

# Bayesian Evidence Synthesis:Opioid Crisis

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Abstract

# 1 Introduction

The opioid crisis is a major issue in North America including British Columbia, Canada. British Columbia is in the midst of a drug overdose crisis due to illicit opioids and the circumstances are getting worse. According to the BC Coroners Service, there were more than 930 apparent illicit drug overdose deaths in BC from Jan. 1 to Dec. 31, 2016. This compares to 518 in 2015, an increase of 79.2%. [fDC19] The goal of this project is to apply Bayesian evidence synthesis to better understand the opioid crisis in British Columbia, Canada.

One difficulty in coping with the opioid crisis is that the total number of overdoses is unknown. The reason is that the use of prescription opioids often leads to use of illicit opioids and the extent of illicit usage cannot be known. Many of those who become addicted to opioids do so after initially receiving a prescription. The highly addictive nature of these pain relievers makes it easy for the human brain to crave more. It is only after their prescription ends that many users realize they've become dependent on the effects of opioids to function "normally." At that point, they are either forced to get clean and endure the pain that comes with the withdrawal symptoms of opioids or look for another means of getting their high. This is often the time where people will turn to illicit drugs or other analogues. Because prescription opioids are so expensive, many users turn to heroin. It is often cheaper, more potent, and easier to locate than what they were taking before. In fact, about 80% of people using heroin started with a prescription to another opioid. After using heroin, however, 23% of individuals develop opioid addiction.[Smi19]

Since the total number of overdoses is unknown, this number needs to be estimated. Bayesian statistics can be a way to approach the problem and give us a good estimate of the number with a defensible range of uncertainty. Bayesian statistics is a theory in the field of statistics based on the Bayesian interpretation of probability where probability expresses a degree of belief in an event.

[wik20] Further explanation is provided at the following section. [todo: explain Bayesian statistics a little bit more here]

## 2 Methods

The number of overdoses is our ultimate target of estimation. To achieve the goal in context of Bayesian statistics, we use our prior belief and available data set about the target of estimation. The posterior belief is coming from both of these sources of the information. Let  $\theta$  denote all the unknown variables and let  $Y$  denote all the related data sets. The following equation express the idea of Bayesian generative models in a mathematical form:

$$p(\theta|Y) = \frac{p(\theta, Y)}{p(Y)} = \frac{p(\theta)p(Y|\theta)}{p(Y)} \propto p(\theta)p(Y|\theta), \quad (1)$$

where  $p(\cdot)$  is probability density (or mass) function. The equation 1 states that the posterior belief of unobserved variables,  $p(\theta|Y)$ , is proportional to the mixture of the prior belief of the variables,  $p(\theta)$ , and the related data sets given the prior belief,  $p(Y|\theta)$ . Notice that the vector  $\theta$  from equation 1 includes parameters and latent variables, and hence it contains the total number of overdoses.

Let us assume that we are interested in the opioid crisis from month 1 to month T. Let the subscript 1:T for any variable denote the variable for all target months and subscript t for any variable denote the variable of the month t. Then the target variable, the total number of overdoses from month 1 to month T in British Columbia, is denoted by  $O_{1:T}$ . In Bayesian concepts, the variable of interest  $O_{1:T}$  is a vector of fixed numbers but the numbers are yielded from a certain distribution where the distribution describes knowledge about  $O_{1:T}$ ; that is, each  $O_t$  is considered to be a sample of a random variable. The simplest conceptual model is to take an underlying log-rate

$z_t$  for each month  $t$  where  $z_t$  is independent and identically distributed across months according to a normal distribution with mean  $\mu$  and variance  $\sigma^2$ . [IM19] Then let  $\lambda_t^{OD}$  denote the rate of overdose at month  $t$ . It is assumed that the number of overdoses in the month,  $O_t$ , follows Poisson distribution with parameter  $\lambda_t^{OD}N$  where the size of the population in the region of interest is  $N$ . Note that the population size,  $N$ , is unknown and vary over time in reality but it is assumed known and constant for the investigator here for simplicity. The mechanism is named as the overdose model and is illustrated in mathematical form below:

$$\left. \begin{aligned} z_t &\sim N(\mu, \sigma^2), \\ \lambda_t^{OD} &= \exp(z_t), \\ O_t &\sim Poi(\lambda_t^{OD}N) \end{aligned} \right\} \text{(overdose model).} \quad (2)$$

