Combinatorial Game Theory

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What is a Combinatorial Game?

In a combinatorial game:

- there are two players
- there is a set of states (usually finite)
- there is a set of moves between states for each player
  - If both players can make the same moves from any state, the game is said to be *impartial*, or *partisan* otherwise. We will only consider impartial games.
- the players alternate moving
- the game ends when the player whose turn it is cannot make a move (we say the state is *terminal*)
  - Under the *normal play rule*, the player who is unable to make a move loses. However, we could instead use the *misère play rule*, where this player wins.
- the game must end within finitely many moves

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Winning and Losing Positions

- A winning state (or *N-position*) is one from which the player can guarantee a win, regardless of the other player's moves.
- Since there are no draws, a state which is not winning must be losing (*P-position*).
- We can represent the states as vertices in a directed graph, with moves corresponding to directed edges. Assuming the vertex set is finite, the graph must be acyclic, to prevent non-terminating games.

Winning and Losing Positions

With this knowledge, we can use dynamic programming to classify the states as winning or losing:

1. Topologically sort the graph.
2. Traverse the vertices in reverse order, marking each as:
   - winning if it has an edge to a losing vertex, or
   - losing otherwise.

Example: Subtraction Game

Problem

Alice and Bob have a pile of N stones. They take turns to remove 1, 2 or 3 stones from the pile, with the player who takes the last stone winning.

If Alice moves first, who wins?

Solution

- 0 is a losing state.
- 1, 2 and 3 are winning states, as they can reach 0.
- 4 is a losing state, as it can only reach winning states.
- 5, 6 and 7 are winning states, as they can reach 4.

In this way, we see that N is a losing state iff it is divisible by 4, i.e. Bob wins if N is divisible by 4, and Alice wins otherwise.

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Problem

Alice and Bob have N piles of stones, with the ith pile containing a_i stones. They take turns to remove any number of stones from a single pile, with the player who takes the last stone winning.

If Alice moves first, who wins?

Solution

Denote bitwise xor by $\oplus$. We refer to this as *nim-sum*.

Claim

$(a_1, a_2, \ldots, a_N)$ is a losing state iff

$$a_1 \oplus a_2 \oplus \cdots \oplus a_N = 0.$$

Note that $\oplus$ is associative and commutative, so we may evaluate the expression above in any order. Also,

$$x \oplus x = 0,$$

and

$$x \oplus y = x \oplus z \implies y = z.$$

The only terminal state in this game is $(0, 0, \ldots, 0)$, which indeed satisfies this condition. It remains to check the non-terminal states.

Proof

Consider the case where $a_1 \oplus a_2 \oplus \cdots \oplus a_N = 0$ (and the a_i are not all zero). Then for any j,

$$\bigoplus_{i \neq j} a_i = a_j.$$

If our move takes all but b_j stones from pile j, our new nim-sum is

$$a_1 \oplus \cdots \oplus a_{j-1} \oplus b_j \oplus a_{j+1} \oplus \cdots \oplus a_N = a_j \oplus b_j,$$

which is certainly nonzero since $b_j < a_j$.
Thus from any non-terminal state of zero nim-sum, we can only reach states with nonzero nim-sum.

Proof (continued)

Now consider the alternative case, where

$$a_1 \oplus a_2 \oplus \cdots \oplus a_N = s_N \neq 0.$$

Consider the most significant bit of s_N. There must be an odd number of a_i with this bit set, so in particular there is at least one, say j.

Now reduce pile j from a_j stones to $a_j \oplus s_N$ stones. Since this unsets the most significant bit of s_N in a_j, it indeed decreases the pile size and hence is a valid move. Our new nim-sum is

$$a_1 \oplus \cdots \oplus a_{j-1} \oplus (a_j \oplus s_N) \oplus a_{j+1} \oplus \cdots \oplus a_N = s_N \oplus s_N = 0.$$

Thus from any state of nonzero nim-sum, we can reach a state with zero nim-sum.

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Why is Nim important?

Nim is the most famous combinatorial game, as many other games can be reduced to Nim. As we will see, we can analyse every impartial game in terms of Nim.

