Segtrees I

Intro to the S-Tier data structure

Warmup: Range sum query

Given an array of N integers, a[1], a[2] … a[n], support Q queries of the form

“Given 1 <= L <= R <= N, return the sum a[L] + a[L+1] + … + a[R]”

Target complexity: O(N) precalculation, O(1) per query.

Range min query?

Given an array of N integers, a[1], a[2] … a[n], support Q queries of the form

“Given 1 <= L <= R <= N, return the minimum min(a[L],a[L+1] … a[R])”

What techniques do you know to solve this?

(*answer withheld before lecture*)

Range min query + point updates

Given an array of N integers, a[1], a[2] … a[n], support Q queries of the form

(Query) L R: Returns a[L] + a[L+1] + … + a[R]
(Update) X Y: Change the value of a[X] to Y

Idea: Precalc blocks of size 2^i

Given an array of N integers, a[1], a[2] … a[n], support Q queries of the form

(Query) L R: Return a[L] + a[L+1] + … + a[R]
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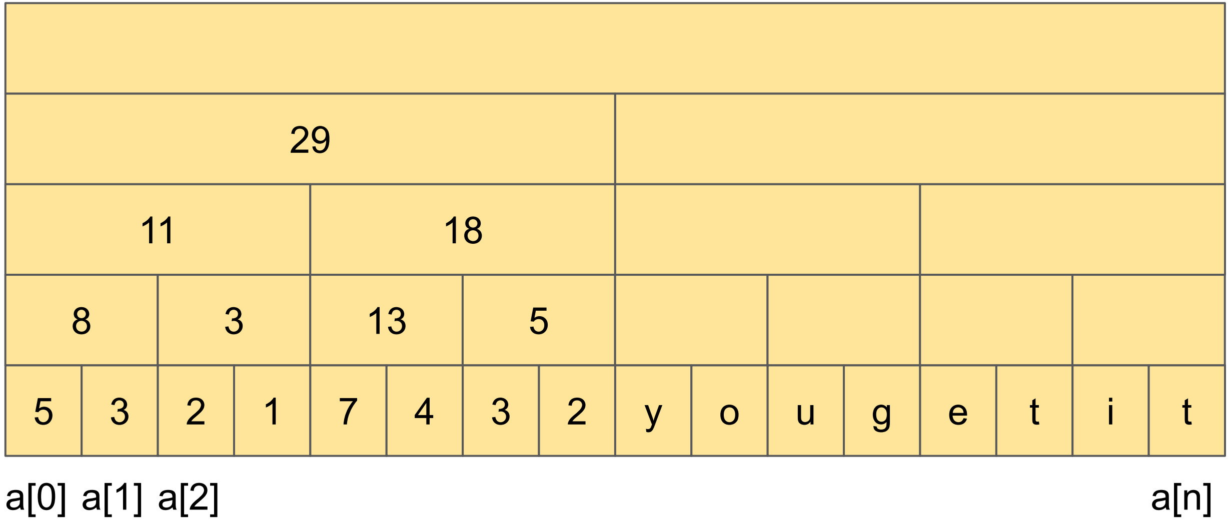

Number of things you have to precalc?

Given an array of N integers, a[1], a[2] … a[n], support Q queries of the form

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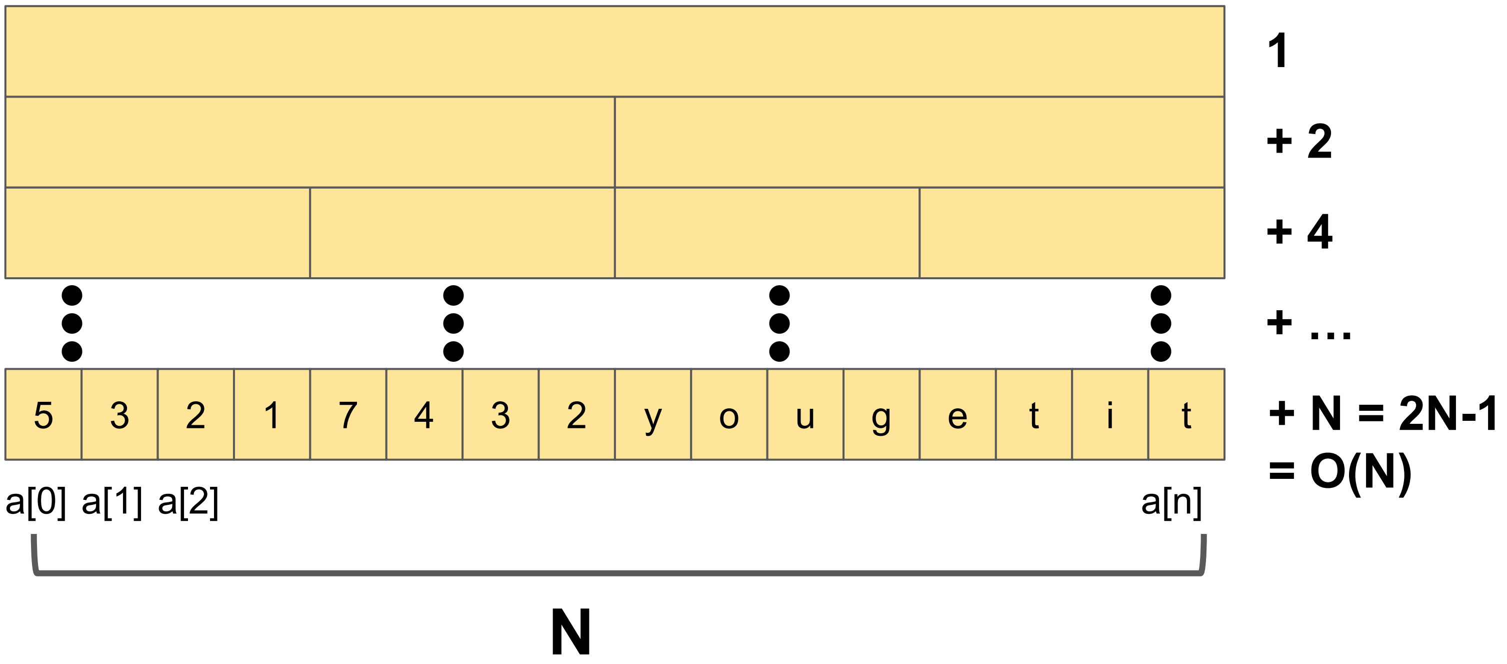