Since estimation of  $O_t$  is not straightforward as some of the variables ( $\mu$ ,  $\sigma$ ) determining  $O_t$  is not known, we assume there was a survey conducted to help to support the inference about  $O_t$ . The survey asked witnesses of opioid overdoses whether someone called an ambulance, for each month. This provides an estimate of the proportion of ambulance call  $p_{A,t}$  among the overdoses. For the moment  $p_{A,t}$  is assumed constant across time for simplicity. Let  $n_{A,t}$  be the sample size of the survey at month  $t$ , and  $x_{A,t}$  be the total number who confirmed that an ambulance was called at the month  $t$ . It is assumed that  $x_{A,t}$  follows a Binomial distribution with number of trials,  $n_{A,t}$ , and success probability for each trial,  $p_{A,t}$ . The mechanism is named as the ambulance call-outs model and is illustrated in mathematical form below:

$$x_{A,t} \sim Bin(n_{A,t}, p_{A,t}) \text{ (ambulance call-outs model).} \quad (3)$$

Now the estimated  $p_{A,t}$  can help to infer  $O_t$  if there is a data regarding the number of ambulance-attended overdoses. Let  $U_t$  be the number of ambulance-attended overdoses at a time point  $t$ . In general, the data of ambulance-attended overdoses  $U_t$  can be obtained. It is assumed that  $U_t$  follows

a Binomial distribution:

$$U_t \sim \text{Bin}(O_t, p_{A,t}). \quad (4)$$

Note that  $O_t$  can be estimated because  $p_{A,t}$  can be inferred from the survey data and the data regarding  $U_t$  is given. Figure 1 shows an example of how the estimation can proceed. The variables that can be observed ( $N$ ,  $U_{1:T}$ , and  $x_{A,1:T}$ ) are drawn in squares and the parameters ( $\mu, \sigma^2$ ) and the latent variables ( $\lambda_{1:T}, O_{1:T}, p_{A,1:T}$ ) that are not observable from possible data sets are drawn in circles. Variables that need prior distributions are indicated with asterisks. The arrows indicate which variables affects which variables; for example, there is an arrow from  $(\mu, \sigma^2)$  to  $\lambda_{1:T}$  since  $\mu, \sigma^2$  are the parameters of  $\lambda_{1:T}$ , and the two arrows from  $\lambda_{1:T}$  and  $N$  to  $O_{1:T}$  indicate that both  $N$  and  $\lambda_{1:T}$  affects the value of  $O_{1:T}$  since the product of  $\lambda_{1:T}$  and  $N$  is the parameter of the distribution of  $O_{1:T}$ .

We suggest a simple model in figure 1 as a start where the model only combines Overdose Model 2 and Ambulance Call-outs Model 3 with the relation 4. The next step is to run some simulations to figure out how different types of inputs lead some changes of output.

## 2.1 Generative Models

The amalgamation of the prior distribution and the statistical model can be referred to as the generative model.[reference] A generative model described in figure 1 at a month  $t$  can be written as follows given some a priori assumptions:

$$\begin{aligned} p(\theta, Y) &= p(\mu, \sigma^2, \lambda_t, N, O_t, p_{(A,t)}, n_{(A,t)}, x_{(A,t)}, U_t) \\ &= p(\mu)p(\sigma^2|\mu)p(\lambda_t|\mu, \sigma^2)p(N)p(O_t|\lambda_t, N)p(p_{(A,t)})p(n_{(A,t)})p(x_{(A,t)}|p_{(A,t)}, n_{(A,t)})p(U_t|O_t, p_{(A,t)}) \\ &\propto p(\mu)p(\sigma^2)p(\lambda_t|\mu, \sigma^2)p(O_t|\lambda_t, N)p(p_{(A,t)})p(x_{(A,t)}|p_{(A,t)}, n_{(A,t)})p(U_t|O_t, p_{(A,t)}), \end{aligned} \quad (5)$$

