The Proof about the Existence and Uniqueness of the NE in the JC Game

Meng Xie, Student Member, IEEE

The NE of the JC game exists and is unique, which is given by

$$P_{J_i}^{ne} = \frac{R(M-1)\left(\sum_{j=1}^{M}\frac{D_{J_i}}{D_{J_j}} - M + 1\right)}{c\cdot\left(\sum_{j=1}^{M}\frac{D_{J_i}}{D_{J_j}}\right)^2} = R\cdot\kappa_i. \tag{1}$$

Here we give the proofs of the above statement.

We use $G = (\mathcal{J}, \{\mathbf{P}_J\}, \{U_{J_i}\})$ to denote the JC game, where $\mathcal{J}$, $\{\mathbf{P}_J\}$ and $\{U_{J_i}\}$ are the sets of jammers, strategy profiles and utility functions, respectively.

Proposition 1 *An NE exists in the JC game $G = (\mathcal{J}, \{\mathbf{P}_J\}, \{U_{J_i}\})$, if: 1) $\{\mathbf{P}_J\}$ is a nonempty, compact, and convex subset of the M-dimensional Euclidean space $\mathbf{R}^M$ and 2) U_{J_i} is concave on P_{J_i}, for every $J_i \in \mathcal{J}$.*

For the first part of the proof, since the strategy of player $P_{J_i} \geq 0$, the strategy space of the JC game $\{\mathbf{P}_J\}$ is a nonempty, compact, and convex subset of the M-dimensional Euclidean space $\mathbf{R}^M$. Taking the first- and second-order derivatives of U_{J_i} with respect to P_{J_i} yields

$$\frac{\partial U_{J_i}}{\partial P_{J_i}} = \frac{Rr_i}{\sum_{J_j\in\mathcal{J}} P_{J_j}r_j} - \frac{RP_{J_i}r_i^2}{\left(\sum_{J_j\in\mathcal{J}} P_{J_j}r_j\right)^2} - c \tag{2}$$

$$\frac{\partial^2 U_{J_i}}{\partial P_{J_i}^2} = -2\frac{Rr_i^2\sum_{J_j\in\mathcal{J}\setminus\{J_i\}} P_{J_j}r_j}{\left(\sum_{J_j\in\mathcal{J}} P_{J_j}r_j\right)^3} < 0. \tag{3}$$

We could get that U_{J_i} is continuous and differentiable on P_{J_i} and its second-order derivative is negative. So U_{J_i} is a concave function on P_{J_i}. Then we get the following theorem.

Theorem 1 *There exists at least an NE in the JC game $G = (\mathcal{J}, \{\mathbf{P}_J\}, \{U_{J_i}\})$.*

As we know, each player will play the strategy which could achieve its maximal utility in an NE. We refer to it as the best response strategy, which is defined as follows.

Definition 1 *(Best Response Strategy): Given $\mathbf{P}_{-J_i}$, a strategy is the best response strategy of jammer J_i, denoted as $b_i(\mathbf{P}_{-J_i})$, if it satisfies $U_{J_i}(b_i(\mathbf{P}_{-J_i}),\ \mathbf{P}_{-J_i}) \geq U_{J_i}(P_{J_i}, \mathbf{P}_{-J_i})$ for all $P_{J_i} \geq 0$.*

Now we check the uniqueness of NE in the game. Considering the best response correspondence of the game, i.e., $\mathbf{b}(\mathbf{P}_J) = (b_1(\mathbf{P}_{-J_1}), b_2(\mathbf{P}_{-J_2}), \ldots,\ b_M(\mathbf{P}_{-J_M}))$. Then an NE is actually a fixed point of the best response correspondence $\mathbf{b}(\mathbf{P}_J)$. Therefore, the uniqueness of NE is equivalent to that the function $\mathbf{b}(\mathbf{P}_J)$ has a unique fixed point.

Proposition 2 *The fixed point of function $\mathbf{b}(\mathbf{P}_J)$ is unique if for all $P_{J_i} \geq 0$: 1) $\mathbf{b}(\mathbf{P}_J) > 0$. 2) $\mathbf{b}(\mathbf{P}_J)$ is monotonicity. 3) for all $\beta > 1$, $\beta\mathbf{b}(\mathbf{P}_J) > \mathbf{b}(\beta\mathbf{P}_J)$.*

We know that U_{J_i} is a concave function on P_{J_i}, so we could get the best response strategy $b_i(\mathbf{P}_{-J_i})$ by solving the zero of the first-order derivative of U_{J_i} with respect to P_{J_i}, there is,

$$P_{J_i} = \sqrt{\frac{R\sum_{J_j\in\mathcal{J}\setminus\{J_i\}} P_{J_j}r_j}{cr_i}} - \frac{1}{r_i}\sum_{J_j\in\mathcal{J}\setminus\{J_i\}} P_{J_j}r_j. \tag{4}$$