Later in implementation: how to precalc in O(N)

Query examples…

Given an array of N integers, a[1], a[2] … a[n], support Q queries of the form

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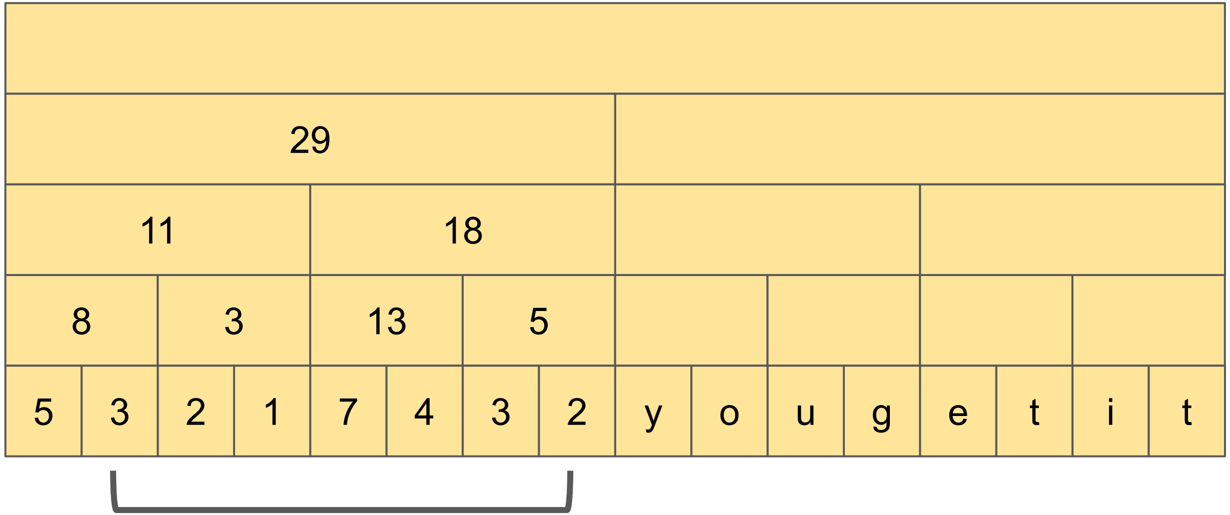

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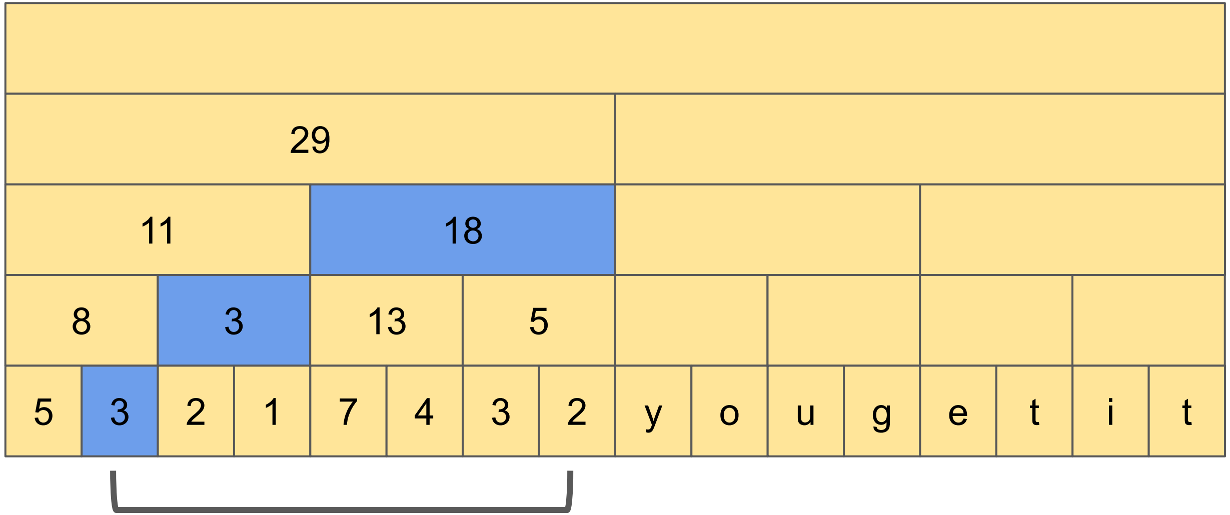

Querying sum of 7 elements, requires 3 blocks.

