Why use a hash function instead of just distributing requests randomly?

LOAD BALANCING

Suppose we have n servers and m requests, $x_1, \ldots, x_m$. Let s_i be the number of requests sent to server $i \in \{1, \ldots, n\}$:

$$s_i = \sum_{j=1}^{m} \mathbb{1}[h(x_j) = i].$$

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If we distribute requests in a round robin fashion, $S \approx \frac{m}{n}$. But we have to repeat work.

LOAD BALANCING

A good first step is to first think about expectations. If we have n servers and m requests, and a uniformly random hash function, for any $i \in \{1, \ldots, n\}$, what is $\mathbb{E}[s_i]$?

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$$\mathbb{E}[s_i] = \sum_{j=1}^{m} \mathbb{E}\left[\mathbb{1}[h(x_j) = i]\right] = \frac{m}{n}.$$

But it's unclear what the expectation of $S = \max_{i \in \{1,\dots,n\}} s_i$ is... in particular, $\mathbb{E}[S] \neq \max_{i \in \{1,\dots,n\}} \mathbb{E}[s_i]$.

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Exercise: Convince yourself that for two random variables A and B, $\mathbb{E}[\max(A, B)] \neq \max(\mathbb{E}[A], \mathbb{E}[B])$ even if those random variable are independent.

SIMPLIFYING ASSUMPTIONS

Number of servers: To reduce notation and keep the math simple, let's assume that $m = n$. I.e., we have exactly the same number of servers and requests.

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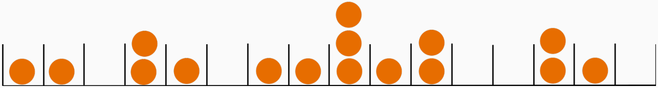
Hash function: Continue to assume a fully (uniformly) random hash function h.



Often called the "balls-into-bins" model.

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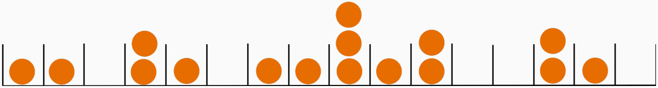

Often called the "balls-into-bins" model.

$\mathbb{E}[s_i]$ = expected number of balls per bin $= \frac{m}{n} = 1$. We would like to prove a bound of the form:

$$\Pr[\max_i s_i \geq C] \leq \frac{1}{10}.$$

for as tight a value of C. I.e., something much better than $C = n$.

BOUNDING A UNION OF EVENTS

Goal: Prove that for some C,

$$\Pr[\max_i s_i \geq C] \leq \frac{1}{10}.$$

Equivalent statement: Prove that for some C,

$$\Pr[(s_1 \geq C) \cup (s_2 \geq C) \cup \ldots \cup (s_n \geq C)] \leq \frac{1}{10}.$$

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These events are not independent, but we can apply union bound!

$$\Pr[(s_1 \geq C) \cup (s_2 \geq C) \cup \ldots \cup (s_n \geq C)] \leq \sum_{i=1}^{n} \Pr[s_i \geq C]$$

$n =$ number of balls and number of bins. s_i is number of balls in bin i.
$C =$ upper bound on maximum number of balls in any bin.

APPLICATION OF UNION BOUND

We want to prove that:

$$\Pr[\max_i s_i \geq C] = \Pr[(s_1 \geq C) \cup (s_2 \geq C) \cup \ldots \cup (s_n \geq C)] \leq \frac{1}{10}.$$

To do so, it suffices to prove that for all i:

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To do so, it suffices to prove that for all i:

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Why? Because then by the union bound,

$$\begin{aligned} \Pr[\max_i s_i \geq C] &\leq \sum_{i=1}^{n} \Pr[s_i \geq C] \quad \text{(Union bound)} \\ &\leq \sum_{i=1}^{n} \frac{1}{10n} = \frac{1}{10}. \quad \square \end{aligned}$$

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NEW GOAL

Prove that for some C,

$$\Pr[s_i \geq C] \leq \frac{1}{10n}.$$

Let's try doing this with Markov's, Chebyshev, and exponential concentration.

ATTEMPT WITH MARKOV'S INEQUALITY

Goal: Prove that $\Pr[s_i \geq C] \leq \frac{1}{10n}$.

- **Step 1.** Verify we can apply Markov's: s_i takes on non-negative values only. Good to go!

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- **Step 2.** Apply Markov's: $\Pr[s_i \geq C] \leq \frac{\mathbb{E}[s_i]}{C} = \frac{1}{C}$.

To prove our target statement, need to see $C = 10n$.

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To prove our target statement, need to see $C = 10n$.

Meaningless! There are only n balls, so of course there can't be more than $10n$ in the most overloaded bin.

ATTEMPT WITH CHEBYSHEV'S INEQUALITY

Goal: Prove that $\Pr[s_i \geq C] \leq \frac{1}{10n}$.

- **Step 1.** To apply Chebyshev's inequality, we need to understand $\sigma^2 = \mathrm{Var}[s_i]$.

