Rotation Matrix Reconstruction in Quaternions with an Optimal Observer within a Navigation Algorithm Based on Nonlinear Complementary Observers

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***Abstract*—This work proposes a variation for a earlier navigation algorithm in the literature composed by SO(3) complementary observers. Such modification involves the inclusion of a quaternion optimal observer to determine the rotation matrix instead using a vectorial reconstruction approach, in particular this proposal involves an experimental comparison between the original and modified method. The results show about 40% increase for estimation quality while 21% more complexity using this new approach.**

I. Introduction

Roughly speaking the process of moving between two points implies determine the position and direction throughout all the trajectory. This problem is generally solved by interpreting measurements from fixed references or variables revealing the internal states of the motion body. Hereby, navigation can defined as the art of calculate or interpret this variables and the set elements inferring such results are known as Navigation System. Furthermore, a navigation system incorporates a navigation algorithm that tweaks the motion models optimizing the predictions much like the way of a control systems. In fact, the famous Kalman Optimal Observer solving the Winner-Hoft optimal observer problem presents itself as a consequence of the duality control-observer principle [1], thus the author extends a spectral statistical filter to make predictions in time recalling principles from the Optimal Control Theory. Moreover, most recent trends for designing this kind of observers also explode tools from control theory that is nonlinear filters developed in the framework of Lyapunov theory e.g. [2], [3]. Further, some authors worked to reduce the complexity involved in navigation equations by dividing the process in separated observers, for example in [4] two observers are combined first to determine the rotation matrix and then to calculate the position, or in [5] who combine to complementary filters to determine rotation and position in two separated processes. Therefore, based on this initial idea, in this work we propose the calculation of the rotational matrix in quaternions using an optimal observer by modifying the initial navigation algorithm proposed in [5].

One of the first formal approaches dealing the navigation problem was addressed as an application of the famous Kalman Filter which has been extended through the linear perturbation equation to solve the trajectory estimation for a nonlinear model of motion and its was used to unravel the trajectory from motion signals in navigation computer of the Apollo project [6]. In the Kalman Filter's original paper, the authors re-casted the optimal estimation methods developed by Norbert Wiener (Stationary Stochastic Signals) in terms of first and second momentum matrices of conditional probabilities in the states space form which allots the same solution to non-stationaty signals embedded in the discrete time version of the Riccati equation [7]. Since that milestone event many other authors have issued an enormous quantity of ideas diverging from Kalman's core paper in the field of navigation algorithms e.g. [8], [9]. Some versions of the Kalman Filter work with powerful representation of motion based on atypical algebraic structures like quaternions, [10] pre-calculates the quaternion using a Gauss-Newton iteration algorithm to find the best quaternion that relates the measured accelerations and earth magnetic field in the body coordinate frame, they addressed with a similar approach to what is followed in this work in the sense that we pre-compute the rotation matrix as they did.

There exists a high cohesion between the ideas of design of Luenberger Observers and the optimal approach followed by the Kalman Filter, in deed the target of the Luenberger Observer is to determine unknown states as in the Kalman Filter with the difference of that Luenberger works within dynamical system framework and avoids to use statistical terminology [11]. Luenberger develops theoretical foundation through the use of modern control applied to the observer problem, which triggers the engagement of non-linear control theory [12][13] in this field. Further trends in this approach work to implement non linear observers when the system's dynamic entails impossible linearize nonlinear terms as the navigation equations, hence this approach has been used to develop navigation algorithms founded in the design of Lyapunov and exponential stability, and again they grip to strong rugged representation of rotation using quaternions [14] [15] and [16]. Quaternion algebra plays a mayor role in nonlinear observers simplifying calculations and offering good algebraic structure that allows to recover angles without singularities.