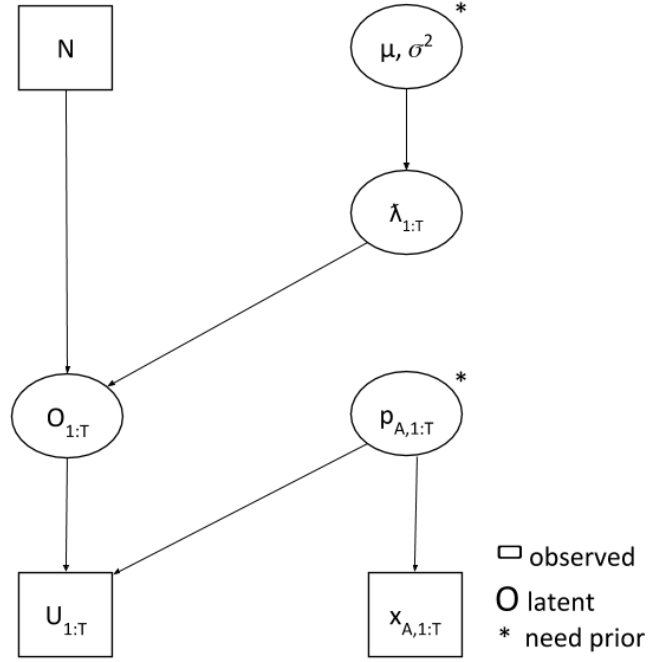


Figure 1: Example of estimating the total number of overdoses indirectly given two data sets: ambulance attended overdoses and survey data.

where  $\theta = (\mu, \sigma^2, \lambda_t, O_t, p_{A,t})$  and  $Y = (N, n_{(A,t)}, x_{(A,t)}, U_t)$ . The chain rule for random variables [Cha20] is used to the joint distribution of all variables at the first line of equation 5 such that the second line of the equation is derived. A priori independence assumptions and conditional independence assumptions were presumed so that the second line of the equation is simplified. For example, it is asumed that  $\sigma^2$  and  $\mu$  are independent to each other and  $p_{A,t}$  and also all the variables from overdose model 2,  $(\mu, \sigma^2, \lambda_t, N, O_t)$ , are independent to each other. Note that as a function of unknown variables, the function from the second line of the equation 5 is proportional to the function from the third line of the equation since  $N$  and  $n_{(A,t)}$  are presumed to be known and constants such that the density functions of the known variables are merely constants. Therefore the third line of the equation is the ultimate generative model of interest.

## 2.2 Simulation

Simulation can be used to predict the performance of the Bayesian models. In order to measure the performance of the simplified model, a data set was simulated and the process is described in this section.

### 2.2.1 Likelihood

There exist two data sets: the survey data  $(n_{A,1:T}, x_{A,1:T})$ , and the ambulance attended overdose data  $(U_{1:T})$ . The two data sets were simulated as follows. It was assumed that the two data sets were collected for a year ( $T=12$ ). In terms of the survey data, it is assumed that we have 12 survey data sets where each survey data set was collected for each month. The true value of  $p_{A,t}$  was set as  $p_{A,t} = 0.8$  and the survey sample size  $n_{A,t}$  was set as  $n_{A,t} = 1000$  for every month ( $t=1:T$ ). The  $x_{A,t}$  values for each month were independently generated from the Binomial distribution (3). In terms of the overdose data, it is assumed that the total population size was  $N = 10000$ , and the true values of parameters for the overdose model were  $\mu = \log 0.05$ ,  $\sigma = 1$ . The vector of  $O_t$  was generated following the overdose model (2). The vector of  $U_t$  was generated from the Binomial relation of the two variables (4). The two generated vectors have the same length as the survey data ( $T=12$ ). Note that only  $U_t$  and  $x_{A,t}$  are observable in the real-world application and  $p_{A,t}$  needs to be estimated first in order to estimate  $O_t$  which is the ultimate interest of the research.

### 2.2.2 Prior Distributions

Noninformative prior distributions are presumed as a start for simplicity.

$$p(p_A) \sim \text{Beta}(1, 1) \text{ noninformative prior for ambulance model} \quad (6)$$

$$\left. \begin{array}{l} \mu \sim U(-10, 0) \\ \sigma \sim U(0, 5) \end{array} \right\} \text{noninformative prior for overdose model} \quad (7)$$

This leads the posterior distribution of variables of interest to heavily depend on the likelihood. Later, the noninformative priors will be changed and the impact of the changes to posteriors will be investigated.

### 2.2.3 Markov Chain Monte Carlo

[Why do we need MCMC? Because we don't have platinum, golden, and silver grails.]