Query examples…

Given an array of N integers, a[1], a[2] … a[n], support Q queries of the form

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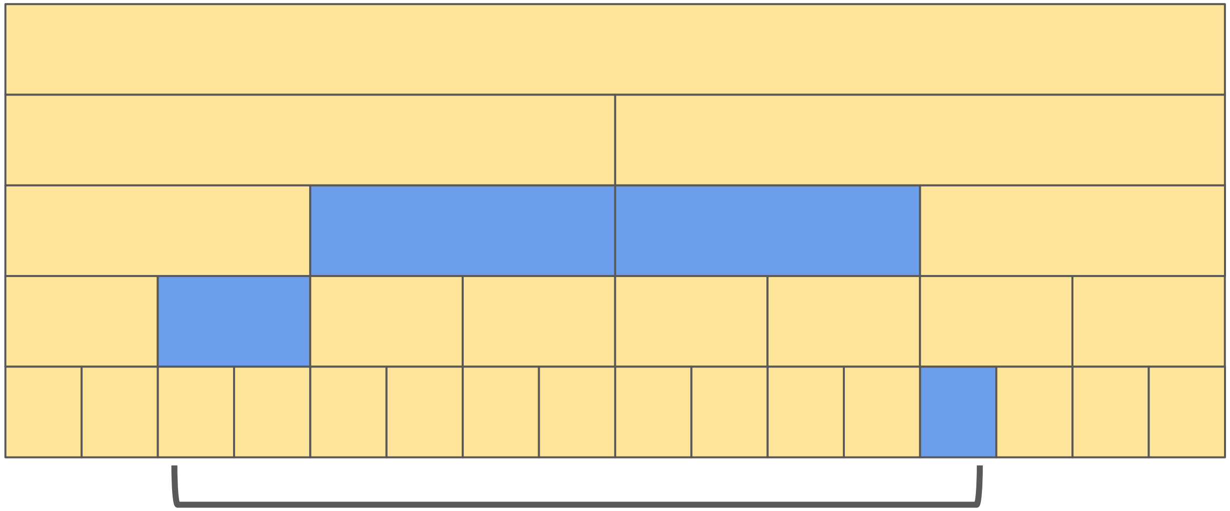

Querying sum of 11 elements, requires 4 blocks.

Query examples…

Given an array of N integers, a[1], a[2] … a[n], support Q queries of the form

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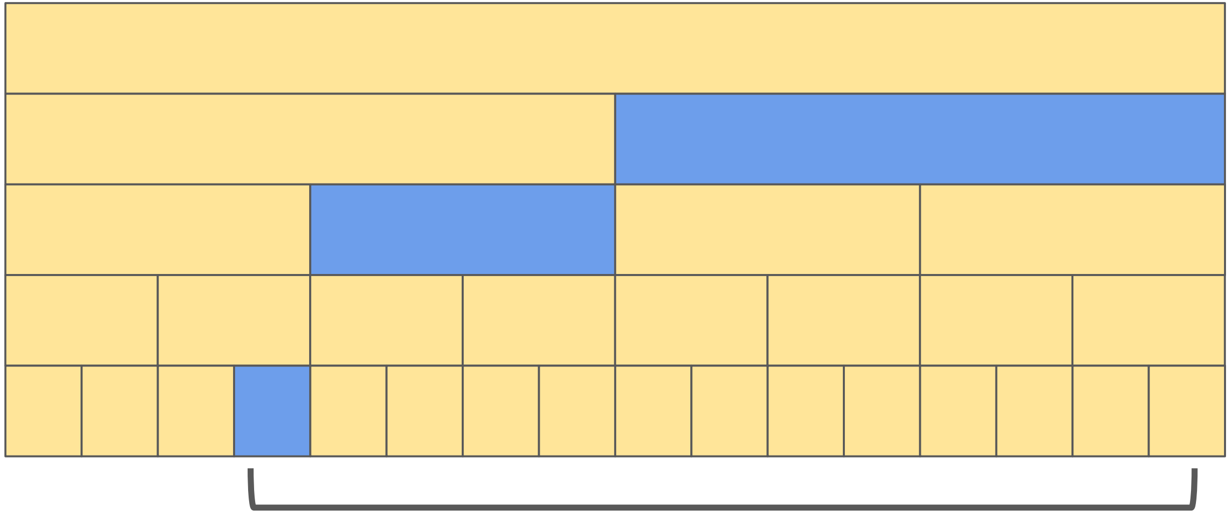
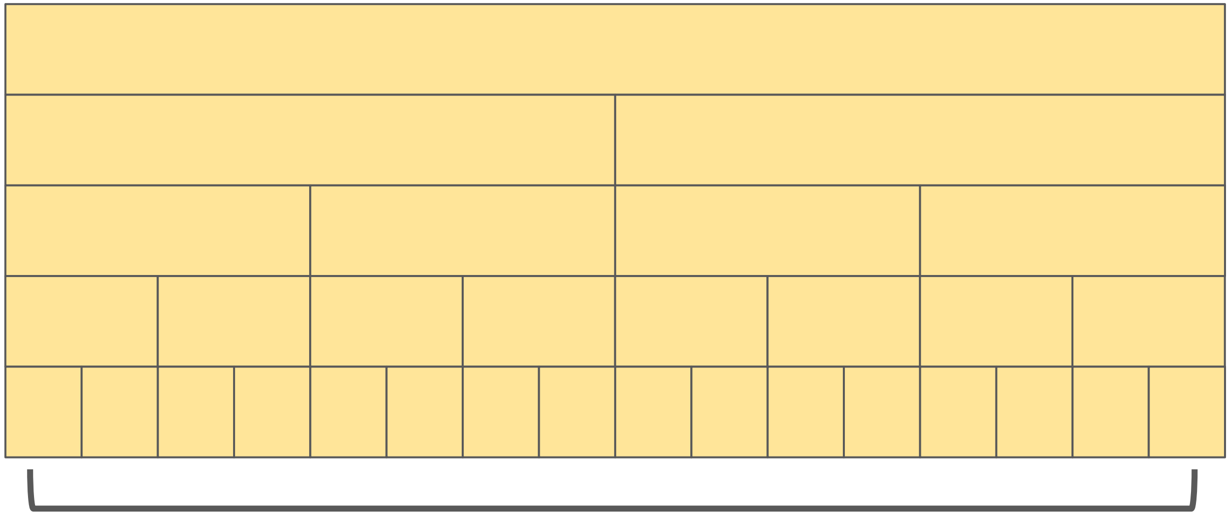


Querying sum of 13 elements, requires 3 blocks.

Query examples…

Given an array of N integers, a[1], a[2] … a[n], support Q queries of the form

(Query) L R: Return a[L] + a[L+1] + … + a[R]
(Update) X Y: Change the value of a[X] to Y



Querying sum of 16 elements, which blocks are required?
(*Answer withheld before lecture*)

Query time complexity?

