Beta Mini: Approximation Algorithms

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What is NP?

- NP is short for non-deterministic polynomial time.
- Informally: NP is the set of decision problem (yes/no problems) where a 'yes' answer can be verified in polynomial time.
- Polynomial time means $O(n^k)$ for some constant k. For example, $O(n), O(n^2), O(n^{100})$.
- Examples:
  - Any decision problem that can be solved in polynomial time.
  - Hamiltonian cycle: "Given a graph, does there exist a cycle which visits every vertex exactly once?".
  - Travelling Salesperson (decision problem): "Given a complete graph with weighted edges, is there a cycle which visits every vertex exactly once and has a length $\leq K$?".
- P vs NP: Can every problem in NP be solved in polynomial time?

NP-Hardness

- A problem is NP-Hard if a polynomial time algorithm to solve it would give a polynomial time algorithm to solve every problem in NP.
- If a problem is NP-Hard and in NP, it is NP-Complete.
- Hamiltonian Cycle and Travelling Salesperson (decision problem) are NP-Complete.

Optimisation Problems

- Problems where you want to find the "optimal" solution, rather than a yes/no answer.
- Travelling Salesperson (optimisation problem): “Given a complete graph with weighted edges, what is the shortest cycle which visits every vertex exactly once?”
- This version of TSP is NP-Hard but not NP-Complete (because it is not in NP!).
- From here on, we will just consider optimisation problems.

Approximation Algorithms

- An α-approximation algorithm is an algorithm which is guaranteed to find an answer of at most $\alpha \times opt$, where opt is the optimal answer.

Metric Travelling Salesperson

- You are given a completed undirected weighted graph, where the distance between two vertices i, j is $d(i, j)$. All edge weights satisfy the triangle inequality, meaning that for any three vertices i, j, k:

$$d(i, j) + d(j, k) \geq d(i, k)$$

- This is NP-hard, but a 2-approximation can be found in $O(n^2)$!
- A 1.5-approximation can also be found in polynomial time (not covered in this lecture).
- In 2020, this was improved to a $(1.5 - 10^{-36})$-approximation (also not covered).

Vertex Cover

- Given a graph, pick the smallest subset of vertices so that for every edge, at least one of its endpoints is chosen.
- This is NP-hard ... but there is a 2-approximation!

General (not metric) travelling salesperson

- There is no polynomial time α-approximation for any $\alpha > 1$ unless P = NP.
- This means no 10^{100}-approximation algorithm exists!