There are also many generalisations of Nim, induced by rule changes. An example is Moore's Nim_k, where a move consists of removing any number of stones from exactly k piles.

Exercise

Solve Moore's Nim_k.

Example: Turning Turtles

Problem

Alice and Bob have a row of N coins, each with either heads or tails up. They take turns to flip a head to a tail, and flip at most one other coin anywhere to its left. The first player unable to make a move loses.

If Alice moves first, who wins?

Example: Turning Turtles

Solution

We can transform this game to Nim!

- A head in position n corresponds to a pile of n stones.
- Flipping a single head to a tail corresponds to removing all the stones from that pile.
- Flipping a head to a tail in position n and also the coin in position $m < n$ corresponds to removing $n - m$ stones from that pile.
  - If coin m was also a head, we now have two piles of size m, which do not affect the nim-sum and hence the classification of the state as winning or losing.

So Alice wins iff the starting position is a winning state, i.e. the nim-sum of the locations of the heads is nonzero.

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The Sprague-Grundy Function

Definition

Define the *minimal excludant* (mex) of a set $S \subset \mathbb{N}$ as the smallest natural number not in S.

Denote $F(x)$ for the set of states that can be reached by a move from state x (the *followers* of x).

Definition

The Sprague-Grundy function is defined on states x as follows:

$$g(x) = \text{mex}\{g(y) : y \in F(x)\}.$$

Winning and Losing States from the Sprague-Grundy Function

It is easy to see that $g(x) = 0$ iff x is a losing state:

- If x is terminal, $g(x) = \text{mex}(\emptyset) = 0$.
- If x is winning, there exists some losing $y \in F(x)$, i.e. $g(y) = 0$ for some $y \in F(x)$, so $g(x) > 0$.
- If x is losing, there is no losing $y \in F(x)$, i.e. $g(y) > 0$ for all $y \in F(x)$, so $g(x) = 0$.

However, the real power of the Sprague-Grundy function is that it gives more information about a state, which allows us to solve sums of graph games.

Example: Lasker's Nim

Problem

Alice and Bob have a pile of N stones. They take turns to remove any number of stones from a single pile, or split a pile into two nonempty piles, with the player who takes the last stone winning.

Compute the Sprague-Grundy function for this game.

Example: Lasker's Nim

Solution

Since the only terminal state is 0, $g(0) = 0$. We claim that for $k \geq 0$,

- $g(4k+1) = 4k+1$
- $g(4k+2) = 4k+2$
- $g(4k+3) = 4k+4$
- $g(4k+4) = 4k+3$

Exercise

Complete the proof (by induction).

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Sums of Graph Games

Lasker's Nim, and indeed Nim itself, is very boring with one pile. You can just take all the stones and win immediately!

But if we have multiple piles, this is no longer a viable strategy, and we need to take a more refined approach.

The Sprague-Grundy Theorem

Denote $G = G_1 + G_2 + \ldots + G_N$ for a sum of graph games, where states of G are tuples of states of the games G_i (e.g. $x = (x_1, x_2, \ldots, x_N)$) and a move consists of making a move in one of the G_i while leaving the others unchanged.

Theorem (Sprague-Grundy)

$$g(x_1, x_2, \ldots, x_N) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_N(x_N),$$

where g_i is the Sprague-Grundy function of G_i.

This remarkable theorem allows us to solve sums of graph games by just computing the Sprague-Grundy function of each component game. Indeed, we do not even need the G_i to be the same game!

Example: Lasker's Nim

Problem

Alice and Bob have three piles of 2, 5 and 7 stones. They take turns to remove any number of stones from a single pile, or split a pile into two nonempty piles, with the player who takes the last stone winning.

If Alice moves first, who wins?

If it is Alice, what is her first move?

Example: Lasker's Nim

Solution

Using the Sprague-Grundy function found earlier,

$$\begin{aligned} g(2,5,7) &= g(2) \oplus g(5) \oplus g(7) \\ &= 2 \oplus 5 \oplus 8 \\ &= 15. \end{aligned}$$

This is nonzero, so Alice wins.

