Divide and Conquer

Patrick Moore

Australian Mathematics Trust

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Divide and Conquer

Centroid Decomposition

Problems

Divide and Conquer

Divide and Conquer involves two things. Dividing and
Conquering.

1. Dividing
    - Split N objects into groups, perhaps 2 groups, maybe more.
2. Conquering
    - Solve the smaller problems
    - Combine the results of the smaller problems to come up
      with the answer to the whole problem (This is the hard
      part).

Divide and Conquer - Merge Sort

Divide and Conquer - Inversion Counting

Given an array with N unique elements from 1 to N. Two
elements A_i and A_j are inverted if $i < j$ and $A_i > A_i$. Count the
number of inversions in the array.

Segtrees are illegal.

Divide and Conquer - Inversion Counting

Divide and Conquer - Closest Pair of Points

Given N points in the plane, output the smallest euclidean distance between two points.

Divide and Conquer - Closest Pair of Points

Centroids

In a tree T of N nodes, a node X is a centroid of T iff, once X is removed, it splits T into subtrees such that each subtree has at most $N/2$ nodes in it.

Finding the Centroid

Perform a DFS from a root node to create the subtree sizes, and move in the direction with more than $N/2$ nodes until all subtree sizes are $\leqslant N/2$.

Proof a centroid must always exist

Theorem: A centroid always exists in a tree Proof: Consider the
sum of the subtree sizes of the node we just came from

Centroid Decomposition

If we repeatedly find the centroid of a tree and then split the tree off from the centroid, we can decompose the tree into a "Centroid Tree". This can be useful for visualisation, storing information, or even performing DP.

Motivating Problem - Ciel the Commander (CF)

Ciel the Commander has an officer at each node of a tree with N nodes. He must assign a rank from 'A' to 'Z' to each node such that on the path between any two officers of the same rank, there is an officer of a higher rank monitoring their communications.
$N \leqslant 100,000$

Motivating Problem - Ciel the Commander

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Why the centroid?

There are many ways to define the 'center' of a tree. Depending on the problem these may have breaking cases (in time or memory) if you use them. Here are a couple:

- The node where the distance to the furthest node is minimal

- The node with the highest degree

Problems

- Divide and Conquer
    - Inversion Counting
    - Panorama
    - Negotiations
    - Skysurfing
    - Arranging Heaps
    - Scootchhop
- Centroid Decomposition
    - Ciel the Commander
    - Race
    - Xenia and Tree