Satisfiability Problems

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A refresher on boolean logic

- Literals
  - Symbols (e.g. x_1) which are either *true* or *false*
- Disjunction
  - Equivalent to the OR operator, denoted by the symbol $\vee$
- Conjunction
  - Equivalent to the AND operator, denoted by the symbol $\wedge$
- Exclusive-Or
  - Equivalent to the XOR operator, denoted by the symbol $\oplus$
- Negation
  - Equivalent to the NOT operator denoted by the symbol $\neg$
- Boolean Formula
  - A combination of literals, disjunction, conjunction and negation to create a formula which can evaluate to either *true* or *false* (e.g. $(x_1 \vee x_2) \wedge (x_3)$). ⇒ $(x_1$ OR $x_2)$ AND x_3

0 1
1 1

0 0
0 1

0 1
1 0

Conjunctive Normal Form

A boolean formula is in Conjunctive Normal Form (CNF) if it is the conjunction (AND, $\wedge$) of a list of clauses, each of which is a disjunction (OR, $\vee$) of literals (symbols, either *true* or *false*).

e.g. $\overbrace{(x_1 \vee x_2 \vee \neg x_3)}^{\text{clause}} \wedge \overbrace{(\neg x_1 \vee x_3)}^{\text{clause}} \wedge \overbrace{(\neg x2)}^{\text{clause}}$
$(x_1 \text{ OR } x_2 \text{ OR } \neg x_3) \text{ AND } (\neg x_1 \text{ OR } x_3) \text{ AND } (\neg x2)$

$\neg(x_1 \text{ OR } x_2)$

SAT

Definition

There are N literals, $x_1 \ldots x_N$.

We are given a boolean formula in CNF, with the literals $x_1 \ldots x_N$.

The formula is **satisfiable** if there is some assignment of $x_1 \ldots x_N$ to *true* or *false* such that the boolean formula evaluates to *true*.

The problem is to decide whether or not such an assignment exists, and to output the assignment if it exists.

SAT Example

$x_1 = \text{TRUE}$
$x_2 = \text{FALSE}$ — ✓
$x_3 = \text{TRUE}$

F_1 $(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3) \wedge (\overset{T}{\neg x2})$

$(x_1 \text{ OR } x_2 \text{ OR } \neg x_3) \text{ AND } (\neg x_1 \text{ OR } x_3) \text{ AND } (\neg x2)$

$x_1 = \text{FALSE}$

$x_2 =$ "

$x_3 =$ "

Solving SAT

There is no known algorithm to solve SAT in polynomial time.

Solving SAT in polynomial time has been shown to be equivalent to to solving P = NP.

However, there are many heuristic algorithms to solve SAT fairly quickly and consistently with reasonable amounts of clauses and symbols.

$N \le 1,000,000$

$m \le 1000000$

We will be focussing on a couple variations and sub-problems of SAT.

Solving SAT

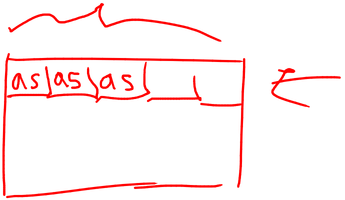
Brute Force

SAT can be solved in $O(2^N \times M)$, where N is the number of literals and M is the length of the formula in CNF.

Randomised

k

M



Definition of the 3-SAT Problem

- This is a slight variation on the general SAT problem.
  - Setup is the same, but the number of literals in each clause is at most 3.

$(x_1 \lor x_2 \lor x_3 \lor x_4) \land$

Reducing 3-SAT to SAT

3-SAT $\subseteq$ SAT

- Given M clauses with at most 3 literals per clause, give a single boolean formula in CNF that is equisatisfiable to the original clauses.

Reducing 3-SAT to SAT

Reducing SAT to 3-SAT

$F: (\quad) \wedge (\quad)$

Given a boolean formula in CNF, provide a set of clauses, each with at most 3 literals, that is equisatisfiable (not necessarily logically equivalent) to the original formula.

$(x_1 \vee x_2 \vee z_3)$

$(\quad \vee \quad \vee \quad)$

$(\quad \vee \quad \vee \quad)$

new variables

Reducing SAT to 3-SAT

$x_1 \dots x_L$

$$(x_1 \vee x_2 \vee \dots \vee x_L) \wedge$$

$$\overset{F}{x_1}\ \ \overset{F}{x_2}\ \ \overset{T}{c_1}$$

$$(x_1 \vee x_2 \vee c_1) \wedge$$

$$(\neg c_1 \vee x_3 \vee c_2) \wedge$$

$$(\neg c_2 \vee x_4 \vee c_3) \wedge$$

$$(\neg c_3 \vee x_5 \vee \dots) \wedge \dots$$

F F F

$$(\neg c_{L-2} \vee x_{n-1} \vee x_n)$$

2-Satisfiability

- This is very similar to 3-SAT, but now each clause is limited to at most 2 literals (e.g. $(x_1 \vee x_2) \wedge (\neg x_2 \vee x_3)$).