Example: Lasker's Nim

Solution (continued)

Now we construct a winning first move.

The most significant bit in 15 is the 8 bit, which is set in 8 but not in 2 or 5. So we need to make a move to change the 8 to $8 \oplus 15 = 7$.

- Note that such a move must be possible from our definition of $g(x)$ as the mex of the g-values of followers of x.

This can be achieved by splitting the 7 pile into 1 and 6, 2 and 5 or 3 and 4. After any of these moves, the nim-sum of the Sprague-Grundy function values of the four piles will be zero, putting Bob in a losing position.

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What is a Coin-Turning Game?

Coin-turning games have a finite number of coins in a row. A move consists of flipping some subset of coins, where the rightmost flipped coin must flip from heads to tails (to ensure that the game terminates).

Theorem

In these games, the same decomposition from Turning Turtles can be used:

$$g(THHTH) = g(TH) \oplus g(TTH) \oplus g(TTTTH).$$

That is, we can solve each head independently and consider a state as a sum of these single-head games.

Example: Mock Turtles

Problem

Alice and Bob have a row of N coins. They take turns to flip up to three coins, with the rightmost going from heads to tails. The first player who is unable to move loses.

If Alice moves first, who wins?

Example: Mock Turtles

Solution

Number the coins from 0. We use the number k to represent a state with k tails followed by a single head at position k.

- $g(0) = g(H) = 1$, since we must flip the only coin to reach a position of all tails, which is terminal and hence has g-value 0.
- $g(1) = g(TH) = 2$, since we can either:
  - flip both coins to reach g-value 1, or
  - flip just the second to reach g-value 0.

Example: Mock Turtles

Solution (continued)

- $g(2) = g(TTH) = 4$, since we can either:
  - flip all three coins to reach $g(HH) = g(H) \oplus g(TH) = 1 \oplus 2 = 3$,
  - flip the first and third coins to reach g-value 1,
  - flip the second and third coins to reach g-value 2, or
  - flip just the third to reach g-value 0.

Example: Mock Turtles

Solution (continued)

Continuing thus, we obtain:

x	0	1	2	3	4	5	6	7	8	9
$g(x)$	1	2	4	7	8	11	13	14	16	19

So $g(x)$ seems to always be either $2x$ or $2x + 1$, but which one?

Example: Mock Turtles

Solution (continued)

We say that n is *odious* if its binary representation has an odd number of ones, and *evil* otherwise.

$$\text{evil} \oplus \text{evil} = \text{odious} \oplus \text{odious} = \text{evil}$$

$$\text{evil} \oplus \text{odious} = \text{odious}$$

We claim that $g(x)$ is whichever of $2x$ and $2x + 1$ is odious, which again can be proven by induction (omitted here).

Example: Mock Turtles

Solution (continued)

Then for the state with heads at $x_1, x_2, \dots, x_N$, we have $g(x_1, x_2, \dots, x_N) = 0$ iff

$$g(x_1) \oplus g(x_2) \oplus \cdots \oplus g(x_N) = 0.$$

Each of these numbers is odious, and they have an evil nim-sum, so there must be an even number of them, i.e. N is even. Also, if we discard the last bit from all terms of the nim-sum, we recover

$$x_1 \oplus x_2 \oplus \cdots x_N = 0.$$

Example: Mock Turtles

Solution (continued)

So the losing positions in Mock Turtles correspond exactly to the losing positions in Nim with an *even* number of piles, i.e. Bob wins iff there are an even number of heads, and the 0-based positions of the heads have zero nim-sum.

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Two-Dimensional Coin Turning Games

Suppose a game has coins in a rectangular array, numbered from (0,0). Moves again flip some subset of coins, but now every move must have a southeast coin (one which is not below or to the left of any other coin) which is flipped heads to tails.

Such games can again be analysed as a sum of single-head games.

Tartan Games

A special subclass of these games are Tartan games.

Definition

A Tartan game is a product of coin-turning games

$$G = G_1 \times G_2,$$

with states (x, y) corresponding to pairs of states x and y of G_1 and G_2 respectively.