The posterior distribution of the unseen variables, parameters  $(\mu, \sigma^2, p_{A,1:T})$  and the latent variables  $(\lambda_{1:T}, O_{1:T})$ , given the seen variables,  $Y=(N, n_{(A,1:T)}x_{(A,1:T)}, U_{1:T})$ , is the key to make any inference for any of the unseen variables,  $\theta$ . For the inference about the variable of interest,  $O_{1:T}$ , the marginal posterior distribution,  $p(O_{1:T}|Y)$ , is essential which can be obtained by integrating the joint posterior distribution,  $p(\theta|Y)$ , over all the other unseen variables except  $O_{1:T}$ . However, in general it is severely complicated and almost impossible to perform such multiple integrations because  $p(\theta|Y)$  is a complex distribution of high dimension. Moreover,  $p(O_{1:T}|Y)$  may not have closed-form nor follow a known parametric distribution, which makes the general inference procedure laborious.

Markov Chain Monte Carlo (MCMC) is a detour to obtain posterior samples from  $p(\theta|Y)$  without explicitly knowing the distribution. MCMC technique allows us to generate Monte Carlo realizations of samples,  $\theta_{(i)}$ , which are dependent, with the distribution of each sample,  $p(\theta_{(i)})$ , converging to  $p(\theta|Y)$ . [Gus20]. [general explanation of what is MC and MC, how it works.]



Generative model auto pilot conceptually there's only one distribution of unseen stuff given the seen stuff we can use the distribution to make whatever inference claims that we need to make, such as point estimation, credibal interval, inference about parameters or latent variables.

Assessing the properties of a target distribution by generating representative random values is a case of a Monte Carlo simulation. Any simulation that samples a lot of random values from a distribution is called a Monte Carlo simulation, named after the dice and spinners and shufflings of the famous casino locale. This appellation is attributed to the mathematician von Neumann, or sometimes to Metropolis [Rat02].

The Metropolis algorithm is a specific type of Monte Carlo process. It generates a random walk such that each step in the walk is completely independent of the steps before the current position. The proposed next step has no dependence on where the walk has been before, and the decision to reject or accept the proposed step has no dependence on where the walk has been before. Any such process in which each step has no memory for states before the current one is called a (first order) Markov process, and a succession of such steps is a Markov chain. The Metropolis algorithm is an example of a Markov chain Monte Carlo (MCMC) process. [Kru10]

[Two types of bronze grails, NUTS, metropolis]

[explain all the figures below was derived by NUTS, and python]

The achievement of the simulation was done by Markov chain Monte Carlo (MCMC). The technique is a popular method to obtain posterior samples from distributions where analytic forms do not exist or are hard to be derived. **[Further explanation is going to be added.]** [attempt:brief explanation of MCMC and related topics]

## 2.3 Initial Result

All examples here were performed in Python 3.8 using the library pyMC [JTC16] and JAGS 4.3.0[Den16] and R 3.6.2 [R C13]. Training was performed using No U-Turn Sampling (NUTS) over two chains with 1000 iterations. Fitting was performed on a GHz Intel Core i5 with 8GM of LPDD3 RAM and typically had wall times under ten minutes. Code for all examples in this study are provided.

### 2.3.1 Posterior Distribution

Figure 2 is the boxplot of posterior samples of  $O_t$ . It is shown that our posterior estimates of  $O_t$  are fairly accurate since (1), the boxplots contain actual values of  $O_t$  within their interquartile range (IQR), and (2), the ranges of IQR and 95% range seem narrow. Notice that the range of the boxplot from a higher  $O_t$  values ( $t=1, 7, 11$ ) is wider than the other ranges of the boxplots from smaller estimates of  $O_t$  (all  $t$  values but 1, 7, 11)