If the right-hand side of (4) is positive, it is the best response strategy of jammer J_i. If it is less than or equal to 0, then jammer J_i will not participate in the game. Now we show the best response strategy there,

$$b_i\left(\mathbf{P}_{-J_i}\right) = \begin{cases} 0, & R \leq \frac{c}{r_i} \sum_{J_j \in \mathcal{J} \setminus \{J_i\}} P_{J_j} r_j \\ \sqrt{\frac{R \sum_{J_j \in \mathcal{J} \setminus \{J_i\}} P_{J_j} r_j}{c r_i}} - \frac{1}{r_i} \sum_{J_j \in \mathcal{J} \setminus \{J_i\}} P_{J_j} r_j. & \text{otherwise} \end{cases} \tag{5}$$

Obviously the best response function $b_i\left(\mathbf{P}_{-J_i}\right)$ is always positive and monotonic. What's more, we have

$$\beta b_i\left(\mathbf{P}_{-J_i}\right) - b_i\left(\beta \mathbf{P}_{-J_i}\right) = (\beta - \sqrt{\beta}) \sqrt{\frac{R \sum_{J_j \in \mathcal{J} \setminus \{J_i\}} P_{J_j} r_j}{c r_i}}. \tag{6}$$

Then we get the following theorem.

Theorem 2 *There exists a unique NE in the JC game $G = (\mathcal{J}, \{\mathbf{P}_J\}, \{U_{J_i}\})$.*

Let $A_i = \sum_{J_j \in \mathcal{J} \setminus \{J_i\}} P_{J_j}^{ne} r_j$. From (2), we can get

$$\begin{aligned} r_i A_i &= \frac{c}{R}\left(A_i + P_{J_i}^{ne} r_i\right)^2 = \frac{c}{R}\left(\sum_{J_k \in \mathcal{J}} P_{J_k}^{ne} r_k\right)^2 \\ &= \frac{c}{R}\left(A_j + P_{J_j}^{ne} r_j\right)^2 = r_j A_j \quad \forall J_i, J_j \in \mathcal{J}. \end{aligned} \tag{7}$$

We can obtain the system of equations:

$$\begin{cases} A_1 = \frac{r_i}{r_1} A_i \\ A_2 = \frac{r_i}{r_2} A_i \\ \vdots \\ A_M = \frac{r_i}{r_M} A_i. \end{cases} \tag{8}$$

By some basic algebraic transformations, we have

$$A_1 + A_2 + \cdots + A_M = \frac{r_i}{r_1} A_i + \frac{r_i}{r_2} A_i + \cdots + \frac{r_i}{r_M} A_i = \sum_{j=1}^{M} \frac{r_i}{r_j} A_i. \tag{9}$$

$$A_j = \sum_{J_k \in \mathcal{J} \setminus \{J_j\}} P_{J_k}^{ne} r_k = A_i + P_{J_i}^{ne} r_i - P_{J_j}^{ne} r_j. \tag{10}$$

Since the above equations, we have

$$\begin{aligned} \sum_{i=1}^{M} A_i &= M\left(A_i + P_{J_i}^{ne} r_i\right) - \sum_{j=1}^{M} P_{J_j}^{ne} r_j \\ &= M\left(A_i + P_{J_i}^{ne} r_i\right) - \left(A_i + P_{J_i}^{ne} r_i\right) \\ &= (M-1)\left(A_i + P_{J_i}^{ne} r_i\right) = \sum_{j=1}^{M} \frac{r_i}{r_j} A_i. \end{aligned} \tag{11}$$

and thus,

$$A_i = \frac{(M-1) r_i P_{J_i}^{ne}}{\sum_{j=1}^{M} \frac{r_i}{r_j} - M + 1}. \tag{12}$$

Substituting (12) into (4),

$$P_{J_i}^{ne} = \frac{R(M-1)\left(\sum_{j=1}^{M} \frac{D_{J_i}}{D_{J_j}} - M + 1\right)}{c \cdot \left(\sum_{j=1}^{M} \frac{D_{J_i}}{D_{J_j}}\right)^2} = R \cdot \kappa_i. \tag{13}$$

We can now summarize the conclusions from the original paper.

Theorem 3 *The NE of the JC game exists and is unique, which is given by*

$$P_{J_i}^{ne} = \frac{R(M-1)\left(\sum_{j=1}^{M} \frac{D_{J_i}}{D_{J_j}} - M + 1\right)}{c \cdot \left(\sum_{j=1}^{M} \frac{D_{J_i}}{D_{J_j}}\right)^2} = R \cdot \kappa_i. \tag{14}$$