Given an array of N integers, a[1], a[2] … a[n], support Q queries of the form

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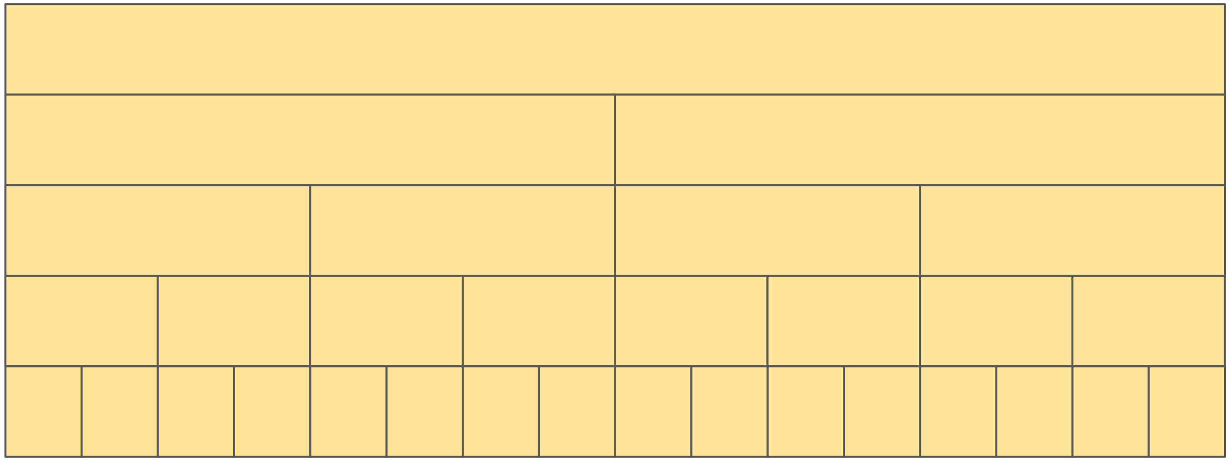

Each query uses at most 2log(n) blocks, and can be calculated in O(logn) time.

This fact will be made a bit more clear in implementation details later on.

What about updates?

Given an array of N integers, a[1], a[2] … a[n], support Q queries of the form

(Query) L R: Return a[L] + a[L+1] + … + a[R]
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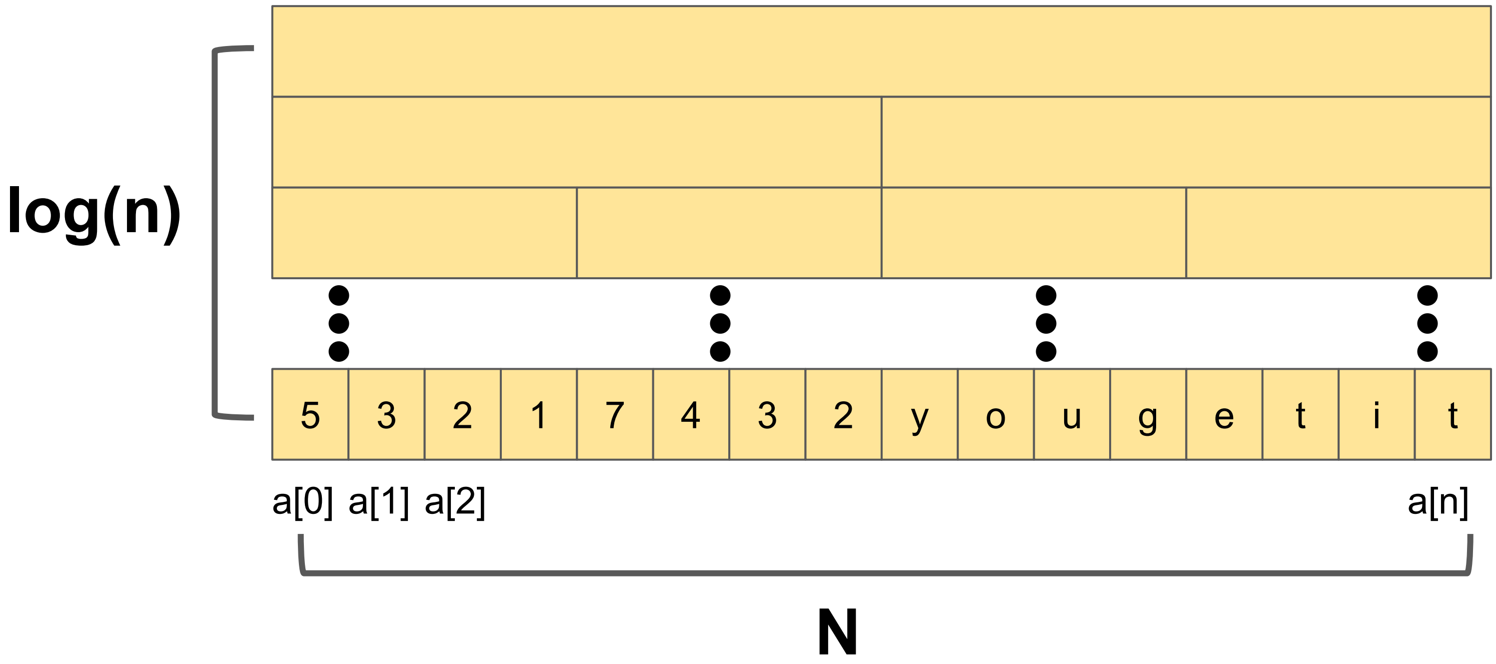

