CPSC 340 Assignment 5 (due Friday March 23 at 9:00pm)

Instructions

Rubric: {mechanics:5}

The above points are allocated for following the general homework instructions. In addition to the usual instructions: if you're embedding your answers in a document that also contains the questions, your answers should be in a colour that clearly stands out, such as green or red. This should hopefully make it much easier for the grader to find your answers. To make something green, you can use the LaTeX macro `\gre{my text}`.

Also, **READ THIS**: Like in a2, you'll need to grab the data from the course website. FYI: this happens because I'm using the GitHub API in a fairly silly way, which limits individual files to 1 MB each.

1 MAP Estimation

Rubric: {reasoning:10}

In class, we considered MAP estimation in a regression model where we assumed that:

- The likelihood $p(y_i|x_i, w)$ is a normal distribution with a mean of $w^T x_i$ and a variance of 1.
- The prior for each variable j, $p(w_j)$, is a normal distribution with a mean of zero and a variance of λ^{-1}.

Under these assumptions, we showed that this leads to the standard L2-regularized least squares objective function:

$$f(w) = \frac{1}{2}\|Xw - y\|^2 + \frac{\lambda}{2}\|w\|^2.$$

For each of the alternate assumptions below, show how the loss function would change (simplifying as much as possible):

1. We use a zero-mean Laplace prior for each variable with a scale parameter of λ^{-1}, so that

$$p(w_j) = \frac{\lambda}{2}\exp(-\lambda|w_j|).$$

   The regularizer changes to

$$\frac{\lambda}{2}\|w\|_1.$$

2. We use a Laplace likelihood with a mean of $w^T x_i$ and a scale of 1, so that

$$p(y_i|x_i, w) = \frac{1}{2}\exp(-|w^T x_i - y_i|).$$

   The data fitting term changes to

$$\frac{1}{2}\|Xw - y\|_1$$

3. We use a Gaussian likelihood where each datapoint has variance σ^2 instead of 1,

$$p(y_i|x_i, w) = \frac{1}{\sqrt{2\sigma^2\pi}} \exp\left(-\frac{(w^T x_i - y_i)^2}{2\sigma^2}\right).$$

The data fitting term changes to

$$\frac{1}{2\sigma^2}\|Xw - y\|^2$$

4. We use a Gaussian likelihood where each datapoint has its own variance σ_i^2,

$$p(y_i|x_i, w) = \frac{1}{\sqrt{2\sigma_i^2\pi}} \exp\left(-\frac{(w^T x_i - y_i)^2}{2\sigma_i^2}\right).$$

The data fitting term changes to

$$\frac{1}{2}(Xw - y)^T Z(Xw - y)$$

where Z is n by n diagonal matrix with diagonal elements $\frac{1}{\sigma_i^2}$ for $i = 1, 2, 3, ...n$.

2 Principal Component Analysis

2.1 PCA by Hand

Rubric: {reasoning:3}

Consider the following dataset, containing 5 examples with 2 features each:

x_1	x_2
-2	-1
-1	0
0	1
1	2
2	3

Recall that with PCA we usually assume that the PCs are normalized ($\|w\| = 1$), we need to center the data before we apply PCA, and that the direction of the first PC is the one that minimizes the orthogonal distance to all data points.

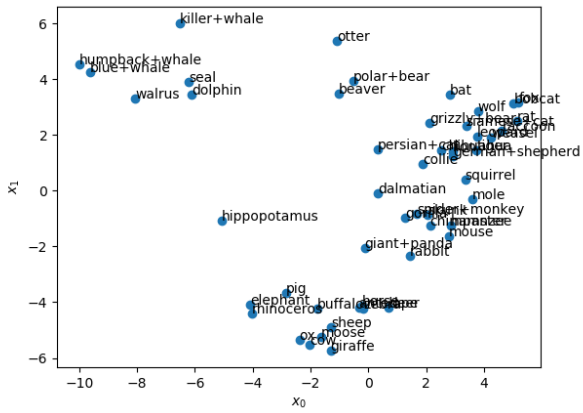
1. What is the first principal component? $w = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
2. What is the (L2-norm) reconstruction error of the point (3,3)?
   First, we scale the vector to be centered. We have $(3, 2)$.
   We multiply w to the scaled vector to get z.
   $z = (3, 2) \cdot w^T = \frac{5}{\sqrt{2}}$
   $\widetilde{x} = \frac{5}{\sqrt{2}} \cdot w + 1 = (2.5, 2.5) + 1 = (2.5, 3.5)$ So the error is $\sqrt{0.5^2 + 0.5^2} = \frac{1}{\sqrt{2}}$
3. What is the (L2-norm) reconstruction error of the point (3,4)? (Show your work.)
   We scale it to be centered. $(3, 3)$
   We multiply w to get the z.
   $z = (3, 3) \cdot w^T = \frac{6}{\sqrt{2}}$.
   $\widetilde{x} = \frac{6}{\sqrt{2}} \cdot w + 1 = (3, 3) + 1 = (3, 4)$. So there is no reconstruction error. It is 0.