Further, an special approach is extended by [17] which proposes a general geometrical framework for the design of observers on finite-dimensional Lie groups. One particular case investigated in [17] explains the general form of non-linear observers on SO(3) where the dynamics and the correction terms are expressed on a tangential Lie Algebras $\mathfrak{so}(3)$, further the orientation is determined from two vectorial measurements depending of the rotational transformation on the Lie group. Such structure, identifies the nonlinear observer proposed in [18] who presents an extensive analysis of non-linear attitude observers on SO(3) in terms of the Lyapunov stability and the reconstruction of the Rotation matrix based also on two vectorial measurements. In addition, they extend a globally stable observer dynamics which global asymptotic stability is proved under the assumption that the observer's error trajectory does not intersect a forward invariant set of instability conditions, further they implicate a sub-optimal solution of the algebraic reconstruction of the rotation equation which we believe can be improved.

What we proposed is to extend the idea of non-linear complementary observer from [5] that combines the passive version of the complementary filter on SO(3) [18]. As a results, we visualized to reduce prediction errors by pre-computing the rotation matrix with an optimal observer in quaternions. Two main points were studied during this work. First, the modification of the original approach of *Mahony-Scandaroli's navigation algorithm of non-linear observers on* $SO(3)$ [18], [5] by including the an EKF/ optimal observer, which was applied to optimally determine the rotation matrix based on vectorial measurements; and second, we studied the behavior of our approach compared with the original method in two different experiments. One sets the rotation on one axis using a camera that tracks the rotation that is used as reference and the other uses a commercial GPS system that allows us to evaluate the estimation of the position

II. Navigation problem: Mahony-Scandarolli navigation algorithm

The navigation problem lays in the determination of an algorithm that possess the ability to determine the set of variables describing the spatial condition of a rigid body using related set of variables which we term *sensory information* (S). In these frame terms the solution as seen through the Mahony-Scandarolli approach derives in the navigation algorithm described in Fig.1 which entails the combination of two nonlinear observers one for attitude and other for position. The measurement of the angular velocity Ω_s and the vectorial reconstruction of the rotation matrix R_y are inputs into the attitude observer which determines:

- $\hat{\Theta}$: Estimation of the Euler angles.
- $\hat{\Omega}$: Estimation of the angular velocity in $\{\mathcal{B}\}$, where $\{\mathcal{B}\}$ denotes the body fixed framework.
- $\tilde{R}$: Rotation matrix error defined as the transformation from $\{\mathcal{E}\} \hookrightarrow \{\mathcal{B}\}$ where $\{\mathcal{E}\}$ denotes the estimation referential framework.
- $\hat{R}$: The estimation of the rotation matrix that also defines a frame transformation $\{\mathcal{E}\} \hookrightarrow \{\mathcal{A}\}$ where $\{\mathcal{A}\}$ denotes the inertial frame.

Finally, the position complementary filter uses the rest of the variables included in S (the measured acceleration a_s and

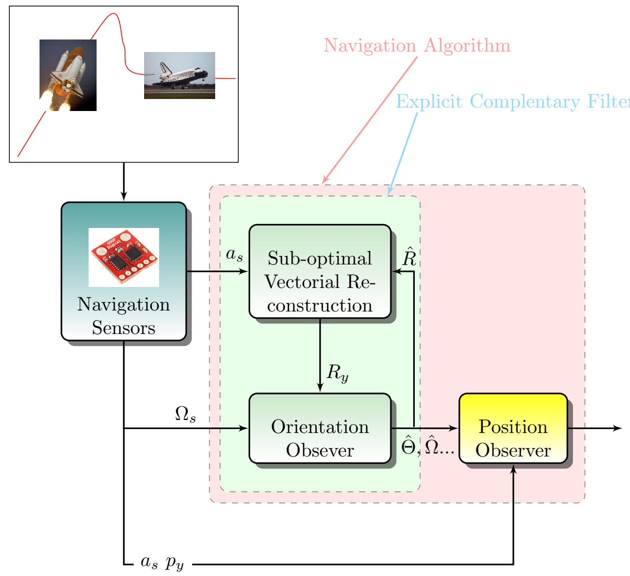


Fig. 1. Simplified scheme of the original navigation algorithm of Mahoni-Scandaroli.
The attitude observer utiliza la medicin de la velocidad angular Ω_s [1] y

position p_y) and the transformation matrices computed by the previous observer; it determines :

- $\hat{p}$: Position estimation in $\{\mathcal{A}\}$.
- $\hat{v}$: Linear velocity estimation in $\{\mathcal{A}\}$.