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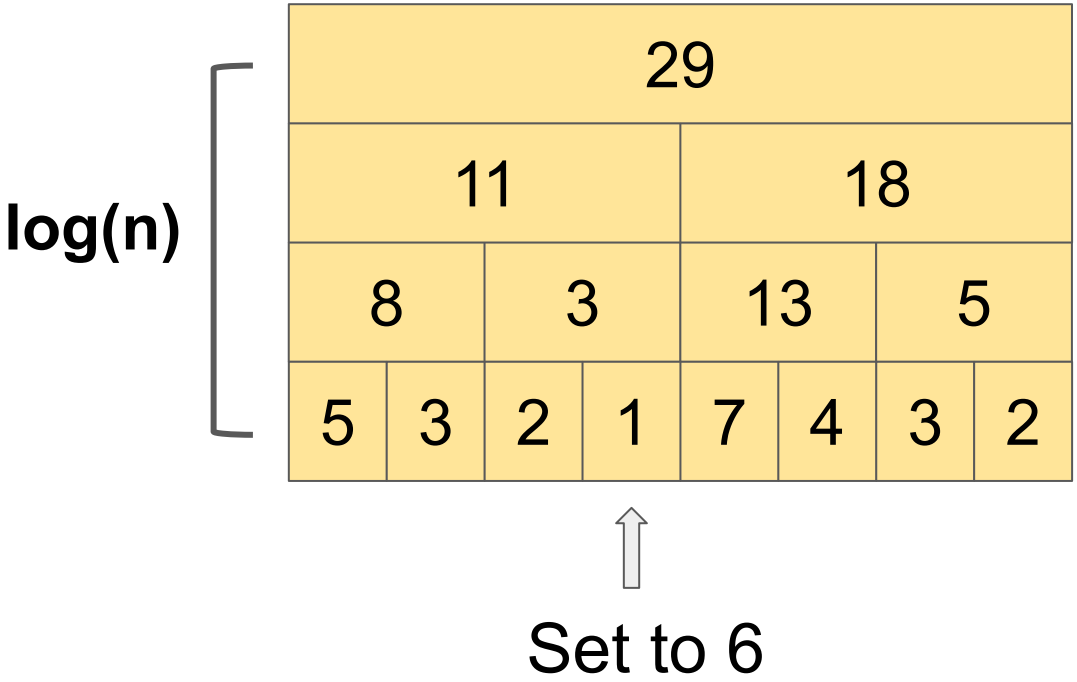

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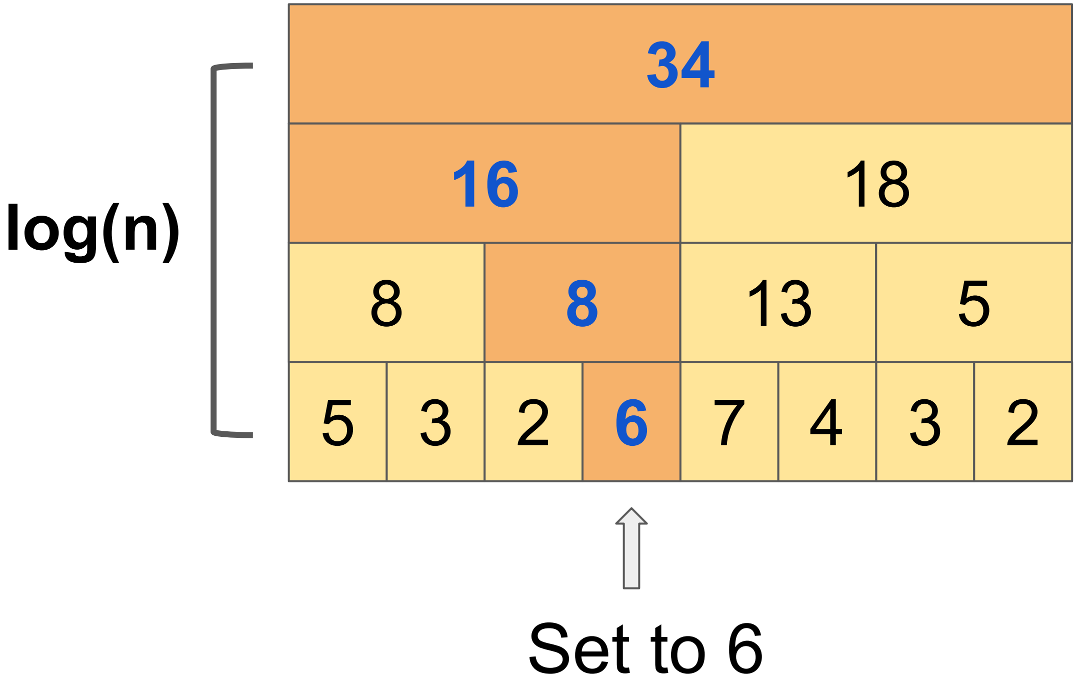