If turning coins $x_1, x_2, \ldots, x_m$ is a move in G_1 (when x_m starts at heads) and turning coins $y_1, y_2, \ldots, y_n$ is a move in G_2 (when y_n starts at heads), then turning all coins (x_i, y_j) for $1 \leq i \leq m$, $1 \leq j \leq n$ is a move in G (when (x_m, y_n) starts at heads).

Nim-Multiplication

Define *nim-multiplication* as follows:

$$x \otimes y = \text{mex}\{(x \otimes b) \oplus (a \otimes y) \oplus (a \otimes b) : 0 \leq a < x, 0 \leq b < y\}.$$

This is indeed a sensible multiplication - so much so that $(\mathbb{N}, \oplus, \otimes)$ is a field!

Nim-multiplication can be evaluated faster using Fermat 2-powers (of the form 2^{2^n}). This algorithm will not be detailed here.

Theorem

The Tartan Theorem states that the Sprague-Grundy function of $G = G_1 \times G_2$ is given by

$$g(x, y) = g_1(x) \otimes g_2(y).$$

Example: Rugs

Problem

Alice and Bob play a game on a two-dimensional array of coins. A move consists of flipping all coins in an axis-aligned rectangle, whose bottom-right corner must be flipped from heads to tails. The players alternate turns, with the first player unable to move losing.

If Alice goes first, who wins?

Example: Rugs

Solution

We must first compute the Sprague-Grundy function of the corresponding one-dimensional game, known as Ruler, in which a move consists of flipping any number of consecutive coins, with the rightmost coin going from heads to tails.

Using 1-based labelling, we find that

$$g(n) = \text{mex}\{0, g(n-1), g(n-1) \oplus g(n-2), \ldots, \\ g(n-1) \oplus g(n-2) \oplus \cdots \oplus g(1)\},$$

which can be shown to be the smallest power of 2 dividing n.

Then for each head at (x, y) in the array, we compute $g(x, y) = g(x) \otimes g(y)$, and nim-sum the results. Alice wins iff this final nim-sum is nonzero.

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The Game of Hackenbush

Alice and Bob play a game. Starting with several undirected graphs attached to the ground, each player takes turns to cut some edge, and remove any part of the graph which is no longer connected to the ground. The first player who is unable to move loses. If Alice moves first, who wins?

Note

The game described above is known as Green Hackenbush - there is a partisan variant known as Blue-Red Hackenbush where some edges can only be cut by a specific player.

Note

The ground can be treated as a single vertex, i.e. all vertices at which a graph is connected to the ground are really the same vertex.

Bamboo Stalks

An easy starting point is to consider the case where each graph is a straight line graph, like a bamboo stalk.

This is evidently equivalent to Nim, with pile sizes given by the number of edges of each stalk. We will henceforth refer to the stalk size by its number of edges.

Green Hackenbush on Trees

Now consider the case where each graph is a tree. The Sprague-Grundy theorem tells us that each tree behaves like a nim pile, or equivalently a bamboo stalk.

Colon Principle

If several stalks branch upward from a single node, they can be replaced by a single stalk of size equal to the nim-sum of the branches, without changing the Sprague-Grundy value of the tree.

By repeatedly applying the Colon Principle, we can eliminate all branching in a tree, to obtain a bamboo stalk. Then the problem is reduced to Nim as before.

Green Hackenbush

Finally, we return to the original problem. It remains to remove cycles - this will leave trees, which we can solve.

Fusion Principle

The vertices of a cycle can be fused without changing the Sprague-Grundy value of the tree.

Applying the Fusion Principle and the Colon Principle will reduce odd cycles to a single edge and even cycles to a single vertex.

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More problems

- Codeforces: Nullify the Matrix, Election Promises
- CodeChef: Dual Nim, Floor Division Game, CAO Stage-3
- Project Euler: Paper-strip Game, Nim Square, Fibonacci tree game
- Google Code Jam: Bacterial Tactics
- ACM-ICPC Jakarta 2010: Playing With Stones
- AtCoder: Strange Nim

Reference

The material covered in this lecture is from Chapter 1 of Thomas S. Ferguson's 2019 book *A Course in Game Theory*.