The group of estimation results are concatenated in what we denote *the Navigation Information Estimation* expressed as $X = [\hat{p}\ \hat{v}\ \hat{\Theta}\ \hat{\Omega}]$.

Under this scheme, the vectorial reconstruction feeds in the orientation observer with the rotation matrix's reconstruction that depends directly of the weighted sum of the vectorial measurements in the body framework $\{\mathcal{B}\}$ (V_i) as well as $\hat{R}$:

$$R_y = \sum_{i=1}^{n} f(v_i, \hat{R}) \tag{1}$$

The main disadvantage in the formulation of the complementary filters on SO(3) is the sensibility to R_y in as much it is involved in the mapping of ω_s to $\{\mathcal{A}\}$, fot that reason the determination of this matrix play a mayor role. They mention that the optimal solution can be determined in the minimization argument of:

$$R^* = \arg \min_R \left\{ \sum_i |V_{0,i} - RV_{m,i}|^2 \right\} \tag{2}$$

Where the optimal solution R^* is function of a loss function defined in the sum of euclidean norms of the errors between the vectorial measurements (R$V_{m,i}$) rotated to $\{\mathcal{A}\}$ with respect to their respective theoretical values ($v_{0,i}$)(Subindexes 0 stays by theoretical and m by measured.). However, Mahony-Scandaroli's algorithm solves R_y through an sub-optimal method depending on $\hat{R}$ so it relies on the Lyapunov statability trajectory. We assume that the optimal solution of R_y could enhance the general performance of the algorithm.

III. Optimal reconstruction of Rotation Matrix

In this section we present the general solution for optimal determination of the rotation matrix that we deem could improve the algorithm's performance. We propose to compute the Rotation Matrix by estimating the inclination of the gravity vector field in $\{\mathcal{B}\}$ with an optimal observer in quaternions in sense that rotation a rigid body in completely described in terms of the Euler angles Θ and the angular velocity Intuitively these given states can be determined through the measurement of: the angular speed Ω_s with a gyroscope and the observation of the attitude Θ with an accelerometer in the declination of the gravity vector. This approach is possible since the rotation matrix can be expressed in terms of the components of a unitary quaternion [19]. Hence, an optimal observer should reduce the deviation measurement of acceleration a_s to its rotated theoretical value g_0 in $\{\mathcal{A}\}$.

Therefore, the optimal observer works to reduce the loss function (3) where R^* depends on unitary quaternion $\breve{q} = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{Q} \subset \mathbb{H}^2$:

$$R^*(\breve{q}) = R\left(q^* = \arg\min_{\breve{q}} \left\{a_s - R^T(\breve{q}) g_0\right\}\right) \tag{3}$$

Thereby the solution is given in terms of the optimal unitary quaternion that rotates g_0 to a_s which is estimated by $\hat{a}_s$ as follows

$$\hat{a}_s = \bar{\breve{q}} \otimes \breve{g}_0 \otimes \breve{q} = R g_0 = g_0 \begin{bmatrix} 2(q_1 q_3 - q_o q_2) \\ 2(q_2 q_3 + q_0 q_1) \\ q_0^2 - q_1^2 - q_2^2 - q_3^2 \end{bmatrix} \tag{4}$$