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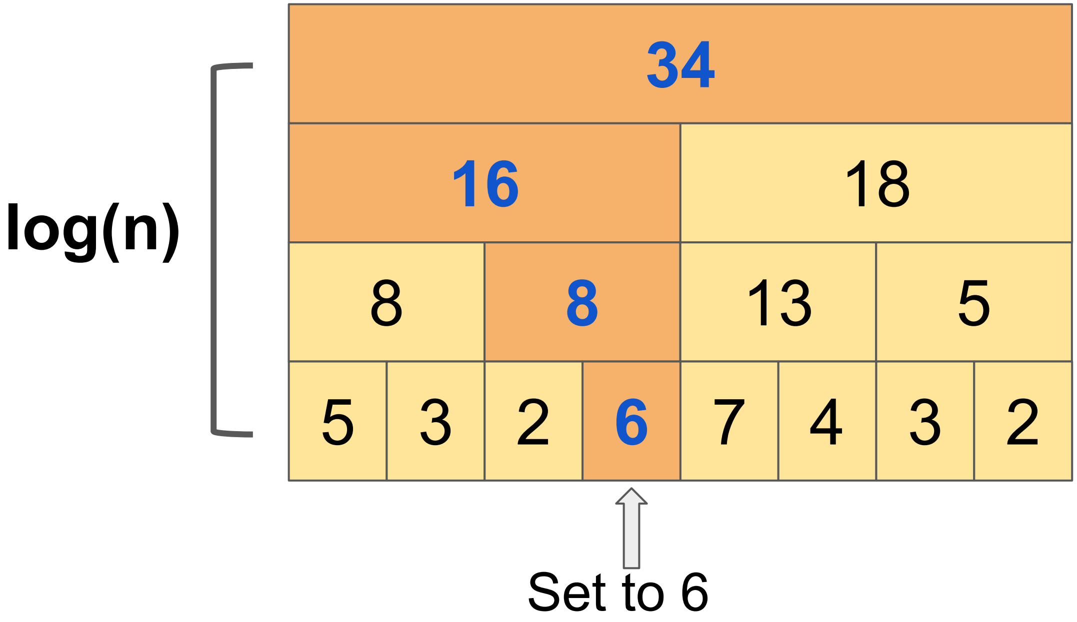
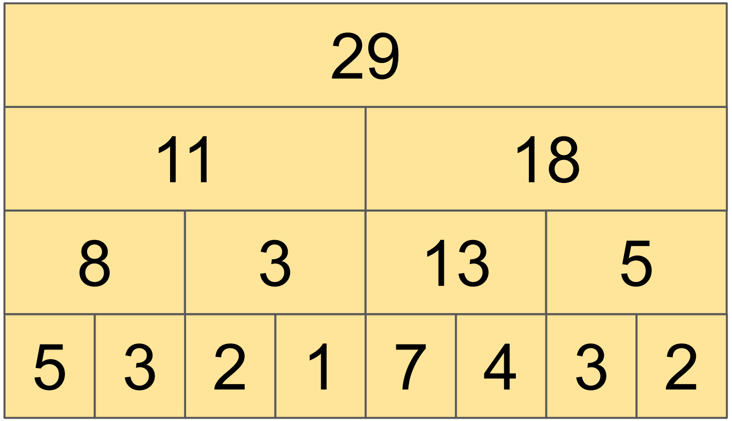
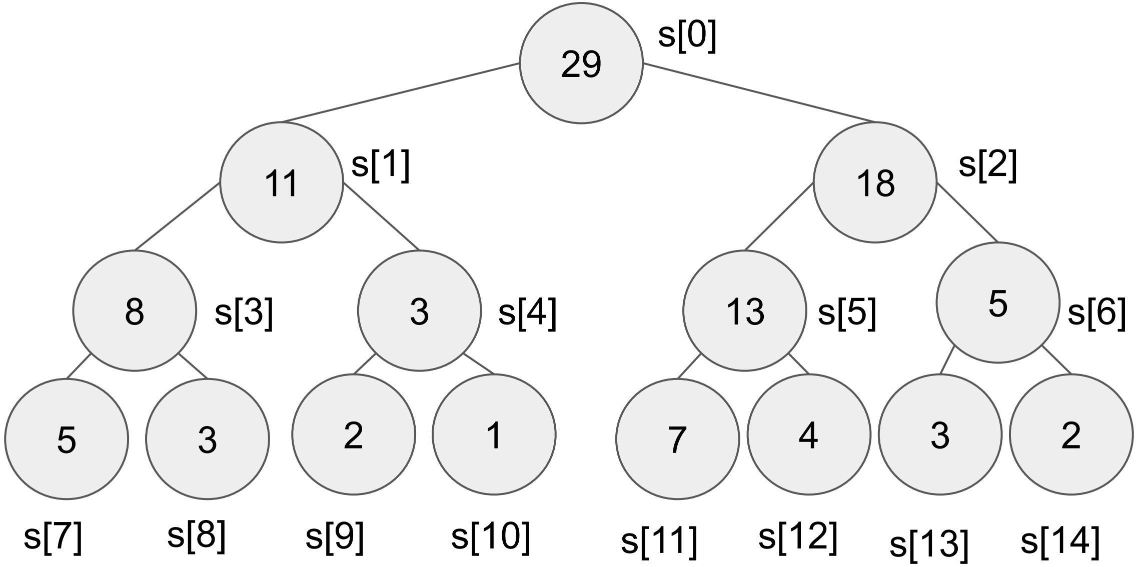


For any update, only O(logn) nodes change!

Implementation…

Store the segment tree as a binary tree, using heap indexing:
Left child of node n is node 2n + 1
Right child is node 2n+2