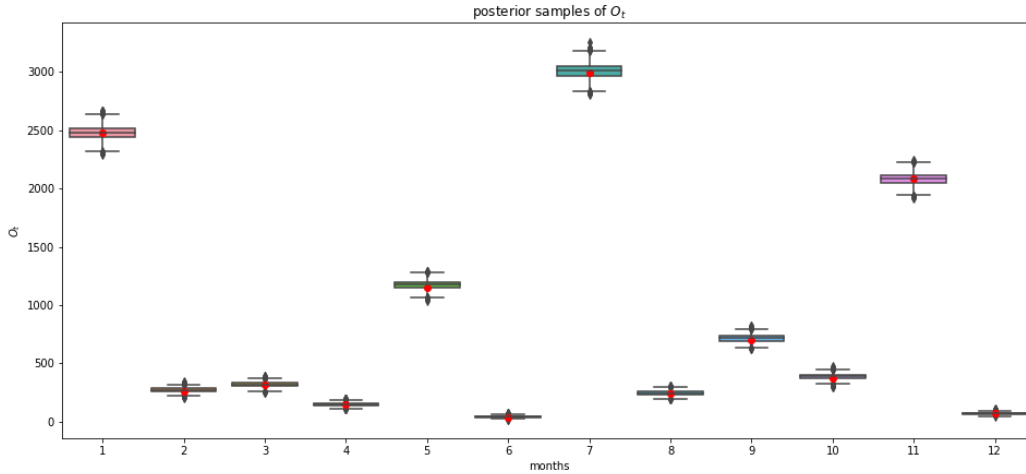


Figure 2: Boxplot of posterior samples of  $O_t$  (2000 samples for each month) with simulated  $O_t$  values as red dots.

Let  $O_+$  denote the total overdose per year ( $O_+ = \sum_{t=1}^{12} O_t$ ). Figure 3 is the histogram of posterior samples of the total number of overdoses  $O_+$  per year. 2000 samples were collected and shown as the histogram. It is seen that the plot is bell-shaped and not skewed. The total number of overdoses per year is inferred to be around 10900 to 11000. [mention about credible interval]

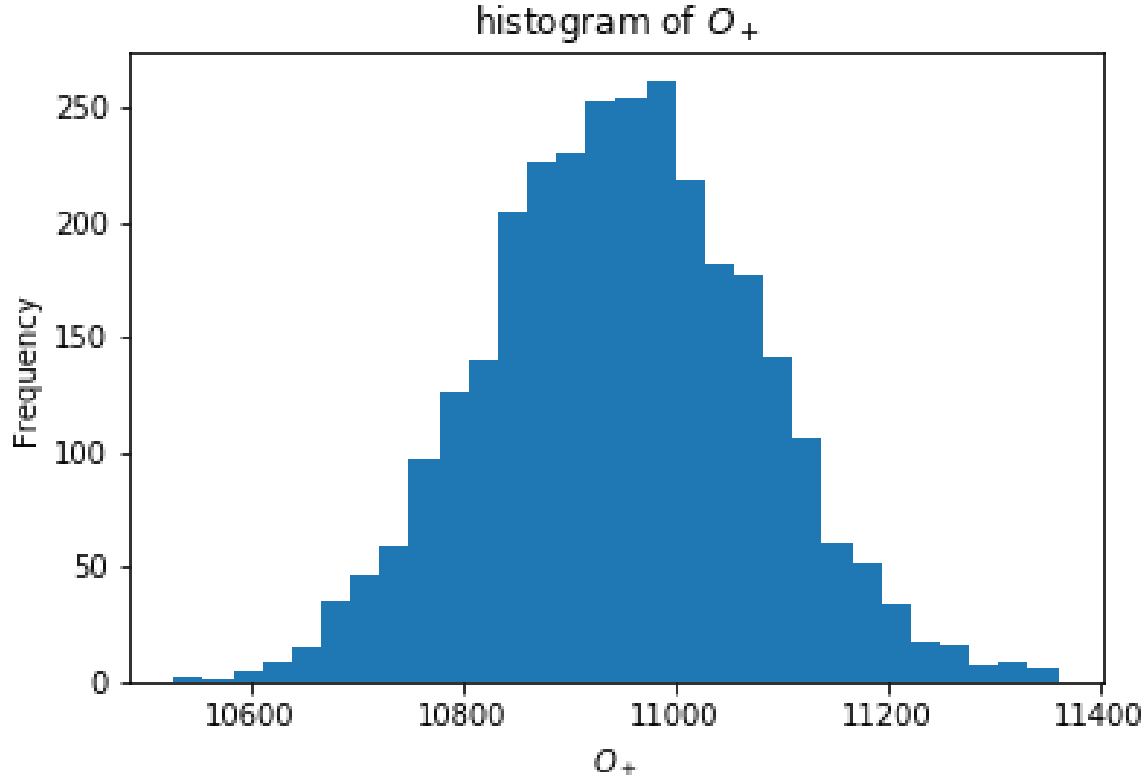
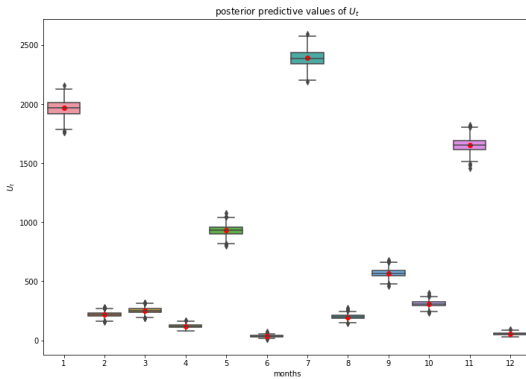