2.2 Data Visualization

Rubric: {reasoning:2}

If you run `python main.py -q 2`, it will load the animals dataset and create a scatterplot based on two randomly selected features. We label some random points, but because of the binary features the scatterplot shows us almost nothing about the data.

The class *pca.PCA* applies the classic PCA method (orthogonal bases via SVD) for a given k. Use this class so that the scatterplot uses the latent features z_i from the PCA model. Make a scatterplot of the two columns in Z, and label a bunch of the points in the scatterplot. Hand in your code and the scatterplot.



https://github.ugrad.cs.ubc.ca/CPSC340-2017W-T2/k6y1b_a5/blob/master/code/main.py

2.3 Data Compression

Rubric: {reasoning:2}

1. How much of the variance is explained by our 2-dimensional representation from the previous question?
   when k=2, explained var is 0.301938151559.
2. How many PCs are required to explain 50% of the variance in the data?
   k=3, 0.38779248564
   k=4, 0.448603643724
   k=5, 0.505901629487
   So we need more than 5 PCs.

3 PCA Generalizations

3.1 Robust PCA

Rubric: {code:10}

If you run `python main -q 3.1` the code will load a dataset X where each row contains the pixels from a single frame of a video of a highway. The demo applies PCA to this dataset and then uses this to reconstruct

the original image. It then shows the following 3 images for each frame (pausing and waiting for input between each frame):

1. The original frame.
2. The reconstruction based on PCA.
3. A binary image showing locations where the reconstruction error is non-trivial.

Recently, latent-factor models have been proposed as a strategy for "background subtraction": trying to separate objects from their background. In this case, the background is the highway and the objects are the cars on the highway. In this demo, we see that PCA does an ok job of identifying the cars on the highway in that it does tend to identify the locations of cars. However, the results aren't great as it identifies quite a few irrelevant parts of the image as objects.

Robust PCA is a variation on PCA where we replace the L2-norm with the L1-norm,

$$f(Z, W) = \sum_{i=1}^{n}\sum_{j=1}^{d} |w_j^T z_i - x_{ij}|,$$

and it has recently been proposed as a more effective model for background subtraction. Complete the class *pca.RobustPCA*, that uses a smooth approximation to the absolute value to implement robust PCA. Comment on the quality of the results.

Hint: most of the work has been done for you in the class *pca.AlternativePCA*. This work implements an alternating minimization approach to minimizing the (L2) PCA objective (without enforcing orthogonality). This gradient-based approach to PCA can be modified to use a smooth approximation of the L1-norm. Note that the log-sum-exp approximation to the absolute value may be hard to get working due to numerical issues, and a numerically-nicer approach is to use the "multi-quadric" approximation:

$$|\alpha| \approx \sqrt{\alpha^2 + \epsilon},$$

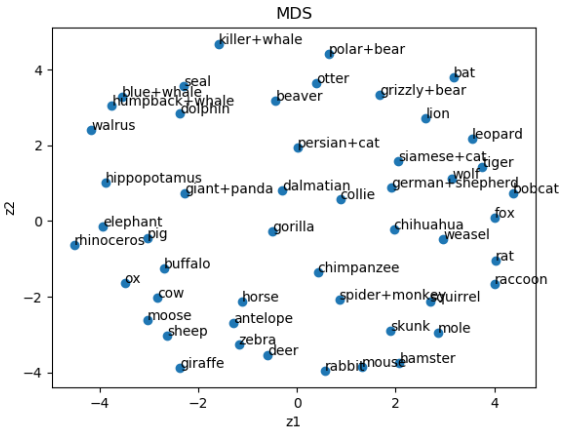
where ϵ controls the accuracy of the approximation (a typical value of ϵ is 0.0001).
`https://github.ugrad.cs.ubc.ca/CPSC340-2017W-T2/k6y1b_a5/blob/master/code/pca.py`
Pictures show that L1 objective function is better for background subtraction.

4 Multi-Dimensional Scaling

If you run `python main.py -q 4`, the code will load the animals dataset and then apply gradient descent to minimize the following multi-dimensional scaling (MDS) objective (starting from the PCA solution):

$$f(Z) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} (\|z_i - z_j\| - \|x_i - x_j\|)^2. \tag{1}$$

The result of applying MDS is shown below.



Although this visualization isn't perfect (with "gorilla" being placed close to the dogs and "otter" being placed close to two types of bears), this visualization does organize the animals in a mostly-logical way.

