Notes on Distributionally Robust Optimization

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We work here with the Cramer-Lundberg model for insurance claims. This is a continuous time stochastic process, where the amount of money in the bank $R(t)$ satisfies

$$R(t) = u + ct - \sum_{i=1}^{N_t} X_i.$$

Here u is the initial money in the bank, c is the premium rate, and the X_i are claim sizes. They come in at a rate that is a Poisson process, N_t. You have to assume the X_i are distributed in some way. Let this distribution have first and second moments m_1 and m_2 respectively. We can assume $m_1 < \infty$ else the insurance company would have no point insuring such a risk. Just to make even, the amount of money coming in has to be the amount of money being paid out on average, so the choice,

$$c = \nu m_1$$

where ν is the rate at which claims come in. We add a safety loading – the so-called risk premium – and so the model is,

$$c = (1 + \eta)\nu m_1.$$

Hence our final model is of the form,

$$R(t) = u + (1 + \eta)\nu m_1 t - \sum_{i=1}^{N_t} X_i.$$

We are interested in the probability of ruin

$$\psi(u, T) \triangleq \mathbf{Pr}\left[\inf_{t \in [0,T]} R(t) \leq 0\right].$$

It turns out that it is computationally intractable to deal with the above so we instead work with the following Brownian motion approximation,

$$\begin{aligned} R_B(t) &\triangleq u + (1 + \eta)\nu m_1 t - (\nu m_1 t + \sqrt{\nu m_2} B(t)) \\ &= u + \eta \nu m_1 t - \sqrt{\nu m_2} B(t) \end{aligned}$$

Our mission is to robustify this estimate by allowing for some non-trivial movement away from the Brownian motion. For this purpose, we identity the Polish space where the stochastic processes of interest live as the *Skorkhod space*,

$$S = D([0,T], \mathbb{R})$$

the space of real valued right-continuous functions with left limits equipped with the J_1 metric. The J_1 metric is supposed to be like a sup metric that is robust against time shifts that go to zero. The formal definition of the J_1 metric is, if we let Λ be the set of strictly increasing functions $\lambda : [0,T] \to [0,T]$ such that $\lambda \in \Lambda$ implies λ, λ^{-1} are continuous up to a set of measure zero, and let e be the identity map on $[0,T]$. Then we define

$$d_{J_1}(x_1, x_2) \triangleq \inf_{\lambda \in \Lambda} \left\{ \|x_1 \circ \lambda - x_2\|_\infty \vee \|\lambda - e\|_\infty \right\}$$

where $\|\cdot\|_\infty$ is the sup norm and $\vee$ is the max function.

Intuitively, this is suppose to capture the sup norm except you're allowed to perturb the function input a litte bit. So it's a bit weaker than the sup norm. Take for example when $T = 1$,

$$x_n = (1 + n^{-1}) \mathbf{1}_{\left\{[\frac{1}{2} + \frac{1}{n}, 1]\right\}}, \quad x = \mathbf{1}_{\left\{[\frac{1}{2}, 1]\right\}}.$$

Then $\|x_n - x\|_\infty \geq 1$ for all n but $d_{J_1}(x_n, x) \to 0$ as $n \to \infty$. To show this notice that,

$$\begin{aligned} d_{J_1}(x_n, x) &= \inf_{\lambda \in \Lambda} \left\{ \|x_n \circ \lambda - x\|_\infty \vee \|\lambda - e\|_\infty \right\} \\ &\leq \max \left\{ \|x_n \circ \lambda_n - x\|_\infty, \|\lambda_n - e\|_\infty \right\}, \end{aligned}$$

for some feasible choices λ_n. Let us choose

$$\lambda_n = \begin{cases} x & x \in [0, \frac{1}{2}] \\ (1 - \frac{2}{n})x + \frac{2}{n} & x \in [\frac{1}{2}, 1], \end{cases}$$

which is effectively the identity from $[0, \frac{1}{2}]$ and then jumps to $\frac{1}{2} + \frac{1}{n}$ and then is the line which connects $(\frac{1}{2}, \frac{1}{2} + \frac{1}{n})$ to $(1,1)$. Notice now that the difference between $x_n \circ \lambda_n$ and x now only sees contribution from the part of the domain $\frac{1}{2} + \frac{1}{n}$, so that,

$$\|x_n \circ \lambda_n - x\|_\infty = \frac{1}{n}.$$

On the other hand we have that the identity map and λ differ the most at the point $x = \frac{1}{2}$ where they differ by $\frac{1}{n}$, so that we conclude,

$$\begin{aligned} d_{J_1}(x_1, x_2) &= \inf_{\lambda \in \Lambda} \left\{ \|x_n \circ \lambda\|_\infty \vee \|\lambda - e\|_\infty \right\} \\ &\leq \max \left\{ \|x_n \circ \lambda_n\|_\infty, \|\lambda_n - e\|_\infty \right\} \\ &\leq \frac{1}{n} \to 0. \end{aligned}$$

Numerical Example

In the numerical example, Jose and Karthyek Murthy, use the Paretto distribution for the claim sizes, with $\alpha = 2.2$. This corresponds to a mean of $\frac{\alpha}{\alpha-1}$. The question