Figure 3: Histogram of posterior samples of total overdoses per year

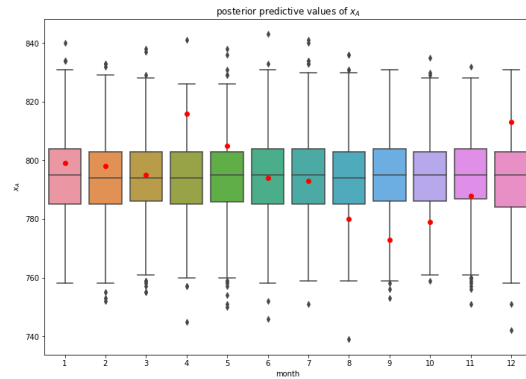
### 2.3.2 Posterior Predictive Check

Posterior predictive checking is a model validation technique when we simulate some replicated data under the fitted model then compare the new data to the observed data. If the model fits, then replicated data generated under the model should look similar to observed data. To put it another way, the observed data should look plausible under the posterior predictive distribution. This is really a self-consistency check: an observed discrepancy can be due to model misfit or chance. [GCS<sup>+</sup>14] [attempted: brief explannation of what is PPC]

Figure 4a is the boxplot of posterior predictive samples of  $U_t$ . It is shown that the posterior predictive estimates of  $U_t$  are failry accurate, for the same two reasons regarding the accuracy of the posterior distribution of  $O_t$ . It is more obvious here that the range of the boxplots from higher  $O_t$  values ( $t=1, 7, 11$ ) is wider than the other ranges of the boxplots from smaller estimates of  $O_t$  (all  $t$  values but 1,7, 11). Notice that the relative ranges of figure 4a follow the ones of figure 2; for months where  $U_t$  values are higher, the values of  $O_t$  are also higher than the average ( $t=1, 7, 11$ ).



(a) ppc of  $U_t$



(b) ppc of  $p_A$

Figure 4: Boxplots of posterior predictive samples of  $U_t$  and  $x_A$  (1000 samples) with the actual data point of simulated  $U_t, x_A$  value. The simulated values are shown as a red dots.

Figure 4b is the boxplots of posterior predictive samples of  $x_A$ . It is shown that the posterior predictive estimates of  $x_A$  are tolerably accurate since (1), the boxplots contain actual values of  $O_t$  within their lines connecting the maximum and the minimum, and (2), the ranges of IQR seem narrow.

## 2.4 Result: Contamination of $p_A$

One focus of this research project is to investigate how robust the model is to contaminations of the data sets. The first inspection is to check an impact of a contamination of  $p_A$ ; what would happen if the estimation of  $p_A$  is biased? It is assumed that the survey data gives us a wrong estimate of  $p_A$  such that it would be underestimated or overestimated. We then want to see how the biased estimation of  $p_A$  affects the estimate of  $O_t$ , the total number of overdoses.

Both underestimation and overestimation were considered for the analysis. In terms of underestimation, the simulated survey data  $(n_A, x_A)$  was generated with  $p_A = 0.6$  while the true value of  $p_A$  is 0.8, and all the other assumptions hold as before. That is,  $x_A$  is generated from  $x_A \sim \text{Bin}(n_A, 0.6)$ , while  $U_t$  is generated from  $U_t \sim \text{Bin}(O_t, 0.8)$  for every  $t$ . For overestimation, the simulated survey data was generated with  $p_A = 0.9$  while the true value of  $p_A$  is 0.8, and all the other assumptions hold the same.