2 2

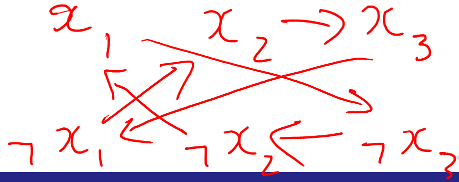
Reducing 2-SAT to an implication graph

T ◻ F
F ◻ T

$(x_1 \vee x_2) \wedge (\cdot \ -)$

- $\neg x_1 \implies x_2$
- $\neg x_2 \implies x_1$

$(x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_3 \vee \neg x_1)$



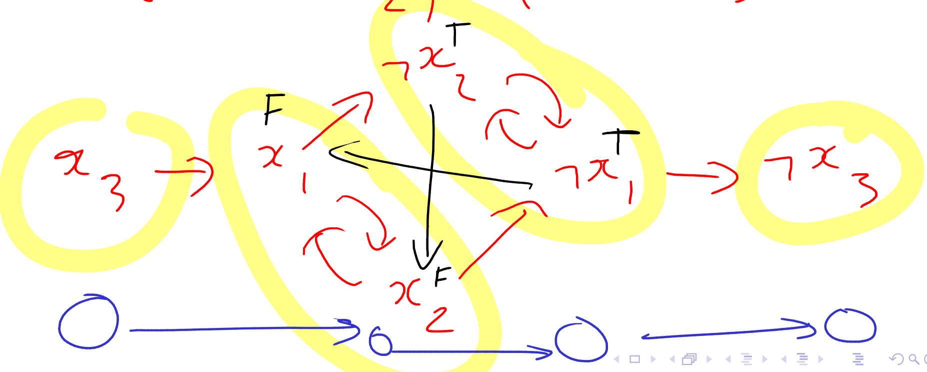
Solving 2-SAT with Strongly Connected Components (SCC)

1 ▸ Create the implication graph from the provided clauses.
2 ▸ Create the SCCs of the implication graph.
3 ▸ The formula is satisfiable iff all x_i and $\neg x_i$ are in different SCCs.

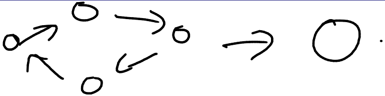
x_i $\neg x_i$

Solving 2-SAT with Strongly Connected Components (SCC)

$$(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2) \wedge (x_1 \vee \neg x_3)$$



Generating an assignment

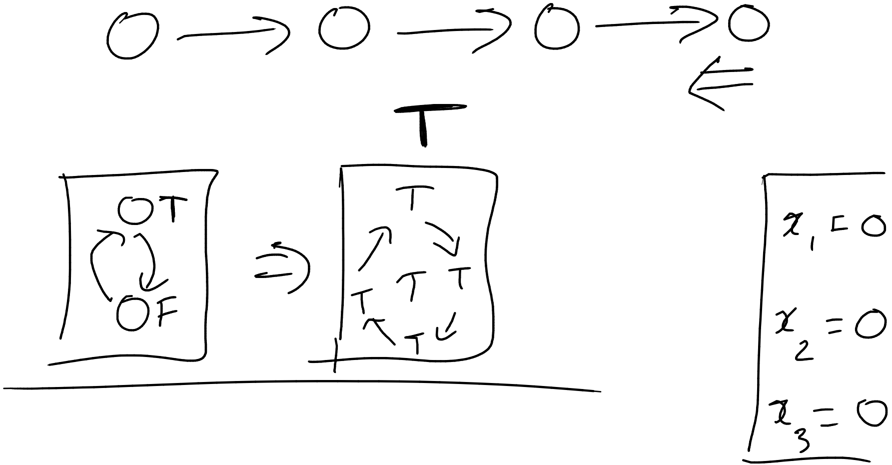


TO ¬TO

SCC

- Get a topological ordering of the nodes in the condensation graph (Use Kosaraju's or Tarjan's or Josh's).
- Traverse the SCC graph in reverse topological order
  - if any of the elements in the SCC are already set, set them to the same state.
  - otherwise, greedily set the SCC to true.

Generating an assignment



Variations on 2-SAT

$L=2$

- Exercises - Create clauses that perform the following things
  1. Force x_1 to be true $(x_1 \vee x_1)$
  2. Force exactly one of x_1 or x_2 to be true $(x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)$
  3. Force x_1 and x_2 to have the same value

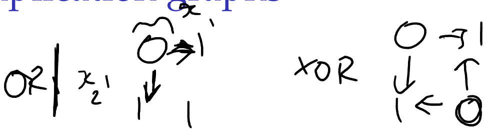
$(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2)$

XOR-SAT

XOR

In XOR-SAT, literals are combined with XORs instead of ORs. (e.g. $(x_1 \oplus x_2) \wedge (x_2 \oplus x_3) \wedge (\neg x_3 \oplus x_4)$). We will be focusing on XOR-2-SAT, where clauses have at most 2 literals.