This allows us to define that the challenge assumed by the observer is to determine the optimal quaternion at the discrete time k considering the following elements: i)the initial conditions uncertainty for the estimated quaternion $\tilde{q}_0$ y and the measurement deviation $\hat{v}_0$, ii) the model uncertainty $\hat{w}_k$ defined in (5)[3] iii) the measurement uncertainty, that defines itself as deviation of $\hat{a}_{s,k}$ to $a_{s,k}$ in the dicrete time k, that also depends on $\hat{\breve{q}}_k$.

$$\hat{\breve{q}}_{k+1} = f_k(\hat{\breve{q}}_k, \Omega_k) + \hat{w}_k \tag{5}$$

$$\hat{v}_k = a_{s,k} - \underbrace{\hat{a}_{s,k}}_{\hat{a}_{s,k} = h_k(\hat{\breve{q}}_k)} \tag{6}$$

The loss function (3) is re-written in terms of two norms in (7) where the first combines $\hat{v}_0$ and $\tilde{q}_0$ y the second combines $\hat{w}_k$ and $\hat{v}_k$ within the interval $i \in \{k_0, ..., k_f\}$ considering k_f current time. Hereby we propose the solution of the optimization problem finding the next $\hat{\breve{q}}_{k+1}$ in terms of successive corrections of $\hat{w}_k$ that can be obtained from dynamical programming approach [20].

$$q_k^* = \arg \min_{q_k} \left\{ norm_0(\tilde{q}_0, \hat{v}_0) + \sum_{i=0}^{k} norm_1(w_i, v_i) \right\} \tag{7}$$

IV. Experimental evaluation Methodology

In general, the experimental evaluation aims to evaluate reduction of estimation errors by including the optimal observer stage into the original navigation algorithm, that results in what we name the modified navigation algorithm. We studied each case by providing the same input and collecting the results in $\hat{X}_{mdf}$ for the modified algorithm and $\hat{X}_{mh}$ for the original algorithm these are compared with a reference measurement X_{real} obtained from an external source. By means of simplicity we reduce the whole experiment into two different study cases:

1) Estimation of Euler angles for independent axis. In this case the rotations were performed for one particular axis at the time, the measurements were simultaneously introduced to both navigation algorithms, and the reference was obtained with a camera that tracked the trajectory of two black marks placed on the box containing the navigation sensors.
2) Estimation of p osition in a closed trajectory in a urban terrain. In this case the experiments were conducted along long closed tracks in a vehicle were we mount the sensors and a commercial GPS tracker, the former was used as input for the navigation algorithms and the latter as reference signal.

V. Experimental results

We present a sample of many experiments in Fig.2a for the estimation of the Euler angles, where we observe above the errors moving around within the range $9.04° \cdot 10^{-3}$ to $10.58°$ for the modified approach in contrast to $7.33° \cdot 10^{-3}$ to 18.97 for the original algorithm. In the second case angles rates are within $\pm 45[/s]$ range, and also for this case we observe that the modified algorithm catches the reference nearer that original after the 13 seconds. Finally, for the third case we experiment with abrupt changes of direction, in addition the gravity vector is parallel to rotating axis what difficulties the vectorial reconstruction. In this graph the error is progressively reduced by the modified algorithm in contrast to the original algorithm which seems to remain in 1.5 along at least for the time window that encloses this particular essay.