### 2.4.1 Posterior Distribution

Figure 5 is the boxplots of posterior samples where  $p_A$  is contaminated from the survey data. It is seen that the underestimation of  $p_A$  (5a) leads to an overestimation of  $O_t$  as the boxplots are above

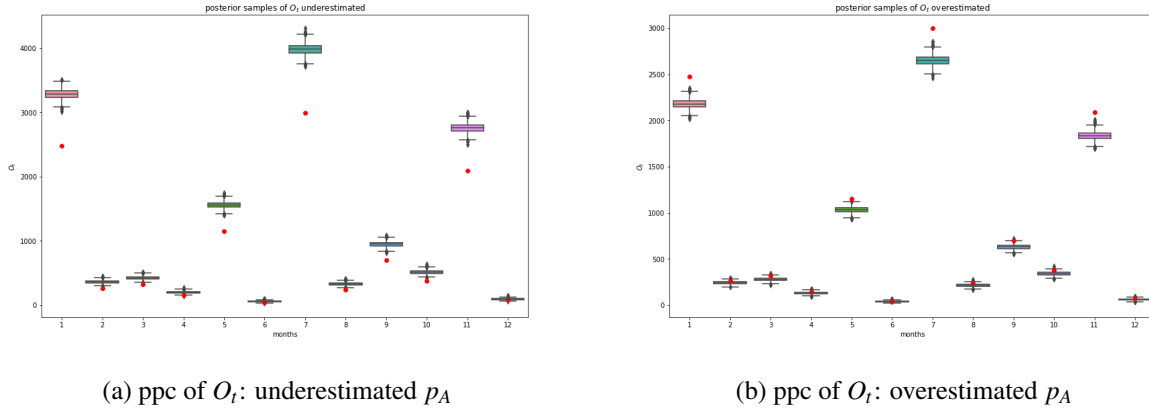


Figure 5: Boxplot of posterior samples of  $O_t$  (1000 samples) where survey data is contaminated. The actual data point of simulated  $O_t$  values are shown as red dots.

the red dots. This is explainable considering the given data sets (likelihoods) and the relationship between the two models (4); given the likelihood of the ambulance attended overdoses,  $U_t$ , a value of  $O_t$  is an unknown parameter of the binomial distribution,  $Bin(O_t, p_A)$ . Let  $o_t, u_t$  denote posterior samples from  $O_t$  and the observed value of  $U_t$  respectively. Then  $o_t$  is proportional to the quantity generated by multiplying  $u_t$  and the inverse of  $\hat{p}_A$  where  $\hat{p}_A$  is underestimated estimator of  $p_A$ ; hence  $\frac{1}{\hat{p}_A}$  is overestimated which leads overestimation of  $O_t$ :

$$o_t \propto u_t \frac{1}{\hat{p}_A}. \quad (8)$$

Figure 5b shows the opposite case. Overestimation of  $p_A$  leads underestimation of the inverse of  $p_A$  which causes underestimation of  $O_t$ . From both figures it is seen that the bias increases as the estimated values and the actual values get large.

## 2.4.2 Posterior Predictive Check

Figure 6 gives the boxplots of posterior predictive samples where  $p_A$  is underestimated and overestimated respectively from the survey data. It is seen that none of the contaminations of  $p_A$  lead to an contamination effect on  $U_t$  but only affect the  $O_t$  estimation.

The possible explanation is that the contaminations of  $p_A$  and  $O_t$  (from  $p_A$ ) cancel out the bias component as combined so that  $U_t$  has no bias. Notice that  $U_t$  is a likelihood (data set) of  $O_t$  estimation. Hence it plays the role of  $y$  in Bayes theorem (??) and the likelihood,  $U_t$ , is used to obtain the biased posterior samples of  $O_t$  with the contaminated estimates of  $p_A$ . Here the pair of  $(O_t, p_A)$  are both biased and the pair is fitted by the likelihood,  $U_t$ . To obtain the posterior predictive samples, MCMC algorithm help obtain posterior samples of biased  $O_t$  and then the samples will be combined with biased  $p_A$  such that the new samples of  $\tilde{U}_t$  from  $p(\tilde{U}_t|U_t)$  is obtained. The result also matches with our intuition. Posterior predictive checks use the existing data points twice; firstly it uses the data to obtain the posterior samples of parameters, and secondly the samples are used to produce posterior predictive samples. Given the fact that there could be an overfitting issue due to using the data twice, the result shown here seems reasonable.