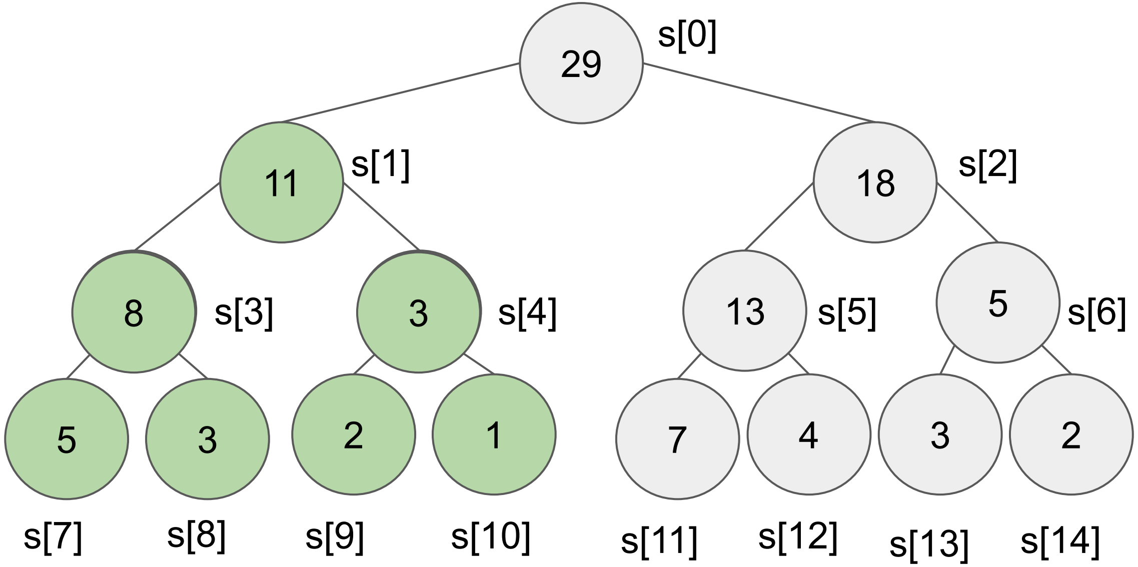
Implementation…

Each node is “**responsible**” for a range of indices that in its subtree

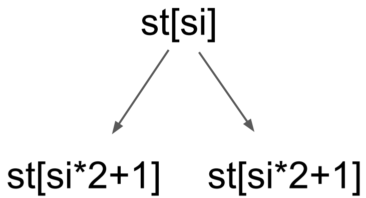
This information **won’t** be stored in these nodes in implementation: but rather passed through the recursive function calls.

Also don’t worry about when it’s not a perfect power of 2, the implementation later will just handle that.

<table>
<tr><td colspan="8">29</td></tr>
<tr><td colspan="4">11</td><td colspan="4">18</td></tr>
<tr><td colspan="2">8</td><td colspan="2">3</td><td colspan="2">13</td><td colspan="2">5</td></tr>
<tr><td>5</td><td>3</td><td>2</td><td>1</td><td>7</td><td>4</td><td>3</td><td>2</td></tr>
</table>

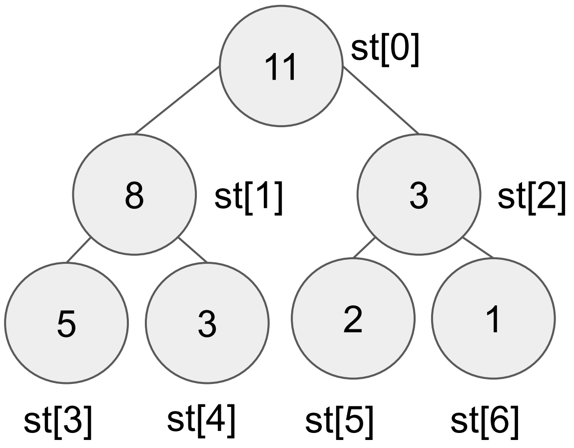


Implementation: Initialising the segtree!




```
// This function, given an array a,
// builds a range sum segtree in the array st
void build(int si, int sl, int sr){
    // si = current segtree left
    // sl = segtree left
    // sr = segtree right
    // Precondition: node si is responsible for indices [sl,sr]

    // Base case: we're at a leaf node.
    if (sl == sr){
        st[si] = a[sl];
        return;
    }
    int mid = (sl + sr)/2;
    // Initialise the children!
    build(si * 2 + 1, sl, mid);
    build(si * 2 + 2, mid+1,sr);
    st[si] = st[si*2+1] + st[si*2+2]
}
```



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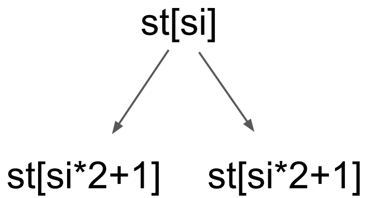


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        return;
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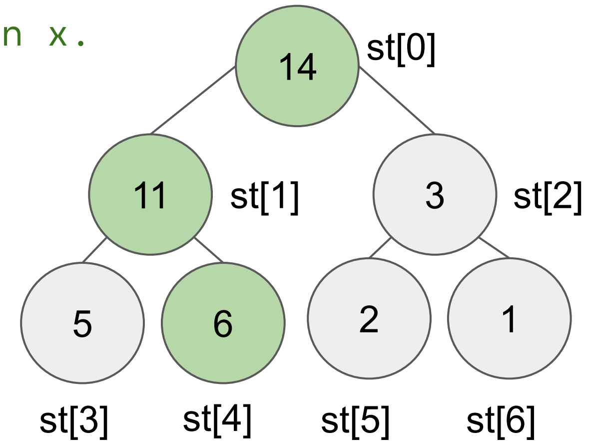
    int mid = (sl + sr)/2;
    // Initialise the children!
    build(si * 2 + 1, sl, mid);
    build(si * 2 + 2, mid+1,sr);
    st[si] = st[si*2+1] + st[si*2+2]
}
```

Question: If we're initialising a segtree over an array of size N, what arguments do we put in when we call build()?

Another question: what is the time complexity?

Implementation: Updating!

```
// This function updates a[x] to v
void update(int si, int sl, int sr, int x, int v){
     // Note si,sl,sr have the same meanings as in build
     // Base case: leaf node is simple
     if (sl == sr){
          st[si] = v;
          return;
     }
     int mid = (sl + sr)/2;
     // Update only the child that contains position x.
     if (x <= mid){
          update(si*2+1,sl,mid,x,v);
     }
     else{
          update(si*2+2,mid+1,sr,x,v);
     }
     // Update this node.
     st[si] = st[si*2+1] + st[si*2+2]
}
```



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        st[si] = v;
        return;
    }
    int mid = (sl + sr)/2;
    // Update only the child that contains position x.
    if (x <= mid){
        update(si*2+1,sl,mid,x,v);
    }
    else{
        update(si*2+2,mid+1,sr,x,v);
    }
    // Update this node.
    st[si] = st[si*2+1] + st[si*2+2]
}
```

Question: What arguments do we give to update() in order for it to update position x to value y?

Another question: what is the time complexity of this?

Implementation: Query!