XOR-2-SAT implication graphs

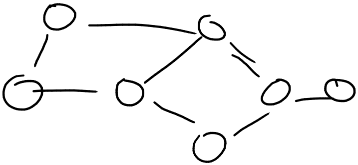


Since the state of one literal in XOR-2-SAT uniquely determines the state of the other, the implication graph of each clause in XOR-2-SAT is much stronger than in regular 2-SAT.

e.g. $x_1 \oplus x_2$

$$x_1 \to \neg x_2$$
$$\neg x_1 \to x_2$$
$$x_2 \to \neg x_1$$
$$\neg x_2 \to x_1$$

XOR-2-SAT: King Arthur II

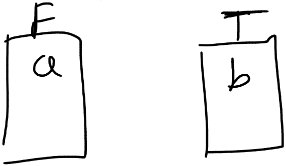


The king is holding a gathering of N knights in two rooms. You are given a pairwise list of enemies between the knights. Determine if it is possible to allocate the knights to rooms so that no two enemies are in the same room.

$a \leftarrow F0 \rightarrow B$

$E \rightarrow$ Enemies

$a \oplus b = T$



XOR-2-SAT variations

$x_1 \oplus x_2 = 1$

$x_1 \oplus \neg x_2 = 1 \Rightarrow \boxed{x_1 \oplus x_2 = 0}$

- Exercises - Create clauses/graphs that perform the following things
  - (1) Force two literals to be different $x_1 \oplus x_2$
  - Force two literals to be the same
    - (1) Using 1 clause $x_1 \oplus \neg x_2$
    - (2) Using 2 clauses without additional negation
  - Force a literal to be true/false

$T \longleftrightarrow x_1$

$F \longleftrightarrow x_2$

$\longrightarrow (x_1 \oplus a) \wedge (x \oplus a)$

Problems

- XOR-SAT
  - King Arthur II (Easy)
  - Detective (Hard)
  - ICPC Brazil 2018 - Modifying SAT (OOS)

Gaussian Elim

- 2-SAT
  - Black Mountain (Medium) Vanilla
  - Table Colouring (Hard)

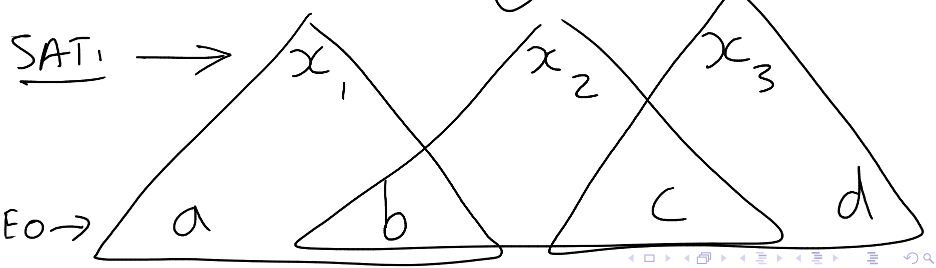
2-sat

- EO-3-SAT, R-SAT, XOR-3-SAT
  - If you run out of problems and want a very interesting variation of SAT

Additional Space

EO-3-SAT: $\Rightarrow$ Exactly one literal is true $\Rightarrow (T,F,F) \vee (F,T,F) \vee (F,F,T)$

$\Downarrow$

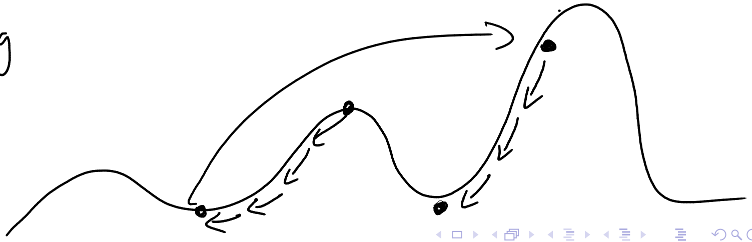


Additional Space

$x_i \in \mathbb{R}$

$\mathbb{R}$-SAT: $(x_1 \leq 10 \vee x_2 \geq 3) \wedge (x_2 \leq 5 \vee x_1 \geq 11) \dots$

solve w gradient descent + basin-hopping



Additional Space

XOR-3-SAT: $(x_1 \oplus x_2 \oplus \neg x_3)$

$(x_1 \oplus x_2 \oplus x_3 = 0)$

Gaussian Elim
% 2

$(\quad \cdots \quad) = 1$

$= 0$

$(\quad \cdots \quad) = 1$