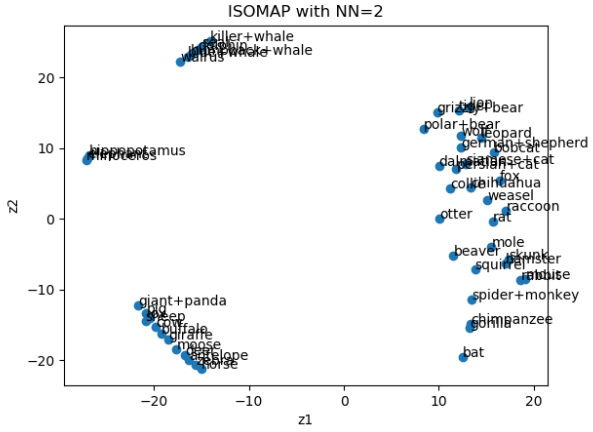
4.1 ISOMAP

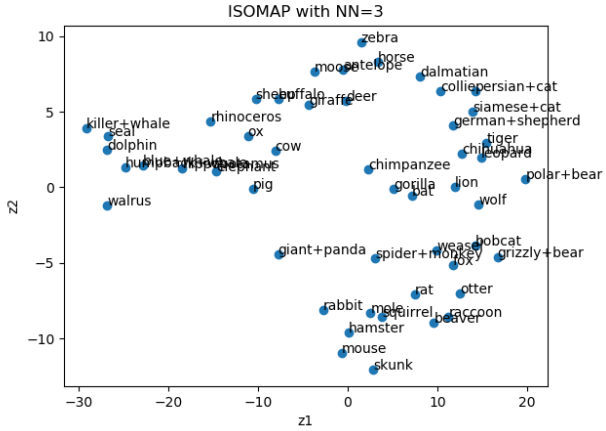
Rubric: {code:10}

Euclidean distances between very different animals are unlikely to be particularly meaningful. However, since related animals tend to share similar traits we might expect the animals to live on a low-dimensional manifold. This suggests that ISOMAP may give a better visualization. Fill in the class *ISOMAP* so that it computes the approximate geodesic distance (shortest path through a graph where the edges are only between nodes that are k-nearest neighbours) between each pair of points, and then fits a standard MDS model (1) using gradient descent. Plot the results using 2 and using 3-nearest neighbours.

Note: when we say 2 nearest neighbours, we mean the two closest neigbours excluding the point itself. This is the opposite convention from what we used in KNN at the start of the course.

The function *utils.dijskstra* can be used to compute the shortest (weighted) distance between two points in a weighted graph. This function requires an $n \times n$ matrix giving the weights on each edge (use 0 as the weight for absent edges). Note that ISOMAP uses an undirected graph, while the k-nearest neighbour graph might be asymmetric. One of the usual heuristics to turn this into a undirected graph is to include an edge i to j if i is a KNN of j or if j is a KNN of i. (Another possibility is to include an edge only if i and j are mutually KNNs.)
`https://github.ugrad.cs.ubc.ca/CPSC340-2017W-T2/k6y1b_a5/blob/master/code/manifold.py`





4.2 Reflection

Rubric: {reasoning:2}

Briefly comment on PCA vs. MDS vs. ISOMAP for dimensionality reduction on this particular data set. In your opinion, which method did the best job and why?
ISOMAP with 2 nearest neighbor is the best of reducing dimentionality for this dataset. The clusters identified are far away from one another. ISOMAP is latent factor model for visualizing data on manifolds. On the other hand, PCA and MDS cannot discover non-linear manifolds. It seems that the data lives on a low-dimentional "manifold". Moreover, PCA and MDS relatively focuses more on large distances, which let them likely to lose local structure and lead to "crowding" effect.

5 Very-Short Answer Questions

Rubric: {reasoning:10}

1. Why is the kernel trick often better than explicitly transforming your features into a new space? If you have too many features, storing the basis can be painfil while kernel trick enable us not need to do that, which means faster!

2. Why is the kernel trick more popular for SVMs than with logistic regression? Because of the support vectors.

3. What is the key advantage of stochastic gradient methods over gradient descent methods? in n large case, gradient descent is expensive, while stochastic gradient is independent of the n.

4. Does stochastic gradient descent with a fixed α converge to the minimum of a convex function in general? No, it makes progess until variance is too large.

5. What is the difference between multi-label and multi-class classification? multi-label is that one object has more than one labels. multi-class classification is that we have more than two categories where each of object can go into.

6. What is the difference between MLE and MAP?
MLE is like a loss function and MAP is like loss function plus regularizer term.

7. Linear regression with one feature and PCA with 2 features (and $k = 1$) both find a line in a two-dimensional space. Do they find the same line? Briefly justify your answer. No. PCA line will always come across the origin. Linear regression only minimize the distance between the line and y(label) values while PCA with 2 features will minimize squared distance in both dimension.

8. Are the vectors minimizing the PCA objective unique? Briefly justify your answer No. If the vectors in W are made by SVD, it may seem to be unique but we cannot ignore the direction.

9. Name two methods for promoting sparse solutions in a linear regression model that result in convex problems. We use L1 norm regularization or put non-negative constraint.

10. Can we use the normal equations to solve non-negative least squares problems? Yes. we just get 0 if the optimal value is negative value, which makes it sparse solution.