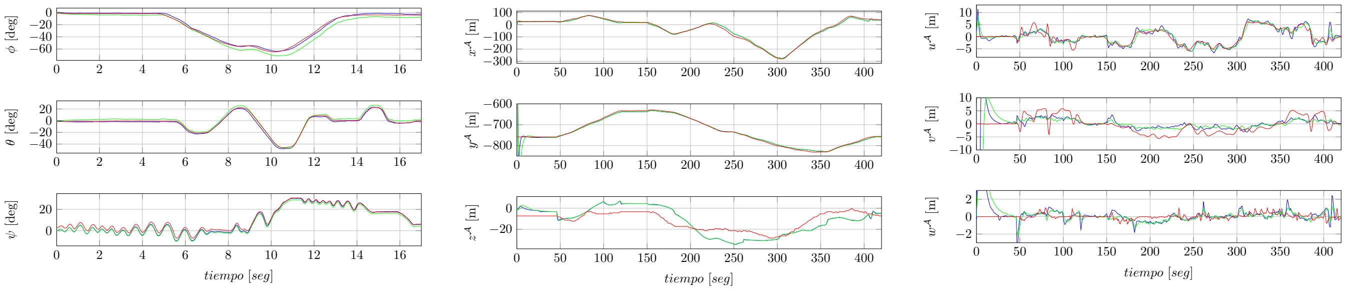
The results in the case of the linear movement are presented in Fig.2b for position and in Fig.2c for velocity. We observed that the errors varies around before the transitory stage $2[m]$ y $10[m]$ in the axis x and y, further in the horizontal axis tend to be bigger above $20[m]$ which typically are bigger. Regarding the linear velocity some spikes in both algorithms but they seem to follow the reference.

VI. Comparative Analysis

In order to compare the results, we used two different metrics: the mean absolute percentage error (MAPE) for

[2] $\mathbb{H}$ denotes the Hamiltonian space of quaternion complex numbers $\mathbb{Q}$ is the sub-space of all quaternions with unitary norm.

[3] f_k represents the quaternion kinematics which defined in terms the quaternion product of the unitary quaternion and the angular velocity expressed as pure quaternion.



(a) Case I (b) Case II (c) Case III

Fig. 2. Simulation results for the network.

TABLE I
MODIFIED AND ORIGINAL ALGORITHMS COMPARATIVE TABLE.

Estimation	**MAPE modified**	**MAPE original**	**PRD modified vs. original.**
X position (x)	5.8862[m]	5.9427[m]	0.9596%
Y position (y)	3.5445[m]	4.7739[m]	34.6828%
Z position (z)	6.1698[m]	6.3369[m]	2.7080%
X-velocity (v_x)	0.8214[m/s]	0.7871[m/s]	-12.1305%
Y-velocity (v_y)	3.1564[m/s]	3.3291[m/s]	5.4710%
Z-velocity (v_z)	0.4445[m/s]	0.5697[m/s]	28.1664%
Pitching (ϕ)	1.3579 [°]	3.7560 [°]	177.97%
Rolling (θ)	1.4635 [°]	2.801 [°]	91.394%
Yawing (ψ)	1.3268 [°]	2.1855 [°]	44.53%

each variable's performance , and the relative difference in percentage (PRD) to compare results between algorithms. The results are arrange in the table I where we identify a better performance for the modified algorithm compared to the original version. The mean of all PRDs (MPRD) reinforces this statement placing itself in 40.92%, namely we obtained an average 42.92% better performance by including a optimal pre-computing of R^* compared with the original approach. In addition, notice that best relations within the MAPE pairs rest on y i.e. $1.22[m]$ MAPE and the pitch and roll angles i.e. 2.39 and 1.73 MAPE respectively.

We also perform a processing time comparison where we measure the times to process samples sets of 2957 points within a $49.4311[s]$ time window. We obtained and average time for ten essays of $1.0158[ms]$ for the modified algorithm in contrast to $0.7954[ms]$ for the original algorithm, i.e a MPRD:

$$MPRD_{\texttt{processing time}} = -21.6933\% \tag{8}$$

This expression points out that the original algorithm uses 21.6933% less time than the modified.

VII. Conclusion

In this work we proposed a observer architecture composed by non-linear complementary filters on the $SO(3)$ and an optimal observer of the rotation matrix, this considers the signals coming from an inertial measurement unit (IMU) and a GPS receptor as inputs. In addition, we carried out an experimental exploration of the properties of both algorithms in the same conditions in order to obtain comparable information. Based on the experimental results we deem important to point out that the navigation algorithms including an optimal estimation of the rotation matrix does better in estimation than the algorithm computing the rotation matrix with sub-optimal vectorial reconstruction, however such modification involves a considerable increment of complexity.

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