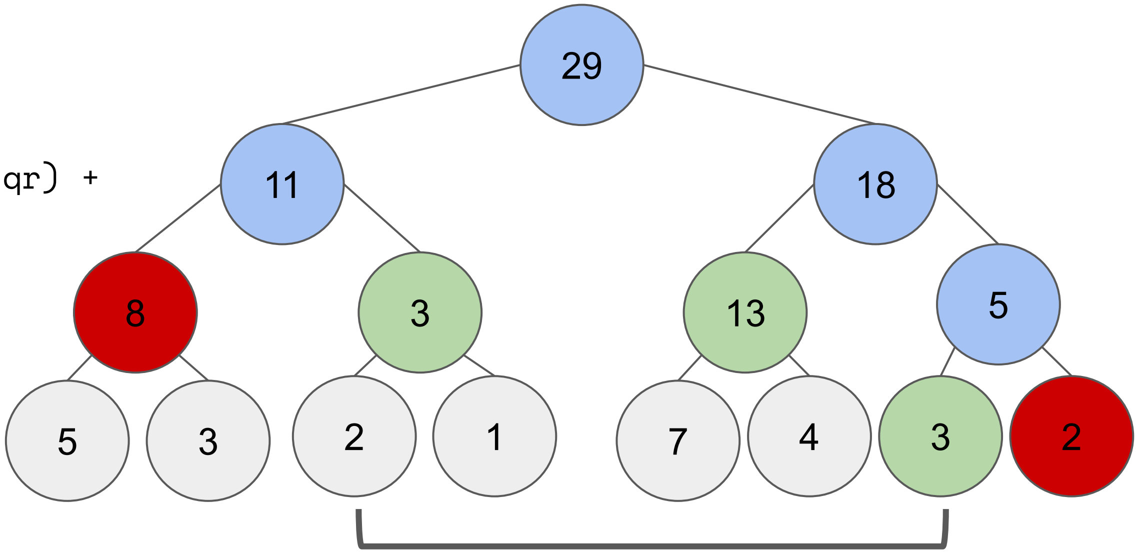
```
// This function returns the sum of the values in the intersection of
// [sl, sr] and [ql,qr] (Calling query(0,1,n,ql,qr) returns the desired query)
void query(int si, int sl, int sr, int ql, int qr){
    // If [sl,sr] and [ql,qr] don't intersect, the sum is 0.
    if (sl > qr || sr < ql){
        return 0;
    }
    // If [sl,sr] is fully contained in [ql,qr], then the answer is the sum
    from sl to sr (conveniently stored in st[si])
    if (ql <= sl && sr <= qr){
        return st[si];
    }
    // Otherwise, idk, ask our children for the answer.
    int mid = (sl + sr)/2;
    return query(si*2+1,sl,mid,ql,qr) + query(si*2+2,mid+1,sr,ql,qr);
}
```

Query time complexity…

```
void query(int si, int sl, int sr, int ql, int qr){
     if (sl > qr || sr < ql){
          // Point A
          return 0;
     }
     if (ql <= sl && sr <= qr){
          // Point B
          return st[si];
     }
     // Point C
     int mid = (sl + sr)/2;
     return query(si*2+1,sl,mid,ql,qr) +
query(si*2+2,mid+1,sr,ql,qr);
}
```

It can be shown that the query will access at **most 4 nodes** at each level in the segtree: at most two greens bookended by blue or red nodes.

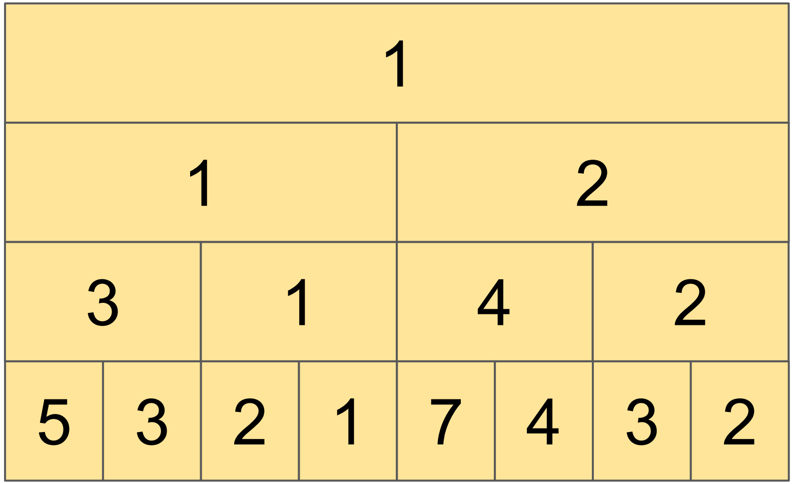
Hence at most 4logn nodes are considered, so time complexity is O(logn)



Implementation: Flexibility!!!

What if we want the segtree to solve range minimum query instead? How much do we have to modify the segtree?

(*Answer withheld before lecture*)



Application: Inversion counting

Given an array of N integers a[1] … a[N],
count the number of (i,j) such that 1 <= i < j <= N and a[i] > a[j]

Can you find a solution in O(nlogn)?

Remember: you just learnt what a segtree is! A magical black box that can do arbitrary point updates (change any value in the array) and range queries (find the sum/min/whatever of any (you can even choose the array you build the segtree on)!

Application / New Technique: Range update, point query…

Given an array of N integers, a[1], a[2] … a[n], support Q queries of the form

(Update) L,R,V: Add V to a[L] + a[L+1] + … + a[R]
(Query) X: Return the value of x

Can you find a solution in O(nlogn)?

(*Answer withheld before lecture*)

Bonus! Maximum subarray sum, with updates!

Given an array of N (possibly negative) integers a[1] … a[N], there are Q updates of the form:

Change a[i] to V

After each update, calculate the maximum subarray sum of the array.

The maximum subarray sum is the largest value of a[L] + a[L+1] + … + a[R] for any 1 <= L, R <= N

Find an O(nlogn) solution.

(*Answer withheld before lecture*)