Worksheet

This problem are taken from the excellent text book Cover and Thomas and is not for submission. It is actually hard to find reasonably doable questions on differential entropy and these would be hard to get done in time in an exam.

Q1 - differential entropy

What is the entropy of $X_1 + X_2$ where X_1 and X_2 are independent normal variables with means μ_i and variances σ_i^2?

Q2 - channel capacity

If $Z = X + Y$ where X is uniform on $[-1/2, 1/2]$ and Y is independent of X and uniform on $[-a/2, a/2]$ with $a < 1$, what is $I(X; Z)$ as a function of a?

Q1 - outline solution

You can use the convolution,

$$p_Z(z) = \int_{-\infty}^{\infty} p_X(x)p_Y(z-x)dx \tag{1}$$

for $Z = X+Y$, to show that the sum of two Gaussians gives another Gaussian with $\mu = \mu_1 + \mu_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$; you can then apply the usual formula for the entropy of a Gaussian:

$$h(X_1 + X_2) = \frac{1}{2}\log 2\pi e\sigma^2 \tag{2}$$

Q2 - outline solution

So for a random variable uniform on a region of width a, say $[y_0, y_0 + a]$

$$h(Y) = -\frac{1}{a}\int_{y_0}^{y_o+a} \log\frac{1}{a}dy = \log a \tag{3}$$

Now working out $p_Z(z)$ is harder because, again using the convolution theorem we get

$$p_Z(z) = \begin{cases} \frac{1}{2a}\left(z + \frac{1+a}{2}\right) & -\frac{1+a}{2} < z < -\frac{1-a}{2} \\ 1 & -\frac{1-a}{2} < z < \frac{1-a}{2} \\ \frac{1}{2a}\left(-z - \frac{1+a}{2}\right) & \frac{1-a}{2} < z < \frac{1+a}{2} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

This would allow $h(Z)$ to be calculated by splitting up the integral and doing the relevant integrations by parts. This gives

$$h(Z) = \frac{a}{2} \tag{5}$$

Finally if X is fixed Z is just the uniform distribution so

$$h(Z|X) = \log a \tag{6}$$

and hence

$$I(Z;X) = h(Z) - h(Z|X) = \frac{a}{2} - \log a \tag{7}$$