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Goal: Prove that $\Pr[s_i \geq C] \leq \frac{1}{10n}$.

- **Step 1.** To apply Chebyshev's inequality, we need to understand $\sigma^2 = \text{Var}[s_i]$.

Let $s_{i,j}$ be a $\{0, 1\}$ indicator random variable for the event that ball j falls in bin i. We have:

$$s_i = \sum_{j=1}^{n} s_{i,j} = \sum_{j=1}^{n} \mathbb{1}[\text{ball j falls in bin i}].$$

VARIANCE ANALYSIS

$$s_{i,j} = \begin{cases} 1 \text{ with probability } \frac{1}{n} \\ 0 \text{ otherwise.} \end{cases}$$

$$\mathbb{E}[s_{i,j}] = \frac{1}{n} \tag{1}$$

$$\mathbb{E}[s_{i,j}^2] = \frac{1}{n} - \frac{1}{n^2} \approx \frac{1}{n} \tag{2}$$

So:

$$\mathrm{Var}[s_i] = \mathrm{Var}\left[\sum_{j=1}^{n} s_{i,j}\right] \approx \sum_{j=1}^{n} \frac{1}{n} \approx 1.$$

APPLYING CHEBYSHEV'S

Goal: Prove that $\Pr[s_i \geq C] \leq \frac{1}{10n}$.

Step 1. To apply Chebyshev's inequality, we need to understand $\sigma^2 = \mathrm{Var}[s_i]$.

$$\mathrm{Var}[s_i] \approx 1.$$

Step 2. Apply Chebyshev's inequality:

$$\Pr\left[|s_i - \mathbb{E}[s_i]| \geq k \cdot 1\right] \leq \frac{1}{k^2}$$

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$$\Pr\left[|s_i - \mathbb{E}[s_i]| \geq k \cdot 1\right] \leq \frac{1}{k^2}$$

Setting $k = \sqrt{10n}$,

$$\Pr[|s_i - 1| \geq \sqrt{10n}] \leq \frac{1}{10n}.$$

APPLYING CHEBYSHEV'S

Goal: Prove that $\Pr[s_i \geq C] \leq \frac{1}{10n}$.

We just proved that, for any k: $\Pr[|s_i - 1| \geq \sqrt{10n}] \leq \frac{1}{10n}$.

So, we have that:

$$\Pr[s_i \geq \sqrt{10n} + 1] \leq \frac{1}{10n}.$$

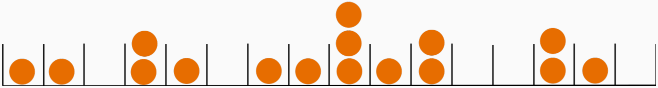
By the union bound argument from earlier, it thus holds that:

$$\Pr[\max_{i \in \{1,\dots,n\}} s_i \geq \sqrt{10n} + 1] \leq \frac{1}{10}.$$

So with probability at least 90%, $S = \max_{i \in \{1,\dots,n\}} s_i \leq \sqrt{10n} + 1$.

FINAL RESULT FOR CHEBYSHEV'S

When hashing n balls into n bins, the maximum bin contains $O(\sqrt{n})$ balls with probability $\frac{9}{10}$.



Much better than the trivial bound of n!

ATTEMPT WITH EXPONENTIAL CONCENTRATION

Goal: Prove that $\Pr[s_i \geq C] \leq \frac{1}{10n}$.

Recall: $s_i = \sum_{j=1}^{n} s_{i,j}$, where $s_{i,j} = \mathbb{1}[\text{ball } j \text{ lands in bin } i]$.

What bound might we use?

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Chernoff bound!

ATTEMPT WITH EXPONENTIAL CONCENTRATION

Theorem (Chernoff Bound)

Let $X_1, X_2, \ldots, X_n$ be independent $\{0,1\}$-valued random variables and let $p_i = \mathbb{E}[X_i]$, where $0 < p_i < 1$. Then the sum $S = \sum_{j=1}^{n} X_i$, which has mean $\mu = \sum_{j=1}^{n} p_i$, satisfies

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Apply with $S = s_i$, $X_j = s_{i,j}$. Set $\epsilon = c \cdot \log(n)$,

$$\begin{aligned}\Pr[S \geq (1 + c\log n)\mu] &\leq 2e^{\frac{-c^2(\log n)^2}{2+c\log n}} \leq 2e^{\frac{-c^2(\log n)^2}{2c\log n}} \\ &\leq 2e^{\frac{-c\log n}{2}} = 2 \cdot \left(\frac{1}{n}\right)^{c/2} \leq \frac{1}{10n}\end{aligned}$$

So max load for randomized load balancing is $O(\log n)$! Best we could prove with Chebyshev's was $O(\sqrt{n})$.

POWER OF TWO CHOICES

Power of 2 Choices: Instead of assigning job to random server, choose 2 random servers and assign to the least loaded. With probability 1/10 the maximum load is bounded by:

(a) $O(\log n)$ (b) $O(\sqrt{\log n})$ (c) $O(\log \log n)$ (d) $O(1)$