Figure 7 gives the boxplots of posterior predictive samples of  $x_A$  where  $p_A$  is underestimated and overestimated respectively from the survey data. It is seen that none of the contaminations of  $p_A$  leads an effect of contamination on  $x_A$ ; the actual simulated points (red dots) are close to the medians from the two boxplots. However, notice that the range of the estimated values are different between the two boxplots; the median from Figure 7a is around 600 whereas the median from Figure 7b is around 900. [???? I think this should go to the previous of the previous paragraph] This is because overdose model (2) does not affect the result from the ambulance model (3); only the ambulance model has an effect on the overdose model.

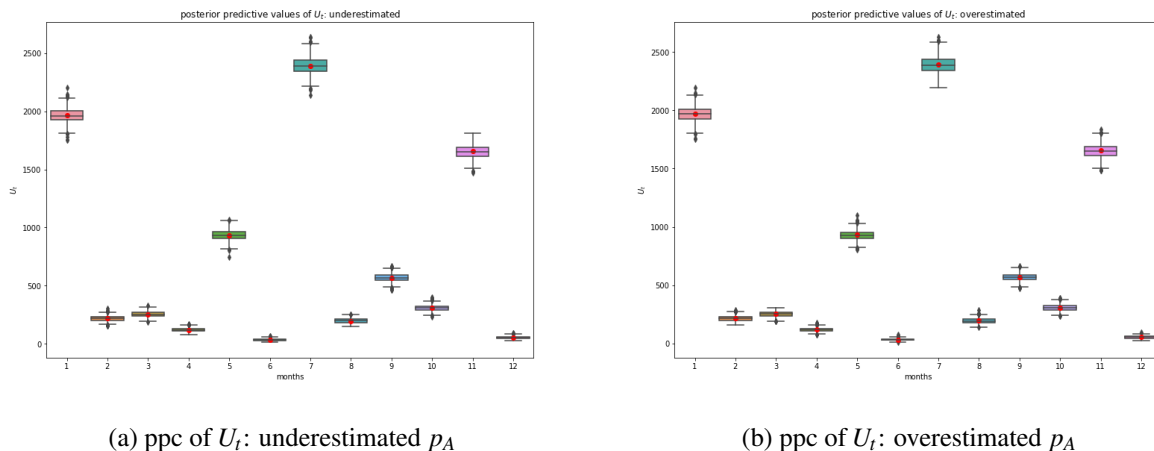


Figure 6: Boxplot of posterior samples of  $U_t$  (1000 samples) where survey data is contaminated. The actual data point of simulated  $U_t$  values are shown as red dots.

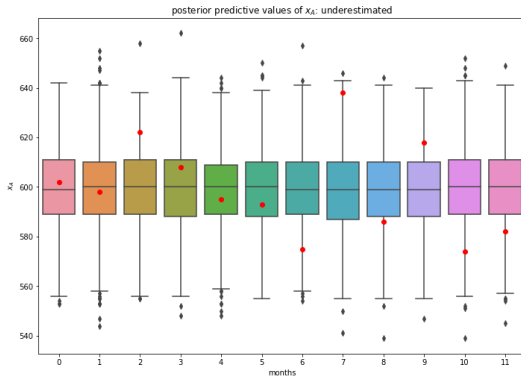
## 2.5 Varying the Assumptions

The assumptions will change later to see the impact of the likelihood over the posterior distributions of variables of interest; the total population size for a region,  $N$ , could vary over time or it can be stratified for a better realization of the reality. The survey size could also vary such as  $n_A = 100$  or  $n_A = 10000$  in later examples.

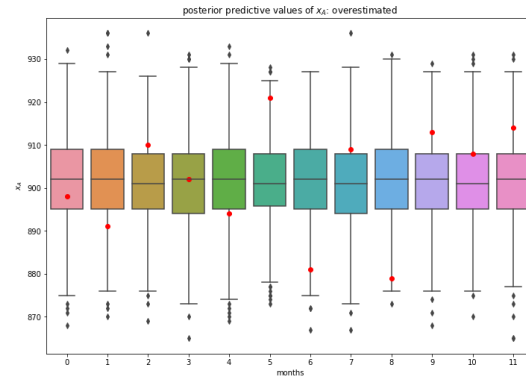
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(a) ppc of  $x_A$ : underestimated  $p_A$



(b) ppc of  $x_A$ : overestimated  $p_A$

Figure 7: Boxplot of posterior samples of  $x_A$  (1000 samples) where survey data is contaminated. The actual data point of simulated  $x_A$  values are shown as red dots.

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